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A DUAL-RECIPROCITY BOUNDARY ELEMENT METHOD FOR ANISOTROPIC HEAT CONDUCTION IN INHOMOGENEOUS SOLIDS

W. T. Ang and K. K. Choo

Division of Engineering Mechanics
School of Mechanical and Production Engineering
Nanyang Technological University
Republic of Singapore
e-mail: mwtang@ntu.edu.sg, web page: http://www.ntu.edu.sg/home/mwtang/

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Abstract. *A dual-reciprocity boundary element method is presented for the numerical solution of a thermal problem involving non-steady two-dimensional anisotropic heat conduction in inhomogeneous solids. Certain terms containing the temperature and the heat flux on the boundary of the solid are approximated using discontinuous linear elements. The method formulates the problem in terms of an initial value problem governed by a system of first-order linear ordinary differential equations. The unknown functions (of time) in the differential equations are given by either the unknown temperature or heat flux at boundary nodal points and also the unknown temperature at selected interior nodal points. To reduce the differential equations to a system of linear algebraic equations, each of the unknown functions giving the temperature at the nodal points is approximated using a polynomial function in time over a small time interval. Once the linear algebraic equations are solved, the temperature can be determined at any desired point in the interior of the solution domain. Numerical results for a specific test problem are given.*

1 INTRODUCTION

According to the classical theory of heat conduction, the non-steady two-dimensional anisotropic flow of heat energy in a solid is governed by the parabolic partial differential equation

$$\frac{\partial}{\partial x_i}(k_{ij}\frac{\partial T}{\partial x_j}) = \rho c\frac{\partial T}{\partial t}, \tag{1}$$

where T is the temperature which depends on the Cartesian co-ordinates x_1 and x_2 and time t, k_{ij} are heat conductivity coefficients such that the symmetry relation $k_{12} = k_{21}$ is satisfied and the strict inequality $(k_{12})^2 - k_{11}k_{22} < 0$ holds at all points in the solid, and ρ and c are respectively the density and the specific heat capacity of the solid. The Einsteinian convention of summing over a repeated index holds for latin subscripts from 1 to 2.

Equation (1) is to be solved in a two-dimensional region R bounded by a simple closed curve C subject to the initial-boundary conditions

$$\begin{aligned} &T(x_1,x_2,0) = f(x_1,x_2) \text{ for } (x_1,x_2)\in R\cup C,\\ &T(x_1,x_2,t) = p(x_1,x_2,t) \text{ for } (x_1,x_2)\in C_1 \text{ and } t>0,\\ &q(x_1,x_2,t) = h(x_1,x_2,t) \text{ for } (x_1,x_2)\in C_2 \text{ and } t>0, \end{aligned} \tag{2}$$

where f, p and h are suitably given functions, C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$ and q is the heat flux defined by $q = -k_{ij}n_i\partial T/\partial x_j$ with $[n_1, n_2]$ being the components of the unit normal vector to C pointing away from R.

The boundary element method for the numerical solution of (1) and (2) for the special case in which the temperature is time-independent and k_{ij} are constants (i.e. the case of steady-state heat conduction in homogeneous solids) is well established, see e.g. Clements [1]. In general, even for the case of steady-state heat conduction, if the coefficients k_{ij} vary spatially (e.g. as in functionally graded materials), it is mathematically difficult to derive a suitable fundamental solution which can be employed to obtain a (strictly) boundary integral formulation for (1). If the fundamental solution for the steady-state two-dimensional heat conduction in homogeneous solids is used instead, the resulting integral formulation includes not only a boundary integral but also a domain integral. To deal with the domain integral in an effective manner or to obtain alternative formulations that do not require the solution domain to be discretized, various approaches were proposed for inhomogeneous isotropic media, see e.g. Clements [2], Ang, Kusuma and Clements [3], Kassab and Divo [4], Park and Ang [5] and Tanaka, Matsumoto and Suda [6].

In the present paper, we consider the task of solving (1) and (2) for the case in which thermal conductivity coefficients take the rather general form

$$k_{ij}(x_1,x_2) = k_{ij}^{(0)} g(x_1,x_2), \tag{3}$$

where g is a given positive (grading) function that can be partially differentiated at least twice with respect to x_1 and $k_{ij}^{(0)}$ are non-negative constants satisfying $k_{ij}^{(0)} = k_{ji}^{(0)}$ and $(k_{12}^{(0)})^2 - k_{11}^{(0)}k_{22}^{(0)} < 0$. The corresponding steady-state problem was recently solved by Ang, Clements and Vahdati [7] using a dual-reciprocity boundary element method. Following closely the work in [7], we employ the fundamental solution for the steady-state two-dimensional heat conduction in a homogeneous anisotropic solid, which takes the form of a simple logarithmic function, to derive an integro-differential formulation for the partial differential equation (1). With such a fundamental solution, the formulation inevitably contains a domain integral over the region R. The dual-reciprocity method proposed by Brebbia and Nardini [8] is applied to convert the domain integral into a line integral approximately. The method requires us to collocate at points in $R\cup C$, but the discretization of the region R into tiny elements is not needed. Thus, only the curve boundary C has to be discretized.

If the boundary C is discretized into elements and certain terms containing the boundary temperature and heat flux approximated as spatially linear functions over the elements, the integro-differential equation can be used to reformulate (1) and (2) together with (3) as an initial value problem governed by a system of first-order linear ordinary differential equations. The unknown functions (of time) in the differential equations are given by either the unknown temperature or heat flux at boundary nodal points and also the unknown temperature at interior nodal points. To reduce the differential equations to a system of linear algebraic equations, each of the unknown functions giving the temperature at the nodal points is approximated using a polynomial function in time over a small time interval. Once the linear algebraic equations are solved, the temperature can be determined at any desired point in the interior of the solution domain. Numerical results for a specific problem solved using the dual-reciprocity boundary element method described here are presented.

2 INTEGRO-DIFFERENTIAL EQUATION

Proceeding as in Ang, Clements and Vahdati [7], we find that (1) together with (3) gives rise to the integro-differential equation

$$\begin{aligned}
&\gamma(\xi_1,\xi_2)\sqrt{g(\xi_1,\xi_2)}T(\xi_1,\xi_2,t) \\
&= \iint_R \left\{ \kappa(x_1,x_2)T(x_1,x_2,t) + \frac{\rho c}{\sqrt{g(x_1,x_2)}} \frac{\partial}{\partial t}\left[T(x_1,x_2,t)\right] \right\} \Phi(x_1,x_2,\xi_1,\xi_2)\,dx_1 dx_2 \\
&+ \int_C \left\{ \Gamma(x_1,x_2,\xi_1,\xi_2)\sqrt{g(x_1,x_2)}T(x_1,x_2,t) \right. \\
&\left. - \left[k_{ij}^{(0)} n_i(x_1,x_2)T(x_1,x_2,t)\frac{\partial}{\partial x_j}\sqrt{g(x_1,x_2)} - \frac{q(x_1,x_2,t)}{\sqrt{g(x_1,x_2)}} \right] \Phi(x_1,x_2,\xi_1,\xi_2) \right\} ds(x_1,x_2),
\end{aligned} \tag{4}$$

where $\gamma(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R\cup C$, $\gamma(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R$, $0 < \gamma(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C$ [$\gamma(\xi_1,\xi_2) = 1/2$ if (ξ_1,ξ_2) lies on a smooth part of C] and

$$\Phi(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)} - (k_{12}^{(0)})^2}} \operatorname{Re}\{\ln[(x_1-\xi_1)+\tau(x_2-\xi_2)]\},$$

$$\Gamma(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)} - (k_{12}^{(0)})^2}} \operatorname{Re}\left[\frac{(k_{11}^{(0)}+\tau k_{12}^{(0)})n_1(x_1,x_2) + (k_{21}^{(0)}+\tau k_{22}^{(0)})n_2(x_1,x_2)}{(x_1-\xi_1)+\tau(x_2-\xi_2)} \right],$$

$$\tau=\frac{-k_{12}^{(0)}+i\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}{k_{22}^{(0)}}\quad (i=\sqrt{-1}),\quad \kappa=k_{ij}^{(0)}\frac{\partial^2}{\partial x_i x_j}\left(\sqrt{g(x_1,x_2)}\right). \tag{5}$$

3 DISCONTINUOUS LINEAR ELEMENTS

The boundary C is discretized into N straight line elements denoted by $C^{(1)}$, $C^{(2)}$, … , $C^{(N-1)}$ and $C^{(N)}$. The starting and ending points of the line element $C^{(m)}$ are given by $(a_1^{(m)},a_2^{(m)})$ and $(b_1^{(m)},b_2^{(m)})$ respectively.

For an accurate evaluation of the line integral in (4) alter the discretization of C into the straight line elements, discontinuous linear boundary elements are to be employed. Some details on the implementation of such boundary elements may be found in Paris and Canas [9]

For the discontinuous linear boundary elements, two points $(\eta_1^{(m)},\eta_2^{(m)})$ and $(\eta_1^{(N+m)},\eta_2^{(N+m)})$ on $C^{(m)}$ are chosen as follows:

$$\left.\begin{aligned}\eta_i^{(m)}&=a_i^{(m)}+r_0(b_i^{(m)}-a_i^{(m)})\\ \eta_i^{(N+m)}&=b_i^{(m)}-r_0(b_i^{(m)}-a_i^{(m)})\end{aligned}\right\}\text{ for a given } r_0\in(0,\frac{1}{2}). \tag{6}$$

After C is discretized into line elements, if the temperature T has values $T^{(m)}(t)$ and $T^{(N+m)}(t)$ at $(\eta_1^{(m)},\eta_2^{(m)})$ and $(\eta_1^{(N+m)},\eta_2^{(N+m)})$ respectively, the terms $\sqrt{g(x_1,x_2)}T(x_1,x_2,t)$ and $T(x_1,x_2,t)\frac{\partial}{\partial x_j}\sqrt{g(x_1,x_2)}$ on the third and fourth lines of (4) respectively are *separately* approximated to vary linearly across each of the line elements according to

$$\begin{aligned}\sqrt{g(x_1,x_2)}T(x_1,x_2,t)\approx{}&\sqrt{g(\eta_1^{(m)},\eta_2^{(m)})}T^{(m)}(t)[1-d^{(m)}(x_1,x_2)]\\ &+\sqrt{g(\eta_1^{(N+m)},\eta_2^{(N+m)})}T^{(N+m)}(t)d^{(m)}(x_1,x_2)\quad\text{for }(x_1,x_2)\in C^{(m)},\end{aligned}$$

$$\begin{aligned}T(x_1,x_2,t)\frac{\partial}{\partial x_j}\sqrt{g(x_1,x_2)}\approx{}&\left.\frac{\partial}{\partial x_j}\sqrt{g(x_1,x_2)}\right|_{(x_1,x_2)=(\eta_1^{(m)},\eta_2^{(m)})}T^{(m)}(t)[1-d^{(m)}(x_1,x_2)]\\ &+\left.\frac{\partial}{\partial x_j}\sqrt{g(x_1,x_2)}\right|_{(x_1,x_2)=(\eta_1^{(N+m)},\eta_2^{(N+m)})}T^{(N+m)}(t)d^{(m)}(x_1,x_2)\quad\text{for }(x_1,x_2)\in C^{(m)},\end{aligned} \tag{7}$$

where

$$d^{(m)}(x_1,x_2)=\frac{\sqrt{(x_1-a_1^{(m)})^2+(x_2-a_2^{(m)})^2}-r_0\ell^{(m)}}{(1-2r_0)\ell^{(m)}}, \tag{8}$$

with $\ell^{(m)}$ being the length of the line element $C^{(m)}$.

Similarly, for the heat flux q, if its values are given by $q^{(m)}(t)$ and $q^{(N+m)}(t)$ at $(\eta_1^{(m)},\eta_2^{(m)})$ and $(\eta_1^{(N+m)},\eta_2^{(N+m)})$ respectively, then the term $\frac{1}{\sqrt{g(x_1,x_2)}}q(x_1,x_2,t)$ on the fourth line of (4) is approximated to vary linearly across each line element as

$$\begin{aligned}\frac{1}{\sqrt{g(x_1,x_2)}}q(x_1,x_2,t)\approx{}&\frac{1}{\sqrt{g(\eta_1^{(m)},\eta_1^{(m)})}}q^{(m)}(t)[1-d^{(m)}(x_1,x_2)]\\ &+\frac{1}{\sqrt{g(\eta_1^{(N+m)},\eta_1^{(N+m)})}}q^{(N+m)}(t)d^{(m)}(x_1,x_2)\quad\text{for }(x_1,x_2)\in C^{(m)}.\end{aligned} \tag{9}$$

On $C^{(m)}$, either T or q is specified. It follows that, for a given m, either $(T^{(m)}(t), T^{(N+m)}(t))$ or $(q^{(m)}(t), q^{(N+m)}(t))$ is known. This gives rise to $2N$ unknown functions at the boundary nodes $(\eta_1^{(m)},\eta_2^{(m)})$ and

$(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ (for m = 1,2,…,N). The approximations in (7) and (9) for the appropriate terms on the third and the fourth lines of (4) are to be substituted into the line integral after C is replaced by the line elements. The calculation of the coefficients of $T^{(n)}(t)$ and $q^{(n)}(t)$ (n = 1, 2, … , 2N) in the resulting approximate expression for the line integral involves integrations over the line elements. The integrations over the elements can be done analytically using the exact formulae given in Ang [10].

4 DUAL-RECIPROCITY METHOD

To apply the dual-reciprocity method to transform the domain integral in (4) into a line integral, we first make the approximation

$$\kappa(x_1,x_2)T(x_1,x_2,t)+\frac{\rho c}{\sqrt{g(x_1,x_2)}}\frac{\partial}{\partial t}\left[T(x_1,x_2,t)\right]\approx\sum_{p=1}^{2N+L}a^{(p)}\sigma^{(p)}(x_1,x_2)\,, \tag{10}$$

where $a^{(p)}$ are constants to be determined and

$$\begin{aligned}\sigma^{(p)}(x_1,x_2)=1&+\left(\left[x_1-\eta_1^{(p)}+\mathrm{Re}(\tau)(x_2-\eta_2^{(p)})\right]^2+\left[\mathrm{Im}(\tau)(x_2-\eta_2^{(p)})\right]^2\right)\\&+\left(\left[x_1-\eta_1^{(p)}+\mathrm{Re}(\tau)(x_2-\eta_2^{(p)})\right]^2+\left[\mathrm{Im}(\tau)(x_2-\eta_2^{(p)})\right]^2\right)^{3/2},\end{aligned} \tag{11}$$

where $(\eta_1^{(1)},\eta_2^{(1)})$, $(\eta_1^{(2)},\eta_2^{(2)})$, … , $(\eta_1^{(2N-1)},\eta_2^{(2N-1)})$ and $(\eta_1^{(2N)},\eta_2^{(2N)})$ are the 2N points on the boundary elements as defined by (6) and $(\eta_1^{(2N+1)},\eta_2^{(2N+1)})$, $(\eta_1^{(2N+2)},\eta_2^{(2N+2)})$, … , $(\eta_1^{(2N+L-1)},\eta_2^{(2N+L-1)})$ and $(\eta_1^{(2N+L)},\eta_2^{(2N+L)})$ are L selected points in the interior of R.

We can let (x_1, x_2) in (10) be given by $(\eta_1^{(m)},\eta_2^{(m)})$ for m = 1, 2, … , 2N+L, to set up a system of linear algebraic equations in $a^{(p)}$. The algebraic equations can then be inverted to obtain

$$a^{(p)}=\sum_{m=1}^{2N+L}\left\{\kappa(\eta_2^{(m)},\eta_2^{(m)})T^{(m)}(t)+\frac{\rho c}{\sqrt{g(\eta_2^{(m)},\eta_2^{(m)})}}\frac{d}{dt}\left[T^{(m)}(t)\right]\right\}\chi^{(mp)}, \tag{12}$$

where $T^{(m)}=T(\xi_1^{(1)},\xi_2^{(1)},t)$ (m = 1, 2, … , 2N+L) and $\chi^{(mp)}$ are constants defined by

$$\sum_{m=1}^{2N+L}\sigma^{(p)}(\eta_2^{(m)},\eta_2^{(m)})\chi^{(mr)}=\begin{cases}1 & \text{if } p=r,\\ 0 & \text{if } p\neq r.\end{cases} \tag{13}$$

Using (10) and (12) and applying the dual-reciprocity method, we find that the double integral in (4) can be approximately re-written as

$$\begin{aligned}&\iint_R\left\{\kappa(x_1,x_2)T(x_1,x_2,t)+\frac{\rho c}{\sqrt{g(x_1,x_2)}}\frac{\partial}{\partial t}\left[T(x_1,x_2,t)\right]\right\}\Phi(x_1,x_2,\xi_1,\xi_2)dx_1dx_2\\&\approx\sum_{m=1}^{N+L}\left\{\kappa(\eta_2^{(m)},\eta_2^{(m)})T^{(m)}(t)+\frac{\rho c}{\sqrt{g(\eta_2^{(m)},\eta_2^{(m)})}}\frac{d}{dt}\left[T^{(m)}(t)\right]\right\}\sum_{p=1}^{N+L}\chi^{(mp)}\Psi^{(p)}(\xi_1,\xi_2),\end{aligned} \tag{14}$$

where

$$\begin{aligned}&\Psi^{(p)}(\xi_1,\xi_2)\\&=\gamma(\xi_1,\xi_2)\theta^{(p)}(\xi_1,\xi_2)+\int_C\left[\Phi(x_1,x_2,\xi_1,\xi_2)\beta^{(p)}(x_1,x_2)-\Gamma(x_1,x_2,\xi_1,\xi_2)\theta^{(p)}(x_1,x_2)\right]ds(x_1,x_2)\end{aligned} \tag{15}$$

with

$$\left(\frac{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}{k_{22}^{(0)}}\right)\theta^{(p)}(x_1,x_2)$$
$$=\frac{1}{4}\left(\left[(x_1-\eta_1^{(p)})+\mathrm{Re}(\tau)\,(x_2-\eta_2^{(p)})\right]^2+\left[\mathrm{Im}(\tau)(x_2-\eta_2^{(p)})\right]^2\right)$$
$$+\frac{1}{16}\left(\left[(x_1-\eta_1^{(p)})+\mathrm{Re}(\tau)\,(x_2-\eta_2^{(p)})\right]^2+\left[\mathrm{Im}(\tau)(x_2-\eta_2^{(p)})\right]^2\right)^2 \tag{16}$$
$$+\frac{1}{25}\left(\left[(x_1-\eta_1^{(p)})+\mathrm{Re}(\tau)\,(x_2-\eta_2^{(p)})\right]^2+\left[\mathrm{Im}(\tau)(x_2-\eta_2^{(p)})\right]^2\right)^{5/2}$$

and

$$\beta^{(p)}(x_1,x_2)=k_{ij}^{(0)}n_i(x_1,x_2)\frac{\partial\theta^{(p)}}{\partial x_j}. \tag{17}$$

Note that $\Psi^{(p)}(\xi_1, \xi_2)$ can be evaluated numerically by discretizing C into boundary elements.

The dual-reciprocity method gives rise to an additional L unknown functions of time, as given by the temperature at the L interior points, namely $T^{(n)}(t)$ for $n = 2N+1, 2N+2, \ldots, 2N+L$. As pointed out in the preceding section, with the discontinuous linear elements, there are $2N$ unknown functions at points on the boundary. Thus, if we choose (ξ_1,ξ_2) in (4) to be given in turn by $(\eta_1^{(p)},\eta_2^{(p)})$ for $p = 1, 2, \ldots, 2N+L$, the dual-reciprocity boundary element method here approximately reduces the integro-differential equation into a system of $2N+L$ first order linear ordinary differential equations to be solved subject to known values of $T^{(n)}(t)$ at $t = 0$ as given by the initial temperature in (2).

5 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

To solve the system of ordinary differential equations, we approximate the nodal temperature $T^{(n)}(t)$ ($n = 1, 2, \ldots, 2N+L$) as an M-th order polynomial function of t over the interval $t_0 < t < t_0+M\Delta t$, where Δt is a small positive number. Specifically,

$$T^{(n)}(t)\approx\sum_{\ell=1}^{M+1}\left(T^{(n)}(t^{(\ell)})\frac{\prod_{j=1,j\neq\ell}^{M+1}(t-t^{(j)})}{\prod_{k=1,k\neq\ell}^{M+1}(t^{(\ell)}-t^{(k)})}\right), \tag{18}$$

where $t^{(\ell)} = t_0 + (\ell - 1)\Delta t$ (for $\ell = 1, 2, \ldots, M+1$).

Differentiating (18) with respect to t, we obtain:

$$\frac{d}{dt}\left[T^{(n)}(t)\right]\approx\sum_{\ell=1}^{M+1}\left(T^{(n)}(t^{(\ell)})\frac{\sum_{m=1,\,m\neq\ell}^{M+1}\prod_{j=1,\,j\neq\ell,j\neq m}^{M+1}(t-t^{(j)})}{\prod_{k=1,k\neq\ell}^{M+1}(t^{(\ell)}-t^{(k)})}\right). \tag{19}$$

If (18)-(19) is substituted into the system of ordinary differential equations and if t (in the system of differential equations) is chosen to be given in turn by $t^{(1)}, t^{(2)}, \ldots, t^{(M)}$ and $t^{(M+1)}$, we obtain a system of linear algebraic equations containing $T^{(n)}(t^{(\ell)})$ (for $n = 1, 2, \ldots, 2N+L$ and $\ell = 1, 2, \ldots, M+1$) and $q^{(m)}(t^{(\ell)})$ (for $m = 1, 2, \ldots, 2N$ and $\ell = 1, 2, \ldots, M+1$). If we let $t_0 = 0, M\Delta t, 2M\Delta t, \ldots$, the linear algebraic equations can be solved together with the initial-boundary conditions (2) for the unknown thermal quantities at different time levels. For example, when $t_0 = 0$, we can solve for the unknown quantities at the time levels $t = \Delta t, 2\Delta t, \ldots, M\Delta t$, making use of the known temperature (hence the known boundary heat flux) at $t=0$. If we press on with $t_0 = M\Delta t$, we can find the solution at the time levels $t = (M+1)\Delta t, (M+2)\Delta t, \ldots, 2M\Delta t$, using the just determined temperature at $t = M\Delta t$. The task can be repeated for $t_0 = 2M\Delta t, 3M\Delta t, \ldots$, in order to determine the temperature at higher and higher time levels.

6 A TEST PROBLEM

Consider the partial differential equation (1) with

$$\frac{1}{2}k_{11} = k_{12} = k_{21} = \frac{2}{3}k_{22} = 3 + \cos^2(\frac{x_1}{2} + \frac{x_2}{3}),$$
$$\rho c = 3\ [1 + \cos^2(\frac{x_1}{2} + \frac{x_2}{3})]. \tag{20}$$

For a specific problem governed by the partial differential equation (1) with coefficients given by (20), we take the solution domain to be the square region $0 < x_1 < 1$, $0 < x_2 < 1$.

We use a particular solution of the partial differential equation, the one given by

$$T(x_1, x_2, t) = x_1 - \frac{4}{3}x_2 + 2 + \exp(-t)\cos(\frac{x_1}{2} + \frac{x_2}{3})\ , \tag{21}$$

to generate the initial temperature $T(x_1, x_2, 0)$ at all points in the solution domain and the temperature and heat flux on respectively the vertical and the horizontal sides of the square. The dual-reciprocity boundary element method is applied to solve (1) with coefficients given by (20) inside the square domain subject to the initial-boundary conditions generated using (21). If the dual-reciprocity boundary element method really works, it should be able to recover approximately the temperature given by (21).

Table 1

t	Exact	DRBEM $(N, L) = (20, 9)$, $r_0 = 0.22$, $\Delta t = 0.10$		
		$M = 1$	$M = 2$	$M = 3$
0.05	2.113752	2.112520	2.113648	2.113709
0.35	2.006515	2.005921	2.006497	2.006444
0.65	1.927072	1.926664	1.926982	1.927012
0.95	1.868219	1.867913	1.868196	1.868167
1.25	1.824620	1.824386	1.824557	1.824574
1.55	1.792321	1.792140	1.792295	1.792279
1.85	1.768393	1.768252	1.768346	1.768355
2.05	1.755995	1.755874	1.755951	1.755959
2.45	1.737535	1.737444	1.737496	1.737501
2.85	1.725160	1.725090	1.725125	1.725129
3.25	1.716865	1.716809	1.716832	1.716835
3.65	1.711305	1.711258	1.711274	1.711275
4.05	1.707578	1.707537	1.707548	1.707549
4.45	1.705080	1.705043	1.705050	1.705051
4.85	1.703405	1.703371	1.703376	1.703376
5.25	1.702282	1.702250	1.702253	1.702254
5.65	1.701530	1.701499	1.701501	1.701501
6.05	1.701026	1.700995	1.700997	1.700997
Norm of error		3.55795×10^{-4}	4.62594×10^{-5}	3.99648×10^{-5}

To obtain some numerical results, we firstly discretize the square boundary into 20 boundary elements and select 9 well-spaced out collocation points in the interior of the solution domain, that is, $(N, L) = (20, 9)$. In addition, the parameter r_0 in (6) and Δt are taken to be 0.22 and 0.10 respectively. In Table 1, the numerical values of the temperature at the point (0.50, 0.60) as obtained by using $M = 1$, 2 and 3 (that is, by approximating $T^{(n)}(t)$ respectively as a linear, a quadratic, and a cubic function of time) are compared with the exact solution given by (21) at various time levels. The norm of error of the numerical values in each column of the table is also computed and given in the last row. In Table 2, the results obtained by using a finer discretization of the boundary and more interior collocation points with $(N, L) = (40, 16)$ are presented. From the norm of the error, it is obvious that there is an improvement in the numerical results in Table 2 over those in Table 1. The results obtained using $M = 3$ (cubic approximation of the nodal temperature) is also significantly more accurate than those from $M = 1$ (linear approximation).

Table 2

t	Exact	DRBEM $(N, L) = (40, 16)$, $r_0 = 0.22$, $\Delta t = 0.10$		
		$M = 1$	$M = 2$	$M = 3$
0.05	2.113752	2.112553	2.113678	2.113738
0.35	2.006515	2.005965	2.006545	2.006492
0.65	1.927072	1.926703	1.927023	1.927053
0.95	1.868219	1.867947	1.868232	1.868204
1.25	1.824620	1.824417	1.824590	1.824607
1.55	1.792321	1.792169	1.792325	1.792309
1.85	1.768393	1.768279	1.768374	1.768383
2.05	1.755995	1.755901	1.755979	1.755986
2.45	1.737535	1.737470	1.737522	1.737527
2.85	1.725160	1.725115	1.725149	1.725153
3.25	1.716865	1.716833	1.716856	1.716859
3.65	1.711305	1.711281	1.711297	1.711299
4.05	1.707578	1.707560	1.707571	1.707572
4.45	1.705080	1.705066	1.705073	1.705073
4.85	1.703405	1.703394	1.703398	1.703399
5.25	1.702282	1.702273	1.702276	1.702276
5.65	1.701530	1.701522	1.701524	1.701524
6.05	1.701026	1.701018	1.701019	1.701020
Norm of error		3.36986×10^{-4}	2.49179×10^{-5}	1.10779×10^{-5}

Table 3

t	Exact	DRBEM $(N, L) = (20, 9)$, $r_0 = 0.22$, $\Delta t = 0.10$		
		$M = 4$	$M = 5$	$M = 7$
0.05	2.113752	2.113714	2.113714	2.113715
0.35	2.006515	2.006448	2.006447	2.006447
0.65	1.927072	1.927014	1.927014	1.927014
0.95	1.868219	1.868169	1.868169	1.868169
1.25	1.824620	1.824575	1.824575	1.824575
1.55	1.792321	1.792280	1.792280	1.792280
1.85	1.768393	1.768355	1.768355	1.768355
2.05	1.755995	1.755959	1.755959	1.755959
2.45	1.737535	1.737501	1.737501	1.737501
2.85	1.725160	1.725128	1.725128	1.725128
3.25	1.716865	1.716835	1.716835	1.716835
3.65	1.711305	1.711275	1.711275	1.711275
4.05	1.707578	1.707549	1.707549	1.707549
4.45	1.705080	1.705051	1.705051	1.705051
4.85	1.703405	1.703376	1.703376	1.703376
5.25	1.702282	1.702254	1.702254	1.702254
5.65	1.701530	1.701501	1.701501	1.701501
6.05	1.701026	1.700997	1.700997	1.700997
Norm of error		3.89354×10^{-5}	3.89104×10^{-5}	3.88801×10^{-5}

For a further comparison of the numerical values of the temperature at (0.50, 0.60) with the exact solution, additional results are obtained by using M=4, 5 and 7 (with (N, L) = (20,9) and (40, 16) as before). Refer to Tables 3 and 4. From the tables, it is obvious that more accurate numerical values can be obtained using a higher value of M. However, there is only a very slight improvement in the accuracy if M is increased beyond a particular value. As shown in Tables 3 and 4, the reduction in the norm of error is not very significant when the value of M is increased from 4 to 7.

Table 4

t	Exact	DRBEM $(N, L) = (40, 16)$, $r_0 = 0.22$, $\Delta t = 0.10$		
		$M = 4$	$M = 5$	$M = 7$
0.05	2.113752	2.11374182	2.11374235	2.11374256
0.35	2.006515	2.00649506	2.00649490	2.00649489
0.65	1.927072	1.92705519	1.92705522	1.92705526
0.95	1.868219	1.86820517	1.86820504	1.86820505
1.25	1.824620	1.82460772	1.82460781	1.82460781
1.55	1.792321	1.79231021	1.79231017	1.79231018
1.85	1.768393	1.76838350	1.76838350	1.76838350
2.05	1.755995	1.75598654	1.75598658	1.75598658
2.45	1.737535	1.73752692	1.73752694	1.73752694
2.85	1.725160	1.72515306	1.72515308	1.72515308
3.25	1.716865	1.71685862	1.71685863	1.71685863
3.65	1.711305	1.71129868	1.71129869	1.71129869
4.05	1.707578	1.70757175	1.70757176	1.70757176
4.45	1.705080	1.70507351	1.70507352	1.70507352
4.85	1.703405	1.70339889	1.70339889	1.70339889
5.25	1.702282	1.70227636	1.70227636	1.70227636
5.65	1.701530	1.70152391	1.70152391	1.70152391
6.05	1.701026	1.70101952	1.70101952	1.70101952
Norm of error		1.00532×10^{-5}	1.00401×10^{-5}	1.00248×10^{-5}

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