A boundary integral method for the anti-plane harmonic vibration of an electrically loaded ceramic slab: application to piezoelectric transformers

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Abstract

A boundary integral method for analysing the anti-plane harmonic vibration of a ceramic slab which is subject to an electrical load is outlined here. The problem under consideration has application in the analysis of piezoelectric transformers. Quantities of practical interest, such as the transforming ratio and the efficiency of the ceramic slab as a piezoelectric transformer, are computed.

Keywords: boundary integral method, polarised ceramic slab, harmonic vibrations, piezoelectric transformers

1. Introduction

Of interest here is the analysis of the anti-plane harmonic vibration of a piezoelectric slab of a polarised ceramic which is subject to an electrical load.

With reference to a Cartesian coordinate system $Ox_1x_2x_3$, the piezoelectric slab occupies the region given by $0 < x_1 < \ell$, $-h < x_2 < h$, $-\infty < x_3 < \infty$, where ℓ and h are positive real numbers. The electrical poling direction of the ceramic is along the x_3 direction. All the sides of the slab are traction-free. Certain parts of the boundary of the slab are electroded as shown in Fig. 1. More precisely, the parts $0 < x_1 < a$, $x_2 = \pm h$ are occupied by input electrodes with voltages $\pm V_{\text{in}} \exp(i\omega t)$ ($i^2 = -1$), while the parts $b < x_1 < \ell$, $x_2 = \pm h$ are covered by output electrodes with voltages $\mp V_{\text{out}} \exp(i\omega t)$, where t denotes time, V_{in} and V_{out} are constants and ω is the driving frequency of the input voltage. The output electrodes are connected by an electric circuit with impedance Z. The parts $a < x_1 < b$, $x_2 = \pm h$ and the vertical sides of the slab are unelectroded. Note that $a < b$.

Under the input voltage $V_{\text{in}} \exp(i\omega t)$, the slab is driven into an anti-plane harmonic vibration such that the only non-zero component of its displacement is the one along the x_3 direction and given by $u_3 = u(x_1, x_2) \exp(i\omega t)$. The electric potential is given by $\phi = \psi(x_1, x_2) \exp(i\omega t)$ and the non-zero x_3 component of the traction on the boundary by $T = q(x_1, x_2)\exp(i\omega t)$.

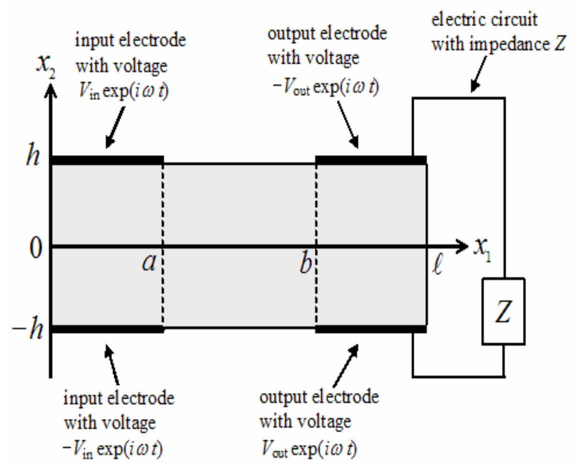

Figure 1: A sketch of the problem

For given V_{in}, ω and Z, the problem is to determine V_{out} and the normal electric displacement $D = p(x_1, x_2) \exp(i\omega t)$ on the electroded parts of the boundary of the slab. Once the problem is solved, quantities of interest, such as the transforming ratio and the efficiency of the piezoelectric slab as a transformer, can be computed.

2. Mathematical formulation

The following non-dimensionalised quantities are defined:

$$x_j^* = \frac{x_j}{\ell}, a^* = \frac{a}{\ell}, h^* = \frac{h}{\ell}, u^*(x_1^*, x_2^*) = \frac{u(x_1, x_2)}{\ell}, \omega^* = \frac{2h\omega}{\pi}\sqrt{\frac{\rho\varepsilon_{11}}{\varepsilon_{11}c_{44} + e_{15}^2}},$$
$$q^*(x_1^*, x_2^*) = \frac{q(x_1, x_2)}{c_{44}}, p^*(x_1^*, x_2^*) = \frac{p(x_1, x_2)}{e_{15}}, \psi^*(x_1^*, x_2^*) = \frac{\varepsilon_{11}\psi(x_1, x_2)}{\ell e_{15}},$$
$$V_{\text{in}}^*(x_1^*, x_2^*) = \frac{\varepsilon_{11}V_{\text{in}}(x_1, x_2)}{\ell e_{15}}, V_{\text{out}}^*(x_1^*, x_2^*) = \frac{\varepsilon_{11}V_{\text{out}}(x_1, x_2)}{\ell e_{15}}, Z^* = \frac{i\omega\varepsilon_{11}\ell Z}{2h},$$

where c_{44} is the shear modulus of the material, e_{15} and ε_{11} are respectively the piezoelectric and dielectric constants and ρ is the density. Henceforth, for convenience, the superscript $*$ for the non-dimensionalised quantities will be omitted.

The problem in Section 1 requires solving for u and ψ from

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial x_j \partial x_j} + \left(\dfrac{\pi\omega}{2h}\right)^2 u = 0 \\ \dfrac{\partial^2}{\partial x_j \partial x_j}(u - \psi) = 0 \end{array}\right\} \text{for } 0 \le x_1 \le 1, 0 \le x_2 \le h, \tag{1}$$

subject to

$$\left.\begin{array}{l} q(0, x_2) = q(1, x_2) = 0 \\ p(0, x_2) = p(1, x_2) = 0 \end{array}\right\} \text{for } 0 < x_2 < h,$$
$$\left.\begin{array}{l} q(x_1, h) = 0 \\ u(x_1, 0) = 0 \\ \psi(x_1, 0) = 0 \end{array}\right\} \text{for } 0 < x_1 < 1,$$
$$\psi(x_1, h) = V_{\text{in}} \text{ for } 0 < x_1 < a,$$
$$\psi(x_1, h) = -V_{\text{out}} \text{ for } b < x_1 < 1,$$
$$p(x_1, h) = 0 \text{ for } a < x_1 < b, \tag{2}$$

and

$$V_{\text{out}} = -Zh\int_b^1 p(x_1, h)\, dx_1. \tag{3}$$

Further details, such as on how s and p are related to u and ψ, may be found in Ref. [2]. The usual convention of summing over latin subscripts which run from 1 to 2 is adopted here.

As V_{out} is not known in Eqn. (2), the additional condition in Eqn. (3) which gives the relation between the output voltage and the current in the circuit connecting the output electrodes is required to complete the formulation of the problem.

3. Boundary integral equations

Equations (1) and (2) give rise to the boundary integral equation (see Ref. [1])

$$\begin{aligned}
&\lambda(\xi_1,\xi_2)u(\xi_1,\xi_2) \\
&= \int_C [u(x_1,x_2)n_j(x_1,x_2)\frac{\partial\Omega(x_1,x_2;\xi_1,\xi_2)}{\partial x_j} \\
&\quad - \Omega(x_1,x_2;\xi_1,\xi_2)\frac{k^2}{1+k^2}p(x_1,x_2)]ds(x_1,x_2), \\
&\lambda(\xi_1,\xi_2)(u(\xi_1,\xi_2)-\psi(\xi_1,\xi_2)) \\
&= \int_C [(u(x_1,x_2)-\psi(x_1,x_2))n_j(x_1,x_2)\frac{\partial\Phi(x_1,x_2;\xi_1,\xi_2)}{\partial x_j} \\
&\quad - \Phi(x_1,x_2;\xi_1,\xi_2)p(x_1,x_2)]ds(x_1,x_2),
\end{aligned} \tag{4}$$

where C is the path which comprises three straight lines defined by $\{(x_1, x_2) : x_1 = 0, 0 \leq x_2 < h\}$, $\{(x_1, x_2) : x_2 = h, 0 \leq x_1 \leq 1\}$ and $\{(x_1, x_2) : x_1 = 1, 0 \leq x_2 < h\}$, $n_j(x_1, x_2)$ are the components of the unit normal vector to C pointing away from the interior of the slab, $\lambda(\xi_1, \xi_2) = 1/2$ if (ξ_1, ξ_2) lies on a smooth part of C and is not an endpoint of C and $\lambda(\xi_1, \xi_2) = 1$ if (ξ_1, ξ_2) lies in the interior of the slab, $k^2 = (e_{15})^2/(c_{44}\varepsilon_{11})$ and

$$\begin{aligned}
\Omega(x_1,x_2;\xi_1,\xi_2) &= \frac{1}{4}[Y_0(\frac{\pi\omega}{2h}\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}) \\
&\quad - Y_0(\frac{\pi\omega}{2h}\sqrt{(x_1-\xi_1)^2+(x_2+\xi_2)^2})], \\
\Phi(x_1,x_2;\xi_1,\xi_2) &= \frac{1}{2\pi}[\ln(\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}) \\
&\quad - \ln(\sqrt{(x_1-\xi_1)^2+(x_2+\xi_2)^2})],
\end{aligned}$$

where Y_0 is the zero-th order Bessel function of the second kind.

Equations (3) and (4) can be discretised together with Eqn. (2) as explained in Ref. [1] to determine numerically $u(x_1, x_2)$ on C, and $p(x_1, x_2)$ and $\psi(x_1, x_2)$ on respectively the electroded and unelectroded parts of C, and the unknown output voltage V_{out}. Once these unknown quantities are determined, the transforming ratio of the ceramic slab as a piezoelectric transformer can be computed by $|V_{\text{out}}/V_{\text{in}}|$ and the efficiency by

$$\text{efficiency} = \frac{\mathrm{Re}\{i\overline{V}_{\text{out}}\omega\sqrt{\varepsilon_{11}c_{44}+e_{15}^2}\int_b^1 p(x_1,h)dx_1\}}{\mathrm{Re}\{i\overline{V}_{\text{in}}\omega\sqrt{\varepsilon_{11}c_{44}+e_{15}^2}\int_0^a p(x_1,h)dx_1\}},$$

where the overhead bar denotes the complex conjugate.

4. Numerical results

The transformer is taken to be made of polarised ceramic PZT-5H with damping in the elastic material constant c_{44}. As in Ref. [3], the coefficients of the ceramic are chosen to be $c_{44} = (2.3+0.023i)\times10^{10}$ N/m^2, $\rho = 7500$ kg/m^3, $e_{15} = 17$ C/m^2 and $\varepsilon_{11} = 1.505\times10^{-8}$ C/(Vm). Here the non-dimensionalised height h is taken to be given by 0.05 and a and b by 0.40 and 0.60 respectively.

The boundary C is discretized into equal length straight line elements. The calculation is refined by doubling the number of elements until convergence is observed in the numerical values. For a larger ω or a very small h, more elements may be needed. Up to 900 constant elements are used here.

The numerical values of the transforming ratio $|V_{\text{out}}/V_{\text{in}}|$ are calculated for a range of the non-dimensionalised driving frequency ω for a given Z. For $Z = 2$, the transforming ratio appears to peak near $\omega = 0.904$. Table 1 records the values of $|V_{\text{out}}/V_{\text{in}}|$ for selected values of ω near 0.904.

Table 1: Numerical values of $|V_{\text{out}}/V_{\text{in}}|$ ($Z = 2$).

ω	0.901	0.903	0.904	0.905	0.907	0.909
$\lvert V_{\text{out}}/V_{\text{in}}\rvert$	0.988	2.13	2.78	1.74	0.725	0.424

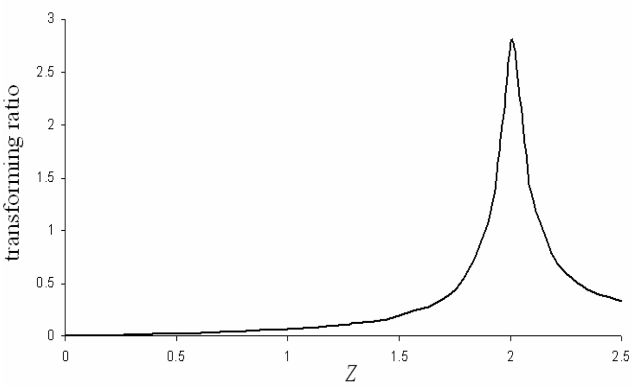

Figure 2: Plot of $|V_{\text{out}}/V_{\text{in}}|$ against Z ($\omega = 0.904$)

For $\omega = 0.904$, $|V_{\text{out}}/V_{\text{in}}|$ is plotted against Z in Fig. 2. It is observed $|V_{\text{out}}/V_{\text{in}}|$ is maximum at $Z = 2.01$ (approximately) and appears to tend to a fixed value (approximately 0.066) as Z increases. Also, for $\omega = 0.904$, the efficiency of the transformer is plotted against Z/i in Fig. 3. The efficiency reaches a peak value of approximately 0.0762 at close to $Z/i = 2$ and then decreases as the Z/i increases. The plots in Fig. 2 and 3 exhibit qualitative features similar to those in Ref. [3].

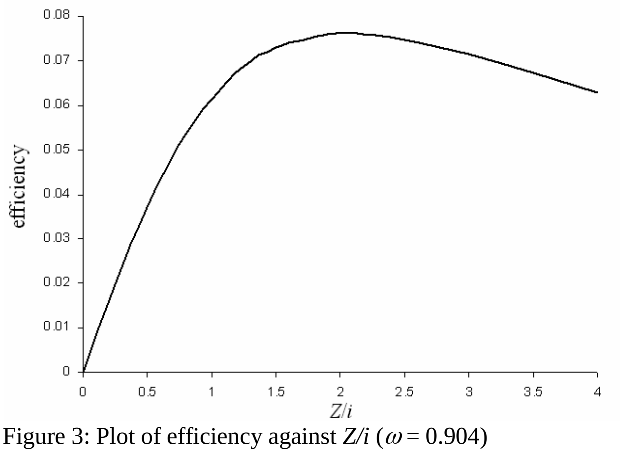

Figure 3: Plot of efficiency against Z/i ($\omega = 0.904$)

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