Additional Note for Chapter 1 of "Hypersingular Integral Equations in Fracture Analysis"

<u>*Definitions of Cauchy principal integrals in (1.62) and (1.63) (page 20)*</u>

Show that the definitions in (1.62) and (1.63) are equivalent.

For $a < x < b$, the definitions are given by

$$\fint_a^b \frac{f(t)dt}{t-x} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+}\{\int_a^{x-\epsilon} \frac{f(t)dt}{t-x} + \int_{x+\epsilon}^b \frac{f(t)dt}{t-x}\}, \tag{1.62}$$

$$\fint_a^b \frac{f(t)dt}{t-x} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\int_a^b \frac{(t-x)f(t)dt}{(t-x)^2+\epsilon^2}. \tag{1.63}$$

Assume that $f(t)$ can be expressed in terms of Taylor series about $t = x$, that is,

$$f(t) = f(x) + \sum_{m=1}^{\infty} \frac{(t-x)^m}{m!} f^{(m)}(x). \tag{1}$$

Substitute (1) into (1.62):

$$\begin{aligned}
\fint_a^b \frac{f(t)dt}{t-x} &= f(x)\lim_{\epsilon\to 0^+}\{\int_a^{x-\epsilon}\frac{dt}{t-x} + \int_{x+\epsilon}^b \frac{dt}{t-x}\} \\
&\quad + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\lim_{\epsilon\to 0^+}\{\int_a^{x-\epsilon}(t-x)^{m-1}dt + \int_{x+\epsilon}^b (t-x)^{m-1}dt\} \\
&= f(x)\lim_{\epsilon\to 0^+}\{\ln|\epsilon| - \ln|a-x| + \ln|b-x| - \ln|\epsilon|\} \\
&\quad + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!m}\lim_{\epsilon\to 0^+}\{(-\epsilon)^m - (a-x)^m + (b-x)^m - \epsilon^m\} \qquad (2) \\
&= f(x)\{\ln|b-x| - \ln|a-x|\} \\
&\quad + \sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!m}\{(b-x)^m - (a-x)^m\}.
\end{aligned}$$

Substitute (1) into (1.63):

$$\begin{aligned}
-\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{t-x} &= f(x)\lim_{\epsilon\to 0}\int_a^b \frac{(t-x)dt}{(t-x)^2+\epsilon^2} \\
&\quad +\sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\lim_{\epsilon\to 0}\int_a^b \frac{(t-x)^{m+1}dt}{(t-x)^2+\epsilon^2} \\
&= \frac{f(x)}{2}\lim_{\epsilon\to 0}\{\ln((b-x)^2+\epsilon^2)-\ln((a-x)^2+\epsilon^2)\} \\
&\quad +\sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!}\int_a^b (t-x)^{m-1}dt \qquad (3) \\
&= f(x)\{\ln|b-x|-\ln|a-x|\} \\
&\quad +\sum_{m=1}^{\infty}\frac{f^{(m)}(x)}{m!m}\{(b-x)^m-(a-x)^m\}.
\end{aligned}$$

Note that we can interchange the order of the working out the limit and the integral in the second line of (3) because the integrand of the integral is not singular for $\epsilon = 0$.

Derivation of the alternative definition in (1.78) (page 23)

From (1.62), we obtain

$$\begin{aligned}
\frac{d}{dx}\Big[-\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{t-x}\Big] &= \frac{d}{dx}\Big[\lim_{\epsilon\to 0^+}\Big\{\int_a^{x-\epsilon}\frac{f(t)dt}{t-x}+\int_{x+\epsilon}^b \frac{f(t)dt}{t-x}\Big\}\Big] \\
&= f(x)\Big\{\frac{1}{a-x}-\frac{1}{b-x}\Big\} \\
&\quad +f'(x)\{\ln|b-x|-\ln|a-x|\} \\
&\quad +\sum_{m=1}^{\infty}\frac{f^{(m+1)}(x)}{(m+1)!m}\{(b-x)^m-(a-x)^m\}.
\end{aligned} \tag{1.73}$$

Use (1), (1.73) (page 22) and (1.77) to show that

$$\begin{aligned}
=\!\!\!\!\!\!\int_a^b \frac{f(t)dt}{(t-x)^2} &= \lim_{\epsilon\to 0^+}\Big\{\int_a^b \frac{(t-x)^2 f(t)dt}{[(t-x)^2+\epsilon^2]^2}-\frac{\pi}{2\epsilon}f(x)\Big\} \\
&\quad -\frac{f(x)}{2}\Big[\frac{1}{b-x}-\frac{1}{a-x}\Big] \text{ for } a<x<b.
\end{aligned} \tag{1.78*}$$

Using (1) and

$$\begin{aligned}\int \frac{(t-x)^2 dt}{((t-x)^2+\epsilon^2)^2} &= -\frac{t-x}{2\,((t-x)^2+\epsilon^2)} + \frac{1}{2\epsilon}\arctan(\frac{1}{2}\frac{2t-2x}{\epsilon}),\\ \int \frac{(t-x)^3 dt}{((t-x)^2+\epsilon^2)^2} &= \frac{\epsilon^2}{2\,((t-x)^2+\epsilon^2)} + \frac{1}{2}\ln\left((t-x)^2+\epsilon^2\right), \qquad (4)\end{aligned}$$

we find that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^2 f(t)dt}{[(t-x)^2+\epsilon^2]^2} &= f(x)\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^2 dt}{[(t-x)^2+\epsilon^2]^2}\\ &\quad +f'(x)\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^3 dt}{[(t-x)^2+\epsilon^2]^2}\\ &\quad +\sum_{m=1}^{\infty}\frac{f^{(m+1)}(x)}{(m+1)!}\lim_{\epsilon\to 0^+}\int_a^b \frac{(t-x)^{3+m}dt}{[(t-x)^2+\epsilon^2]^2}\\ &= \lim_{\epsilon\to 0^+}\frac{\pi}{2\epsilon}f(x) - \frac{f(x)}{2(b-x)} + \frac{f(x)}{2(a-x)} \qquad (5)\\ &\quad +f'(x)\{\ln|b-x| - \ln|a-x|\}\\ &\quad +\sum_{m=1}^{\infty}\frac{f^{(m+1)}(x)}{(m+1)!}\int_a^b (t-x)^{m-1}dt.\end{aligned}$$

Note that we can interchange the order of evaluating the limit and the integral on the third line of (5).

From (5) (after evaluating the integral on the last line), (1.73) and (1.77), we obtain (1.78*).

Correction of (1.78) in the book

The term $-\frac{f(x)}{2}[\frac{1}{b-x} - \frac{1}{a-x}]$ on the right hand side of (1.78*) above is missing in (1.78) in the book. Thus, (1.78) on page 23 of the book should be replaced by (1.78*).

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