A dual-reciprocity boundary element approach for solving axisymmetric heat equation subject to specification of energy

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Abstract

A dual-reciprocity boundary element procedure is presented for the numerical solution of an axisymmetric heat conduction problem subject to a non-local condition. The non-local condition specifies the total amount of heat energy stored inside the solid under consideration. An unknown control function (of time) which governs the temperature on a certain part of the boundary of the solid is to be determined in the process of solving the axisymmetric heat equation. To check the validity of the numerical procedure, specific problems with known exact solutions are solved.

Keywords: Axisymmetric heat conduction, non-local condition, dual-reciprocity method, boundary element method.

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1 Introduction

Of interest here is the non-classical heat conduction problem which requires solving the parabolic heat equation subject to a non-local condition. The non-local condition is given by a domain integral which specifies the total amount of heat energy stored in the solid under consideration. On a certain part of the boundary of the solid, the temperature is specified in terms of an unknown control function (of time) to be determined.

Such a problem has been solved numerically using the finite difference methods, usually for relatively simple solution domains (like a square) with temperature specified on the whole boundary. For some examples of finite difference solutions, one may refer to the works of Wang and Lin [1], Cannon, Lin and Matheson [2], Noye, Dehghan and van der Hoek [3], Gumel, Ang and Twizell [4], Dehghan [5], and many other references therein.

Ang [6] and Ang and Gumel [7] applied the boundary element method together with the Laplace transformation to solve the problem for two- and three-dimensional solution domains. The domain integral in the non-local condition was reduced to an integral involving only the heat flux on the boundary of the solution domain. The physical solution was recovered from the Laplace transformation domain through the use of a numerical technique for inverting Laplace transformation. More recently, Ang [8] presented a dual-reciprocity boundary element method together with discontinuous linear elements and a time-stepping scheme for solving a generalized (anisotropic) heat equation subject to the non-local condition.

In the present paper, the dual-reciprocity boundary element approach described in Wang, Mattheij and ter Morsche [9] is applied to compute numerically the temperature field which varies axially and radially in an axisymmetric body which contains a prescribed amount of heat energy. Discontinuous linear elements are employed in the approximations of temperature and heat flux on the boundary. In general, the discontinuous linear elements have been found to perform better than the continuous ones, as the former can model more accurately heat flux which is discontinuous at sharp edges of the boundary of the solution domain. Furthermore, in general, the temperature prescribed on the boundary may also be discontinuous at certain points. The first order time derivative of the temperature is approximated using a central finite difference formula. In order to check its validity, the method presented is used to solve two specific problems whose exact solutions are known.

2 The problem

With reference to an $Oxyz$ Cartesian coordinate system, consider a thermally isotropic solid whose geometry is symmetrical about the z-axis. If r and θ denote the polar coodinates defined by $x = r\cos\theta$ and $y = r\sin\theta$, the temperature distribution inside the solid is assumed to be independent of θ, given by $T(r, z, t)$, where t denotes time. For a homogeneous solid, the conservation of energy and the Fourier's law of heat conduction requires the temperature to satisfy the axisymmetric heat equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{\rho c}{\kappa}\frac{\partial T}{\partial t} \text{ in } R \text{ for } t \geq 0, \tag{1}$$

where R denotes the region occupied by the solid and the positive constants ρ, c and κ are respectively the density, specific heat capacity and thermal conductivity of the solid.

Equation (1) is to be solved subject to the initial-boundary conditions

$$\begin{aligned} T(r, z, 0) &= f_0(r, z) \text{ in } R, \\ T(r, z, t) &= p(t) f_1(r, z) \text{ on } S_1 \text{ for } t > 0, \\ T(r, z, t) &= f_2(r, z, t) \text{ on } S_2 \text{ for } t > 0, \\ \frac{\partial T}{\partial n} &= f_3(r, z, t) \text{ on } S_3 \text{ for } t > 0, \end{aligned} \tag{2}$$

and the non-local condition

$$\rho c \iiint_R [T(r,z,t) - T_0] r dr d\theta dz = \varepsilon(t) \text{ for } t > 0, \tag{3}$$

where S_1, S_2 and S_3 are non-intersecting surfaces such that $S_1 \cup S_2 \cup S_3 = S$, S is the (surface) boundary of the region R, $\partial T/\partial n$ denotes the outward normal derivative of T on S, $f_0(r,z)$, $f_1(r,z)$, $f_2(r,z,t)$, $f_3(r,z,t)$ and $\varepsilon(t)$ are suitably given functions, $p(t)$ is a control function to be determined, and T_0 is the temperature corresponding to absolute zero (for example, $T_0 \simeq -273.13^{\circ}$C if $T(r,z,t)$ is given using the Celcius scale).

Note that (3) implies that the total heat energy which is stored inside the solid is known and given by $\varepsilon(t)$. In a physical sense, the control function $p(t)$ allows one to regulate the total amount of heat energy through manipulating the temperature on a certain of the boundary.

The problem is to find $T(r,z,t)$ and $p(t)$ from (1) subject to (2) and (3).

The mathematical problem described here may also arise in the diffusion of substance subject to specification of mass. The class of problems defined by the equations in (1)-(3) has practical applications in heat transfer, control theory, thermoelasticity and medical sciences. A specific application which involves the use of the absorption of light to measure the concentration of a diffusing chemical is described in Noye and Dehghan [10]. Another example of applications is concerned with controlling the quantity of drug in biological tissues.

3 Integro-differential formulation

Assume that the solution domain R and its boundary S are obtained by rotating respectively the two-dimensional region Ω and the open curve Γ in Figure 1 by an angle of 360° about the z-axis. In Figure 1, Γ is an open curve having endpoints A and B on the z-axis. In general, Γ may also be a closed curve, as in, for example, the case in which R is the hollow cylindrical region defined by $u < r < v$, $0 < z < w$, where u, v and w are positive constants.

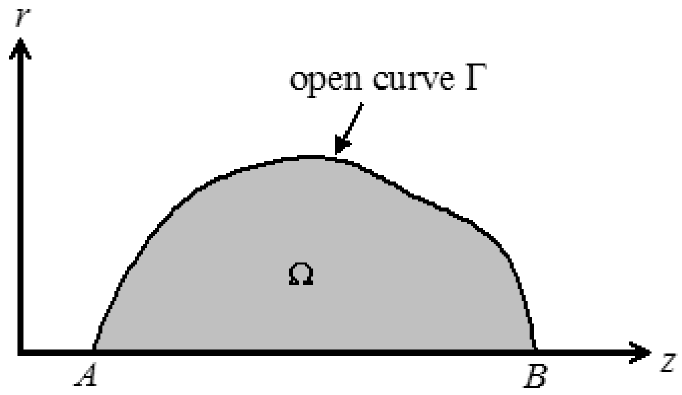

Figure 1

An integro-differential equation in terms of integrals over Γ and Ω can be derived from (1), that is,

$$
\begin{aligned}
&\gamma(\xi,\eta)T(\xi,\eta,t) \\
&= \frac{\rho c}{\kappa}\iint\limits_{\Omega} G(r,z;\xi,\eta)\frac{\partial}{\partial t}[T(r,z,t)]dA(r,z) \\
&+ \int\limits_{\Gamma}(T(r,z,t)\frac{\partial}{\partial n}[G(r,z;\xi,\eta)] - G(r,z;\xi,\eta)\frac{\partial}{\partial n}[T(r,z,t)])rds(r,z) \\
&\qquad\qquad\text{for } (\xi,\eta)\in\Omega\cup\Gamma, \qquad (4)
\end{aligned}
$$

where $\gamma(\xi,\eta) = 1$ if (ξ,η) lies in the interior of Ω, $\gamma(\xi,\eta) = 1/2$ if (ξ,η) lies on a smooth part of Γ, $dA(r,z)$ denotes the area of an infinitesimal portion of the region Ω, $ds(r,z)$ denotes the length of an infinitesimal part of the curve Γ, $\partial T/\partial n$ is the outward normal derivative of T on the curve Γ, and

$$
\begin{aligned}
G(r,z;\xi,\eta) &= -\frac{K(m(r,z;\xi,\eta))}{\pi\sqrt{a(r,z;\xi,\eta)+b(r;\xi)}},\\
\frac{\partial}{\partial n}[G(r,z;\xi,\eta)] &= -\frac{1}{\pi\sqrt{a(r,z;\xi,\eta)+b(r;\xi)}}\\
&\quad\times\{\frac{n_r}{2r}[\frac{\xi^2-r^2+(\eta-z)^2}{a(r,z;\xi,\eta)-b(r;\xi)}E(m(r,z;\xi,\eta))\\
&\qquad -K(m(r,z;\xi,\eta))]\\
&\quad+n_z\frac{\eta-z}{a(r,z;\xi,\eta)-b(r;\xi)}E(m(r,z;\xi,\eta))\},\\
m(r,z;\xi,\eta) &= \frac{2b(r;\xi)}{a(r,z;\xi,\eta)+b(r;\xi)},\\
a(r,z;\xi,\eta) &= \xi^2+r^2+(\eta-z)^2,\quad b(r;\xi)=2r\xi,
\end{aligned}
\tag{5}
$$

where n_r and n_z are the components of the outward unit normal vector on Γ (Figure 1) in the r and z direction respectively, and K and E denote the complete elliptic integral of the first and second kind respectively (as defined in Abramowitz and Stegun [17]).

Details on the derivation of (4) may be found in Brebbia, Telles and Wrobel [16].

If we differentiate both sides of (3) partially with respect to t and apply the divergence theorem, we obtain

$$2\pi\int\limits_{\Gamma}\kappa r\frac{\partial}{\partial n}[T(r,z,t)]ds(r,z)=\frac{d}{dt}(\varepsilon(t))\text{ for }t>0. \tag{6}$$

The problem stated in Section 2 can now be reformulated as one which requires finding $T(r,z,t)$ and $p(t)$ from (4) together with (2) and (6).

4 Dual-reciprocity boundary element method

For a numerical procedure, the curve Γ in Figure 1 is discretized into N straight line elements denoted by $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\cdots$, $\Gamma^{(N-1)}$ and $\Gamma^{(N)}$. The starting and ending points of a typical element $\Gamma^{(k)}$ are given by $(r^{(k)}, z^{(k)})$ and $(r^{(k+1)}, z^{(k+1)})$ respectively. On the element $\Gamma^{(k)}$, choose two points

$$
\begin{aligned}
(\xi^{(k)}, \eta^{(k)}) &= (r^{(k)}, z^{(k)}) + \tau(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}), \\
(\xi^{(N+k)}, \eta^{(N+k)}) &= (r^{(k)}, z^{(k)}) + (1-\tau)(r^{(k+1)} - r^{(k)}, z^{(k+1)} - z^{(k)}),
\end{aligned}
\tag{7}
$$

where τ is a chosen number such that $0 < \tau < 1/2$.

If the temperature T at $(\xi^{(k)}, \eta^{(k)})$ and $(\xi^{(N+k)}, \eta^{(N+k)})$ is denoted by $T^{(k)}(t)$ and $T^{(N+k)}(t)$ respectively, then the boundary temperature is approximated using

$$
T(r, z, t) \simeq \frac{[s^{(k)}(r, z) - (1-\tau)\ell^{(k)}]T^{(k)}(t) - [s^{(k)}(r, z) - \tau\ell^{(k)}]T^{(N+k)}(t)}{(2\tau - 1)\ell^{(k)}} \quad \text{for } (r, z) \in \Gamma^{(k)}, \tag{8}
$$

where $\ell^{(k)} = s^{(k)}(r^{(k+1)}, z^{(k+1)})$ and $s^{(k)}(r, z)$ is the arc length along the element $\Gamma^{(k)}$ as defined by

$$
s^{(k)}(r, z) = \sqrt{(r - r^{(k)})^2 + (z - z^{(k)})^2}. \tag{9}
$$

Similarly, $q(r, z, t) = \partial T/\partial n$ is approximated using

$$
q(r, z, t) \simeq \frac{[s^{(k)}(r, z) - (1-\tau)\ell^{(k)}]q^{(k)}(t) - [s^{(k)}(r, z) - \tau\ell^{(k)}]q^{(N+k)}(t)}{(2\tau - 1)\ell^{(k)}} \quad \text{for } (r, z) \in \Gamma^{(k)}, \tag{10}
$$

if $q^{(k)}(t) = q(\xi^{(k)}, \eta^{(k)}, t)$ and $q^{(N+k)}(t) = q(\xi^{(N+k)}, \eta^{(N+k)}, t)$.

Note that the approximations in (8) and (10) do not guarantee that $T(r, z, t)$ and $q(r, z, t)$ are continuous from one element to the next and are known as discontinuous linear elements in the literature (see, for example, París and Cañas [18]).

With (8) and (10), the integro-differential equation (4) can be approximately written as

$$
\begin{aligned}
&\gamma(\xi,\eta)T(\xi,\eta,t)\\
&= \frac{\rho c}{\kappa}\iint\limits_{\Omega} G(r,z;\xi,\eta)\frac{\partial}{\partial t}[T(r,z,t)]dA(r,z)\\
&+ \sum_{k=1}^{N}\frac{1}{(2\tau-1)\ell^{(k)}}\{[-(1-\tau)\ell^{(k)}\mathcal{F}_2^{(k)}(\xi,\eta)+\mathcal{F}_4^{(k)}(\xi,\eta)]T^{(k)}(t)\\
&\qquad\qquad + [\tau\ell^{(k)}\mathcal{F}_2^{(k)}(\xi,\eta)-\mathcal{F}_4^{(k)}(\xi,\eta)]T^{(N+k)}(t)\\
&- [-(1-\tau)\ell^{(k)}\mathcal{F}_1^{(k)}(\xi,\eta)+\mathcal{F}_3^{(k)}(\xi,\eta)]q^{(k)}(t)\\
&\qquad\qquad - [\tau\ell^{(k)}\mathcal{F}_1^{(k)}(\xi,\eta)-\mathcal{F}_3^{(k)}(\xi,\eta)]q^{(N+k)}(t)\},
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
\mathcal{F}_1^{(k)}(\xi,\eta) &= \int\limits_{\Gamma^{(k)}} G(r,z;\xi,\eta)r\,ds(r,z),\\
\mathcal{F}_2^{(k)}(\xi,\eta) &= \int\limits_{\Gamma^{(k)}} \frac{\partial}{\partial n}[G(r,z;\xi,\eta)]r\,ds(r,z),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_3^{(k)}(\xi,\eta) &= \int\limits_{\Gamma^{(k)}} s(r,z)G(r,z;\xi,\eta)r\,ds(r,z),\\
\mathcal{F}_4^{(k)}(\xi,\eta) &= \int\limits_{\Gamma^{(k)}} s(r,z)\frac{\partial}{\partial n}[G(r,z;\xi,\eta)]r\,ds(r,z).
\end{aligned}
\tag{12}
$$

The integrals over $\Gamma^{(k)}$ in (12) may be evaluated using numerical integration formula such as the Gaussian quadratures.

The integral over the domain Ω in (11) is treated using the dual-reciprocity method. To do this, L well-spaced out collocation points are chosen in the interior of Ω. Denote the selected points by $(\xi^{(2N+1)},\eta^{(2N+1)})$, $(\xi^{(2N+2)},\eta^{(2N+2)})$, $\cdots$, $(\xi^{(2N+L-1)},\eta^{(2N+L-1)})$ and $(\xi^{(2N+L)},\eta^{(2N+L)})$. The points $(\xi^{(k)},\eta^{(k)})$ and

$(\xi^{(N+k)}, \eta^{(N+k)})$ on the element $\Gamma^{(k)}$ $(k = 1,\ 2,\ \cdots,\ N)$ are also used as collocation points.

The domain integral is approximated as follows:

$$\begin{aligned}
&\iint\limits_{\Omega} G(r,z;\xi,\eta)\frac{\partial}{\partial t}[T(r,z,t)]dA(r,z) \\
&\simeq \sum_{k=1}^{2N+P} \frac{d}{dt}[T^{(k)}(t)] \sum_{j=1}^{2N+P} W^{(kj)}\Psi^{(j)}(\xi,\eta) \qquad (13)
\end{aligned}$$

where $T^{(k)}(t) = T(\xi^{(k)}, \eta^{(k)})$ for $k = 1,\ 2,\ \cdots,\ 2N + L$, the coefficients $W^{(kj)}$ are defined implicitly by

$$\sum_{j=1}^{2N+L} W^{(kj)}\phi^{(p)}(\xi^{(j)}, \eta^{(j)}) = \begin{cases} 0 & \text{if} \quad p \neq k \\ 1 & \text{if} \quad p = k \end{cases} \quad \text{for } p,\ k = 1, 2, \cdots, 2N + L,$$

and

$$\begin{aligned}
\Psi^{(j)}(\xi,\eta) &= \gamma(\xi,\eta)\chi^{(j)}(\xi,\eta) + \int\limits_{\Gamma} rG(r,z;\xi,\eta)\frac{\partial}{\partial n}[\chi^{(j)}(\xi,\eta)]ds(r,z) \\
&- \int\limits_{\Gamma} r\chi^{(j)}(\xi,\eta)\frac{\partial}{\partial n}[G(r,z;\xi,\eta)]ds(r,z) \\
&\text{for } j = 1, 2, \cdots, 2N + L,
\end{aligned}$$

$$\phi^{(p)}(r,z) = 4E(m(r,z;\xi^{(p)},\eta^{(p)}))\sqrt{a(r,z;\xi^{(p)},\eta^{(p)}) + b(r;\xi^{(p)})},$$

$$\begin{aligned}
\chi^{(p)}(\xi,\eta) &= \frac{1}{9}(a(r,z;\xi^{(p)},\eta^{(p)}) + b(r;\xi^{(p)}))\sqrt{a(r,z;\xi^{(p)},\eta^{(p)}) + b(r;\xi^{(p)})} \\
&\times [(m(r,z;\xi^{(p)},\eta^{(p)}) - 1)K(m(r,z;\xi^{(p)},\eta^{(p)})) \\
&\quad + (4 - 2m(r,z;\xi^{(p)},\eta^{(p)}))E(m(r,z;\xi^{(p)},\eta^{(p)}))]. \qquad (14)
\end{aligned}$$

Details on the local interpolating functions $\phi^{(p)}(r,z)$ and the corresponding functions $\chi^{(p)}(\xi,\eta)$ are given in Wang, Mattheij and ter Morsche [9].

It may be worth mentioning here that other approximating functions have been proposed in the literature for treating the domain integral. Examples are the global approximation functions in Partridge, Brebbia and Wrobel [11], Bai and Lu [12] and Benz and Rencis[13], the axisymmetric augmented thin plate splines in Šarler [14] and the axisymmetric multiquadrics in Šarler, Jelić, Kovačević, Lakner and Perko [15]. The approximation functions in [9] are used here as they give rise to a relatively less complicated formulation.

If one makes the approximations

$$\begin{aligned} T^{(k)}(t) &\simeq \frac{1}{2}[T^{(k)}(t+\frac{1}{2}\Delta t)+T^{(k)}(t-\frac{1}{2}\Delta t)], \\ \frac{d}{dt}[T^{(k)}(t)] &\simeq \frac{1}{\Delta t}[T^{(k)}(t+\frac{1}{2}\Delta t)-T^{(k)}(t-\frac{1}{2}\Delta t)], \end{aligned} \tag{15}$$

uses (13) and lets (ξ,η) in (11) be given by $(\xi^{(n)},\eta^{(n)})$ for $n=1,\ 2,\ \cdots,$ $2N+L$, one obtains

$$\begin{aligned} &\frac{1}{2}\gamma(\xi^{(n)},\eta^{(n)})[T^{(n)}(t+\frac{1}{2}\Delta t)+T^{(n)}(t-\frac{1}{2}\Delta t)] \\ &= \frac{\rho c}{\kappa\Delta t}\sum_{k=1}^{2N+P}[T^{(k)}(t+\frac{1}{2}\Delta t)-T^{(k)}(t-\frac{1}{2}\Delta t)]\sum_{j=1}^{2N+P}W^{(kj)}\Psi^{(j)}(\xi^{(n)},\eta^{(n)}) \\ &+\sum_{k=1}^{N}\frac{1}{(2\tau-1)\ell^{(k)}}\{\frac{1}{2}[-(1-\tau)\ell^{(k)}\mathcal{F}_2^{(k)}(\xi^{(n)},\eta^{(n)})+\mathcal{F}_4^{(k)}(\xi^{(n)},\eta^{(n)})] \\ &\times[T^{(k)}(t+\frac{1}{2}\Delta t)+T^{(k)}(t-\frac{1}{2}\Delta t)] \\ &+\frac{1}{2}[\tau\ell^{(k)}\mathcal{F}_2^{(k)}(\xi^{(n)},\eta^{(n)})-\mathcal{F}_4^{(k)}(\xi^{(n)},\eta^{(n)})] \\ &\times[T^{(N+k)}(t+\frac{1}{2}\Delta t)+T^{(N+k)}(t-\frac{1}{2}\Delta t)] \\ &-[-(1-\tau)\ell^{(k)}\mathcal{F}_1^{(k)}(\xi^{(n)},\eta^{(n)})+\mathcal{F}_3^{(k)}(\xi^{(n)},\eta^{(n)})]q^{(k)}(t) \\ &-[\tau\ell^{(k)}\mathcal{F}_1^{(k)}(\xi^{(n)},\eta^{(n)})-\mathcal{F}_3^{(k)}(\xi^{(n)},\eta^{(n)})]q^{(N+k)}(t)\} \\ &\qquad\text{for } n=1,2,\cdots,2N+L. \end{aligned} \tag{16}$$

Note that Δt is a small positive number.

At each of the boundary collocation points $(\xi^{(1)}, \eta^{(1)})$, $(\xi^{(2)}, \eta^{(2)})$, $\cdots$, $(\xi^{(2N-1)}, \eta^{(2N-1)})$ and $(\xi^{(2N)}, \eta^{(2N)})$, either the temperature or its normal derivative (not both) is specified in accordance with the boundary conditions given in (2). If the temperature is given by either the second or third line of (2) at $(\xi^{(k)}, \eta^{(k)})$ $(k = 1, 2, \cdots, 2N)$ then $q^{(k)}(t)$ is unknown. Otherwise, if the normal derivative of the temperature is given by the fourth line of (2) at $(\xi^{(k)}, \eta^{(k)})$ $(k = 1, 2, \cdots, 2N)$ then $T^{(k)}(t)$ is unknown. The temperature is not known at all L interior collocation points.

Thus, if $T^{(n)}(t - \frac{1}{2}\Delta t)$ $(n = 1, 2, \cdots, 2N + L)$ is assumed known, then (16) constitutes a system of $2N + L$ linear algebraic equations in $2N + L + 1$ unknown functions of t. (One should not forget that the control function $p(t)$ which appears in the second line of (2) is an unknown function yet to be determined.) Another equation is needed to complete the system.

The last linear algebraic equation is obtained from (6) using (10), that is,

$$2\pi\kappa \sum_{k=1}^{N} [\mathcal{H}_1^{(k)} - (1-\tau)\mathcal{H}_2^{(k)}](q^{(k)} + q^{(N+k)}) = \frac{d}{dt}[\varepsilon(t)], \qquad (17)$$

where

$$\begin{aligned}
\mathcal{H}_1^{(k)} &= \frac{1}{(2\tau - 1)}(\frac{1}{2}\ell^{(k)} r^{(k)} + \frac{1}{3}[\ell^{(k)}]^2 n_z^{(k)}), \\
\mathcal{H}_2^{(k)} &= \frac{1}{(2\tau - 1)}(\ell^{(k)} r^{(k)} + \frac{1}{2}[\ell^{(k)}]^2 n_z^{(k)}).
\end{aligned} \qquad (18)$$

Note that $n_z^{(k)}$ is the z component of the outward unit normal vector to the boundary element $\Gamma^{(k)}$.

The linear algebraic equations in (16) and (17) may be solved as follows. If one lets $t = \frac{1}{2}\Delta t$, then one can determine $T^{(k)}(0)$ $(k = 1, 2, \cdots, 2N + L)$ using the initial condition in (2) and solve the linear algebraic equations for $p(\frac{1}{2}\Delta t)$, $T^{(2N+k)}(\Delta t)$ $(k = 1, 2, \cdots, L)$ and either $T^{(i)}(\Delta t)$ or $q^{(i)}(\frac{1}{2}\Delta t)$ $(i = 1, 2, \cdots, 2N)$. With $T^{(k)}(\Delta t)$ $(k = 1, 2, \cdots, 2N + L)$ now known and $t = \frac{3}{2}\Delta t$ in (16) and (17), the unknowns $p(\frac{3}{2}\Delta t)$, $T^{(2N+k)}(2\Delta t)$ $(k = 1, 2,$

$\cdots$, L) and either $T^{(i)}(2\Delta t)$ or $q^{(i)}(\frac{3}{2}\Delta t)$ ($i = 1, 2, \cdots , 2N$) can be found. Letting $t = \frac{5}{2}\Delta t$, $\frac{7}{2}\Delta t$, $\cdots$ in a consecutive manner, one can solve for the unknowns at higher and higher time levels.

5 Specific problems

For the mere purpose of checking its validity, the numerical procedure outlined in Section 4 is now applied to solve two specific problems with known exact solutions.

Problem 1. The region R is given by $0 \le r < 1$, $0 < z < 1$. The axisymmetric heat equation given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{3}\frac{\partial T}{\partial t}, \tag{19}$$

is to be solved in R subject to the initial-boundary conditions

$$\begin{aligned} T(r,z,0) &= \cos(\frac{\pi z}{2}) \text{ for } 0 \le r < 1,\ 0 < z < 1, \\ T(r,0,t) &= p(t) \text{ for } 0 \le r < 1 \text{ and } t > 0, \\ T(r,\ell,t) &= 0 \text{ for } 0 \le r < 1 \text{ and } t > 0, \\ \frac{\partial T}{\partial n} &= 0 \text{ on } r = 1,\ 0 < z < 1 \text{ and } t > 0, \end{aligned} \tag{20}$$

and the non-local condition

$$\int_0^1\int_0^{2\pi}\int_0^1 T(r,z,t)rdrd\theta dz = 2\exp(-\frac{3\pi^2 t}{4}) \equiv \varepsilon(t) \text{ for } t > 0. \tag{21}$$

It is easy to check that the exact solution of this problem is

$$\begin{aligned} T(r,z,t) &= \cos(\frac{\pi z}{2})\exp(-\frac{3\pi^2 t}{4}), \\ p(t) &= \exp(-\frac{3\pi^2 t}{4}). \end{aligned} \tag{22}$$

For this particular problem, the boundary Γ comprises three line segments of unit length. For the numerical solution of the problem, Γ is discretized into $3M$ elements, each of length $1/M$ and the L interior collocation points are chosen to be equally spaced out in the interior of the region R. (Note that $N = 3M$.)

In Table 1, the numerical values of the temperature obtained using $M = 10$, $L = 16$, $\tau = 0.25$ (for the discontinuous linear elements) and $\Delta t = 0.01$ are compared with the exact temperature in (22) at selected interior points and at $t = 0.05$. The numerical and exact values are in reasonably good agreement with each other. When the computation is refined using $M = 20$, $L = 81$ and $\Delta t = 0.001$, it is obvious that the numerical values show convergence to the exact solution.

Table 1. Comparison of numerical and exact values of the temperature T at selected interior points and at $t = 0.05$.

(r, z)	$M = 10, L = 16$ $\Delta t = 0.01$	$M = 20, L = 81$ $\Delta t = 0.001$	Exact
(0.25, 0.10)	0.6968	0.6838	0.6822
(0.25, 0.20)	0.6696	0.6581	0.6569
(0.25, 0.30)	0.6257	0.6164	0.6154
(0.50, 0.40)	0.5661	0.5595	0.5588
(0.50, 0.50)	0.4934	0.4889	0.4884
(0.50, 0.60)	0.4096	0.4063	0.4060
(0.75, 0.70)	0.3160	0.3138	0.3136
(0.75, 0.80)	0.2149	0.2136	0.2134
(0.75, 0.90)	0.1086	0.1081	0.1080

In Figure 2, the control function $p(t)$ obtained numerically using $M = 20$, $L = 81$ and $\Delta t = 0.001$ is compared with the exact one given in (22) over the interval $0 \leq t \leq 0.50$. It appears that the numerical procedure is capable of recovering the control function $p(t)$ with good accuracy.

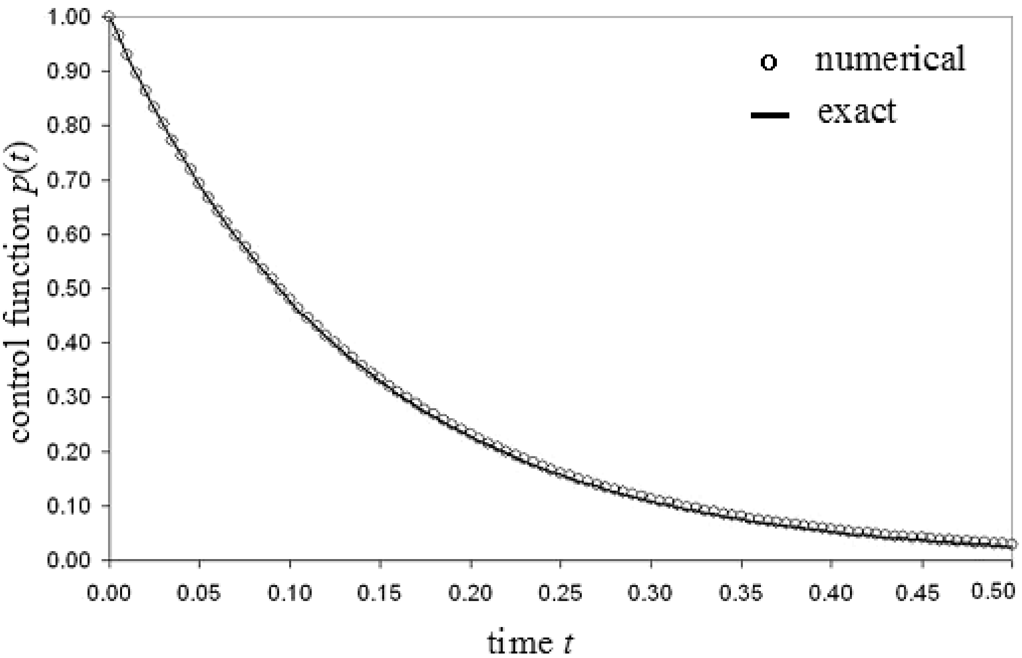

Figure 2

Problem 2. Here R is taken to be given by $r^2 + z^2 < 4$, $1 < z < 2$ (that is, a portion of the region inside a sphere of radius 2). The axisymmetric heat equation (19) is to be solved in R subject to the initial-boundary conditions

$$\begin{aligned}
T(r,z,0) &= -1 + z + [r^2 + z^2]^{-1/2}\ \sin(\frac{\pi}{4}\sqrt{r^2+z^2}) \\
&\quad \text{for } r^2 + z^2 < 4, 1 < z < 2, \\
T(r,1,t) &= p(t)[r^2+1]^{-1/2}\ \sin(\frac{\pi}{4}\sqrt{r^2+1}) \\
&\quad \text{for } 0 \leq r \leq \sqrt{3} \text{ and } t > 0, \\
T(r,z,t) &= -1 + z + \frac{1}{2}\exp(-\frac{3\pi^2 t}{16}) \\
&\quad \text{on } r^2 + z^2 = 4 \text{ for } 1 < z < 2 \text{ and } t > 0,
\end{aligned} \tag{23}$$

and the non-local condition

$$\int_{1}^{2}\int_{0}^{2\pi}\int_{0}^{\sqrt{4-z^2}} T(r,z,t) r dr d\theta dz$$
$$= \frac{7}{12}\pi + \frac{16(2-\sqrt{2})}{\pi}\exp(-\frac{3\pi^2 t}{16}) \equiv \varepsilon(t) \text{ for } t > 0. \tag{24}$$

The exact solution of the problem here is given by

$$\begin{aligned} T(r,z,t) &= -1 + z + [r^2+z^2]^{-1/2} \\ &\quad \times \sin(\frac{\pi}{4}\sqrt{r^2+z^2})\exp(-\frac{3\pi^2 t}{16}), \\ p(t) &= \exp(-\frac{3\pi^2 t}{16}). \end{aligned} \tag{25}$$

The curve Γ consists of two parts: $r^2 + z^2 = 4$, $1 < z \leq 2$ and $z = 1$, $0 \leq r \leq \sqrt{3}$. Each of the parts is discretized into M equal length straight line elements, so that the total number of elements (N) is given by $2M$. The L interior collocation points are selected to be well spaced out throughout the region R.

Table 2. Comparison of numerical and exact values of the temperature at the point $(r, z) = (0.50, 1.50)$ and at selected time levels.

Time t	$M = 10, L = 19$ $\Delta t = 0.02$	$M = 20, L = 58$ $\Delta t = 0.002$	Exact
0.02	1.0769	1.0768	1.0768
0.04	1.0561	1.0558	1.0558
0.06	1.0362	1.0357	1.0356
0.08	1.0170	1.0162	1.0162
0.10	0.9985	0.9975	0.9974
0.12	0.9807	0.9794	0.9793
0.14	0.9635	0.9620	0.9619
0.16	0.9469	0.9452	0.9451
0.18	0.9309	0.9291	0.9290
0.20	0.9156	0.9135	0.9134

In Table 2, two sets of numerical values of the temperature T are compared with the exact temperature in (25) at the point $(r, z) = (0.50, 1.50)$ and at selected time levels. The first set is obtained by using $M = 10$, $L = 19$ and $\Delta t = 0.02$, while the second by $M = 20$, $L = 58$ and $\Delta t = 0.002$. In both sets, τ is chosen to be 0.25 (for the discontinuous linear elements). Accuracy of up to 4 decimal places is achieved in the numerical calculation. Like in the first problem above, the control function $p(t)$ is recovered successfully here by the dual-reciprocity boundary element method.

6 Summary

A procedure based on the dual-reciprocity boundary element method is presented for the numerical solution of the axisymmetric heat equation subject to specification of energy. The domain integral prescribing the energy in the solution domain is converted to a line integral. No discretization of the solution domain is required. The procedure eventually reduces the problem under consideration to a system of linear algebraic equations to be solved at consecutive time levels.

Numerical results obtained by solving specific problems indicate that the procedure may be used to obtain accurate approximate solutions. Convergence of the numerical solutions to the exact ones is observed when the calculation is refined (for example, by increasing the number of boundary elements).

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