Chapters 1 and 5 in
"A Beginner's Course in Boundary Element Methods"

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Comments on this document may be sent to the author at wtang@pmail.ntu.edu.sg.

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Chapter 1
TWO–DIMENSIONAL LAPLACE'S EQUATION

1.1 Introduction

Perhaps a good starting point for introducing boundary element methods is through solving boundary value problems governed by the two-dimensional Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{1.1}$$

The Laplace's equation occurs in the formulation of problems in many diverse fields of studies in engineering and physical sciences, such as thermostatics, elastostatics, electrostatics, magnetostatics, ideal fluid flow and flow in porous media.

An interior boundary value problem which is of practical interest requires solving Eq. (1.1) in the two-dimensional region R (on the Oxy plane) bounded by a simple closed curve C subject to the boundary conditions

$$\begin{aligned} \phi &= f_1(x, y) \text{ for } (x, y) \in C_1, \\ \frac{\partial\phi}{\partial n} &= f_2(x, y) \text{ for } (x, y) \in C_2, \end{aligned} \tag{1.2}$$

where f_1 and f_2 are suitably prescribed functions and C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$. Refer to Figure 1.1 for a geometrical sketch of the problem.

The normal derivative $\partial\phi/\partial n$ in Eq. (1.2) is defined by

$$\frac{\partial\phi}{\partial n} = n_x\frac{\partial\phi}{\partial x} + n_y\frac{\partial\phi}{\partial y}, \tag{1.3}$$

where n_x and n_y are respectively the x and y components of a unit normal vector to the curve C. Here the unit normal vector $[n_x, n_y]$ on C is taken to be pointing away from the region R. Note that the normal vector may vary from point to point on C. Thus, $[n_x, n_y]$ is a function of x and y.

The boundary conditions given in Eq. (1.2) are assumed to be properly posed so that the boundary value problem has a unique solution, that is, it is assumed that one can always find a function $\phi(x, y)$ satisfying Eqs. (1.1)-(1.2) and that there is only one such function.

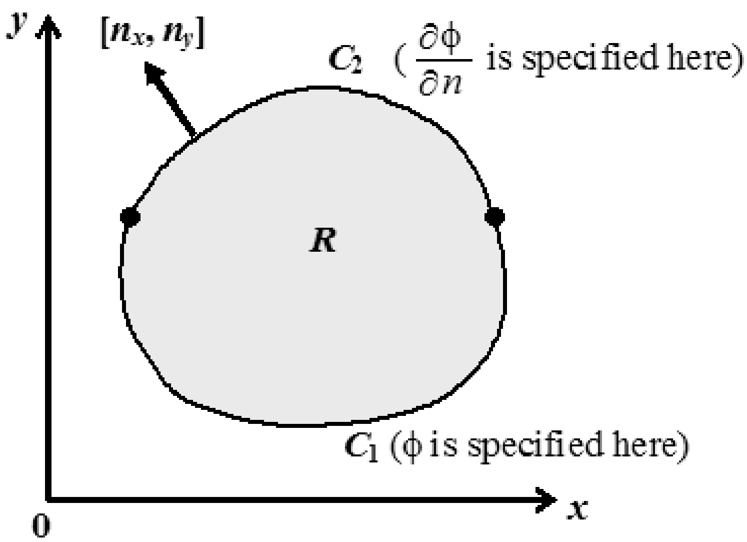

Figure 1.1

For a particular example of practical situations involving the boundary value problem above, one may mention the classical heat conduction problem where ϕ denotes the steady-state temperature in an isotropic solid. Eq. (1.1) is then the temperature governing equation derived, under certain assumptions, from the law of conservation of heat energy together with the Fourier's heat flux model. The heat flux out of the region R across the boundary C is given by $-\kappa\partial\phi/\partial n$, where κ is the thermal heat conductivity of the solid. Thus, the boundary conditions in Eq. (1.2) imply that at each and every given point on C either the temperature or the heat flux (but not both) is known. To determine the temperature field in the solid, one has to solve Eq. (1.1) in R to find the solution that satisfies the prescribed boundary conditions on C.

In general, it is difficult (if not impossible) to solve exactly the boundary value problem defined by Eqs. (1.1)-(1.2). The mathematical complexity involved depends on the geometrical shape of the region R and the boundary conditions given in Eq. (1.2). Exact solutions can only be found for relatively simple geometries of R (such as a square region) together with particular boundary conditions. For more complicated geometries or general boundary conditions, one may have to resort to numerical (approximate) techniques for solving Eqs. (1.1)-(1.2).

This chapter introduces a boundary element method for the numerical solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2). We show how a boundary integral solution can be derived for Eq. (1.1) and applied to obtain a simple boundary element procedure for approximately solving the boundary value problem under consideration. The implementation of the numerical procedure on the computer, achieved through coding in FORTRAN 77, is discussed in detail.

1.2 Fundamental Solution

If we use polar coordinates r and θ centered about $(0,0)$, as defined by $x = r\cos\theta$ and $y = r\sin\theta$, and introduce $\psi(r,\theta) = \phi(r\cos\theta, r\sin\theta)$, we can rewrite Eq. (1.1) as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0. \quad (1.4)$$

For the case in which ψ is independent of θ, that is, if ψ is a function of r alone, Eq. (1.4) reduces to the ordinary differential equation

$$\frac{d}{dr}(r\frac{d}{dr}[\psi(r)]) = 0 \text{ for } r \neq 0. \quad (1.5)$$

The ordinary differential equation in Eq. (1.5) can be easily integrated twice to yield the general solution

$$\psi(r) = A\ln(r) + B, \quad (1.6)$$

where A and B are arbitrary constants.

From (1.6), it is obvious that the two-dimensional Laplace's equation in Eq. (1.1) admits a class of particular solutions given by

$$\phi(x,y) = A\ln\sqrt{x^2+y^2} + B \text{ for } (x,y) \neq (0,0). \quad (1.7)$$

If we choose the constants A and B in (1.7) to be $1/(2\pi)$ and 0 respectively and shift the center of the polar coordinates from $(0,0)$ to the general point (ξ,η), a particular solution of Eq. (1.1) is

$$\phi(x,y) = \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2} \text{ for } (x,y) \neq (\xi,\eta). \quad (1.8)$$

As we shall see, the particular solution in Eq. (1.8) plays an important role in the development of boundary element methods for the numerical solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2). We specially denote this particular solution using the symbol $\Phi(x,y;\xi,\eta)$, that is, we write

$$\Phi(x,y;\xi,\eta) = \frac{1}{4\pi}\ln[(x-\xi)^2+(y-\eta)^2]. \quad (1.9)$$

We refer to $\Phi(x,y;\xi,\eta)$ in Eq. (1.9) as the fundamental solution of the two-dimensional Laplace's equation. Note that $\Phi(x,y;\xi,\eta)$ satisfies Eq. (1.1) everywhere except at (ξ,η) where it is not well defined.

1.3 Reciprocal Relation

If ϕ_1 and ϕ_2 are any two solutions of Eq. (1.1) in the region R bounded by the simple closed curve C then it can be shown that

$$\int\limits_C (\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n}) ds(x,y) = 0. \tag{1.10}$$

Eq. (1.10) provides a reciprocal relation between any two solutions of the Laplace's equation in the region R bounded by the curve C. It may be derived from the two-dimensional version of the Gauss-Ostrogradskii (divergence) theorem as explained below.

According to the divergence theorem, if $\underline{\mathbf{F}}= u(x,y)\underline{\mathbf{i}} + v(x,y)\underline{\mathbf{j}}$ is a well defined vector function such that $\underline{\nabla} \cdot \underline{\mathbf{F}} = \partial u/\partial x + \partial v/\partial y$ exists in the region R bounded by the simple closed curve C then

$$\int\limits_C \underline{\mathbf{F}} \cdot \underline{\mathbf{n}} \; ds(x,y) = \iint\limits_R \underline{\nabla} \cdot \underline{\mathbf{F}} \; dxdy,$$

that is,

$$\int\limits_C [u n_x + v n_y] ds(x,y) = \iint\limits_R [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] dxdy,$$

where $\underline{\mathbf{n}} = [n_x, n_y]$ is the unit normal vector to the curve C, pointing away from the region R.

Since ϕ_1 and ϕ_2 are solutions of Eq. (1.1), we may write

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} &= 0, \\ \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} &= 0. \end{aligned}$$

If we multiply the first equation by ϕ_2 and the second one by ϕ_1 and take the difference of the resulting equations, we obtain

$$\frac{\partial}{\partial x}(\phi_2 \frac{\partial \phi_1}{\partial x} - \phi_1 \frac{\partial \phi_2}{\partial x}) + \frac{\partial}{\partial y}(\phi_2 \frac{\partial \phi_1}{\partial y} - \phi_1 \frac{\partial \phi_2}{\partial y}) = 0,$$

which can be integrated over R to give

$$\iint\limits_R [\frac{\partial}{\partial x}(\phi_2 \frac{\partial \phi_1}{\partial x} - \phi_1 \frac{\partial \phi_2}{\partial x}) + \frac{\partial}{\partial y}(\phi_2 \frac{\partial \phi_1}{\partial y} - \phi_1 \frac{\partial \phi_2}{\partial y})] dxdy = 0.$$

Application of the divergence theorem to convert the double integral over R into a line integral over C yields

$$\int_C [(\phi_2 \frac{\partial \phi_1}{\partial x} - \phi_1 \frac{\partial \phi_2}{\partial x}) n_x + (\phi_2 \frac{\partial \phi_1}{\partial y} - \phi_1 \frac{\partial \phi_2}{\partial y}) n_y] ds(x,y) = 0$$

which is essentially Eq. (1.10).

Together with the fundamental solution given by Eq. (1.9), the reciprocal relation in Eq. (1.10) can be used to derive a useful boundary integral solution for the two-dimensional Laplace's equation.

1.4 Boundary Integral Solution

Let us take $\phi_1 = \Phi(x,y;\xi,\eta)$ (the fundamental solution as defined in Eq. (1.9)) and $\phi_2 = \phi$, where ϕ is the required solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2).

Since $\Phi(x,y;\xi,\eta)$ is not well defined at the point (ξ,η), the reciprocal relation in Eq. (1.10) is valid for $\phi_1 = \Phi(x,y;\xi,\eta)$ and $\phi_2 = \phi$ only if (ξ,η) does not lie in the region $R \cup C$. Thus,

$$\int_C [\phi(x,y) \frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}(\phi(x,y))] ds(x,y) = 0$$
$$\text{for } (\xi,\eta) \notin R \cup C. \quad (1.11)$$

A more interesting and useful integral equation than Eq. (1.11) can be derived from Eq. (1.10) if we take the point (ξ,η) to lie in the region $R \cup C$.

For the case in which (ξ,η) lies in the interior of R, Eq. (1.10) is valid if we replace C by $C \cup C_\varepsilon$, where C_ε is a circle of center (ξ,η) and radius ε as shown in Figure 1.2*. This is because $\Phi(x,y;\xi,\eta)$ and its first order partial derivatives (with respect to x or y) are well defined in the region between C and C_ε. Thus, for C and C_ε in Figure 1.2, we can write

$$\int_{C \cup C_\varepsilon} [\phi(x,y) \frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}(\phi(x,y))] ds(x,y) = 0,$$

that is,

$$\int_C [\phi(x,y) \frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}(\phi(x,y))] ds(x,y)$$
$$= -\int_{C_\varepsilon} [\phi(x,y) \frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}(\phi(x,y))] ds(x,y). \quad (1.12)$$

*The divergence theorem is not only applicable for simply connected regions but also for multiply connected ones such as the one shown in Figure 1.2. For the region in Figure 1.2, the unit normal vector to C_ε (the inner boundary) points towards the center of the circle.

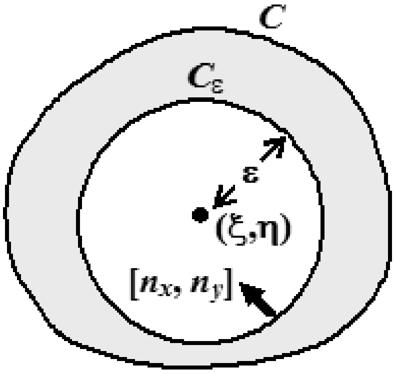

Figure 1.2

Eq. (1.12) holds for any radius $\varepsilon > 0$, so long as the circle C_ε (in Figure 1.2) lies completely inside the region bounded by C. Thus, we may let $\varepsilon \to 0^+$ in Eq. (1.12). This gives

$$\begin{aligned}&\int\limits_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y)\\ = & -\lim_{\varepsilon\to 0^+}\int\limits_{C_\varepsilon}[\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y).\end{aligned} \quad (1.13)$$

Using polar coordinates r and θ centered about (ξ,η) as defined by $x-\xi = r\cos\theta$ and $y-\eta = r\sin\theta$, we may write

$$\begin{aligned}\Phi(x,y;\xi,\eta) &= \frac{1}{2\pi}\ln(r),\\ \frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta)] &= n_x\frac{\partial}{\partial x}[\Phi(x,y;\xi,\eta)] + n_y\frac{\partial}{\partial y}[\Phi(x,y;\xi,\eta)]\\ &= \frac{n_x\cos\theta + n_y\sin\theta}{2\pi r}.\end{aligned} \quad (1.14)$$

The Taylor's series of $\phi(x,y)$ about the point (ξ,η) is given by

$$\phi(x,y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m\phi}{\partial x^k\partial y^{m-k}})\bigg|_{(x,y)=(\xi,\eta)}\frac{(x-\xi)^k(y-\eta)^{m-k}}{k!(m-k)!}.$$

On the circle C_ε, $r=\varepsilon$. Thus,

$$\phi(x,y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k\partial y^{m-k}}[\phi(x,y)])\bigg|_{(x,y)=(\xi,\eta)}\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!}$$
$$\text{for } (x,y)\in C_\varepsilon. \quad (1.15)$$

Similarly, we may write

$$\frac{\partial}{\partial n}[\phi(x,y)] = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m}{\partial x^k \partial y^{m-k}}\{\frac{\partial}{\partial n}[\phi(x,y)]\}\right)\Big|_{(x,y)=(\xi,\eta)} \times \frac{\varepsilon^m \cos^k\theta \sin^{m-k}\theta}{k!(m-k)!} \quad \text{for } (x,y)\in C_\varepsilon. \tag{1.16}$$

Using Eqs. (1.14), (1.15) and (1.16) and writing $ds(x,y) = \varepsilon d\theta$ with θ ranging from 0 to 2π, we may now attempt to evaluate the limit on the right hand side of Eq. (1.13). On C_ε, the normal vector $[n_x, n_y]$ is given by $[-\cos\theta, -\sin\theta]$. Thus,

$$\begin{aligned}
&\int_{C_\varepsilon} \phi(x,y)\frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta]ds(x,y) \\
=\;& -\frac{1}{2\pi}\phi(\xi,\eta)\int_0^{2\pi} d\theta \\
& -\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}\frac{\varepsilon^m}{k!(m-k)!}\left(\frac{\partial^m \phi}{\partial x^k\partial y^{m-k}}\right)\Big|_{(x,y)=(\xi,\eta)}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta \\
\to\;& -\phi(\xi,\eta) \quad \text{as } \varepsilon\to 0^+,
\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}
&\int_{C_\varepsilon} \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}[\phi(x,y)]ds(x,y) \\
=\;& \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\left(\frac{\partial}{\partial n}[\phi(x,y)]\right)\right)\Big|_{(x,y)=(\xi,\eta)} \\
& \times\frac{\varepsilon^{m+1}\ln(\varepsilon)}{k!(m-k)!}\int_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta \\
\to\;& 0 \quad \text{as } \varepsilon\to 0^+,
\end{aligned} \tag{1.18}$$

since $\varepsilon^{m+1}\ln(\varepsilon)\to 0$ as $\varepsilon\to 0^+$ for $m = 0, 1, 2, \cdots$.

Consequently, as $\varepsilon\to 0^+$, Eq. (1.13) yields

$$\phi(\xi,\eta) = \int_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \quad \text{for } (\xi,\eta)\in R. \tag{1.19}$$

Together with Eq. (1.9), Eq. (1.19) provides us with a boundary integral solution for the two-dimensional Laplace's equation. If both ϕ and $\partial\phi/\partial n$ are known at all points on C, the line integral in Eq. (1.19) can be evaluated (at least in theory) to calculate ϕ at any point (ξ, η) in the interior of R. From the boundary conditions (1.2), at any given point on C, either ϕ or $\partial\phi/\partial n$, not both, is known, however.

To solve the interior boundary value problem, we must find the unknown ϕ and $\partial\phi/\partial n$ on C_2 and C_1 respectively. As we shall see later on, this may be done through manipulation of data on the boundary C only, if we can derive a boundary integral formula for $\phi(\xi, \eta)$, similar to the one in Eq. (1.19), for a general point (ξ, η) that lies on C.

For the case in which the point (ξ, η) lies on C, Eq. (1.10) holds if we replace the curve C by $D \cup D_\varepsilon$, where the curves D and D_ε are as shown in Figure 1.3. (If C_ε is the circle of center (ξ, η) and radius ε, then D is the part of C that lies outside C_ε and D_ε is the part of C_ε that is inside R.) Thus,

$$\begin{aligned}
&\int\limits_{D} [\phi(x, y)\frac{\partial}{\partial n}(\Phi(x, y; \xi, \eta)) - \Phi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, y))]ds(x, y) \\
= &-\int\limits_{D_\varepsilon} [\phi(x, y)\frac{\partial}{\partial n}(\Phi(x, y; \xi, \eta)) - \Phi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, y))]ds(x, y). \quad (1.20)
\end{aligned}$$

Let us examine what happens to Eq. (1.20) when we let $\varepsilon \to 0^+$.
As $\varepsilon \to 0^+$, the curve D tends to C. Thus, we may write

$$\begin{aligned}
&\int\limits_{C} [\phi(x, y)\frac{\partial}{\partial n}(\Phi(x, y; \xi, \eta)) - \Phi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, y))]ds(x, y) \\
= &-\lim_{\varepsilon \to 0^+} \int\limits_{D_\varepsilon} [\phi(x, y)\frac{\partial}{\partial n}(\Phi(x, y; \xi, \eta)) - \Phi(x, y; \xi, \eta)\frac{\partial}{\partial n}(\phi(x, y))]ds(x, y).
\end{aligned}$$
(1.21)

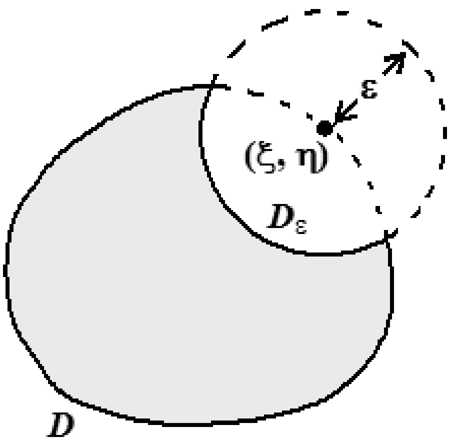

Figure 1.3

Note that, unlike in Eq. (1.13), the line integral over C in Eq. (1.21) is improper as its integrand is not well defined at (ξ, η) which lies on C. Strictly speaking, the line integration should be over the curve C without an infinitesimal segment that contains the point (ξ, η), that is, the line integral over C in Eq. (1.21) has to be interpreted in the Cauchy principal sense if (ξ, η) lies on C.

To evaluate the limit on the right hand side of Eq. (1.21), we need to know what happens to D_ε when we let $\varepsilon \to 0^+$. Now if (ξ, η) lies on a smooth part of C (not at where the gradient of the curve changes abruptly, that is, not at a corner point, if there is any), one can intuitively see that the part of C inside C_ε approaches an infinitesimal straight line as $\varepsilon \to 0^+$. Thus, we expect D_ε to tend to a semi-circle as $\varepsilon \to 0^+$, if (ξ, η) lies on a smooth part of C. It follows that in attempting to evaluate the limit on the right hand side of Eq. (1.21) we have to integrate over only half a circle (instead of a full circle as in the case of Eq. (1.13)).

Modifying Eqs. (1.17) and (1.18), we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \int\limits_{D_\varepsilon} \phi(x,y) \frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta)] ds(x,y) &= -\frac{1}{2}\phi(\xi,\eta), \\
\lim_{\varepsilon \to 0^+} \int\limits_{D_\varepsilon} \Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}[\phi(x,y)] ds(x,y) &= 0.
\end{aligned}$$

Hence Eq. (1.21) gives

$$\frac{1}{2}\phi(\xi,\eta) = \int\limits_{C} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))] ds(x,y)$$

$$\text{for } (\xi,\eta) \text{ lying on a smooth part of } C. \quad (1.22)$$

Together with the boundary conditions in Eq. (1.2), Eq. (1.22) may be applied to obtain a numerical procedure for determining the unknown ϕ and/or $\partial\phi/\partial n$ on the boundary C. Once ϕ and $\partial\phi/\partial n$ are known at all points on C, the solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2) is given by Eq. (1.19) at any point (ξ, η) inside R. More details are given in Section 1.5 below.

For convenience, we may write Eqs. (1.11), (1.19) and (1.22) as a single equation given by

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int\limits_{C} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))] ds(x,y), \quad (1.23)$$

if we define

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta) \notin R \cup C, \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } C, \\ 1 & \text{if } (\xi,\eta) \in R. \end{cases} \quad (1.24)$$

1.5 Boundary Element Solution with Constant Elements

We now show how Eq. (1.23) may be applied to obtain a simple boundary element procedure for solving numerically the interior boundary value problem defined by Eqs. (1.1)-(1.2).

The boundary C is discretized into N very small straight line elements $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$, that is,

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N-1)} \cup C^{(N)}. \tag{1.25}$$

The boundary elements $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$ are constructed as follows. We put N well spaced out points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(N-1)}, y^{(N-1)})$ and $(x^{(N)}, y^{(N)})$ on C, in the order given, following the counter clockwise direction. Defining $(x^{(N+1)}, y^{(N+1)}) = (x^{(1)}, y^{(1)})$, we take $C^{(k)}$ to be the boundary element from $(x^{(k)}, y^{(k)})$ to $(x^{(k+1)}, y^{(k+1)})$ for $k = 1, 2, \cdots , N$.

As an example, in Figure 1.4, the boundary $C = C_1 \cup C_2$ in Figure 1.1 is approximated using 5 boundary elements denoted by $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, $C^{(4)}$ and $C^{(5)}$.

For a simple approximation of ϕ and $\partial\phi/\partial n$ on the boundary C, we assume that these functions are constants over each of the boundary elements. Specifically, we make the approximation:

$$\phi \simeq \overline{\phi}^{(k)} \text{ and } \frac{\partial \phi}{\partial n} = \overline{p}^{(k)} \text{ for } (x, y) \in C^{(k)} \ (k = 1, 2, \cdots , N), \tag{1.26}$$

where $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ are respectively the values of ϕ and $\partial\phi/\partial n$ at the midpoint of $C^{(k)}$.

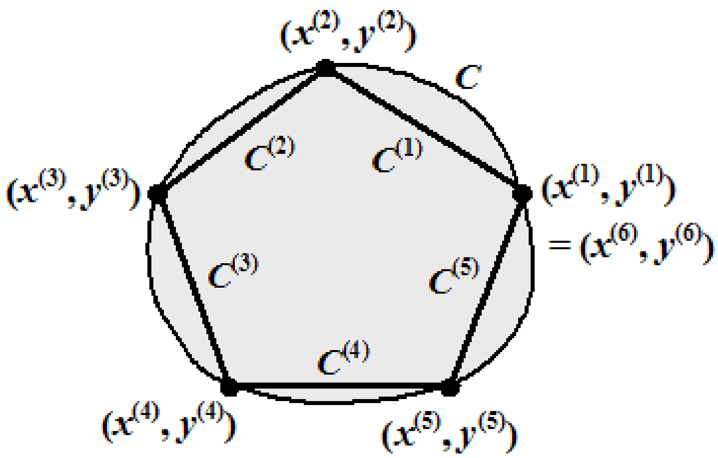

Figure 1.4

With Eqs. (1.25) and (1.26), we find that Eq. (1.23) can be approximately written as

$$\lambda(\xi,\eta)\phi(\xi,\eta)=\sum_{k=1}^{N}\{\overline{\phi}^{(k)}\mathcal{F}_2^{(k)}(\xi,\eta)-\overline{p}^{(k)}\mathcal{F}_1^{(k)}(\xi,\eta)\},\tag{1.27}$$

where

$$\begin{aligned}\mathcal{F}_1^{(k)}(\xi,\eta)&=\int_{C^{(k)}}\Phi(x,y;\xi,\eta)ds(x,y),\\ \mathcal{F}_2^{(k)}(\xi,\eta)&=\int_{C^{(k)}}\frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta)]ds(x,y).\end{aligned}\tag{1.28}$$

For a given k, either $\overline{\phi}^{(k)}$ or $\overline{p}^{(k)}$ (not both) is known from the boundary conditions in Eq. (1.2). Thus, there are N unknown constants on the right hand side of Eq. (1.27). To determine their values, we have to generate N equations containing the unknowns.

If we let (ξ,η) in Eq. (1.27) be given in turn by the midpoints of $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$, we obtain

$$\frac{1}{2}\overline{\phi}^{(m)}=\sum_{k=1}^{N}\{\overline{\phi}^{(k)}\mathcal{F}_2^{(k)}(\overline{x}^{(m)},\overline{y}^{(m)})-\overline{p}^{(k)}\mathcal{F}_1^{(k)}(\overline{x}^{(m)},\overline{y}^{(m)})\}\quad\text{for } m=1,2,\cdots,N,\tag{1.29}$$

where $(\overline{x}^{(m)},\overline{y}^{(m)})$ is the midpoint of $C^{(m)}$.

In the derivation of Eq. (1.29), we take $\lambda(\overline{x}^{(m)},\overline{y}^{(m)})=1/2$, since $(\overline{x}^{(m)},\overline{y}^{(m)})$ being the midpoint of $C^{(m)}$ lies on a smooth part of the approximate boundary $C^{(1)}\cup C^{(2)}\cup\cdots\cup C^{(N-1)}\cup C^{(N)}$.

Eq. (1.29) constitutes a system of N linear algebraic equations containing the N unknowns on the right hand side of Eq. (1.27). We may rewrite it as

$$\sum_{k=1}^{N}a^{(mk)}z^{(k)}=\sum_{k=1}^{N}b^{(mk)}\text{ for } m=1,2,\cdots,N,\tag{1.30}$$

where $a^{(mk)}$, $b^{(mk)}$ and $z^{(k)}$ are defined by

$$a^{(mk)} = \begin{cases} -\mathcal{F}_1^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) & \text{if } \phi \text{ is specified over } C^{(k)}, \\ \mathcal{F}_2^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) - \frac{1}{2}\delta^{(mk)} & \text{if } \partial\phi/\partial n \text{ is specified over } C^{(k)}, \end{cases}$$

$$b^{(mk)} = \begin{cases} \overline{\phi}^{(k)}(-\mathcal{F}_2^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) + \frac{1}{2}\delta^{(mk)}) & \text{if } \phi \text{ is specified over } C^{(k)}, \\ \overline{p}^{(k)}\mathcal{F}_1^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) & \text{if } \partial\phi/\partial n \text{ is specified over } C^{(k)}, \end{cases}$$

$$\delta^{(mk)} = \begin{cases} 0 & \text{if } m \neq k, \\ 1 & \text{if } m = k, \end{cases}$$

$$z^{(k)} = \begin{cases} \overline{p}^{(k)} & \text{if } \phi \text{ is specified over } C^{(k)}, \\ \overline{\phi}^{(k)} & \text{if } \partial\phi/\partial n \text{ is specified over } C^{(k)}. \end{cases} \tag{1.31}$$

Note that $z^{(1)}$, $z^{(2)}$, $\cdots$, $z^{(N-1)}$ and $z^{(N)}$ are the N unknown constants on the right hand side of Eq. (1.27), while $a^{(mk)}$ and $b^{(mk)}$ are known coefficients.

Once Eq. (1.30) is solved for the unknowns $z^{(1)}$, $z^{(2)}$, $\cdots$, $z^{(N-1)}$ and $z^{(N)}$, the values of ϕ and $\partial\phi/\partial n$ over the element $C^{(k)}$, as given by $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ respectively, are known for $k = 1, 2, \cdots , N$. Eq. (1.27) with $\lambda(\xi, \eta) = 1$ then provides us with an explicit formula for computing ϕ in the interior of R, that is,

$$\phi(\xi, \eta) \simeq \sum_{k=1}^{N}\{\overline{\phi}^{(k)}\mathcal{F}_2^{(k)}(\xi, \eta) - \overline{p}^{(k)}\mathcal{F}_1^{(k)}(\xi, \eta)\} \text{ for } (\xi, \eta) \in R. \tag{1.32}$$

To summarize, a boundary element solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2) is given by Eq. (1.32) together with Eqs. (1.28), (1.30) and (1.31). Because of the approximation in Eqs. (1.25) and (1.26), the solution is said to be obtained using constant elements. Analytical formulae for calculating $\mathcal{F}_1^{(k)}(\xi, \eta)$ and $\mathcal{F}_2^{(k)}(\xi, \eta)$ in Eq. (1.28) are given in Eqs. (1.37), (1.38), (1.40) and (1.41) (together with Eq. (1.35)) in the section below.

1.6 Formulae for Integrals of Constant Elements

The boundary element solution above requires the evaluation of $\mathcal{F}_1^{(k)}(\xi, \eta)$ and $\mathcal{F}_2^{(k)}(\xi, \eta)$. These functions are defined in terms of line integrals over $C^{(k)}$ as given in Eq. (1.28). The line integrals can be worked out analytically as follows.

Points on the element $C^{(k)}$ may be described using the parametric equations

$$\left.\begin{array}{l} x = x^{(k)} - t\ell^{(k)}n_y^{(k)} \\ y = y^{(k)} + t\ell^{(k)}n_x^{(k)} \end{array}\right\} \text{ from } t = 0 \text{ to } t = 1, \tag{1.33}$$

where $\ell^{(k)}$ is the length of $C^{(k)}$ and $[n_x^{(k)}, n_y^{(k)}] = [y^{(k+1)} - y^{(k)}, x^{(k)} - x^{(k+1)}]/\ell^{(k)}$ is the unit normal vector to $C^{(k)}$ pointing away from R.

For $(x,y)\in C^{(k)}$, we find that $ds(x,y)=\sqrt{(dx)^2+(dy)^2}=\ell^{(k)}dt$ and

$$(x-\xi)^2+(y-\eta)^2=A^{(k)}t^2+B^{(k)}(\xi,\eta)t+E^{(k)}(\xi,\eta), \tag{1.34}$$

where

$$\begin{aligned} A^{(k)} &= [\ell^{(k)}]^2,\\ B^{(k)}(\xi,\eta) &= [-n_y^{(k)}(x^{(k)}-\xi)+(y^{(k)}-\eta)n_x^{(k)}](2\ell^{(k)}),\\ E^{(k)}(\xi,\eta) &= (x^{(k)}-\xi)^2+(y^{(k)}-\eta)^2. \end{aligned} \tag{1.35}$$

The parameters in Eq. (1.35) satisfy $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2\geq 0$ for any point (ξ,η). To see why this is true, consider the straight line defined by the parametric equations $x=x^{(k)}-t\ell^{(k)}n_y^{(k)}$ and $y=y^{(k)}+t\ell^{(k)}n_x^{(k)}$ for $-\infty<t<\infty$. Note that $C^{(k)}$ is a subset of this straight line (given by the parametric equations from $t=0$ to $t=1$). Eq. (1.34) also holds for any point (x,y) lying on the extended line. If (ξ,η) does not lie on the line then $A^{(k)}t^2+B^{(k)}(\xi,\eta)t+E^{(k)}(\xi,\eta)>0$ for all real values of t (that is, for all points (x,y) on the line) and hence $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2>0$. On the other hand, if (ξ,η) is on the line, we can find exactly one point (x,y) such that $A^{(k)}t^2+B^{(k)}(\xi,\eta)t+E^{(k)}(\xi,\eta)=0$. As each point (x,y) on the line is given by a unique value of t, we conclude that $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0$ for (ξ,η) lying on the line.

From Eqs. (1.28), (1.33) and (1.34), $\mathcal{F}_1^{(k)}(\xi,\eta)$ and $\mathcal{F}_2^{(k)}(\xi,\eta)$ may be written as

$$\begin{aligned} \mathcal{F}_1^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}}{4\pi}\int_0^1 \ln[A^{(k)}t^2+B^{(k)}(\xi,\eta)t+E^{(k)}(\xi,\eta)]dt,\\ \mathcal{F}_2^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}}{2\pi}\int_0^1 \frac{n_x^{(k)}(x^{(k)}-\xi)+n_y^{(k)}(y^{(k)}-\eta)}{A^{(k)}t^2+B^{(k)}(\xi,\eta)t+E^{(k)}(\xi,\eta)}dt. \end{aligned} \tag{1.36}$$

The second integral in Eq. (1.36) is the easiest one to work out for the case in which $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0$. For this case, the point (ξ,η) lies on the straight line of which the element $C^{(k)}$ is a subset. Thus, the vector $[x^{(k)}-\xi,y^{(k)}-\eta]$ is perpendicular to $[n_x^{(k)},n_y^{(k)}]$, that is, $n_x^{(k)}(x^{(k)}-\xi)+n_y^{(k)}(y^{(k)}-\eta)=0$, and we obtain

$$\mathcal{F}_2^{(k)}(\xi,\eta)=0 \text{ for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0. \tag{1.37}$$

From the integration formula

$$\int\frac{dt}{at^2+bt+c} = \frac{2}{\sqrt{4ac-b^2}}\arctan(\frac{2at+b}{\sqrt{4ac-b^2}})+\text{ constant}$$
$$\text{for real constants } a,\ b \text{ and } c \text{ such that } 4ac-b^2>0,$$

we find that

$$\begin{aligned}
\mathcal{F}_2^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}[n_x^{(k)}(x^{(k)}-\xi)+n_y^{(k)}(y^{(k)}-\eta)]}{\pi\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}} \\
&\quad\times[\arctan(\frac{2A^{(k)}+B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}}) \\
&\quad-\arctan(\frac{B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}})] \\
&\quad\text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2>0. \qquad (1.38)
\end{aligned}$$

If $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0$, we may write

$$A^{(k)}t^2+B^{(k)}(\xi,\eta)t+E^{(k)}(\xi,\eta)=A^{(k)}(t+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})^2.$$

Thus,

$$\begin{aligned}
\mathcal{F}_1^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}}{4\pi}\int_0^1 \ln[A^{(k)}(t+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})^2]dt \\
&\quad\text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0. \qquad (1.39)
\end{aligned}$$

Now if (ξ,η) lies on a smooth part of $C^{(k)}$, the integral in Eq. (1.39) is improper, as its integrand is not well defined at the point $t=t_0\equiv -B^{(k)}(\xi,\eta)/(2A^{(k)})\in(0,1)$. Strictly speaking, the integral should then be interpreted in the Cauchy principal sense, that is, to evaluate it, we have to integrate over $[0,t_0-\varepsilon]\cup[t_0+\varepsilon,1]$ instead of $[0,1]$ and then let $\varepsilon\to 0$ to obtain its value. However, in this case, it turns out that the limits of integration $t=t_0-\varepsilon$ and $t=t_0+\varepsilon$ eventually do not contribute anything to the integral. Thus, for $4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0$, the final analytical formula for $\mathcal{F}_1^{(k)}(\xi,\eta)$ is the same irrespective of whether (ξ,η) lies on $C^{(k)}$ or not. If (ξ,η) lies on $C^{(k)}$, we may ignore the singular behaviour of the integrand and apply the fundamental theorem of integral calculus as usual to evaluate the definite integral in Eq. (1.39) directly over $[0,1]$.

The integration required in Eq. (1.39) can be easily done to give

$$\begin{aligned}
\mathcal{F}_1^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}}{2\pi}\{\ln(\ell^{(k)})+(1+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})\ln|1+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}| \\
&\quad-\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\ln|\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}|-1\} \\
&\quad\text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2=0. \qquad (1.40)
\end{aligned}$$

Using

$$\int \ln(at^2+bt+c)dt = t[\ln(a)-2]+(t+\frac{b}{2a})\ln[t^2+\frac{b}{a}t+\frac{c}{a}] + \frac{1}{a}\sqrt{4ac-b^2}\arctan(\frac{2at+b}{\sqrt{4ac-b^2}}) + \text{ constant}$$

for real constants a, b and c such that $4ac-b^2>0$,

we obtain

$$\begin{aligned}\mathcal{F}_1^{(k)}(\xi,\eta) = & \frac{\ell^{(k)}}{4\pi}\{2[\ln(\ell^{(k)})-1]-\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}}\ln|\frac{E^{(k)}(\xi,\eta)}{A^{(k)}}| \\ & +(1+\frac{B^{(k)}(\xi,\eta)}{2A^{(k)}})\ln|1+\frac{B^{(k)}(\xi,\eta)}{A^{(k)}}+\frac{E^{(k)}(\xi,\eta)}{A^{(k)}}| \\ & +\frac{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}}{A^{(k)}} \\ & \times[\arctan(\frac{2A^{(k)}+B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}}) \\ & -\arctan(\frac{B^{(k)}(\xi,\eta)}{\sqrt{4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2}})]\} \\ & \text{for } 4A^{(k)}E^{(k)}(\xi,\eta)-[B^{(k)}(\xi,\eta)]^2>0. \end{aligned} \quad (1.41)$$

1.7 Implementation on Computer

We attempt now to develop double precision FORTRAN 77 codes which can be used to implement the boundary element procedure described in Section 1.5 on the computer. In our discussion here, syntaxes, variables and statements in FORTRAN 77 are written in typewriter fonts, for example, `xi`, `eta` and `A=L**2d0`.

One of the tasks involved is the setting up of the system of linear algebraic equations given in Eqs. (1.30) and (1.31). To do this, the functions $\mathcal{F}_1^{(k)}(\xi,\eta)$ and $\mathcal{F}_2^{(k)}(\xi,\eta)$ have to be computed using the formulae in Section 1.6. We create a subroutine called `CPF` which accepts the values of ξ, η, $x^{(k)}$, $y^{(k)}$, $n_x^{(k)}$, $n_y^{(k)}$ and $\ell^{(k)}$(stored in the real variables `xi`, `eta`, `xk`, `yk`, `nkx`, `nky` and `L`) in order to calculate and return the values of $\pi\mathcal{F}_1^{(k)}(\xi,\eta)$ and $\pi\mathcal{F}_2^{(k)}(\xi,\eta)$ (in the real variables `PF1` and `PF2`).

The subroutine `CPF` is listed below.

```
      subroutine CPF(xi,eta,xk,yk,nkx,nky,L,PF1,PF2)

      double precision xi,eta,xk,yk,nkx,nky,L,PF1,PF2,
     & A,B,E,D,BA,EA
```

```
      A=L**2d0
      B=2d0*L*(-nky*(xk-xi)+nkx*(yk-eta))
      E=(xk-xi)**2d0+(yk-eta)**2d0
      D=dsqrt(dabs(4d0*A*E-B**2d0))
      BA=B/A
      EA=E/A

      if (D.lt.0.0000000001d0) then
      PF1=0.5d0*L*(dlog(L)
    & +(1d0+0.5d0*BA)*dlog(dabs(1d0+0.5d0*BA))
    & -0.5d0*BA*dlog(dabs(0.5d0*BA))-1d0)
      PF2=0d0
      else
      PF1=0.25d0*L*(2d0*(dlog(L)-1d0)-0.5d0*BA*dlog(dabs(EA))
    & +(1d0+0.5d0*BA)*dlog(dabs(1d0+BA+EA))
    & +(D/A)*(datan((2d0*A+B)/D)-datan(B/D)))
      PF2=L*(nkx*(xk-xi)+nky*(yk-eta))/D
    & *(datan((2d0*A+B)/D)-datan(B/D))
      endif

      return
      end
```

CPF is repeatedly called in the subroutine CELAP1. CELAP1 reads in the number of boundary elements (N) in the real variable N, the midpoints $(\overline{x}^{(k)}, \overline{y}^{(k)})$ in the real arrays xm(1:N) and ym(1:N), the boundary points $(x^{(k)}, y^{(k)})$ in the real arrays xb(1:N+1) and yb(1:N+1), the normal vectors $(n_x^{(k)}, n_y^{(k)})$ in the real arrays nx(1:N) and ny(1:N), the lengths of the boundary elements in the real array lg(1:N) and the types of boundary conditions (on the boundary elements) in the integer array BCT(1:N) together with the corresponding boundary values in the real array BCV(1:N), set up and solve Eq. (1.30), and return all the values of $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ in the arrays phi(1:N) and dphi(1:N) respectively. (More details on the arrays BCT(1:N) and BCV(1:N) will be given later on in Section 1.8.) Thus, a large part of the boundary element procedure (with constant elements) for the numerical solution of the boundary value problem is executed in CELAP1.

The subroutine CELAP1 is listed as follows.

```
      subroutine CELAP1(N,xm,ym,xb,yb,nx,ny,lg,BCT,BCV,phi,dphi)

      integer m,k,N,BCT(1000)

      double precision xm(1000),ym(1000),xb(1000),yb(1000),
```

```
     & nx(1000),ny(1000),lg(1000),BCV(1000),A(1000,1000),
     & B(1000),pi,PF1,PF2,del,phi(1000),dphi(1000),F1,F2,
     & Z(1000)

       pi=4d0*datan(1d0)

       do 10 m=1,N
       B(m)=0d0
       do 5 k=1,N
       call CPF(xm(m),ym(m),xb(k),yb(k),nx(k),ny(k),lg(k),PF1,PF2)
       F1=PF1/pi
       F2=PF2/pi
       if (k.eq.m) then
       del=1d0
       else
       del=0d0
       endif
       if (BCT(k).eq.0) then
       A(m,k)=-F1
       B(m)=B(m)+BCV(k)*(-F2+0.5d0*del)
       else
       A(m,k)=F2-0.5d0*del
       B(m)=B(m)+BCV(k)*F1
       endif
     5 continue
    10 continue

       call solver(A,B,N,1,Z)

       do 15 m=1,N
       if (BCT(m).eq.0) then
       phi(m)=BCV(m)
       dphi(m)=Z(m)
       else
       phi(m)=Z(m)
       dphi(m)=BCV(m)
       endif
    15 continue

      return
      end
```

The values of $a^{(mk)}$ in Eq. (1.30) are kept in the real array `A(1:N,1:N)`, the sum $b^{(m1)} + b^{(m2)} + \cdots + b^{(mN)}$ on the right hand side of the equation in the real array `B(1:N)` and the solution $z^{(k)}$ in the real array `Z(1:N)`. To solve for $z^{(k)}$, an *LU* decomposition is performed on the matrix containing the coefficients $a^{(mk)}$ to obtain a simpler system that may be easily solved by backward substitutions. This is done in the subroutine `SOLVER` (listed below together with supporting subprograms `DAXPY`, `DSCAL` and `IDAMAX`†) which accepts the integer `N` (giving the number of unknowns), the real arrays `A(1:N,1:N)` and `B(1:N)`and the integer `lud` to return `Z(1:N)`. In general, the integer `lud` may be given any value except 0. However, if we are solving two different systems of linear algebraic equations with the same square matrice $[a^{(mk)}]$, one after the other, `lud` may be given the value 0 the second time `SOLVER` is called. This is because it is not necessary to perform the *LU* decomposition on the same square matrix again to solve the second system after solving the first. If `lud` is given the value 0, `SOLVER` assumes that the square matrix has already been properly decomposed before and avoids the time consuming decomposition process. In `CELAP1`, since the square matrix has not been decomposed yet, the value of 1 is passed into `lud` when we call `SOLVER`.

The subroutine `SOLVER` and its supporting programs are listed as follows.

```
      subroutine SOLVER(A,B,N,lud,Z)

      integer lda,N,ipvt(1000),info,lud,IDAMAX,
     & j,k,kp1,l,nm1,kb

      double precision A(1000,1000),B(1000),Z(1000),t,AMD(1000,1000)

      common /ludcmp/ipvt,AMD

      nm1=N-1

      do 5 i=1,N
      Z(i)=B(i)
    5 continue

      if (lud.eq.0) goto 99
```

†The main part of `SOLVER` for decomposing the square matrix A and solving AX = B is respectively taken from the codes in the LINPACK subroutines `DGEFA` and `DGESL` written by Cleve Moler. The supporting subprograms `DAXPY`, `DSCAL` and `IDAMAX` written by Jack Dongarra are also from LINPACK. `DGEFA`, `DGESL`, `DAXPY`, `DSCAL` and `IDAMAX` are all in the public domain and may be downloaded from Netlib website at http://www.netlib.org. Permission for reproducing the codes here was granted by Netlib's editor-in-chief Jack Dongarra.

```
      do 6 i=1,N
      do 6 j=1,N
      AMD(i,j)=A(i,j)
   6 continue

      info=0

      if (nm1.lt.1) go to 70

      do 60 k=1,nm1
      kp1=k+1
      l=IDAMAX(N-k+1,AMD(k,k),1)+k-1
      ipvt(k)=l
      if (AMD(l,k).eq.0.0d0) goto 40
      if (l.eq.k) goto 10
      t=AMD(l,k)
      AMD(l,k)=AMD(k,k)
      AMD(k,k)=t
   10 continue
      t=-1.0d0/AMD(k,k)
      call DSCAL(N-k,t,AMD(k+1,k),1)
      do 30 j=kp1,N
      t=AMD(l,j)
      if (l.eq.k) go to 20
      AMD(l,j)=AMD(k,j)
      AMD(k,j)=t
   20 continue
      call DAXPY(N-k,t,AMD(k+1,k),1,AMD(k+1,j),1)
   30 continue
      goto 50
   40 continue
      info=k
   50 continue
   60 continue

   70 continue

      ipvt(N)=N

      if (AMD(N,N).eq.0.0d0) info=N
      if (info.ne.0) pause 'Division by zero in SOLVER!'
```

```
 99 continue

    if (nm1.lt.1) goto 130

    do 120 k=1,nm1
    l=ipvt(k)
    t=Z(l)
    if (l.eq.k) goto 110
    Z(l)=Z(k)
    Z(k)=t
110 continue
    call DAXPY(N-k,t,AMD(k+1,k),1,Z(k+1),1)
120 continue

130 continue

    do 140 kb=1,N
    k=N+1-kb
    Z(k) = Z(k)/AMD(k,k)
    t=-Z(k)
    call DAXPY(k-1,t,AMD(1,k),1,Z(1),1)
140 continue

    return
    end

    subroutine DAXPY(N,da,dx,incx,dy,incy)

    double precision dx(1000),dy(1000),da

    integer i,incx,incy,ix,iy,m,mp1,N

    if(N.le.0) return
    if (da .eq.  0.0d0) return
    if(incx.eq.1.and.incy.eq.1) goto 20

    ix=1
    iy=1

    if(incx.lt.0) ix=(-N+1)*incx+1
    if(incy.lt.0) iy=(-N+1)*incy+1
```

```
   do 10 i=1,N
   dy(iy)=dy(iy)+da*dx(ix)
   ix=ix+incx
   iy=iy+incy
10 continue

   return

20 m=mod(N,4)

   if( m.eq.  0 ) go to 40

   do 30 i=1,m
   dy(i)=dy(i)+da*dx(i)
30 continue

   if(N.lt.4) return

40 mp1=m+1

   do 50 i=mp1,N,4
   dy(i)=dy(i)+da*dx(i)
   dy(i+1)=dy(i+1)+da*dx(i+1)
   dy(i+2)=dy(i+2)+da*dx(i+2)
   dy(i+3)=dy(i+3)+da*dx(i+3)
50 continue

   return
   end

   subroutine DSCAL(N,da,dx,incx)

   double precision da,dx(1000)

   integer i,incx,m,mp1,N,nincx

   if(N.le.0.or.incx.le.0) return
   if(incx.eq.1) goto 20
   nincx = N*incx

   do 10 i=1,nincx,incx
   dx(i)=da*dx(i)
```

```
 10 continue

    return

 20 m=mod(N,5)

    if(m.eq.0) goto 40

    do 30 i=1,m
    dx(i) = da*dx(i)
 30 continue

    if(N.lt.5) return

 40 mp1=m+1

    do 50 i=mp1,N,5
    dx(i)=da*dx(i)
    dx(i+1)=da*dx(i+1)
    dx(i+2)=da*dx(i+2)
    dx(i+3)=da*dx(i+3)
    dx(i+4)=da*dx(i+4)
  50 continue

    return
    end

    function IDAMAX(N,dx,incx)

    double precision dx(1000),dmax

    integer i,incx,ix,N,IDAMAX

    IDAMAX = 0
    if(N.lt.1.or.incx.le.0 ) return
    IDAMAX = 1

    if(N.eq.1)return
    if(incx.eq.1) goto 20
    ix = 1
    dmax = dabs(dx(1))
    ix = ix + incx
```

```
   do 10 i=2,N
   if(dabs(dx(ix)).le.dmax) goto 5
   IDAMAX=i
   dmax=dabs(dx(ix))
5  ix=ix+incx
10 continue

   return

20 dmax=dabs(dx(1))

   do 30 i=2,N
   if(dabs(dx(i)).le.dmax) goto 30
   IDAMAX=i
   dmax=dabs(dx(i))
30 continue

   return
   end
```

Once the values of $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ are returned in the arrays `phi(1:N)` and `dphi(1:N)` by `CELAP1`, they can be used by the subroutine `CELAP2` to compute the value of ϕ at any chosen point (ξ, η) in the interior of the solution domain. In the listing of `CELAP2` below, `xi` and `eta` are the real variables which carry the values of ξ and η respectively. The computed value of $\phi(\xi, \eta)$ is returned in the real variable `pint`. Note that the subroutine `CPF` is called in `CELAP2` to compute $\pi\mathcal{F}_1^{(k)}(\xi, \eta)$ and $\pi\mathcal{F}_2^{(k)}(\xi, \eta)$.

```
  subroutine CELAP2(N,xi,eta,xb,yb,nx,ny,lg,phi,dphi,pint)

  integer N,i

  double precision xi,eta,xb(1000),yb(1000),nx(1000),ny(1000),
& lg(1000),phi(1000),dphi(1000),pint,sum,pi,PF1,PF2

  pi=4d0*datan(1d0)
  sum=0d0

  do 10 i=1,N
  call CPF(xi,eta,xb(i),yb(i),nx(i),ny(i),lg(i),PF1,PF2)
  sum=sum+phi(i)*PF2-dphi(i)*PF1
```

```
10 continue

   pint=sum/pi

   return
   end
```

1.8 Numerical Examples

We now show how the subroutines CELAP1 and CELAP2 may be used to solve two specific examples of the interior boundary value problem described in Section 1.1.

Example 1.1

The solution domain is the square region $0 < x < 1$, $0 < y < 1$. The boundary conditions are

$$\left.\begin{array}{ll} \phi = 0 & \text{on } x = 0 \\ \phi = \cos(\pi y) & \text{on } x = 1 \end{array}\right\} \text{ for } 0 < y < 1$$

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } y = 0 \text{ and } y = 1 \text{ for } 0 < x < 1.$$

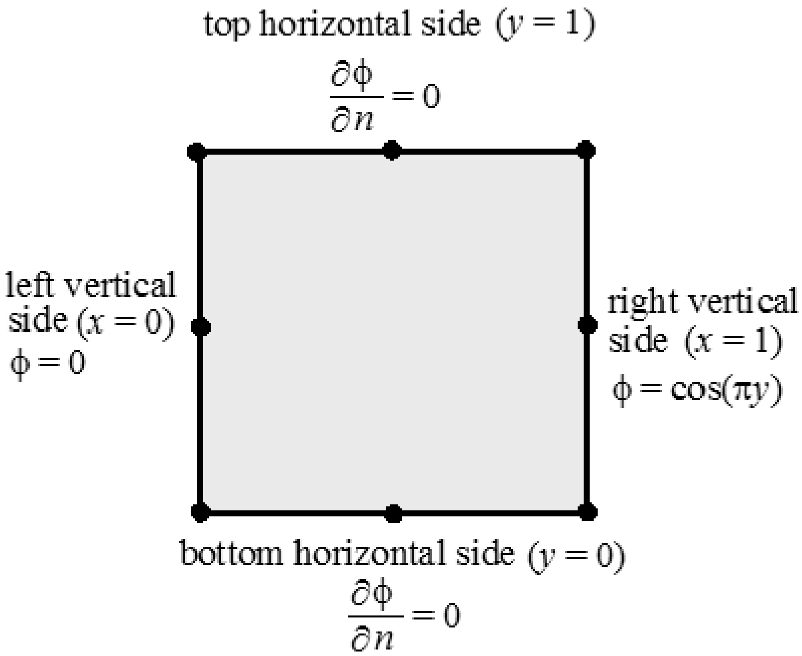

Figure 1.5

The sides of the square are discretized into boundary elements of equal length. To do this, we choose N evenly spaced out points on the sides as follows. The boundary points on the sides $y = 0$ (bottom horizontal), $x = 1$ (right vertical), $y = 1$

(top horizontal) and $x = 0$ (left vertical) are respectively given by $(x^{(m)}, y^{(m)}) = ([m-1]\ell, 0)$, $(x^{(m+N_0)}, y^{(m+N_0)}) = (1, [m-1]\ell)$, $(x^{(m+2N_0)}, y^{(m+2N_0)}) = (1-[m-1]\ell, 1)$ and $(x^{(m+3N_0)}, y^{(m+3N_0)}) = (0, 1-[m-1]\ell)$ for $m = 1, 2, \cdots, N_0$, where N_0 is the number of boundary elements per side (so that $N = 4N_0$) and $\ell = 1/N_0$ is the length of each element. For example, the boundary points for $N_0 = 2$ (that is, 8 boundary elements) are shown in Figure 1.5.

The input points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(N-1)}, y^{(N-1)})$, $(x^{(N)}, y^{(N)})$ and $(x^{(N+1)}, y^{(N+1)})$, arranged in counter clockwise order on the boundary of the solution domain, are stored in the real arrays `xb(1:N+1)` and `yb(1:N+1)`. (Recall that $(x^{(N+1)}, y^{(N+1)}) = (x^{(1)}, y^{(1)})$.) The values in these arrays are input data defining the geometry of the solution domain, to be generated by the user of the subroutines `CELAP1` and `CELAP2`. As the geometry in this example is a simple one, the input data for the boundary points may be generated by writing a simple code as follows.

```
  N=4*N0
  dl=1d0/dfloat(N0)
  do 10 i=1,N0
  xb(i)=dfloat(i-1)*dl
  yb(i)=0d0
  xb(N0+i)=1d0
  yb(N0+i)=xb(i)
  xb(2*N0+i)=1d0-xb(i)
  yb(2*N0+i)=1d0
  xb(3*N0+i)=0d0
  yb(3*N0+i)=1d0-xb(i)
10 continue
  xb(N+1)=xb(1)
  yb(N+1)=yb(1)
```

Note that `N0` is an integer variable which gives the number of boundary elements per side and `dl` is a real variable giving the length of an element. The value of `N0` is a given input. The boundary points in Figure 1.5 may be generated by the code above if we give `N0` the value of 2.

In order to call `CELAP1` and `CELAP2`, the midpoints of the elements (in the real arrays `xm(1:N)` and `ym(1:N)`), the lengths of the elements (in the real array `lg(1:N)`) and the unit normal vectors to the elements (in the real arrays `nx(1:N)` and `ny(1:N)`) are required. These can be calculated from the input data stored in the arrays `xb(1:N+1)` and `yb(1:N+1)`. The general code for the calculation (which is valid for any geometry of the solution domain) is as follows.

```
  do 20 i=1,N
  xm(i)=0.5d0*(xb(i)+xb(i+1))
```

```
  ym(i)=0.5d0*(yb(i)+yb(i+1))
  lg(i)=dsqrt((xb(i+1)-xb(i))**2d0+(yb(i+1)-yb(i))**2d0)
  nx(i)=(yb(i+1)-yb(i))/lg(i)
  ny(i)=(xb(i)-xb(i+1))/lg(i)
20 continue
```

The type of boundary conditions on an element (that is, whether ϕ or $\partial\phi/\partial n$ is specified) and the corresponding specified value of either ϕ or $\partial\phi/\partial n$ are input data. The integer array `BCT(1:N)`is used to keep track of the types of boundary conditions on the elements. If ϕ is specified on the 5-th boundary element $C^{(5)}$ then `BCT(5)` is given the value 0. If `BCT(5)` is not 0, then we know that $\partial\phi/\partial n$ is specified on $C^{(5)}$. The values of either ϕ or $\partial\phi/\partial n$ prescribed on the boundary elements are stored in the real array `BCV(1:N)`. For the boundary points in Figure 1.5, the input boundary values of ϕ on the two elements on the right vertical sides are given by $\cos(\pi\eta)$ with η being the y coordinates of the midpoints of the elements. For the boundary value problem here, the code for generating the input data for `BCT` and `BCV` are as follows.

```
  do 30 i=1,N
  if (i.le.N0) then
  BCT(i)=1
  BCV(i)=0d0
  else if ((i.gt.N0).and.(i.le.(2*N0))) then
  BCT(i)=0
  BCV(i)=dcos(pi*ym(i))
  else if ((i.gt.(2*N0)).and.(i.le.(3*N0))) then
  BCT(i)=1
  BCV(i)=0d0
  else
  BCT(i)=0
  BCV(i)=0d0
  endif
30 continue
```

We may now invoke `CELAP1` using the statement

```
call CELAP1(N, xm, ym, xb, yb, nx, ny, lg, BCT, BCV, phi, dphi)
```

to give us the (approximate) values of ϕ and $\partial\phi/\partial n$ on the boundary elements. The boundary values of ϕ and $\partial\phi/\partial n$ (that is, $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$) are respectively stored in the real arrays `phi(1:N)` and `dphi(1:N)`. For example, if the variable `BCT(5)` has the value 0, we know that ϕ is specified on the 5-th boundary element and hence the variable `dphi(5)` gives us the approximate value of $\partial\phi/\partial n$ on $C^{(5)}$.

Once CELAP1 is called, we may use CELAP2 to calculate the value of ϕ at any interior point inside the square. For example, if we wish to calculate ϕ at $(0.50, 0.70)$, we may use the call statement

```
call CELAP2(N,0.50,0.70,xb,yb,nx,ny,lg,phi,dphi,pint)
```

to return us the approximate value of $\phi(0.50, 0.70)$ in the real variable pint.

An example of a complete program for the boundary value problem presently under consideration is given below.

```
      program EX1PT1

      integer N0,BCT(1000),N,i,ians

      double precision xb(1000),yb(1000),xm(1000),ym(1000),
     & nx(1000),ny(1000),lg(1000),BCV(1000),
     & phi(1000),dphi(1000),pint,dl,xi,eta,pi

      print*,'Enter number of elements per side (<250):'
      read*,N0
      N=4*N0

      pi=4d0*datan(1d0)
      dl=1d0/dfloat(N0)

      do 10 i=1,N0
      xb(i)=dfloat(i-1)*dl
      yb(i)=0d0
      xb(N0+i)=1d0
      yb(N0+i)=xb(i)
      xb(2*N0+i)=1d0-xb(i)
      yb(2*N0+i)=1d0
      xb(3*N0+i)=0d0
      yb(3*N0+i)=1d0-xb(i)
   10 continue
      xb(N+1)=xb(1)
      yb(N+1)=yb(1)

      do 20 i=1,N
      xm(i)=0.5d0*(xb(i)+xb(i+1))
      ym(i)=0.5d0*(yb(i)+yb(i+1))
      lg(i)=dsqrt((xb(i+1)-xb(i))**2d0+(yb(i+1)-yb(i))**2d0)
      nx(i)=(yb(i+1)-yb(i))/lg(i)
```

```
      ny(i)=(xb(i)-xb(i+1))/lg(i)
   20 continue

      do 30 i=1,N
      if (i.le.N0) then
      BCT(i)=1
      BCV(i)=0d0
      else if ((i.gt.N0).and.(i.le.(2*N0))) then
      BCT(i)=0
      BCV(i)=dcos(pi*ym(i))
      else if ((i.gt.(2*N0)).and.(i.le.(3*N0))) then
      BCT(i)=1
      BCV(i)=0d0
      else
      BCT(i)=0
      BCV(i)=0d0
      endif
   30 continue

      call CELAP1(N,xm,ym,xb,yb,nx,ny,lg,BCT,BCV,phi,dphi)

   50 print*,'Enter coordinates xi and eta of an interior point:'

      read*,xi,eta

      call CELAP2(N,xi,eta,xb,yb,nx,ny,lg,phi,dphi,pint)

      write(*,60)pint,(dexp(pi*xi)-dexp(-pi*xi))*dcos(pi*eta)
     & /(dexp(pi)-dexp(-pi))
   60 format('Numerical and exact values are:',
     & F14.6,' and',F14.6,' respectively')

      print*,'To continue with another point enter 1:'
      read*,ians

      if (ians.eq.1) goto 50

      end
```

All the subprograms needed for compiling EX1PT1 into an executable program are the subroutines CELAP1, CELAP2, CPF and SOLVER (together with its supporting subprograms DAXPY, DSCAL and IDAMAX).

It is easy to check that boundary value problem here has the exact solution

$$\phi = \frac{\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}.$$

In the program `EX1PT1` above, the numerical value of ϕ (as calculated by the boundary element procedure with constant elements) at an input interior point (ξ, η) is compared with the exact solution.

Table 1.1

(ξ,η)	20 elements	80 elements	Exact
$(0.10, 0.20)$	0.022605	0.022397	0.022371
$(0.10, 0.30)$	0.016454	0.016279	0.016254
$(0.10, 0.40)$	0.008681	0.008560	0.008545
$(0.50, 0.20)$	0.163153	0.161521	0.161212
$(0.50, 0.30)$	0.118290	0.117325	0.117127
$(0.50, 0.40)$	0.062107	0.061673	0.061577
$(0.90, 0.20)$	0.586250	0.590103	0.589941
$(0.90, 0.30)$	0.427451	0.428609	0.428618
$(0.90, 0.40)$	0.223159	0.225308	0.225338

The numerical values of ϕ at various interior points obtained by `EX1PT1` using 20 and 80 boundary elements are compared with the exact solution in Table 1.1. There is a significant improvement in the accuracy of the numerical results when the number of boundary elements used is increased from 20 to 80.

Table 1.2

a	0.900	0.950	0.990	0.995	0.999
20 elements	0.136%	2.830%	8.504%	9.563%	10.601%
80 elements	0.111%	0.144%	0.716%	1.403%	2.213%

We also examine the accuracy of the numerical value of ϕ at the interior point (a, a) as a approaches 1 from below, that is, as the point (a, a) gets closer and closer to the point $(1, 1)$ on the boundary of the square domain. The percentage errors in the numerical values of ϕ from calculations using 20 and 80 boundary elements are shown in Table 1.2 for various values of a. In each of the two sets of results, it is interesting to note that the percentage error grows as a approaches 1. For a fixed value of a near 1, the percentage error of the numerical value of ϕ calculated with 80 elements are lower than that obtained using 20 elements. It is a well known fact that the accuracy of a boundary element solution may deteriorate significantly at a point whose distance from the boundary is very small compared with the lengths of nearby boundary elements.

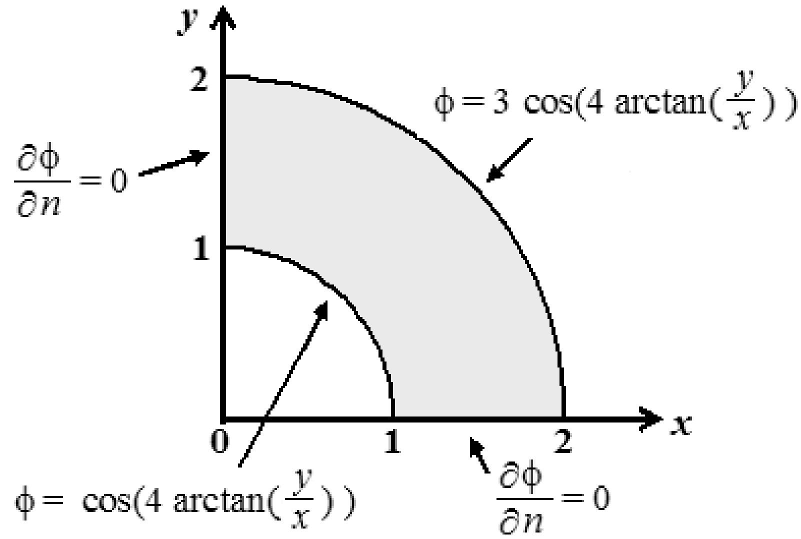

Figure 1.6

Example 1.2

Take the solution domain to be the region bounded between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first quadrant of the Oxy plane as shown in Figure 1.6. The boundary conditions are given by

$$\begin{aligned}
\frac{\partial\phi}{\partial n} &= 0 \text{ on the straight side } x = 0,\ 1 < y < 2, \\
\frac{\partial\phi}{\partial n} &= 0 \text{ on the straight side } y = 0,\ 1 < x < 2, \\
\phi &= \cos(4\arctan(\frac{y}{x})) \text{ on the arc } x^2 + y^2 = 1,\ x > 0,\ y > 0, \\
\phi &= 3\cos(4\arctan(\frac{y}{x})) \text{ on the arc } x^2 + y^2 = 4,\ x > 0,\ y > 0.
\end{aligned}$$

This boundary value problem may be solved numerically using the boundary element procedure with constant elements as in Example 1.1. To do this, we only have to modify the parts in the program `EX1PT1` that generate input data for the arrays `xb(1:N+1)`, `yb(1:N+1)`, `BCT(1:N)` and `BCV(1:N)`. Before we modify the program, we have to work out formulae for the boundary points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(N-1)}, y^{(N-1)})$ and $(x^{(N)}, y^{(N)})$.

Let us discretize each of the straight sides of the boundary into N_0 elements and the arcs on $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ into $2N_0$ and $8N_0$ elements respectively, so that $N = 12N_0$. Specifically, the boundary points are given by

$$\begin{aligned}
(x^{(m)}, y^{(m)}) &= (1 + \frac{[m-1]}{N_0}, 0) \text{ for } m = 1, 2, \cdots, N_0, \\
(x^{(m+N_0)}, y^{(m+N_0)}) &= (2\cos(\frac{[m-1]\pi}{16N_0}), 2\sin(\frac{[m-1]\pi}{16N_0})) \text{ for } m = 1, 2, \cdots, 8N_0,
\end{aligned}$$

$$(x^{(m+9N_0)}, y^{(m+9N_0)}) = (0, 2 - \frac{[m-1]}{N_0}) \text{ for } m = 1, 2, \cdots, N_0,$$

$$(x^{(m+10N_0)}, y^{(m+10N_0)}) = (\sin(\frac{[m-1]\pi}{4N_0}), \cos(\frac{[m-1]\pi}{4N_0})) \text{ for } m = 1, 2, \cdots, 2N_0.$$

Thus, for the boundary value problem presently under consideration, the code for generating the input data for the boundary points in the real arrays `xb(1:N+1)` and `yb(1:N+1)` is as given below. Note that we are required to supply an input value for the integer `N0`.

```
      N=12*N0
      pi=4d0*datan(1d0)

      do 10 i=1,8*N0
      dl=pi/dfloat(16*N0)
      xb(i+N0)=2d0*dcos(dfloat(i-1)*dl)
      yb(i+N0)=2d0*dsin(dfloat(i-1)*dl)
      if (i.le.N0) then
      dl=1d0/dfloat(N0)
      xb(i)=1d0+dfloat(i-1)*dl
      yb(i)=0d0
      xb(i+9*N0)=0d0
      yb(i+9*N0)=2d0-dfloat(i-1)*dl
      endif
      if (i.le.(2*N0)) then
      dl=pi/dfloat(4*N0)
      xb(i+10*N0)=dsin(dfloat(i-1)*dl)
      yb(i+10*N0)=dcos(dfloat(i-1)*dl)
      endif
   10 continue
      xb(N+1)=xb(1)
      yb(N+1)=yb(1)
```

The code for generating the input data for the integer array `BCT(1:N)` and the real array `BCV(1:N)` is as given below.

```
      do 30 i=1,N
      if ((i.le.N0).or.((i.gt.(9*N0)).and.(i.le.(10*N0)))) then
      BCT(i)=1
      BCV(i)=0d0
      else if ((i.gt.N0).and.(i.le.(9*N0))) then
      BCT(i)=0
```

```
      BCV(i)=3d0*dcos(4d0*datan(ym(i)/xm(i)))
      else
      BCT(i)=0
      BCV(i)=dcos(4d0*datan(ym(i)/xm(i)))
      endif
   30 continue
```

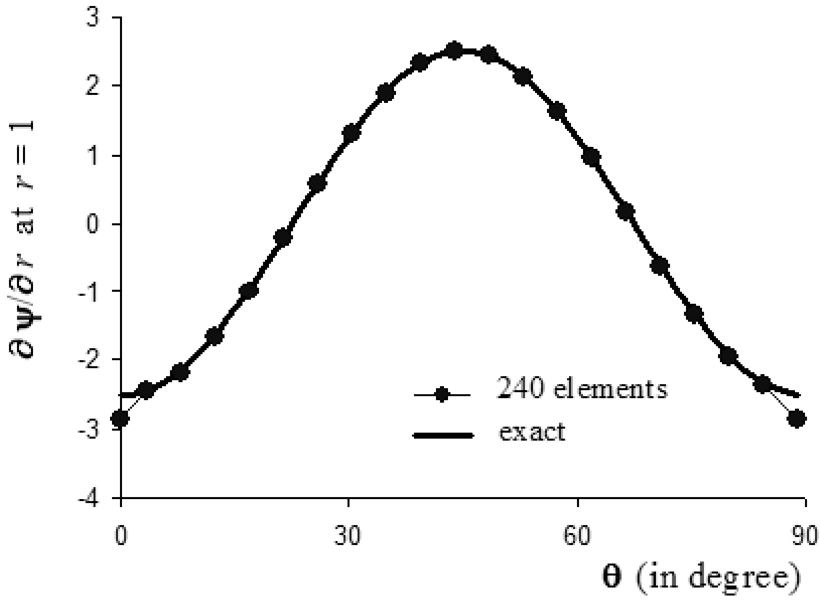

Figure 1.7

As ϕ is specified on the arc $x^2+y^2 = 1$, $x > 0$, $y > 0$, the last $2N_0$ variables in the array `dphi(1:N)`returned by `CELAP1` give us the numerical values of $\partial\phi/\partial n$ at the midpoints of the last $2N_0$ boundary elements, that is, $-\partial\psi/\partial r$ at those midpoints if we define $\psi(r, \theta) = \phi(r\cos\theta, r\sin\theta)$, where the polar coordinates r and θ are given by by $x = r\cos\theta$ and $y = r\sin\theta$. We may print out these variables to obtain the approximate values of $\partial\psi/\partial r$ at the midpoints of the last $2N_0$ boundary elements. In Figure 1.7, the numerical $\partial\psi/\partial r$ at $r = 1$, $0 < \theta < \pi/2$, obtained using 240 elements (that is, using $N_0 = 20$) are compared graphically against the values obtained from the exact solution[‡] given by

$$\phi = [\frac{16}{85}([x^2+y^2]^2 - \frac{1}{[x^2+y^2]^2}) - \frac{16}{255}(\frac{[x^2+y^2]^2}{16} - \frac{16}{[x^2+y^2]^2})]\cos(4\arctan(\frac{y}{x})).$$

The numerical values show a good agreement with the exact ones except at points that are extremely close to the corner points $(0, 1)$ and $(1, 0)$, that is, except at near $\theta = 0$ and $\theta = \pi/2$.

[‡]Refer to page 202 of the book *Partial Differential Equations in Mechanics 1* by APS Selvadurai (Springer-Verlag, 2000).

The numerical values of ϕ at selected points in the interior of the solution domain, obtained using 240 elements, are compared with the exact solution in Table 1.3. There is a good agreement between the two sets of results. The interior points in the last two rows of Table 1.3 are close to the corner point $(1, 0)$. Note that the errors of the numerical values at these two points are higher compared with those at the other points. When we repeat the same calculation using 480 elements ($N_0 = 40$), the numerical values of ϕ are 0.826108 and 0.974111 at $(1.099998, 0.001920)$ and $(1.010000, 0.000176)$ respectively, that is, we observe a significant improvement in the accuracy of the numerical values at the two points.

Table 1.3

(ξ, η)	240 elements	Exact
$(1.082532, 0.625000)$	-0.392546	-0.392045
$(0.875000, 1.515544)$	-0.908254	-0.907816
$(1.060660, 1.060660)$	-1.094489	-1.094211
$(1.099998, 0.001920)$	0.824548	0.826958
$(1.010000, 0.000176)$	0.960174	0.975656

1.9 Summary and Discussion

A boundary element solution for the interior boundary value problem defined by Eqs. (1.1)-(1.2) is given by Eq. (1.32) together with Eqs. (1.28), (1.30) and (1.31). The solution is constructed from the boundary integral solution in Eq. (1.23). Constant elements are used, that is, the boundary (of the solution domain) is discretized into straight line elements and the solution ϕ and its normal derivative $\partial\phi/\partial n$ on the boundary are approximated as constants over a boundary element.

As no discretization of the entire solution domain is required, the boundary element solution may be easily implemented on the computer for problems involving complicated geometries and general boundary conditions. The boundary may be easily discretized into line elements by merely placing on it well spaced out points. We have discussed in detail how the numerical procedure can be coded in FORTRAN 77. In spite of the specific programming language used, our discussion may still be useful to readers who are interested in developing the method using other software tools (such as C++ and MATLAB), as FORTRAN 77 codes are relatively easy to decipher.

The term "direct boundary element method" is often used to describe the boundary element procedure given in this chapter. This is because the unknowns in the formulation given by Eq. (1.30) can be directly interpreted as values of ϕ or $\partial\phi/\partial n$ on the boundary. An alternative boundary element method may be obtained from the simpler boundary integral solution

$$\phi(x,y) = \int_C A(\xi,\eta)\ln([x-\xi]^2+[y-\eta]^2)ds(\xi,\eta),$$

where $A(\xi,\eta)$ is a (boundary) weight function yet to be determined. To determine $A(\xi,\eta)$ approximately, we discretize C into boundary elements $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$as before, and approximate $A(\xi,\eta)$ as a constant $A^{(m)}$ over $C^{(m)}$, in order to obtain the approximation

$$\phi(x,y) \simeq \sum_{m=1}^{N} A^{(m)} \int_{C^{(m)}} \ln([x-\xi]^2+[y-\eta]^2)ds(\xi,\eta).$$

The constants $A^{(m)}$ are to be determined by using the given boundary conditions. We shall not go into further details here other than pointing out that such as an approach gives rise to a so called indirect boundary element method as the unknowns $A^{(m)}$ are not related to ϕ or $\partial\phi/\partial n$ on the boundary in a simple and direct manner.

1.10 Exercises

1. If ϕ satisfies the two-dimensional Laplace's equation in the region R bounded by a simple closed curve C, use the divergence theorem to show that

$$\int_C \frac{\partial}{\partial n}[\phi(x,y)]ds(x,y) = 0.$$

(Note. This implies that if we prescribe $\partial\phi/\partial n$ at all points on C in our boundary value problem we have to be careful to ensure the above equation is satisfied. Otherwise, the boundary value problem does not have a solution.)

2. If ϕ satisfies the two-dimensional Laplace's equation in the region R bounded by the curve C, use the divergence theorem to derive the relation

$$\iint_R |\underline{\nabla}\phi(x,y)|^2 dxdy = \int_C \phi(x,y)\frac{\partial}{\partial n}[\phi(x,y)]ds(x,y).$$

Hence show that: (a) if $\phi = 0$ at all points on C then $\phi = 0$ at all points in R, that is, show that if the boundary conditions are given by $\phi = 0$ on C then the solution of our boundary value problem is uniquely given by $\phi = 0$ for $(x,y) \in R$, and (b) if $\partial\phi/\partial n = 0$ at all points on C then ϕ can be any arbitrary constant function in R, that is, if the boundary conditions are given by $\partial\phi/\partial n = 0$ on C, then our boundary value problem has infinitely many solutions given by $\phi = c$ for $(x,y) \in R$, where c is an arbitrary constant.

3. Use the result in Exercise 2(a) above to show that if the boundary conditions are given by $\phi = f(x, y)$ at all points on the simple closed curve C then the boundary value problem governed by the two-dimensional Laplace's equation in the region R has a unique solution. [Hint. Show that if ϕ_1 and ϕ_2 are any two solutions satisfying the Laplace's equation and the boundary conditions under consideration then $\phi_1 = \phi_2$ at all points in R.] (Notes. (1) In general, for the interior boundary value problem defined by Eqs. (1.1)-(1.2) to have a unique solution, ϕ must be specified at **at least one** point on C. (2) For the case in which $\partial\phi/\partial n$ is specified at all points on C, ϕ is only determined to within an arbitrary constant. In such a case, the boundary element procedure in this chapter may still work to give us one of the infinitely many solutions.)

4. Eq. (1.8) is not the only solution of the two-dimensional Laplace's equation that is not well defined at the single point (ξ, η). By differentiating Eq. (1.8) partially with respect to x and/or y as many times as we like, we may generate other solutions that are not well defined at (ξ, η). An example of these other solutions is

$$\phi(x, y) = \frac{(x - \xi)}{2\pi[(x - \xi)^2 + (y - \eta)^2]}.$$

   If we denote this solution by $\Phi(x, y; \xi, \eta)$ (like what we had done before for the solution in Eq. (1.8)), investigate whether we can still derive the boundary integral solution as given by Eq. (1.19) from the reciprocal relation in Eq. (1.10) or not.

5. Explain why the parameter $\lambda(\xi, \eta)$ in Eq. (1.23) can be calculated using

$$\lambda(\xi, \eta) = \int\limits_C \frac{\partial}{\partial n}(\Phi(x, y; \xi, \eta))ds(x, y).$$

   Taking C to be the boundary of the triangular region $y < -x + 1$, $x > 0$, $y > 0$, evaluate the line integral above to check that: (a) $\lambda(2, 1) = 0$, (b) $\lambda(1, 0) = 1/8$, (c) $\lambda(0, 0) = 1/4$, (d) $\lambda(1/2, 1/2) = 1/2$, and (e) $\lambda(1/2, 1/4) = 1$.

6. The boundary element solution given in this chapter provides us with an approximate but explicit formula for calculating ϕ at any interior point (ξ, η) in the solution domain. We may also be interested in computing the vector quantity $\underline{\nabla}\phi$. Can an approximate explicit formula be obtained for $\underline{\nabla}\phi$ at (ξ, η)? How can we obtain one?

7. Modify the program `EX1PT1` in Section 1.7 to solve numerically the Laplace's equation given by Eq. (1.1) in the region $x^2 + y^2 < 1$, $x > 0$, $y > 0$, subject to

Chapter 5

GREEN'S FUNCTIONS FOR POTENTIAL PROBLEMS

5.1 Introduction

In Chapter 1, for the two-dimensional Laplace's equation, we have derived the boundary integral equation

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y). \quad (5.1)$$

Here ϕ satisfies the Laplace's equation in the two-dimensional region R bounded by a simple closed curve C on the Oxy plane, Φ is the fundamental solution given by

$$\Phi(x,y;\xi,\eta) = \frac{1}{4\pi}\ln([x-\xi]^2 + [y-\eta]^2), \quad (5.2)$$

and λ is the parameter defined by

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta) \notin R \cup C, \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } C, \\ 1 & \text{if } (\xi,\eta) \in R. \end{cases} \quad (5.3)$$

The boundary integral equation in Eq. (5.1) may still be valid with possibly only minor modification of Eq. (5.3), if $\Phi(x,y;\xi,\eta)$ is chosen to take the more general form

$$\Phi(x,y;\xi,\eta) = \frac{1}{4\pi}\ln([x-\xi]^2 + [y-\eta]^2) + \Phi^*(x,y;\xi,\eta), \quad (5.4)$$

where $\Phi^*(x,y;\xi,\eta)$ is any well defined function satisfying

$$\frac{\partial^2}{\partial x^2}[\Phi^*(x,y;\xi,\eta)] + \frac{\partial^2}{\partial y^2}[\Phi^*(x,y;\xi,\eta)] = 0 \text{ for any } (x,y) \text{ and } (\xi,\eta) \text{ in } R. \quad (5.5)$$

Instead of taking $\Phi^* = 0$ (like in Chapters 1 and 2), we may find it advantageous to choose Φ^* that satisfies certain boundary conditions. The function Φ in Eq. (5.4) with a specially chosen Φ^* may be referred to as a Green's function. As we shall see later on, if an appropriately chosen Green's function, instead of the usual fundamental solution in Eq. (5.2), is used in the boundary integral equation, it may be possible to avoid integration over part of the boundary C. The number of boundary elements needed in the discretization of the boundary integral equation (hence the number of unknowns in the resulting system of linear algebraic equations) may then be reduced*.

In this chapter, boundary element solutions of particular potential problems, obtained by using Green's functions for some special domains and boundary conditions, are presented with examples of applications.

5.2 Half Plane

5.2.1 Two Special Green's Functions

For the half plane $y > 0$ in Figure 5.1, two well known Green's functions satisfying particular conditions on the boundary $y = 0$ are derived by the so called method of image. They are then applied to solve specific problems.

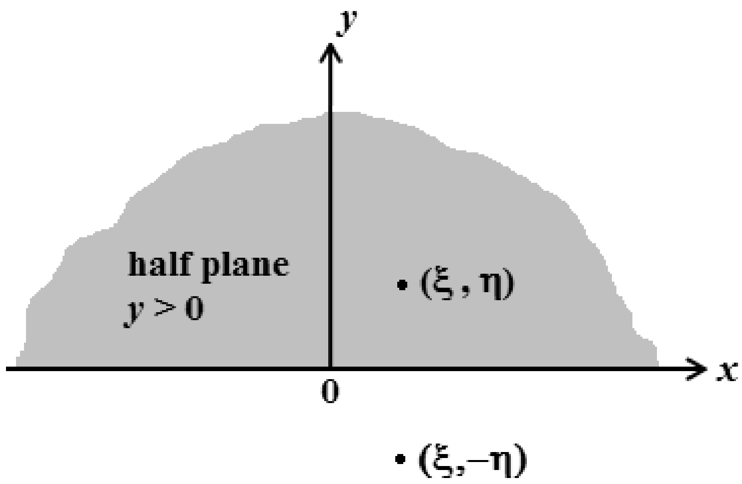

Figure 5.1

We choose $\Phi^*(x, y; \xi, \eta)$ such that it satisfies Eq. (5.5) in the half plane $y > 0$ and the boundary condition

$$\Phi(x, 0; \xi, \eta) = 0 \text{ for } -\infty < x < \infty, \tag{5.6}$$

which may be rewritten as

$$\Phi^*(x, 0; \xi, \eta) = -\frac{1}{4\pi} \ln([x - \xi]^2 + \eta^2) \text{ for } -\infty < x < \infty. \tag{5.7}$$

*An example of potential problems solved using a boundary integral formulation with Green's function, may be found in the article "A method for the numerical solution of some elliptic boundary value problems for a strip" by DL Clements and J Crowe in the International Journal of Computer Mathematics (Volume 8, 1980, pp. 345-355). The term "potential problems" refers to boundary value problems governed by the Laplace's equation.

A suitable $\Phi^*(x, y; \xi, \eta)$ satisfying Eqs. (5.5) and (5.7) is given by

$$\Phi^*(x, y; \xi, \eta) = -\frac{1}{4\pi} \ln([x - \xi]^2 + [y + \eta]^2). \tag{5.8}$$

The function $\Phi^*(x, y; \xi, \eta)$ as given in Eq. (5.8) is well defined at all point (x, y) except at $(\xi, -\eta)$. If the point (ξ, η) is in the half plane then $(\xi, -\eta)$ being its image point obtained through reflection about $y = 0$ should be outside the half plane. Thus, $\Phi^*(x, y; \xi, \eta)$ in Eq. (5.8) satisfies the two-dimensional Laplace's equation at all points (x, y) and (ξ, η) in the interior of the half plane.

Let us denote the Green's function given by Eqs. (5.4) and (5.8) by $\Phi_1(x, y; \xi, \eta)$, that is,

$$\Phi_1(x, y; \xi, \eta) = \frac{1}{4\pi} \ln([x - \xi]^2 + [y - \eta]^2) - \frac{1}{4\pi} \ln([x - \xi]^2 + [y + \eta]^2). \tag{5.9}$$

For another Green's function, in place of the boundary condition in Eq. (5.6), $\Phi^*(x, y; \xi, \eta)$ is chosen to satisfy

$$\frac{\partial}{\partial y}(\Phi(x, y; \xi, \eta))\bigg|_{y=0} = 0 \text{ for } -\infty < x < \infty, \tag{5.10}$$

that is,

$$\frac{\partial}{\partial y}(\Phi^*(x, y; \xi, \eta))\bigg|_{y=0} = \frac{\eta}{2\pi([x - \xi]^2 + \eta^2)} \text{ for } -\infty < x < \infty. \tag{5.11}$$

It is easy to check that Eq. (5.11) is satisfied if

$$\Phi^*(x, y; \xi, \eta) = \frac{1}{4\pi} \ln([x - \xi]^2 + [y + \eta]^2). \tag{5.12}$$

We denote the Green's function given by Eqs. (5.4) and (5.12) by $\Phi_2(x, y; \xi, \eta)$. Thus,

$$\Phi_2(x, y; \xi, \eta) = \frac{1}{4\pi} \ln([x - \xi]^2 + [y - \eta]^2) + \frac{1}{4\pi} \ln([x - \xi]^2 + [y + \eta]^2). \tag{5.13}$$

We show now how the Green's functions in Eqs. (5.9) and (5.13) can be applied to obtain special boundary integral formulations for two particular cases of potential problems.

5.2.2 Applications

Case 5.1

This case involves a finite two-dimensional solution region R which is a subset of the half plane $y > 0$. The curve boundary C of the region R consists of two non-intersecting parts denoted by D and E. The part D is an arbitrarily shaped open curve in the region $y > 0$, while E is a straight line segment on the x axis. For a geometrical sketch of the solution domain, refer to Figure 5.2.

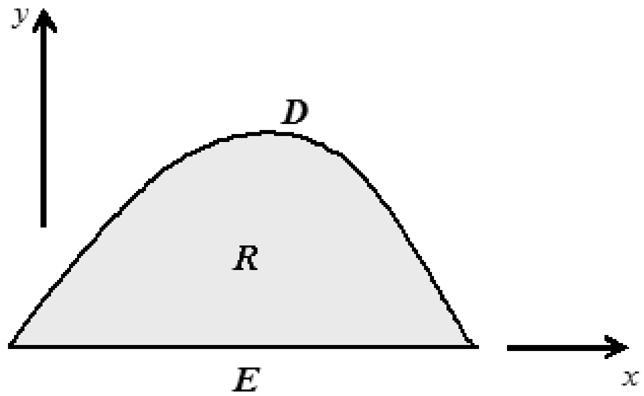

Figure 5.2

Mathematically, we are interested in solving the two-dimensional Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ in } R, \tag{5.14}$$

subject to

$$\begin{aligned} \phi &= 0 \text{ for } (x,y) \in E, \\ \phi &= f_1(x,y) \text{ for } (x,y) \in D_1, \\ \frac{\partial\phi}{\partial n} &= f_2(x,y) \text{ for } (x,y) \in D_2, \end{aligned} \tag{5.15}$$

where f_1 and f_2 are suitably prescribed functions and D_1 and D_2 are non-intersecting curves such that $D_1 \cup D_2 = D$.

If we repeat the analysis in Section 1.4 (page 5, Chapter 1) using the Green's function $\Phi_1(x,y;\xi,\eta)$ in Eq. (5.9) in the place of the usual fundamental solution $\Phi(x,y;\xi,\eta) = (4\pi)^{-1}\ln([x-\xi]^2+[y-\eta]^2)$, we obtain the boundary integral equation

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int\limits_D [\phi(x,y)\frac{\partial}{\partial n}(\Phi_1(x,y;\xi,\eta)) - \Phi_1(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y), \tag{5.16}$$

with the parameter λ defined by

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta) \in E, \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } D, \\ 1 & \text{if } (\xi,\eta) \in R. \end{cases} \tag{5.17}$$

Note that the integral over E does not appear in Eq. (5.16). The conditions that $\phi(x,y)$ and $\Phi_1(x,y;\xi,\eta)$ are both zero for $(x,y) \in E$ are applied in the derivation of Eq. (5.16).

There is no need to discretize the boundary E if Eq. (5.16) is used to derive a boundary element procedure for solving the boundary value problem defined by Eqs. (5.14) and (5.15).

To discretize the integral in Eq. (5.16), let us put $N+1$ well spaced out points $(x^{(1)},y^{(1)})$, $(x^{(2)},y^{(2)})$, $\cdots$, $(x^{(N-1)},y^{(N-1)})$, $(x^{(N)},y^{(N)})$ and $(x^{(N+1)},y^{(N+1)})$ on the open curve D. The end points of the curve D are $(x^{(1)},y^{(1)})$ and $(x^{(N+1)},y^{(N+1)})$ with $x^{(N+1)} < x^{(1)}$. Note that $y^{(1)} = y^{(N+1)} = 0$. The points are arranged such that $(x^{(k)},y^{(k)})$ and $(x^{(k+1)},y^{(k+1)})$ $(k = 1, 2, \cdots , N)$ are two consecutive neighboring points. The line segment between $(x^{(k)},y^{(k)})$ and $(x^{(k+1)},y^{(k+1)})$ forms the element $D^{(k)}$. The curve D is approximated using

$$D \simeq D^{(1)} \cup D^{(2)} \cup \cdots \cup D^{(N-1)} \cup D^{(N)}. \tag{5.18}$$

For constant elements, we may make the approximation

$$\phi \simeq \overline{\phi}^{(k)} \text{ and } \frac{\partial\phi}{\partial n} \simeq \overline{p}^{(k)} \text{ for } (x,y) \in D^{(k)} \ (k = 1,2,\cdots,N), \tag{5.19}$$

in order to rewrite Eq. (5.16) approximately as

$$\begin{aligned} \lambda(\xi,\eta)\phi(\xi,\eta) \ \simeq \ & \sum_{k=1}^{N} \overline{\phi}^{(k)} \int\limits_{D^{(k)}} \frac{\partial}{\partial n}(\Phi_1(x,y;\xi,\eta))ds(x,y) \\ & -\sum_{k=1}^{N} \overline{p}^{(k)} \int\limits_{D^{(k)}} \Phi_1(x,y;\xi,\eta)ds(x,y). \end{aligned} \tag{5.20}$$

Proceeding as before, we may choose (ξ,η) in Eq. (5.20) to be given in turn by each of the midpoints of the boundary elements to set up a system of N linear algebraic equations to solve for either $\overline{\phi}^{(k)}$ or $\overline{p}^{(k)}$ (whichever is unknown on $D^{(k)}$). For setting up the linear algebraic equations, the integrals over $D^{(k)}$ can be evaluated analytically by

$$\begin{aligned} \int\limits_{D^{(k)}} \Phi_1(x,y;\xi,\eta)ds(x,y) &= \mathcal{F}_1^{(k)}(\xi,\eta) - \mathcal{F}_1^{(k)}(\xi,-\eta), \\ \int\limits_{D^{(k)}} \frac{\partial}{\partial n}[\Phi_1(x,y;\xi,\eta)]ds(x,y) &= \mathcal{F}_2^{(k)}(\xi,\eta) - \mathcal{F}_2^{(k)}(\xi,-\eta), \end{aligned} \tag{5.21}$$

where the formulae for $\mathcal{F}_1^{(k)}(\xi,\eta)$ and $\mathcal{F}_2^{(k)}(\xi,\eta)$ are given in Section 1.6 (page 12, Chapter 1).

With Eq. (5.21), the subroutines `CELAP1` and `CELAP2` in Section 1.7 (page 16, Chapter 1) can be easily modified for the problem under consideration here to find unknown values of ϕ or $\partial\phi/\partial n$ on the elements and to compute ϕ at interior points in R. We modify `CELAP1` and `CELAP2` to create the subroutines `G1LAP1` and `G1LAP2` as listed below. In `G1LAP1` or `G1LAP2`, the subroutine `CPF` is called to compute $\mathcal{F}_1^{(k)}(\xi,\eta)$ first and then again to calculate $\mathcal{F}_1^{(k)}(\xi,-\eta)$. The input and output parameters of `G1LAP1` or `G1LAP2` are exactly the same as those of `CELAP1` and `CELAP2` respectively, as detailed in Section 1.7.

```
      subroutine G1LAP1(N,xm,ym,xb,yb,nx,ny,lg,BCT,BCV,phi,dphi)

      integer m,k,N,BCT(1000)

      double precision xm(1000),ym(1000),xb(1000),yb(1000),
     & nx(1000),ny(1000),lg(1000),BCV(1000),A(1000,1000),
     & B(1000),pi,P1F1,P1F2,del,phi(1000),dphi(1000),F1,F2,
     & Z(1000),P2F1,P2F2

      pi=4d0*datan(1d0)

      do 10 m=1,N
      B(m)=0d0
      do 5 k=1,N
      call CPF(xm(m),ym(m),xb(k),yb(k),nx(k),ny(k),lg(k),P1F1,P1F2)
      call CPF(xm(m),-ym(m),xb(k),yb(k),nx(k),ny(k),lg(k),P2F1,P2F2)
      F1=(P1F1-P2F1)/pi
      F2=(P1F2-P2F2)/pi
      if (k.eq.m) then
      del=1d0
      else
      del=0d0
      endif
      if (BCT(k).eq.0) then
      A(m,k)=-F1
      B(m)=B(m)+BCV(k)*(-F2+0.5d0*del)
      else
      A(m,k)=F2-0.5d0*del
      B(m)=B(m)+BCV(k)*F1
      endif
    5 continue
```

```
10 continue

   call solver(A,B,N,1,Z)

   do 15 m=1,N
   if (BCT(m).eq.0) then
   phi(m)=BCV(m)
   dphi(m)=Z(m)
   else
   phi(m)=Z(m)
   dphi(m)=BCV(m)
   endif
15 continue

   return
   end

   subroutine G1LAP2(N,xi,eta,xb,yb,nx,ny,lg,phi,dphi,pint)

   integer N,i

   double precision xi,eta,xb(1000),yb(1000),nx(1000),ny(1000),
  & lg(1000),phi(1000),dphi(1000),pint,sum,pi,P1F1,P1F2,
  & P2F1,P2F2

   pi=4d0*datan(1d0)
   sum=0d0

   do 10 i=1,N
   call CPF(xi,eta,xb(i),yb(i),nx(i),ny(i),lg(i),P1F1,P1F2)
   call CPF(xi,-eta,xb(i),yb(i),nx(i),ny(i),lg(i),P2F1,P2F2)
   sum=sum+phi(i)*(P1F2-P2F2)-dphi(i)*(P1F1-P2F1)
10 continue

   pint=sum/pi

   return
   end
```

Example 5.1

To test the subroutines G1LAP1 or G1LAP2, we use them to solve Eq. (5.14) numerically in the square region $0 < x < 1$, $0 < y < 1$, subject to

$$\begin{aligned}
\phi(x,0) &= 0 \text{ for } 0 < x < 1, \\
\left.\frac{\partial\phi}{\partial n}\right|_{x=0} &= 0 \text{ and } \left.\frac{\partial\phi}{\partial n}\right|_{x=1} = 0 \text{ for } 0 < y < 1, \\
\phi(x,1) &= 4x(1-x) \text{ for } 0 < x < 1.
\end{aligned}$$

The analytical solution for this particular problem is given in series form by

$$\phi(x,y) = \frac{2}{3}y - 4\sum_{n=1}^{\infty} \frac{\sinh(2n\pi y)\cos(2n\pi x)}{n^2\pi^2 \sinh(2n\pi)}.$$

To use G1LAP1 or G1LAP2, we have to discretize only the vertical sides of the square domain and the horizontal side $0 < x < 1$, $y = 1$ into elements. Each of the sides is discretized into N_0 equal length elements. Thus, $N = 3N_0$ and the points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(N-1)}, y^{(N-1)})$, $(x^{(N)}, y^{(N)})$ and $(x^{(N+1)}, y^{(N+1)})$ (on the three sides) are given by

$$\left.\begin{aligned}
(x^{(k)}, y^{(k)}) &= (1, [k-1]/N_0) \\
(x^{(N_0+k)}, y^{(N_0+k)}) &= (1-[k-1]/N_0, 1) \\
(x^{(2N_0+k)}, y^{(2N_0+k)}) &= (0, 1-[k-1]/N_0)
\end{aligned}\right\} \text{ for } k = 1, 2, \cdots, N_0,$$
$$(x^{(3N_0+1)}, y^{(3N_0+1)}) = (0, 0).$$

The main program EX5PT1 which makes use of G1LAP1 and G1LAP2 to solve the problem under consideration is listed below.

```
       program EX5PT1

      integer N0,BCT(1000),N,i,ians

      double precision xb(1000),yb(1000),xm(1000),ym(1000),
     & nx(1000),ny(1000),lg(1000),BCV(1000),
     & phi(1000),dphi(1000),pint,dl,xi,eta,pi,exct

      print*,'Enter number of elements per side (<334):'
      read*,N0
      N=3*N0

      pi=4d0*datan(1d0)
      dl=1d0/dfloat(N0)
```

```
      do 10 i=1,N0
      xb(i)=1d0
      yb(i)=dfloat(i-1)*dl
      xb(N0+i)=1d0-yb(i)
      yb(N0+i)=1d0
      xb(2*N0+i)=0d0
      yb(2*N0+i)=xb(N0+i)
10    continue
      xb(N+1)=0d0
      yb(N+1)=0d0

      do 20 i=1,N
      xm(i)=0.5d0*(xb(i)+xb(i+1))
      ym(i)=0.5d0*(yb(i)+yb(i+1))
      lg(i)=dsqrt((xb(i+1)-xb(i))**2d0+(yb(i+1)-yb(i))**2d0)
      nx(i)=(yb(i+1)-yb(i))/lg(i)
      ny(i)=(xb(i)-xb(i+1))/lg(i)
20    continue

      do 30 i=1,N
      if (i.le.N0) then
      BCT(i)=1
      BCV(i)=0d0
      else if ((i.gt.N0).and.(i.le.(2*N0))) then
      BCT(i)=0
      BCV(i)=4d0*xm(i)*(1d0-xm(i))
      else
      BCT(i)=1
      BCV(i)=0d0
      endif
30    continue

      call G1LAP1(N,xm,ym,xb,yb,nx,ny,lg,BCT,BCV,phi,dphi)

50    print*,'Enter coordinates xi and eta of an interior point:'

      read*,xi,eta

      call G1LAP2(N,xi,eta,xb,yb,nx,ny,lg,phi,dphi,pint)

      exct=2d0*eta/3d0
```

```
  do 55 i=1,1000
  exct=exct-4d0*((dexp(2d0*dfloat(i)*pi*(eta-1d0))
 & -dexp(-2d0*dfloat(i)*pi*(eta+1d0)))
 & *dcos(2d0*dfloat(i)*pi*xi))
 & /((dfloat(i*i)*pi*pi)*(1d0-dexp(-4d0*dfloat(i)*pi)))
55 continue

   write(*,60)pint,exct
60 format('Numerical and exact values are:',
 & F14.6,' and',F14.6,' respectively')

  print*,'To continue with another point enter 1:'
  read*,ians

  if (ians.eq.1) goto 50

  end
```

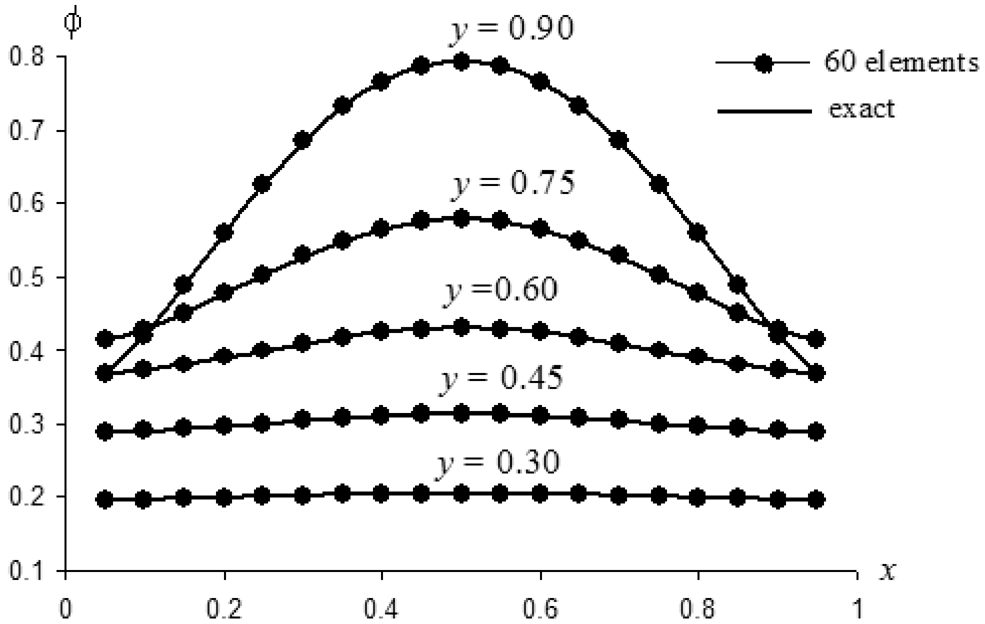

Figure 5.3

In Figure 5.3, we plot the numerical ϕ obtained using $N_0 = 20$ (60 boundary elements altogether) against $0 < x < 1$ for $y = 0.30$, 0.45, 0.60, 0.75 and 0.90. The graphs of the numerical and the exact solutions are in good agreement with each other.

Case 5.2

As sketched in Figure 5.4, the solution domain R is taken to be the half plane $y > 0$ without the finite region bounded by a simple closed curve C. We are interested in solving Eq. (5.14) subject to

$$\left.\frac{\partial\phi}{\partial n}\right|_{y=0} = 0 \text{ for } -\infty < x < \infty,$$
$$\phi = f_1(x,y) \text{ for } (x,y) \in C_1,$$
$$\frac{\partial\phi}{\partial n} = f_2(x,y) \text{ for } (x,y) \in C_2,$$
$$\phi \to 0 \text{ as } x^2 + y^2 \to 0, \quad (5.22)$$

where f_1 and f_2 are suitably prescribed functions and C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$.

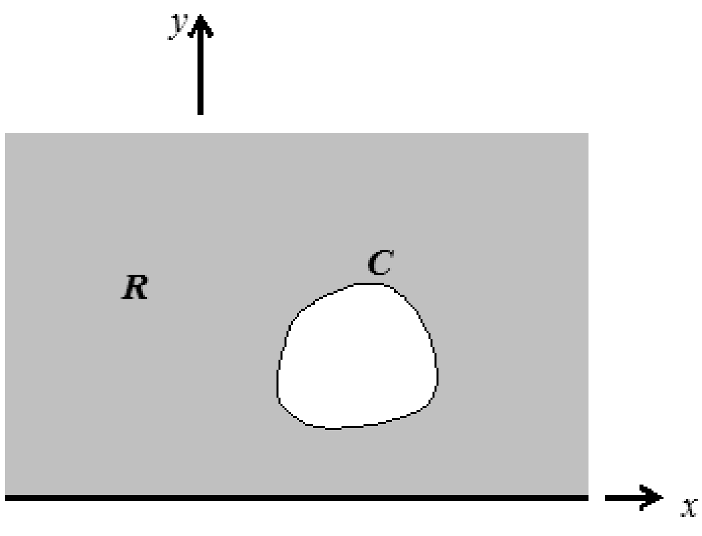

Figure 5.4

Note that $\partial\phi/\partial n = -\partial\phi/\partial y$ on the plane boundary $y = 0$. The last condition in Eq. (5.22) specifies the "far field" behavior of the solution. For our purpose here, we assume that ϕ decays as $O([x^2 + y^2]^{-a} \ln[x^2 + y^2])$ (with a being a positive real number) for large $x^2 + y^2$ (within the half plane).

Particular engineering problems which require the computation of stress around a hole or rigid inclusion or fluid flow past an impermeable body may be formulated in terms of the boundary value problem above. The curve C represents the boundary of the hole or rigid inclusion or impermeable body.

To obtain a boundary integral formulation for the problem under consideration, let us first introduce an artificial boundary S_ρ given by the semi-circle $x^2 + y^2 = \rho^2$, $y > 0$, where ρ is a positive real number. We take R_ρ to be the finite region whose boundary is given by $C \cup S_\rho \cup L_\rho$, where L_ρ is the horizontal straight line between the point $(-\rho, 0)$ and $(\rho, 0)$. We assume that ρ is sufficiently large, so that the simple closed curve C lies wholly in the region R_ρ. Refer to Figure 5.5. We can recover the solution domain R in Figure 5.4 if we let the parameter ρ tend to infinity.

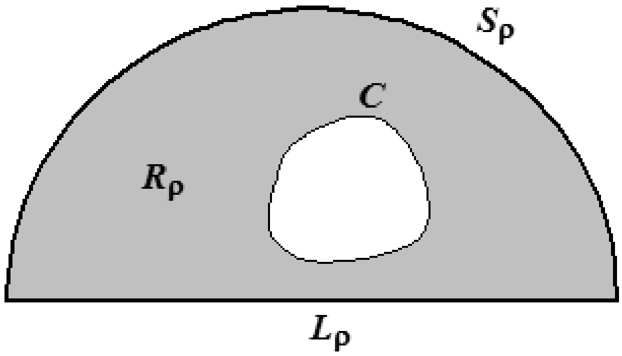

Figure 5.5

If we carry out the analysis in Section 1.4 (page 5, Chapter 1) on the region R_ρ using the Green's function $\Phi_2(x, y; \xi, \eta)$ in Eq. (5.13), we obtain the boundary integral equation

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int\limits_{C\cup S_\rho} [\phi(x,y)\frac{\partial}{\partial n}(\Phi_2(x,y;\xi,\eta)) - \Phi_2(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y), \tag{5.23}$$

with the parameter λ defined by

$$\lambda(\xi,\eta) = \begin{cases} 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } C \cup S_\rho, \\ 1 & \text{if } (\xi,\eta) \in R_\rho \cup L_\rho. \end{cases} \tag{5.24}$$

In deriving Eq. (5.23), we use the boundary conditions $\partial\phi/\partial n = 0$ and $\partial\Phi_2/\partial n = 0$ on L_ρ. The integral over L_ρ does not appear in the formulation.

To examine the integral over S_ρ for ρ tending to infinity, we write

$$\begin{aligned} \ln([x-\xi]^2 + [y \pm \eta]^2) &= \ln(r^2 + \xi^2 + \eta^2 - 2r[\xi\cos\theta \mp \eta\sin\theta]) \\ &= \ln(r^2) + \ln(1 + \frac{\xi^2+\eta^2}{r^2} - \frac{2[\xi\cos\theta \mp \eta\sin\theta]}{r}), \end{aligned} \tag{5.25}$$

where r and θ are the usual polar coordinates.

From Eq. (5.25), it is obvious that $\ln([x-\xi]^2+[y\pm\eta]^2)\simeq\ln(r^2)$ for large r. It follows that

$$\Phi_2 \simeq \frac{1}{\pi}\ln(r) \text{ for large } r.$$

If we further assume that

$$\phi \simeq A(\theta)\frac{\ln(r)}{r^{2a}} \text{ for large } r,$$

where $A(\theta)$ is a well defined function of θ and a is a positive real number, we find that

$$\lim_{\rho\to\infty}\int\limits_{S_\rho}[\phi(x,y)\frac{\partial}{\partial n}(\Phi_2(x,y;\xi,\eta))-\Phi_2(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y)$$

$$= \lim_{\rho\to\infty}\frac{[\ln(\rho)]^2}{\pi\rho^{2a}}\int\limits_0^\pi 2aA(\theta)d\theta = 0.$$

Thus, if we let ρ tend to infinity in Eq. (5.23), for the potential problem here with conditions given in Eq. (5.22), we obtain the boundary integral equation

$$\lambda(\xi,\eta)\phi(\xi,\eta)=\int\limits_C[\phi(x,y)\frac{\partial}{\partial n}(\Phi_2(x,y;\xi,\eta))-\Phi_2(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y), \tag{5.26}$$

with the parameter λ defined by

$$\lambda(\xi,\eta)=\begin{cases}1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } C,\\ 1 & \text{if } (\xi,\eta)\in R \text{ or if } \eta=0.\end{cases} \tag{5.27}$$

Note that in Eq. (5.26) the integral is over the boundary C only. If the usual fundamental solution $\Phi(x,y;\xi,\eta)=(4\pi)^{-1}\ln([x-\xi]^2+[y-\eta]^2)$ is used instead of $\Phi_2(x,y;\xi,\eta)$, the path of integration in the boundary integral equation includes the line $y=0$ for $-\infty<x<\infty$. The advantage of using the special Green's function $\Phi_2(x,y;\xi,\eta)$ is obvious for the particular problem under consideration.

Finally, let us check that the integral expression for $\phi(\xi,\eta)$ in Eq. (5.26) tends to zero as $\xi^2+\eta^2\to\infty$ (within the solution domain). From Eq. (5.13), for $(x,y)\in C$, we find that $\Phi_2(x,y;\xi,\eta)\to(2\pi)^{-1}\ln(\xi^2+\eta^2)$ and $\partial(\Phi_2(x,y;\xi,\eta)/\partial n\to 0$ as $\xi^2+\eta^2\to\infty$. Thus, Eq. (5.26) gives

$$\phi(\xi,\eta)\to-\frac{1}{2\pi}\ln(\xi^2+\eta^2)\int\limits_C\frac{\partial}{\partial n}(\phi(x,y))ds(x,y) \text{ as } \xi^2+\eta^2\to\infty.$$

With reference to Figure 5.5, $\partial\phi/\partial n$ is required to satisfy†

$$\int\limits_{C\cup L_\rho\cup S_\rho} \frac{\partial}{\partial n}(\phi(x,y))ds(x,y) = 0.$$

Now we know that $\partial\phi/\partial n$ is zero on L_ρ. We expect $\partial\phi/\partial n$ to behave as $O([x^2 + y^2]^{-a-1/2}\ln[x^2+y^2])$ on S_ρ for large ρ (since we assume that ϕ decays as $O([x^2 + y^2]^{-a}\ln[x^2+y^2])$ for large x^2+y^2). Thus, for the problem under consideration, if we let $\rho\to\infty$, we find that $\partial\phi/\partial n$ on C satisfies

$$\int\limits_{C} \frac{\partial}{\partial n}(\phi(x,y))ds(x,y) = 0.$$

It follows that $\phi(\xi,\eta)$ as given by Eq. (5.26) tends to zero as $\xi^2+\eta^2\to\infty$. Note that if we prescribe $\partial\phi/\partial n$ at all points on C we must ensure that the condition above is fulfilled.

Since the curve C is an inner boundary (within the half plane), the unit normal vector $[n_x, n_y]$ on C (for computing $\partial\phi/\partial n$ and $\partial\Phi_2/\partial n$) is taken to point into the region bounded by C. To discretize C, we place N well spaced out points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(N-1)}, y^{(N-1)})$ and $(x^{(N)}, y^{(N)})$ on C in the clockwise (instead of counter clockwise) direction. Using the conditions specified on C and constant elements, we may use the computer codes in Chapter 1 to solve the problem under consideration. Since the Green's function Φ_2 is used here (in place of the usual fundamental solution Φ), the subroutines `CELAP1` and `CELAP2` for computing ϕ numerically have to be replaced by `G2LAP1` and `G2LAP2` as listed below.

```
 subroutine G2LAP1(N,xm,ym,xb,yb,nx,ny,lg,BCT,BCV,phi,dphi)

 integer m,k,N,BCT(1000)

 double precision xm(1000),ym(1000),xb(1000),yb(1000),
&nx(1000),ny(1000),lg(1000),BCV(1000),A(1000,1000),
&B(1000),pi,P1F1,P1F2,del,phi(1000),dphi(1000),F1,F2,
&Z(1000),P2F1,P2F2

 pi=4d0*datan(1d0)

 do 10 m=1,N
 B(m)=0d0
 do 5 k=1,N
 call CPF(xm(m),ym(m),xb(k),yb(k),nx(k),ny(k),lg(k),P1F1,P1F2)
```

†Exercise 1 of Chapter 1 on page 34 may be of relevance here.

```
      call CPF(xm(m),-ym(m),xb(k),yb(k),nx(k),ny(k),lg(k),P2F1,P2F2)
      F1=(P1F1+P2F1)/pi
      F2=(P1F2+P2F2)/pi
      if (k.eq.m) then
      del=1d0
      else
      del=0d0
      endif
      if (BCT(k).eq.0) then
      A(m,k)=-F1
      B(m)=B(m)+BCV(k)*(-F2+0.5d0*del)
      else
      A(m,k)=F2-0.5d0*del
      B(m)=B(m)+BCV(k)*F1
      endif
    5 continue
    10 continue

      call solver(A,B,N,1,Z)

      do 15 m=1,N
      if (BCT(m).eq.0) then
      phi(m)=BCV(m)
      dphi(m)=Z(m)
      else
      phi(m)=Z(m)
      dphi(m)=BCV(m)
      endif
    15 continue

      return
      end

      subroutine G2LAP2(N,xi,eta,xb,yb,nx,ny,lg,phi,dphi,pint)

      integer N,i

      double precision xi,eta,xb(1000),yb(1000),nx(1000),ny(1000),
     & lg(1000),phi(1000),dphi(1000),pint,sum,pi,P1F1,P1F2,
     & P2F1,P2F2

      pi=4d0*datan(1d0)
```

```
      sum=0d0

      do 10 i=1,N
      call CPF(xi,eta,xb(i),yb(i),nx(i),ny(i),lg(i),P1F1,P1F2)
      call CPF(xi,-eta,xb(i),yb(i),nx(i),ny(i),lg(i),P2F1,P2F2)
      sum=sum+phi(i)*(P1F2+P2F2)-dphi(i)*(P1F1+P2F1)
10    continue

      pint=sum/pi

      return
      end
```

Example 5.2

For a particular test problem, let us take the inner boundary C to be the circle $x^2 + (y-1)^2 = 1/4$. The boundary condition on the inner boundary is given by

$$\phi = 4x(1 + \frac{1}{16y+1}) \text{ on } C.$$

The condition on $y = 0$ and the far field condition are as described in Eq. (5.22).

It is easy to check that the exact solution of the particular test problem is given by

$$\phi = x(\frac{1}{x^2 + (y-1)^2} + \frac{1}{x^2 + (y+1)^2}).$$

The main program for numerical solution of the test problem is listed in EX5PT2 below. Note that the boundary points on the circular boundary C are arranged in clockwise order.

```
       program EX5PT2

      integer BCT(1000),N,i,ians,j

      double precision xb(1000),yb(1000),xm(1000),ym(1000),
     & nx(1000),ny(1000),lg(1000),BCV(1000),
     & phi(1000),dphi(1000),pint,dl,xi,eta,pi,exct

      print*,'Enter number of elements on the circle (<1000):'
      read*,N
```

```
      pi=4d0*datan(1d0)
      dl=2d0*pi/dfloat(N)

      do 10 i=1,N
      xb(i)=0.5d0*dcos(dfloat(i-1)*dl)
      yb(i)=1d0-0.5d0*dsin(dfloat(i-1)*dl)
10    continue
      xb(N+1)=xb(1)
      yb(N+1)=yb(1)

      do 20 i=1,N
      xm(i)=0.5d0*(xb(i)+xb(i+1))
      ym(i)=0.5d0*(yb(i)+yb(i+1))
      lg(i)=dsqrt((xb(i+1)-xb(i))**2d0+(yb(i+1)-yb(i))**2d0)
      nx(i)=(yb(i+1)-yb(i))/lg(i)
      ny(i)=(xb(i)-xb(i+1))/lg(i)
20    continue

      do 30 i=1,N
      BCT(i)=0
      BCV(i)=4d0*xm(i)*(1d0+1d0/(16d0*ym(i)+1d0))
30    continue

      call G2LAP1(N,xm,ym,xb,yb,nx,ny,lg,BCT,BCV,phi,dphi)

50    print*,'Enter coordinates xi and eta of an interior point:'
      read*,xi,eta

      call G2LAP2(N,xi,eta,xb,yb,nx,ny,lg,phi,dphi,pint)

      exct=xi*(1d0/(xi**2d0+(eta-1d0)**2d0)
     & +1d0/(xi**2d0+(eta+1d0)**2d0))

      write(*,60)pint,exct
60    format('Numerical and exact values are:',
     & F14.6,' and',F14.6,' respectively')

      print*,'To continue with another point enter 1:'
      read*,ians
      if (ians.eq.1) goto 50

      end
```

Discretizing the circular boundary C into 50 elements of equal length, we compute $\phi(x,0)$ numerically and compare the numerical values with the exact ones for $-5 < x < 5$ in Figure 5.6. This verifies that ϕ may be accurately calculated on the exterior boundary $y = 0$ of the solution domain by using Eq. (5.26) with $\lambda = 1$.

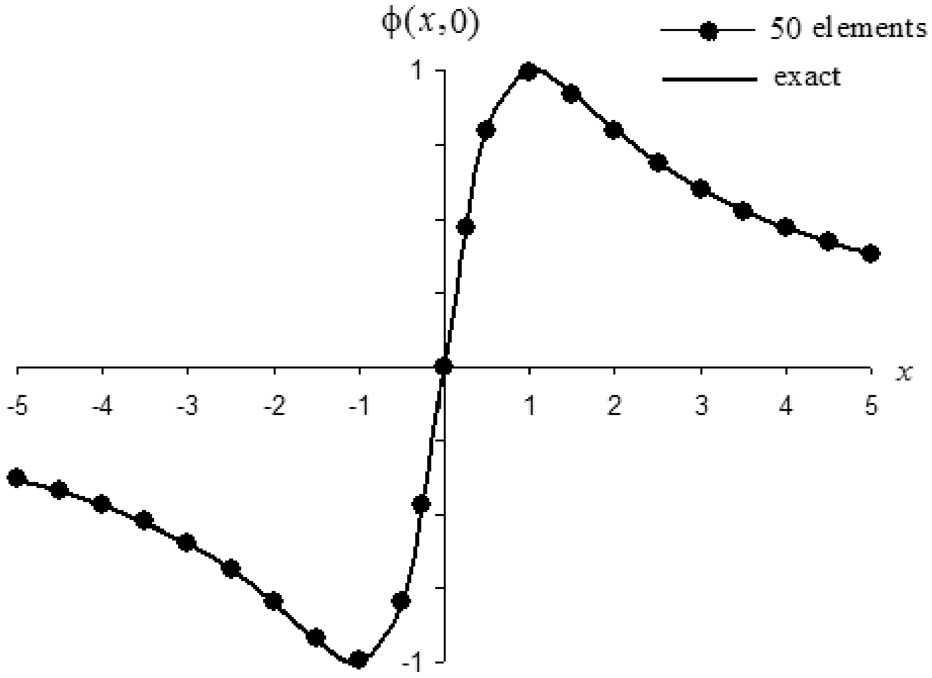

Figure 5.6

5.3 Infinitely Long Strip

5.3.1 Derivation of Green's Functions by Conformal Mapping

Let us consider the infinitely long strip $-\infty < x < \infty$, $0 < y < h$, on the Oxy plane, as shown in Figure 5.7. If we write $z = x + iy$ ($i = \sqrt{-1}$), the conformal mapping $w = \exp(\pi z/h)$ (with $w = u + iv$) can be used to transform the infinitely long strip to the half plane $v > 0$ (on the Ouv plane) in Figure 5.8. The boundary $y = 0$ is mapped to the line $v = 0$, $u > 0$, while $y = h$ to $v = 0$, $u < 0$.

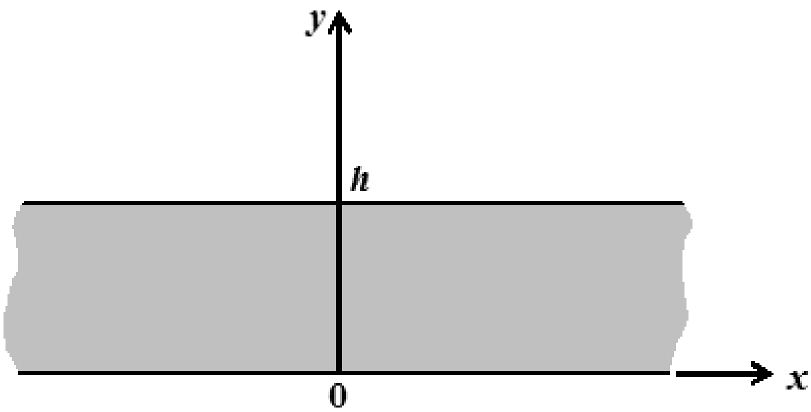

Figure 5.7

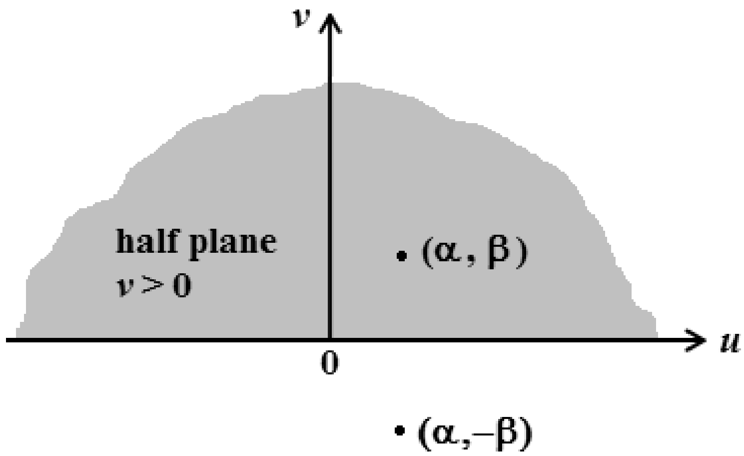

Figure 5.8

In real variables, the conformal mapping $w = \exp(\pi z/h)$ can be expressed in terms of a pair of equations given by

$$\begin{aligned} u &= \exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h}), \\ v &= \exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h}). \end{aligned} \quad (5.28)$$

Let us construct a Green's function $\widetilde{\Psi}(u,v;\alpha,\beta)$ for the half plane $u > 0$ in Figure 5.8 such that $\widetilde{\Psi}$ is singular at $(u,v) = (\alpha,\beta)$ and satisfies the Dirichlet condition

$$\widetilde{\Psi}(u,0;\alpha,\beta) = 0 \text{ for } -\infty < u < \infty. \quad (5.29)$$

From Eq. (5.9), such a Green's function is given by

$$\widetilde{\Psi}(u,v;\alpha,\beta)=\frac{1}{4\pi}\ln([u-\alpha]^2+[v-\beta]^2)-\frac{1}{4\pi}\ln([u-\alpha]^2+[v+\beta]^2). \quad (5.30)$$

If (α,β) is the image point of (ξ,η) (a given point in the infinitely long strip in Figure 5.7), we may use Eq. (5.28) to bring $\widetilde{\Psi}(u,v;\alpha,\beta)$ to the physical $0xy$ plane, that is, we define the function

$$\begin{aligned}\Psi(x,y;\xi,\eta) &= \frac{1}{4\pi}\ln([\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h})-\exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})]^2\\ &\quad+[\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h})-\exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h})]^2)\\ &\quad-\frac{1}{4\pi}\ln([\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h})-\exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})]^2\\ &\quad+[\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h})+\exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h})]^2). \quad (5.31)\end{aligned}$$

According to the theory of conformal mapping[‡], Ψ should satisfy the two-dimensional Laplace's equation everywhere in the infinitely long strip except at $(x,y)=(\xi,\eta)$ and the conditions

$$\Psi(x,0;\xi,\eta)=0 \text{ and } \Psi(x,h;\xi,\eta)=0 \text{ for } -\infty<x<\infty. \quad (5.32)$$

Note that Eq. (5.32) follows directly from Eq. (5.29).

If R is the region bounded by a simple closed curve C which lies in the infinitely long strip, can we use the function $\Psi(x,y;\xi,\eta)$ in Eq. (5.31) in place of $\Phi(x,y;\xi,\eta)$ in Eq. (5.2) to derive the boundary integral equation in Eq. (5.1)?

To find out, let us examine what happens to $\Psi(x,y;\xi,\eta)$ as (x,y) approaches the point (ξ,η).

As (x,y) tends to (ξ,η), the first logarithmic term on the right hand side of Eq. (5.31) blows up but the second logarithmic term is bounded. Examining the argument inside the logarithmic function in the first term, we find that

$$\begin{aligned}&\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h})-\exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})\\ =\ &(\exp(\frac{\pi\xi}{h})+\frac{\pi}{h}\exp(\frac{\pi\xi}{h})[x-\xi]+\frac{\pi^2}{2h^2}\exp(\frac{\pi\xi}{h})[x-\xi]^2+\cdots)\\ &\times(\cos(\frac{\pi\eta}{h})-\frac{\pi}{h}\sin(\frac{\pi\eta}{h})[y-\eta]+\frac{\pi^2}{2h^2}\cos(\frac{\pi\eta}{h})[y-\eta]^2+\cdots)\\ &-\exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})\\ \simeq\ &\frac{\pi}{h}\exp(\frac{\pi\xi}{h})([x-\xi]\cos(\frac{\pi\eta}{h})-[y-\eta]\sin(\frac{\pi\eta}{h}))\\ &\qquad\text{for } (x,y) \text{ very close to } (\xi,\eta),\end{aligned}$$

[‡]For further details, one may refer to the relevant chapters on conformal mapping in the textbook Complex Variables and Applications by RV Churchill and JW Brown (McGraw-Hill, 1990).

and

$$\begin{aligned}
&\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h}) \\
= \; & (\exp(\frac{\pi\xi}{h}) + \frac{\pi}{h}\exp(\frac{\pi\xi}{h})[x-\xi] + \frac{\pi^2}{2h^2}\exp(\frac{\pi\xi}{h})[x-\xi]^2 + \cdots) \\
& \times(\sin(\frac{\pi\eta}{h}) + \frac{\pi}{h}\cos(\frac{\pi\eta}{h})[y-\eta] - \frac{\pi^2}{2h^2}\sin(\frac{\pi\eta}{h})[y-\eta]^2 + \cdots) \\
& -\exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h}) \\
\simeq \; & \frac{\pi}{h}\exp(\frac{\pi\xi}{h})([y-\eta]\cos(\frac{\pi\eta}{h}) + [x-\eta]\sin(\frac{\pi\eta}{h})) \\
& \qquad\qquad \text{for } (x,y) \text{ very close to } (\xi,\eta).
\end{aligned}$$

It follows that

$$\Psi(x,y;\xi,\eta) \simeq \frac{1}{4\pi}\ln([x-\xi]^2 + [y-\eta]^2) \text{ for } (x,y) \text{ very close to } (\xi,\eta),$$

that is, $\Psi(x,y;\xi,\eta)$ behaves like the usual fundamental solution in Eq. (5.2) as (x,y) approaches (ξ,η).

Thus, if R is the region bounded by a simple closed curve C which lies in the infinitely long strip, we can use the function $\Psi(x,y;\xi,\eta)$ in Eq. (5.31) in place of $\Phi(x,y;\xi,\eta)$ in Eq. (5.2) to derive the boundary integral equation in Eq. (5.1).

For the infinitely long strip, let us define

$$\begin{aligned}
\Phi_3(x,y;\xi,\eta) = \; & \frac{1}{4\pi}\ln([\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})]^2 \\
& +[\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h})]^2) \\
& -\frac{1}{4\pi}\ln([\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})]^2 \\
& +[\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h}) + \exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h})]^2), \qquad (5.33)
\end{aligned}$$

as a Green's function (for the two-dimensional Laplace's equation) satisfying the conditions

$$\Phi_3(x,0;\xi,\eta) = 0 \text{ and } \Phi_3(x,h;\xi,\eta) = 0 \text{ for } -\infty < x < \infty. \qquad (5.34)$$

Similarly, we may obtain

$$\begin{aligned}
\Phi_4(x,y;\xi,\eta) = \; & \frac{1}{4\pi}\ln([\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})]^2 \\
& +[\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h})]^2) \\
& +\frac{1}{4\pi}\ln([\exp(\frac{\pi x}{h})\cos(\frac{\pi y}{h}) - \exp(\frac{\pi\xi}{h})\cos(\frac{\pi\eta}{h})]^2 \\
& +[\exp(\frac{\pi x}{h})\sin(\frac{\pi y}{h}) + \exp(\frac{\pi\xi}{h})\sin(\frac{\pi\eta}{h})]^2), \qquad (5.35)
\end{aligned}$$

as a Green's function which satisfies the conditions[§]

$$\frac{\partial}{\partial y}[\Phi_4(x,y;\xi,\eta)]\bigg|_{y=0} = 0 \text{ and } \frac{\partial}{\partial y}[\Phi_4(x,y;\xi,\eta)]\bigg|_{y=h} = 0 \text{ for } -\infty < x < \infty. \tag{5.36}$$

5.3.2 Applications

The Green's functions $\Phi_3(x,0;\xi,\eta)$ and $\Phi_4(x,y;\xi,\eta)$ above are applied to obtain special boundary integral formulations for two specific problems.

Example 5.3

Consider the solution domain R shown in Figure 5.9. The boundary of R comprises four parts denoted by L_1, L_2, L_3 and L_4, where

$$\begin{aligned} L_1 &= \{(x,y) : \ x^2 + (y-1/2)^2 = 1/4, \ x < 0\}, \\ L_2 &= \{(x,y) : \ x = 1, \ 0 < y < 1\}, \\ L_3 &= \{(x,y) : \ 0 \le x < 1, \ y = 1\}, \\ L_4 &= \{(x,y) : \ 0 \le x < 1, \ y = 0\}. \end{aligned}$$

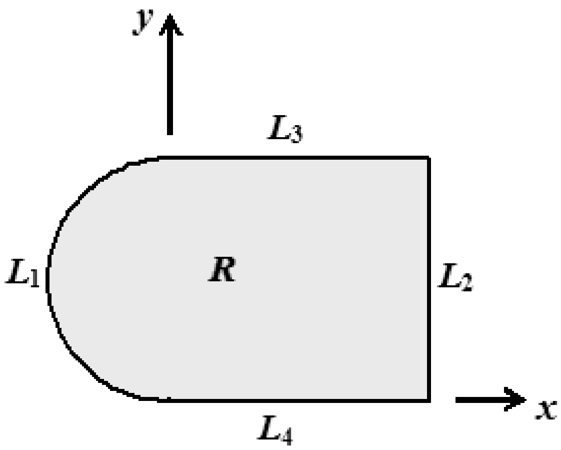

Figure 5.9

The problem of interest here is to solve Eq. (5.14) in R subject to the boundary

§An alternative Green's function satisfying the conditions in Eq. (5.36), expressed in terms of a Fourier integral transform, may be found in the paper "A method for the numerical solution of some elliptic boundary value problems for a strip" by DL Clements and J Crowe in the International Journal of Computer Mathematics (Volume 8, 1980, pp. 345-355).

conditions

$$\begin{aligned}
\frac{\partial\phi}{\partial n} &= 2\pi x\sin(\pi y)\sinh(\pi[x-1]) \\
&\quad +\pi(2y-1)\cos(\pi y)\cosh(\pi[x-1]) \\
&\qquad\qquad\qquad \text{for } (x,y)\in L_1, \\
\frac{\partial\phi}{\partial n} &= 0 \text{ for } (x,y)\in L_2, \\
\phi &= 0 \text{ for } (x,y)\in L_3\cup L_4.
\end{aligned}$$

It is easy to verify that the exact solution is given by

$$\phi(x,y) = \sin(\pi y)\cosh(\pi[x-1]).$$

In view of the condition $\phi = 0$ for $(x,y)\in L_3\cup L_4$, we may use the Green's function $\Phi_3(x,y;\xi,\eta)$ in Eq. (5.33) to obtain the boundary integral equation

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int\limits_{L_1\cup L_2}[\phi(x,y)\frac{\partial}{\partial n}(\Phi_3(x,y;\xi,\eta)) - \Phi_3(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y),$$

with the parameter λ defined by

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta)\in L_3\cup L_4, \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } L_1\cup L_2, \\ 1 & \text{if } (\xi,\eta)\in R. \end{cases}$$

In using the above integral formulation to devise a boundary element procedure, we do not have to discretize L_3 and L_4. We discretize L_1 and L_2 into N straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$. Proceeding as before, we may approximate ϕ and $\partial\phi/\partial n$ as constants over the boundary elements and reduce the task of finding the unknown ϕ on $L_1\cup L_2$ to solving a system of linear algebraic equations. To set up the linear algebraic equations, we have to evaluate the line integrals

$$\int\limits_{C^{(k)}}\Phi_3(x,y;\xi,\eta)ds(x,y) \text{ and } \int\limits_{C^{(k)}}\frac{\partial}{\partial n}[\Phi_3(x,y;\xi,\eta)]ds(x,y).$$

These line integrals may be evaluated numerically using the formula in Eq. (3.23) (page 62, Chapter 3).

We shall not go into details here, but earlier FORTRAN codes such as `CPG`, `CEHHZ1` and `CEHHZ2` (listed on pages 64 to 67, Chapter 3) can be easily modified in an appropriate manner to solve the problem under consideration here through the special boundary integral formulation given above.

To obtain some numerical results, the semi-circle L_1 is discretized into 50 equal length boundary elements and the straight line L_2 into another 50 equal length elements (that is, 100 boundary elements are employed on $L_1 \cup L_2$). As $\partial\phi/\partial n$ is specified on L_2 ($x = 1$, $0 < y < 1$), we compare the numerically obtained values of $\phi(1, y)$ against the exact solution in Figure 5.10. There is a reasonably good agreement between the numerical and the exact values of ϕ on L_2. The validity of the Green's function $\Phi_3(x, y; \xi, \eta)$ is thus verified here.

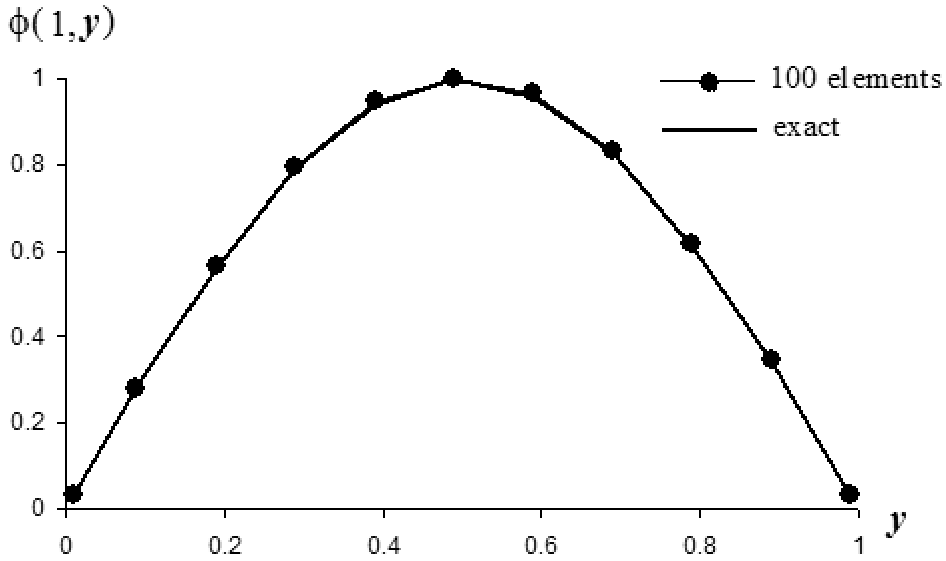

Figure 5.10

Example 5.4

Take the solution domain R to be $0 < x < 3$, $0 < y < 1$. We are interested in solving Eq. (5.14) in R subject to the boundary conditions

$$\begin{aligned}
\left.\frac{\partial\phi}{\partial n}\right|_{y=0} &= 0 \text{ for } 0 < x < 3, \\
\left.\frac{\partial\phi}{\partial n}\right|_{y=1} &= \begin{cases} 1 & \text{for } 1 < x < 2, \\ 0 & \text{otherwise,} \end{cases} \\
\phi(0, y) &= 0 \text{ for } 0 < y < 1, \\
\left.\frac{\partial\phi}{\partial n}\right|_{x=3} &= 0 \text{ for } 0 < y < 1.
\end{aligned}$$

If we use the Green's function $\Phi_4(x, y; \xi, \eta)$ in Eq. (5.35) to derive a boundary integral equation for the problem, we obtain

$$\lambda(\xi,\eta)\phi(\xi,\eta) = -\int_1^2 \Phi_4(x,1;\xi,\eta)dx - \int_0^1 \Phi_4(0,y;\xi,\eta)\left.\frac{\partial\phi}{\partial n}\right|_{x=0} dx + \int_0^1 \frac{\partial}{\partial n}(\Phi_4(x,y;\xi,\eta))\bigg|_{x=3} \phi(3,y)dy, \quad (5.37)$$

where $\lambda(\xi, \eta) = 1/2$ if (ξ, η) lies on the vertical sides ($x = 0$ and $x = 3$) of the rectangular solution domain and $\lambda(\xi, \eta) = 1$ if (ξ, η) lies on the horizontal sides ($y = 0$ and $y = 1$) or in the interior of the solution domain.

With the use of the special Green's function $\Phi_4(x, y; \xi, \eta)$, it is not necessary to integrate over the parts of the horizontal sides ($y = 0$ and $y = 1$) where $\partial\phi/\partial n$ is specified as 0. Proceeding as usual, we may discretize the boundary integral equation above to set up a system of linear algebraic equations to determine $\partial\phi/\partial n$ on $x = 0$ and ϕ on $x = 3$. We assume that $\partial\phi/\partial n$ is a constant over each element. Once the unknown values of $\partial\phi/\partial n$ on the elements are determined, $\phi(\xi, \eta)$ may be computed at any point (ξ, η) in the solution domain.

Table 5.1

(x, y)	With special Green's function	Without special Green's function
$(0.50, 0.25)$	0.4847	0.4858
$(1.50, 0.25)$	1.2666	1.2677
$(2.50, 0.25)$	1.4874	1.4831
$(0.50, 0.50)$	0.4978	0.4983
$(1.50, 0.50)$	1.3322	1.3340
$(2.50, 0.50)$	1.5015	1.4968
$(0.50, 0.75)$	0.5147	0.5129
$(1.50, 0.75)$	1.4556	1.4584
$(2.50, 0.75)$	1.5180	1.5127

It appears that there is no simple analytical solution for the problem under consideration. To check the validity of the special boundary integral equation obtained using the Green's function $\Phi_4(x, y; \xi, \eta)$, we compare its numerical values of ϕ with those obtained using the boundary element procedure outlined in Chapter 1 (that is, without the use of any special Green's function, by using the subroutines `CELAP1` and `CELAP2`). The intervals of integration in the special boundary integral equation in Eq. (5.37) are discretized into 120 straight line elements, each of length 0.0250 units, to set up a system of 80 linear algebraic equations in 80 unknowns. For the boundary element method in Chapter 1, 320 elements, each also of length 0.0250 units, are employed. The numerical values of ϕ thus obtained at selected points in the interior of the solution domain are given in Table 5.1. There is a good agreement between the two sets of numerical values.

5.4 Exterior Region of a Circle

5.4.1 Two Special Green's Functions

Consider the region $x^2 + y^2 > a^2$ (a is a positive real number) as sketched in Figure 5.11. For this region, we give Green's functions¶ satisfying certain conditions on the circular boundary $x^2 + y^2 = a^2$.

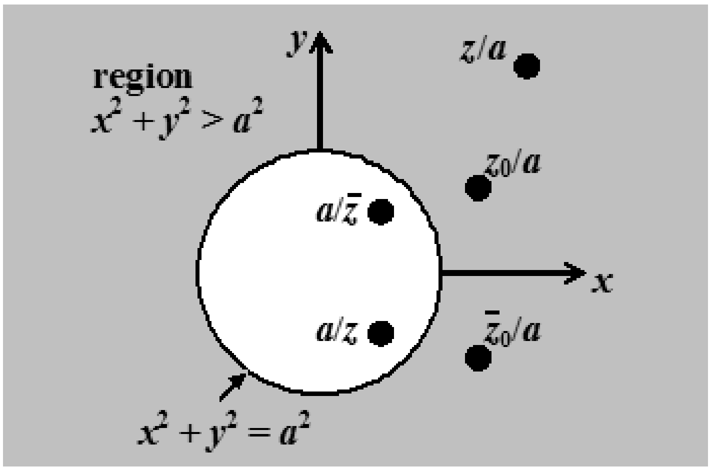

Figure 5.11

Let the required Green's functions take the form

$$\Phi(x, y; \xi, \eta) = \frac{1}{4\pi}\ln([\frac{x}{a} - \frac{\xi}{a}]^2 + [\frac{y}{a} - \frac{\eta}{a}]^2) + \Phi^*(x, y; \xi, \eta), \tag{5.38}$$

such that $\Phi^*(x, y; \xi, \eta)$ satisfies

$$\frac{\partial^2}{\partial x^2}[\Phi^*(x, y; \xi, \eta)] + \frac{\partial^2}{\partial y^2}[\Phi^*(x, y; \xi, \eta)] = 0$$
$$\text{for } x^2 + y^2 > a^2 \text{ and } \xi^2 + \eta^2 > a^2. \tag{5.39}$$

¶Note from the author. I had no prior knowledge of the specific forms of the Green's functions $\Phi_5(x, y; \xi, \eta)$ and $\Phi_6(x, y; \xi, \eta)$ given in Eqs. (5.42) and (5.44). I deduced them by guesswork during the writing of this section. I believe that they must have already been recorded (in equivalent forms) somewhere in the research literature. If you have any information on this, please e-mail me at mwtang@ntu.edu.sg.

For convenience, we define $z = x + iy$ and $z_0 = \xi + i\eta$ ($i = \sqrt{-1}$) and refer to the points (x, y) and (ξ, η) as z and z_0 respectively. We may rewrite $\Phi(x, y; \xi, \eta)$ in Eq. (5.38) given above as[‖]

$$\Phi(x, y; \xi, \eta) = \frac{1}{2\pi} \operatorname{Re}\{\ln(\frac{1}{a}[z - z_0])\} + \Phi^*(x, y; \xi, \eta), \tag{5.40}$$

where Re denotes the real part of a complex number.

Let us take

$$\Phi^*(x, y; \xi, \eta) = -\frac{1}{2\pi} \operatorname{Re}\{\ln(1 - \frac{\overline{z}_0 z}{a^2})\}, \tag{5.41}$$

where the overhead bar denotes the complex conjugate of a complex number.

From Figure 5.11, it is obvious that $a/z \neq \overline{z}_0/a$ for all points z and z_0 in the region $x^2 + y^2 > a^2$. This implies that $1 - \overline{z}_0 z/a^2$ is not zero over the region $x^2 + y^2 > a^2$ and $\Phi^*(x, y; \xi, \eta)$ as given in Eq. (5.41) satisfies Eq. (5.39).

We may use Eqs. (5.40) and (5.41) to define a Green's function for the region $x^2 + y^2 > a^2$, that is,

$$\Phi_5(x, y; \xi, \eta) = \frac{1}{2\pi} \operatorname{Re}\{\ln(\frac{1}{a}[z - z_0])\} - \frac{1}{2\pi} \operatorname{Re}\{\ln(1 - \frac{\overline{z}_0 z}{a^2})\}. \tag{5.42}$$

If we write z in polar form as $z = r\exp(i\theta)$, we obtain

$$\begin{aligned}
&\Phi_5(x, y; \xi, \eta)|_{\text{on the circle } x^2+y^2=a^2} \\
= \;& \frac{1}{2\pi} \operatorname{Re}\{\ln(\exp(i\theta) - \frac{z_0}{a})\} - \frac{1}{2\pi} \operatorname{Re}\{\ln(1 - \frac{\overline{z}_0 \exp(i\theta)}{a})\} \\
= \;& \frac{1}{2\pi} \operatorname{Re}\{\ln(\exp(i\theta) - \frac{z_0}{a})\} - \frac{1}{2\pi} \operatorname{Re}\{\overline{\ln(1 - \frac{\overline{z}_0 \exp(i\theta)}{a})}\} \\
= \;& \frac{1}{2\pi} \operatorname{Re}\{\ln(\exp(i\theta) - \frac{z_0}{a})\} - \frac{1}{2\pi} \operatorname{Re}\{\ln(1 - \frac{z_0 \exp(-i\theta)}{a})\} \\
= \;& \frac{1}{2\pi} \operatorname{Re}\{\ln(\exp(i\theta) - \frac{z_0}{a})\} - \frac{1}{2\pi} \operatorname{Re}\{\ln[\exp(-i\theta)(\exp(i\theta) - \frac{z_0}{a})]\} = 0.
\end{aligned}$$

Note that $r = \sqrt{x^2 + y^2}$, that is, $r = a$ on the circle.

Thus, the Green's function $\Phi_5(x, y; \xi, \eta)$ in Eq. (5.42) satisfies the boundary condition

$$\Phi_5(x, y; \xi, \eta) = 0 \text{ on the circle } x^2 + y^2 = a^2. \tag{5.43}$$

In a similar way, another Green's function for the region $x^2+y^2 > a^2$ as defined by

$$\Phi_6(x, y; \xi, \eta) = \frac{1}{2\pi} \operatorname{Re}\{\ln(\frac{1}{a}[z - z_0])\} + \frac{1}{2\pi} \operatorname{Re}\{\ln(1 - \frac{\overline{z}_0 z}{a^2}) - \ln(\frac{z}{a})\} \tag{5.44}$$

[‖]The complex logarithmic function $\ln(w)$ is defined by $\ln|w| + i\arg(w)$. Thus, $\operatorname{Re}\{\ln(z)\} = \frac{1}{2}\ln(p^2 + q^2)$ if $w = p + iq$, where p and q are real numbers.

can be shown to satisfy the boundary condition

$$\frac{\partial}{\partial n}[\Phi_6(x,y;\xi,\eta)] = 0 \text{ on the circle } x^2+y^2=a^2. \quad (5.45)$$

Note that $\partial[\Phi_6(x,y;\xi,\eta)]/\partial n = -\partial[\Phi_6(r\cos\theta, r\sin\theta;\xi,\eta)]/\partial r$ on $x^2+y^2=a^2$.

5.4.2 Applications

Example 5.5

The solution domain R is taken to be a doubly connected region given by

$$R = \{(x,y):\ x^2+y^2>1,\ -\ell<x<\ell,\ -\ell<y<\ell,\ \ell>1\}.$$

Refer to Figure 5.12.

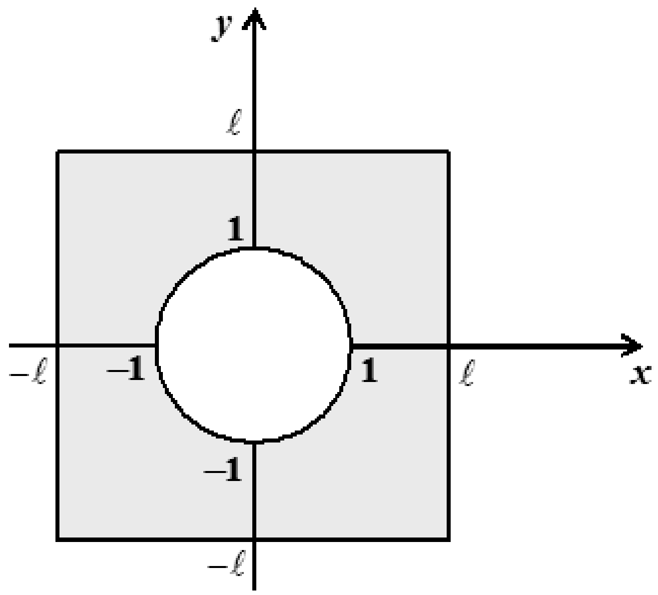

Figure 5.12

We are interested in solving Eq. (5.14) in R subject to the boundary conditions

$$\phi = 0 \text{ on the inner boundary } I \text{ (circle)},$$
$$\phi = 1 \text{ on the outer boundary } E \text{ (sides of square)}.$$

For the boundary value problem above, the Green's function $\Phi_5(x,y;\xi,\eta)$ in Eq. (5.42) may be applied together with the given boundary conditions to obtain the boundary integral equation

$$\lambda(\xi,\eta)\phi(\xi,\eta)=\int\limits_{E}[\frac{\partial}{\partial n}(\Phi_5(x,y;\xi,\eta))-\Phi_5(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y), \qquad (5.46)$$

with the parameter λ defined by

$$\lambda(\xi,\eta)=\begin{cases} 0 & \text{if } (\xi,\eta)\in I, \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } E, \\ 1 & \text{if } (\xi,\eta)\in R. \end{cases}$$

Integration over the inner boundary I is avoided by the use of the special Green's function.

The problem has been solved before using several different approaches**. For $\ell = 1.53499$, a method based on complex variables gives the approximation††

$$\begin{aligned}\phi(x,y) \quad &\simeq \quad 0.99203\cdot\ln(x^2+y^2)+0.01331([x+iy]^4+[x-iy]^4)(1-\frac{1}{[x^2+y^2]^4}) \\ &+0.00007([x+iy]^8+[x-iy]^8)(1-\frac{1}{[x^2+y^2]^8}). \qquad (5.47)\end{aligned}$$

We discretize the outer boundary E into 80 equal length elements to set up a system of 80 linear algebraic equations to determine $\partial\phi/\partial n$ on E. The unknown $\partial\phi/\partial n$ is assumed to be a constant over each boundary element. In Table 5.2, the numerical values of ϕ as obtained using Eq. (5.46) are compared with those computed from Eq. (5.47) at selected points in the interior of R. The two sets of numerical values of ϕ are in quite close agreement with each other.

Table 5.2

(x,y)	Boundary integral equation in Eq. (5.46)	Approximate formula in Eq. (5.47)
$(0.77781, 0.77781)$	0.16856	0.16854
$(-0.60000, 1.0392)$	0.34049	0.34027
$(0.22574, -1.2803)$	0.57292	0.57184
$(1.5000, 0.00000)$	0.94001	0.93756
$(0.00000, 1.4000)$	0.76681	0.76497
$(1.4142, 1.4142)$	0.97920	0.98683

**See, for example, the article "A solution of Laplace's equation for a round hole in a square peg," Journal of the Society for Industrial and Applied Mathematics (Volume 12, 1964, pp. 1-14) by RW Hockney. As pointed in this article, the boundary value problem under consideration arises in the modeling of gas leakage across the graphite brick of a nuclear reactor.

††This is as given in "Laplace's equation in the region bounded by a circle and a square," Technical Report No. M9/97 (Universiti Sains Malaysia, 1997) by KH Chew. Approximate formulae of ϕ for other values of ℓ may also found in this report.

Example 5.6

The solution domain R is as in Figure 5.12. As in Example 5.6, the inner boundary $x^2 + y^2 = 1$ is denoted by I and the outer boundary (the sides of the square) by E. The governing equation in R is the Laplace's equation in Eq. (5.14). We impose the condition $\partial\phi/\partial n = 0$ on the interior boundary I. At each and every point on the exterior boundary E, either ϕ or $\partial\phi/\partial n$ is suitably prescribed.

If the Green's function $\Phi_6(x, y; \xi, \eta)$ in Eq. (5.44) is applied together with the boundary condition on I to obtain the boundary integral equation for the problem under consideration, we obtain

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int\limits_E [\phi(x,y)\frac{\partial}{\partial n}(\Phi_6(x,y;\xi,\eta)) - \Phi_6(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y), \tag{5.48}$$

with the parameter λ defined by

$$\lambda(\xi,\eta) = \begin{cases} 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } E, \\ 1 & \text{if } (\xi,\eta) \in R \cup I. \end{cases}$$

For a particular solution of Eq. (5.14) satisfying the condition $\partial\phi/\partial n = 0$ on I, we take

$$\phi(x,y) = x + \frac{x}{x^2 + y^2}. \tag{5.49}$$

We use Eq. (5.49) to produce boundary data for ϕ on E and discretize the boundary integral equation in Eq. (5.48) to solve numerically the problem under consideration subject to the boundary data of ϕ generated on E. If the numerical procecure really works, we should be able to recover numerically the solution in Eq. (5.49). For $\ell = 2.00000$, in Table 5.3, the numerical values of ϕ on I (where $r = 1$) obtained using 80 boundary elements are compared with the exact solution in Eq. (5.49) for selected values of the polar angle θ.

Table 5.3

θ	Eq. (5.48)	Exact solution
0^o	2.000006	2.000000
15^o	1.931910	1.931852
30^o	1.732099	1.732051
45^o	1.414239	1.414214
60^o	1.000003	1.000000
75^o	0.517632	0.517638
90^o	0.000000	0.000000

5.5 Summary and Discussion

Several Green's functions for the two-dimensional Laplace's equation in a half plane, an infinitely long strip and a region exterior to a circle, which satisfy certain boundary conditions, are given. They are applied to obtain special boundary integral equations for particular boundary value problems.

As is clear from the examples given, if the required solution and the Green's function used satisfy the same homogeneous condition[‡‡] on a certain part of the boundary, it is not necessary to integrate over that part. Furthermore, that boundary condition is automatically satisfied in the boundary integral formulation of the problem under consideration and requires no further treatment. This gives rise to a smaller system of linear algebraic equations in the boundary element procedure.

The smaller system definitely helps to ease the requirement on computer memory storage and precision. There are fewer coefficients to compute in setting up the system and less computer time is needed to invert a smaller matrix. More computer time is, however, required to compute *each* coefficient, as the Green's function assumes a form which is more complicated than the usual fundamental solution. In general, the integration of the Green's function and its normal derivative over an element has to be done numerically. Whether or not there is a significant overall reduction (or increase) in computer time needed to complete the boundary element procedure depends on how complicated the Green's function is.

5.6 Exercises

Unless otherwise stated, the two-dimensional Laplace's equation is the governing partial differential equation in all the exercises below.

1. Consider again the boundary value problem in Exercise 7 of Chapter 1 (page 35). Use the Green's function $\Phi_1(x, y; \xi, \eta)$ in Eq.(5.9) to obtain a special boundary integral equation for the problem. Discretize the boundary integral equation in order to solve for ϕ numerically. Compare the numerical values of ϕ at selected points in the interior of the solution domain with the exact solution.

2. Let $z = x + iy$ and $w = u + iv$, where x, y, u and v are real variables. Check that the mapping $w = z^2$ transforms the quarter plane $x > 0$, $y > 0$ (on the Oxy plane) to the half plane $v > 0$ (on the Ouv plane). Find a Green's function $\Phi(x, y; \xi, \eta)$ for the quarter plane such that

$$\begin{aligned}\Phi(0, y; \xi, \eta) &= 0 \text{ for } 0 < y < \infty,\\ \Phi(x, 0; \xi, \eta) &= 0 \text{ for } 0 < x < \infty.\end{aligned}$$

[‡‡] In the examples given here, the homogeneous conditions involved are given by either $\phi = 0$ or $\partial\phi/\partial n = 0$ on the boundary. The boundary condition given by $\partial\phi/\partial n + k\phi = 0$ may also be considered.

3. For the quarter plane in Exercise 2, construct a Green's function $\Phi(x, y; \xi, \eta)$ such that

$$\begin{aligned} \frac{\partial}{\partial x}[\Phi(x, y; \xi, \eta)]\bigg|_{x=0} &= 0 \text{ for } 0 < y < \infty, \\ \frac{\partial}{\partial y}[\Phi(x, y; \xi, \eta)]\bigg|_{y=0} &= 0 \text{ for } 0 < x < \infty. \end{aligned}$$

4. The mapping $w = \cos(\pi z/a)$ transforms the half strip $0 < x < a$, $y > 0$ to the half plane $v > 0$. Find a Green's function $\Phi(x, y; \xi, \eta)$ for the half strip such that

$$\begin{aligned} \Phi(0, y; \xi, \eta) &= 0 \text{ for } y > 0, \\ \Phi(a, y; \xi, \eta) &= 0 \text{ for } y > 0, \\ \Phi(x, 0; \xi, \eta) &= 0 \text{ for } 0 < x < a. \end{aligned}$$

Use the Green's function to obtain a special boundary integral equation for the boundary value problem in Exercise 3 of Chapter 2 (page 55). Discretize the boundary integral equation in order to solve for ϕ numerically. Compare the numerical values of ϕ at selected points in the interior of the solution domain with the exact solution.

5. For the Helmholtz equation on page 57 in the half plane $y > 0$, construct a Green's function $\Omega(x, y; \xi, \eta)$ satisfying the boundary condition

$$\Omega(x, 0; \xi, \eta) = 0 \text{ for } 0 < x < \infty.$$

6. Repeat Exercise 5 with the boundary condition

$$\frac{\partial}{\partial y}[\Omega(x, y; \xi, \eta)]\bigg|_{y=0} = 0 \text{ for } 0 < x < \infty.$$

7. The Green's function $\Phi_5(x, y; \xi, \eta)$ in Eq. (5.42), but not $\Phi_6(x, y; \xi, \eta)$ in Eq. (5.44), is valid for the circle $x^2 + y^2 < a$. Explain why.

the boundary conditions

$$\begin{aligned}\phi &= 0 \text{ on the horizontal side } y=0 \text{ for } 0<x<1,\\ \phi &= 1 \text{ on the vertical side } x=0 \text{ for } 0<y<1,\\ \frac{\partial\phi}{\partial n} &= 0 \text{ on the quarter circle } x^2+y^2=1,\ x>0,\ y>0.\end{aligned}$$

Discretize each of the three parts of the boundary (the two sides and the quarter circle) into N_0 boundary elements so that $N = 3N_0$, that is, so that we have a total of $3N_0$ boundary elements. Using various values of N_0, run the modified program to calculate the numerical solution at various selected interior points and compare the results obtained with the exact solution $\phi(x,y) = (2/\pi)\arctan(y/x)$. According to the exact solution, ϕ is not well defined at $(0,0)$. This is not surprising, as $(0,0)$ is the point where the value of ϕ suddenly jumps from 0 on the horizontal side to 1 on the vertical side. The normal derivative $\partial\phi/\partial n$ is given by $-\partial\phi/\partial y$ on the horizontal side. From the exact solution, we know that $\partial\phi/\partial n \to -\infty$ on $y=0$ as $x \to 0^+$. This singular behavior of the solution may be a problem for the boundary element procedure. Investigate the numerical solution at points near $(0,0)$. The subroutine `CELAP1` returns the numerical values of $\partial\phi/\partial n$ on the horizontal side in the first N_0 variables of the real array `dphi(1:N)`. Compare the values in these variables with the exact values of $\partial\phi/\partial n$ on the horizontal side. Repeat the same exercise with the vertical side.

8. Modify the program `EX1PT1` in Section 1.7 to solve numerically the Laplace's equation given by Eq. (1.1) in the region $0<x<1$, $0<y<1$, subject to the boundary conditions

$$\begin{aligned}\frac{\partial\phi}{\partial n} &= 5\cos(\pi x) \text{ on the top horizontal side } y=1 \text{ for } 0<x<1,\\ \frac{\partial\phi}{\partial n} &= 0 \text{ on the other three remaining sides},\\ \phi &= 1 \text{ at } (x,y) = (\frac{1}{3},\frac{1}{2}).\end{aligned}$$

By discretizing each side of the square into N_0 equal length elements, run the modified program with various values of N_0 to compute ϕ at some selected interior points. (Notes. (1) With the boundary conditions alone, the boundary value problem does not have a unique solution. Thus, the subroutine `CELAP2` by itself may not return us the desired numerical value of ϕ in the real variable `pint`. Use the additional condition as given by $\phi(1/3,1/2) = 1$ to find the desired numerical solution. (2) The exact solution of this problem is $\phi(x,y) = 5\cos(\pi x)\cosh(\pi y)/(\pi\sinh(\pi)) + 0.827\,103\,138\,436$. Compare your numerical values with the exact solution.)