Hypersingular integral equations for arbitrarily located planar cracks in an anisotropic elastic bimaterial

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Abstract

Hypersingular integral equations are derived for the problem of arbitrarily-located planar cracks lying in the interior of two dissimilar anisotropic elastic half-spaces which adhere perfectly to each other. The unknown functions in the integral equations are the crack-opening displacements. The integral equations are solved numerically for specific examples involving particular transversely-isotropic materials in order to compute physical quantities of interest such as the crack tip stress intensity factors or the crack energy.

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1 INTRODUCTION

During the last ten years or so, there has been a growing interest among many researchers in formulating crack problems using hypersingular integral equations, e.g. Ioakimidis [1], Lin'kov and Mogilevskaya [2], Takakuda *et al.* [3], Nied [4] and Ang and Clements [5]. An advantage of using such an integral formulation is that the unknown functions in the integral equations are directly related to the crack-opening displacements. Furthermore, effective numerical methods for solving the integral equations, such as the collocation technique in Kaya and Erdogan [6], are available.

In the present paper, the plane problem of several arbitrarily-located planar cracks in two dissimilar anisotropic elastic half-spaces which adhere perfectly to each other is formulated in terms of a system of hypersingular integral equations. The cracks are assumed to open up under the action of suitably prescribed internal tractions. In the formulation, the continuity conditions on the interface between the dissimilar materials are enforced exactly. Thus, in the numerical solution of the integral equations, discretisation of the interface is conveniently avoided.

The integral equations derived are valid for the most general anisotropic materials. The materials are not required to possess any symmetries in their anisotropy. However, we solve those equations, numerically, using a collocation technique, for only specific examples of the problem, involving particular transversely-isotropic materials. Numerical results for the crack tip stress intensity factors or the crack energy are obtained.

From a practical standpoint, the problem under consideration may be of useful relevance to composite and anisotropic structures which can now be found in an increasingly wider range of applications in modern technology. The need to assess the reliability and integrity of these structures has indeed generated much interest among many researchers in the analysis of stress in anisotropic layered materials, particularly those containing cracks, e.g. Lahiri *et al.* [7], Ang [8, 9], Clements [10], Clements *et al.* [11] and Willis [12].

2 STATEMENT OF THE PROBLEM

Referring to an $0x_1x_2x_3$ Cartesian coordinate system, consider an infinite elastic medium which comprises two regions: $x_2 > 0$ (region 1) and $x_2 < 0$ (region 2). The regions are occupied by two possibly different anisotropic materials which adhere rigidly to each other along the $x_2 = 0$ interface. There are several arbitrarily-located planar cracks in the elastic medium. It is assumed that the geometries of the cracks do not vary along the x_3-direction and the cracks do not intersect with one another or the interface separating the two regions.

Let us denote those cracks in region 1 (if any) by $C^{(1)}$, $C^{(2)}$, ...,$C^{(N-1)}$ and $C^{(N)}$ and any other remaining cracks in region 2 by $C^{(N+1)}$, $C^{(N+2)}$, ...,$C^{(N+M-1)}$ and $C^{(N+M)}$. (Thus, there are $N + M$ cracks present.) On the $0x_1x_2$ plane, the tips of the crack $C^{(m)}$ are denoted by $(\alpha^{(m)}, \beta^{(m)})$ and

$(\gamma^{(m)}, \delta^{(m)})$.

The cracks are opened up by suitably prescribed internal tractions which are independent of x_3 and time. The displacements and stresses (generated by the presence of the cracks) vanish at infinity. The problem is to determine the displacement and stress fields throughout the composite medium.

3 BASIC EQUATIONS

Here, we present the basic equations of the theory of elasticity, which will be used for the solution of the problem described above.

In the absence of body forces, the equilibrium equations for a homogeneous anisotropic elastic material, are given by the system

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0, \tag{1}$$

where u_k are the Cartesian displacements and c_{ijkl} are the elastic moduli of the material. The latin subscripts i, j, k and l in (1) take the values of 1, 2 and 3. The usual convention of summing over a repeated index is assumed for latin subscripts only.

If the displacements u_k are independent of x_3 then (1) admit solutions of the form

$$u_k(\underset{\sim}{x}) = \text{Re}\{\sum_{\alpha} A_{k\alpha} f_\alpha(z_\alpha)\}, \tag{2}$$

where $\sum_\alpha$ denotes summation over α from 1 to 3, Re denotes the real part of a complex number, $\underset{\sim}{x} = (x_1, x_2)$, $f_\alpha(z_\alpha)$ are holomorphic functions of $z_\alpha = x_1 + \tau_\alpha x_2$, τ_α are roots, having positive imaginary parts, of the sextic equation

$$\det[c_{i1k1} + \tau c_{i1k2} + \tau c_{i2k1} + \tau^2 c_{i2k2}] = 0, \tag{3}$$

and $A_{k\alpha}$ are related to τ_α by

$$[c_{i1k1} + \tau_\alpha c_{i1k2} + \tau_\alpha c_{i2k1} + \tau_\alpha^2 c_{i2k2}]A_{k\alpha} = 0. \tag{4}$$

It can be shown that the roots of (3) cannot be real and they occur in complex conjugate pairs. We will assume that there are 3 distinct complex conjugate pairs of roots for (3). For further details, refer to Clements [13] or Stroh [14].

The Cartesian stresses σ_{kj} which correspond to the displacements in (2) are given by

$$\sigma_{kj}(\underset{\sim}{x}) = \text{Re}\{\sum_{\alpha} L_{kj\alpha} f'_{\alpha}(z_{\alpha})\}, \tag{5}$$

where the prime denotes differentiation with respect to the relevant argument and $L_{kj\alpha} = (c_{kjp1} + \tau_{\alpha} c_{kjp2}) A_{p\alpha}$.

If the system (1) holds in a region R (in the $0x_1x_2$ plane) bounded by a curve D, it can be shown that (see Clements [13])

$$u_k(\underset{\sim}{x}) = \int_D [u_r(\underset{\sim}{\xi})\Gamma_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) - p_r(\underset{\sim}{\xi})\Phi_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi})] dS(\underset{\sim}{\xi}) \text{ for } \underset{\sim}{x} \in R, \tag{6}$$

where $\underset{\sim}{\xi} = (\xi_1, \xi_2)$, p_r are the tractions acting across D and

$$\Phi_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \text{Re}\{\sum_{\alpha} A_{r\alpha} N_{\alpha j} \ln(c_{\alpha} - z_{\alpha})\} d_{jk},$$
$$\Gamma_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \text{Re}\{\sum_{\alpha} L_{rj\alpha} N_{\alpha p} (c_{\alpha} - z_{\alpha})^{-1}\} n_j(\underset{\sim}{\xi}) d_{pk}, \tag{7}$$

where $c_{\alpha} = \xi_1 + \tau_{\alpha}\xi_2$, $n_j(\underset{\sim}{\xi})$ are the components of the unit normal outward vector to D at $\underset{\sim}{\xi}$, $[N_{\alpha j}]$ is the inverse of $[A_{j\beta}]$, $[d_{jk}]$ is defined by the relation

$$-\frac{i}{2} \sum_{\alpha} \{L_{j2\alpha} N_{\alpha p} - \bar{L}_{j2\alpha} \bar{N}_{\alpha p}\} d_{pk} = \delta_{jk},$$

where $i = \sqrt{-1}$, $\bar{z}$ denotes the complex conjugate of z and δ_{jk} is the kronecker-delta.

Now, if R covers the entire $0x_1x_2$ plane and contains several straight cuts (of finite lengths) denoted by L_1,L_2, ...,L_{N-1} and L_N in its interior, and if the displacements $u_k(\underset{\sim}{x})$ behave as $O(|\underset{\sim}{x}|^{-s})$ $(s > 0)$, as $|\underset{\sim}{x}| \to \infty$, then, from (6), the displacements $u_k(\underset{\sim}{x})$ can be written as:

$$u_k(\underset{\sim}{x}) = \sum_{m=1}^{N} \int_{L_m^+} \Delta u_r(\underset{\sim}{\xi}) \Gamma_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) dS(\underset{\sim}{\xi}), \tag{8}$$

where L_m^- and L_m^+ denote respectively the "lower" and "upper" faces of the cut L_m and $\Delta u_r(\underset{\sim}{\xi}) = [u_r(\underset{\sim}{\xi})]^+ - [u_r(\underset{\sim}{\xi})]^-$, with $[u_r(\underset{\sim}{\xi})]^\pm$ denoting the value of $u_r(\underset{\sim}{\xi})$ for $\underset{\sim}{\xi} \in L_m^\pm$. Note that in the derivation of (8) we assume that there is no discontinuity in the stress across the cuts and make use of the fact that $\Phi_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi})$ are single-valued functions for all $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ on the $0x_1x_2$ plane (provided that $\underset{\sim}{\xi} \neq \underset{\sim}{x}$).

4 HYPERSINGULAR INTEGRAL FORMULATION

For the solution of the problem under consideration, guided by (8), we choose the displacements in region n to be given by

$$u_k^{(n)}(\underset{\sim}{x}) = \sum_{m=1}^{N+M} \int_{C_+^{(m)}} \Delta u_p(\underset{\sim}{\xi}) U_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi}) dS(\underset{\sim}{\xi}), \tag{9}$$

where $C_+^{(m)}$ denotes the "upper" face of the crack $C^{(m)}$, Δu_p give the crack-opening displacements and

$$U_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \operatorname{Re}\{\sum_\alpha L_{pj\alpha}^{(n)} \; N_{\alpha r}^{(n)} (c_\alpha^{(n)} - z_\alpha^{(n)})^{-1}\} n_j(\underset{\sim}{\xi}) d_{rk}^{(n)} + G_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi}), \tag{10}$$

where $c_\alpha^{(n)} = \xi_1 + \tau_\alpha^{(n)} \xi_2$ and $z_\alpha^{(n)} = x_1 + \tau_\alpha^{(n)} x_2$. The superscript (n) indicates that the constants $L_{pj\alpha}$, $N_{\alpha r}$, d_{rk} and τ_α, as defined in the preceding section, are to be computed using the elastic moduli of the material in region n.

For (9) to be a solution of (1), the functions $G_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi})$ in (10) must satisfy

$$c_{rskq}^{(n)} \frac{\partial^2 G_{pk}^{(n)}}{\partial x_s \partial x_q} = 0 \text{ for all } \underset{\sim}{x} \text{ in region } n. \tag{11}$$

Now, since we require the different materials making up the composite to adhere rigidly to each other along the $x_2 = 0$ interface, the functions

$G_{pk}^{(n)}(\underset{\sim}{x},\underset{\sim}{\xi})$ must be chosen in such a way that, for $-\infty < x_1 < \infty$,

$$U_{pk}^{(1)}(x_1, 0^+, \underset{\sim}{\xi}) = U_{pk}^{(2)}(x_1, 0^-, \underset{\sim}{\xi}), \tag{12}$$

$$S_{pk2}^{(1)}(x_1, 0^+, \underset{\sim}{\xi}) = S_{pk2}^{(2)}(x_1, 0^-, \underset{\sim}{\xi}), \tag{13}$$

where $S_{pkj}^{(n)} = c_{kjrs}^{(n)} \partial U_{pr}^{(n)}/\partial x_s$. In addition, $G_{pk}^{(n)}(\underset{\sim}{x},\underset{\sim}{\xi})$ are required to vanish as $|\underset{\sim}{x}| \to \infty$ (in region n).

For the solution of (11) subject to (12) and (13), we choose

$$G_{pk}^{(1)}(\underset{\sim}{x},\underset{\sim}{\xi}) = \frac{1}{2\pi}\,\mathrm{Re}\{\sum_\alpha A_{k\alpha}^{(1)} \int_0^\infty E_{p\alpha}(u,\underset{\sim}{\xi}) \exp(iuz_\alpha^{(1)})du\},$$

$$G_{pk}^{(2)}(\underset{\sim}{x},\underset{\sim}{\xi}) = \frac{1}{2\pi}\,\mathrm{Re}\{\sum_\alpha A_{k\alpha}^{(2)} \int_0^\infty F_{p\alpha}(u,\underset{\sim}{\xi}) \exp(-iuz_\alpha^{(2)})du\}, \tag{14}$$

where $E_{p\alpha}(u,\underset{\sim}{\xi})$ and $F_{p\alpha}(u,\underset{\sim}{\xi})$ are functions yet to be determined. The system (11) is satisfied by the choice in (14).

Conditions (12) can be rewritten as

$$\int_{-\infty}^\infty U_{pk}^{(1)}(x_1, 0^+, \underset{\sim}{\xi}) \exp(-i\gamma x_1)dx_1 = \int_{-\infty}^\infty U_{pk}^{(2)}(x_1, 0^-, \underset{\sim}{\xi}) \exp(-i\gamma x_1)dx_1, \tag{15}$$

where $\gamma > 0$ is a real constant.

Using the results (Erdélyi *et al.* [15])

$$\int_{-\infty}^\infty (a - ix)^{-1} \exp(-ixy)dx = H(y)2\pi \exp(-ay),$$

$$\int_{-\infty}^\infty (a + ix)^{-1} \exp(-ixy)dx = -H(-y)2\pi \exp(ay), \tag{16}$$

where a is a constant such that $\mathrm{Re}\{a\} > 0$ and $H(x)$ is the Heaviside unit-step function, we find that (15) becomes

$$\sum_\alpha \{A_{k\alpha}^{(1)} E_{p\alpha}(u,\underset{\sim}{\xi}) - \bar{A}_{k\alpha}^{(2)} \bar{F}_{p\alpha}(u,\underset{\sim}{\xi})\}$$

$$= \sum_\alpha in_j(\underset{\sim}{\xi})\{H(-\xi_2)[T_{pj\alpha k}^{(2)} \exp(-iuc_\alpha^{(2)}) - T_{pj\alpha k}^{(1)} \exp(-iuc_\alpha^{(1)})]$$

$$+ H(\xi_2)[\bar{T}_{pj\alpha k}^{(2)} \exp(-iu\bar{c}_\alpha^{(2)}) - \bar{T}_{pj\alpha k}^{(1)} \exp(-iu\bar{c}_\alpha^{(1)})]\}, \tag{17}$$

where $T^{(n)}_{pj\alpha k} = L^{(n)}_{pj\alpha} N^{(n)}_{\alpha r} d^{(n)}_{rk}$.

Similarly, conditions (13) can be rewritten as

$$\int_{-\infty}^{\infty} S^{(1)}_{pk2}(x_1, 0^+, \underset{\sim}{\xi}) \exp(-i\gamma x_1) dx_1 = \int_{-\infty}^{\infty} S^{(2)}_{pk2}(x_1, 0^-, \underset{\sim}{\xi}) \exp(-i\gamma x_1) dx_1, \tag{18}$$

to give rise to

$$\begin{aligned}
&\sum_\alpha \{ L^{(1)}_{k2\alpha} E_{p\alpha}(u, \underset{\sim}{\xi}) - \bar{L}^{(2)}_{k2\alpha} \bar{F}_{p\alpha}(u, \underset{\sim}{\xi}) \} \\
&= \sum_\alpha i n_l(\underset{\sim}{\xi}) \{ H(-\xi_2) [Q^{(2)}_{pk2l\alpha} \exp(-iuc^{(2)}_\alpha) - Q^{(1)}_{pk2l\alpha} \exp(-iuc^{(1)}_\alpha)] \\
&+ H(\xi_2) [\bar{Q}^{(2)}_{pk2l\alpha} \exp(-iu\bar{c}^{(2)}_\alpha) - \bar{Q}^{(1)}_{pk2l\alpha} \exp(-iu\bar{c}^{(1)}_\alpha)] \},
\end{aligned} \tag{19}$$

where $Q^{(n)}_{pkjl\alpha} = (c^{(n)}_{kjr1} + \tau^{(n)}_\alpha c^{(n)}_{kjr2}) T^{(n)}_{pl\alpha r}$.

Our task now is to solve (17) and (19) for $E_{p\alpha}(u, \underset{\sim}{\xi})$ and $F_{p\alpha}(u, \underset{\sim}{\xi})$. We obtain

$$\begin{aligned}
E_{p\alpha}(u, \underset{\sim}{\xi}) = \sum_{\beta,\gamma} V_{\alpha\beta} \{ & H(-\xi_2) [(\bar{N}^{(2)}_{\beta k} T^{(2)}_{pl\gamma k} - \bar{M}^{(2)}_{\beta k} Q^{(2)}_{pk2l\gamma}) \exp(-iuc^{(2)}_\gamma) \\
& - (\bar{N}^{(2)}_{\beta k} T^{(1)}_{pl\gamma k} - \bar{M}^{(2)}_{\beta k} Q^{(1)}_{pk2l\gamma}) \exp(-iuc^{(1)}_\gamma)] \\
& + H(\xi_2) [(\bar{N}^{(2)}_{\beta k} \bar{T}^{(2)}_{pl\gamma k} - \bar{M}^{(2)}_{\beta k} \bar{Q}^{(2)}_{pk2l\gamma}) \exp(-iu\bar{c}^{(2)}_\gamma) \\
& - (\bar{N}^{(2)}_{\beta k} \bar{T}^{(1)}_{pl\gamma k} - \bar{M}^{(2)}_{\beta k} \bar{Q}^{(1)}_{pk2l\gamma}) \exp(-iu\bar{c}^{(1)}_\gamma)] \} i n_l(\underset{\sim}{\xi}),
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\bar{F}_{p\alpha}(u, \underset{\sim}{\xi}) = \sum_{\beta,\gamma} W_{\alpha\beta} \{ & H(-\xi_2) [(N^{(1)}_{\beta k} T^{(2)}_{pl\gamma k} - M^{(1)}_{\beta k} Q^{(2)}_{pk2l\gamma}) \exp(-iuc^{(2)}_\gamma) \\
& - (N^{(1)}_{\beta k} T^{(1)}_{pl\gamma k} - M^{(1)}_{\beta k} Q^{(1)}_{pk2l\gamma}) \exp(-iuc^{(1)}_\gamma)] \\
& + H(\xi_2) [(N^{(1)}_{\beta k} \bar{T}^{(2)}_{pl\gamma k} - M^{(1)}_{\beta k} \bar{Q}^{(2)}_{pk2l\gamma}) \exp(-iu\bar{c}^{(2)}_\gamma) \\
& - (N^{(1)}_{\beta k} \bar{T}^{(1)}_{pl\gamma k} - M^{(1)}_{\beta k} \bar{Q}^{(1)}_{pk2l\gamma}) \exp(-iu\bar{c}^{(1)}_\gamma)] \} i n_l(\underset{\sim}{\xi}),
\end{aligned} \tag{21}$$

where $[V_{\alpha\beta}]$ and $[W_{\alpha\beta}]$ are obtained from the relations

$$\sum_{\beta} V_{\alpha\beta}[\bar{N}^{(2)}_{\beta k} A^{(1)}_{k\gamma} - \bar{M}^{(2)}_{\beta k} L^{(1)}_{k2\gamma}] = \delta_{\alpha\gamma},$$

$$\sum_{\beta} W_{\alpha\beta}[M^{(1)}_{\beta k} \bar{L}^{(2)}_{k2\gamma} - N^{(1)}_{\beta k} \bar{A}^{(2)}_{k\gamma}] = \delta_{\alpha\gamma}.$$

From (14), (20) and (21), we obtain

$$\begin{aligned}
G^{(1)}_{pk}(\underset{\sim}{x}, \underset{\sim}{\xi}) = -\frac{1}{2\pi} \operatorname{Re}\{ & \sum_{\alpha,\beta,\gamma} A^{(1)}_{k\alpha} V_{\alpha\beta} \\
& \times \{H(-\xi_2)[(\bar{N}^{(2)}_{\beta q} T^{(2)}_{pl\gamma q} - \bar{M}^{(2)}_{\beta q} Q^{(2)}_{pq2l\gamma})[z^{(1)}_{\alpha} - c^{(2)}_{\gamma}]^{-1} \\
& - (\bar{N}^{(2)}_{\beta q} T^{(1)}_{pl\gamma q} - \bar{M}^{(2)}_{\beta q} Q^{(1)}_{pq2l\gamma})[z^{(1)}_{\alpha} - c^{(1)}_{\gamma}]^{-1}] \\
& + H(\xi_2)[(\bar{N}^{(2)}_{\beta q} \bar{T}^{(2)}_{pl\gamma q} - \bar{M}^{(2)}_{\beta q} \bar{Q}^{(2)}_{pq2l\gamma})[z^{(1)}_{\alpha} - \bar{c}^{(2)}_{\gamma}]^{-1} \\
& - (\bar{N}^{(2)}_{\beta q} \bar{T}^{(1)}_{pl\gamma q} - \bar{M}^{(2)}_{\beta q} \bar{Q}^{(1)}_{pq2l\gamma})[z^{(1)}_{\alpha} - \bar{c}^{(1)}_{\gamma}]^{-1}]\} n_l(\underset{\sim}{\xi})\},
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
G^{(2)}_{pk}(\underset{\sim}{x}, \underset{\sim}{\xi}) = -\frac{1}{2\pi} \operatorname{Re}\{ & \sum_{\alpha,\beta,\gamma} A^{(2)}_{k\alpha} \bar{W}_{\alpha\beta} \\
& \times \{H(-\xi_2)[(\bar{N}^{(1)}_{\beta q} \bar{T}^{(2)}_{pl\gamma q} - \bar{M}^{(1)}_{\beta q} \bar{Q}^{(2)}_{pq2l\gamma})[z^{(2)}_{\alpha} - \bar{c}^{(2)}_{\gamma}]^{-1} \\
& - (\bar{N}^{(1)}_{\beta q} \bar{T}^{(1)}_{pl\gamma q} - \bar{M}^{(1)}_{\beta q} \bar{Q}^{(1)}_{pq2l\gamma})[z^{(2)}_{\alpha} - \bar{c}^{(1)}_{\gamma}]^{-1}] \\
& + H(\xi_2)[(\bar{N}^{(1)}_{\beta q} T^{(2)}_{pl\gamma q} - \bar{M}^{(1)}_{\beta q} Q^{(2)}_{pq2l\gamma})[z^{(2)}_{\alpha} - c^{(2)}_{\gamma}]^{-1} \\
& - (\bar{N}^{(1)}_{\beta q} T^{(1)}_{pl\gamma q} - \bar{M}^{(1)}_{\beta q} Q^{(1)}_{pq2l\gamma})[z^{(2)}_{\alpha} - c^{(1)}_{\gamma}]^{-1}]\} n_l(\underset{\sim}{\xi})\}.
\end{aligned} \tag{23}$$

The displacements $u^{(n)}_k(\underset{\sim}{x})$, as given by (9) together with (10), (22) and (23), satisfy (1) in region n, and the continuity conditions on the interface $x_2 = 0$. The remaining conditions to be satisfied are those on the crack faces, specifically given by

$$\sigma_{kj}^{(1)}(\underset{\sim}{x})n_j(\underset{\sim}{x}) \to -\sigma_{kj}^{(0)}(\underset{\sim}{y})n_j(\underset{\sim}{y}) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \in C_+^{(m)} \ (m = 1, 2, ..., N), \tag{24}$$

$$\sigma_{kj}^{(2)}(\underset{\sim}{x})n_j(\underset{\sim}{x}) \to -\sigma_{kj}^{(0)}(\underset{\sim}{y})n_j(\underset{\sim}{y}) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \in C_+^{(m)}$$
$$(m = N + 1, N + 2, ..., N + M), \tag{25}$$

where $\sigma_{kj}^{(0)}$ are suitably prescribed internal stresses.

From (9), (10), (22) and (23), the stresses $\sigma_{kj}^{(n)}(\underset{\sim}{x})$ can be written as

$$\sigma_{kj}^{(n)}(\underset{\sim}{x}) = \frac{1}{2\pi} \sum_{m=1}^{M+N} \int_{C_+^{(m)}} \Delta u_p(\underset{\sim}{\xi})$$
$$\times [\mathrm{Re}\{\sum_{\alpha} Q_{pkjl\alpha}^{(n)}(c_\alpha^{(n)} - z_\alpha^{(n)})^{-2}\} + Z_{pkjl}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi})]n_l(\underset{\sim}{\xi})dS(\underset{\sim}{\xi}), \tag{26}$$

where $Q_{pkjl\alpha}^{(n)}$ are as defined below (19), and

$$\begin{aligned} Z_{pkjl}^{(1)}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \mathrm{Re}\{ & \sum_{\alpha,\beta,\gamma} L_{kj\alpha}^{(1)} V_{\alpha\beta} \\ & \times \{H(-\xi_2)[(\bar{N}_{\beta q}^{(2)} T_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(2)} Q_{pq2l\gamma}^{(2)})[z_\alpha^{(1)} - c_\gamma^{(2)}]^{-2} \\ & - (\bar{N}_{\beta q}^{(2)} T_{pl\gamma q}^{(1)} - \bar{M}_{\beta q}^{(2)} Q_{pq2l\gamma}^{(1)})[z_\alpha^{(1)} - c_\gamma^{(1)}]^{-2}] \\ & + H(\xi_2)[(\bar{N}_{\beta q}^{(2)} \bar{T}_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(2)} \bar{Q}_{pq2l\gamma}^{(2)})[z_\alpha^{(1)} - \bar{c}_\gamma^{(2)}]^{-2} \\ & - (\bar{N}_{\beta q}^{(2)} \bar{T}_{pl\gamma q}^{(1)} - \bar{M}_{\beta q}^{(2)} \bar{Q}_{pq2l\gamma}^{(1)})[z_\alpha^{(1)} - \bar{c}_\gamma^{(1)}]^{-2}]\}\}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} Z_{pkjl}^{(2)}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \mathrm{Re}\{ & \sum_{\alpha,\beta,\gamma} L_{kj\alpha}^{(2)} \bar{W}_{\alpha\beta} \\ & \times \{H(-\xi_2)[(\bar{N}_{\beta q}^{(1)} \bar{T}_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(1)} \bar{Q}_{pq2l\gamma}^{(2)})[z_\alpha^{(2)} - \bar{c}_\gamma^{(2)}]^{-2} \\ & - (\bar{N}_{\beta q}^{(1)} \bar{T}_{pl\gamma q}^{(1)} - \bar{M}_{\beta q}^{(1)} \bar{Q}_{pq2l\gamma}^{(1)})[z_\alpha^{(2)} - \bar{c}_\gamma^{(1)}]^{-2}] \\ & + H(\xi_2)[(\bar{N}_{\beta q}^{(1)} T_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(1)} Q_{pq2l\gamma}^{(2)})[z_\alpha^{(2)} - c_\gamma^{(2)}]^{-2} \\ & - (\bar{N}_{\beta q}^{(1)} T_{pl\gamma q}^{(1)} - \bar{M}_{\beta q}^{(1)} Q_{pq2l\gamma}^{(1)})[z_\alpha^{(2)} - c_\gamma^{(1)}]^{-2}]\}\}. \end{aligned} \tag{28}$$

Notice that $Z^{(n)}_{pkjl}(\underset{\sim}{x}, \underset{\sim}{\xi}) = 0$ if $c^{(1)}_{rskq} = c^{(2)}_{rskq}$, as expected.

For convenience, we may visualise the crack $C^{(m)}$ as being a closed curve which is assigned a clockwise direction and which encloses an elliptical region having an area that tends to zero. We take the "upper" face $C^{(m)}_{+}$ to be the part of the ellipse from the tip $(\alpha^{(m)}, \beta^{(m)})$ to $(\gamma^{(m)}, \delta^{(m)})$.

Conditions (24) and (25) can be written as

$$\begin{aligned}
&\frac{1}{\pi}\,\mathrm{Re}\left\{\sum_{\alpha}\frac{L^{(q)}Q^{(1)}_{pkjl\alpha}n^{(q)}_{j}n^{(q)}_{l}}{\{(\gamma^{(q)}-\alpha^{(q)})+\tau^{(1)}_{\alpha}(\delta^{(q)}-\beta^{(q)})\}^{2}}\right\}\mathcal{H}\int_{-1}^{1}\frac{\Delta u^{(q)}_{p}(s)}{(t-s)^{2}}ds\\
&+\frac{1}{4\pi}\sum_{m=1}^{M+N}L^{(m)}\int_{-1}^{1}\Delta u^{(m)}_{p}(s)n^{(m)}_{l}n^{(q)}_{j}\\
&\times[\mathrm{Re}\{(1-\delta_{mq})\sum_{\alpha}Q^{(1)}_{pkjl\alpha}(\Xi^{(mq)}(s,t)+\tau^{(1)}_{\alpha}\Theta^{(mq)}(s,t))^{-2}\}\\
&+Z^{(1)}_{pkjl}(X^{(q)}(t),Y^{(q)}(t),X^{(m)}(s),Y^{(m)}(s))]ds\\
&=-\sigma^{(0)}_{kj}(X^{(q)}(t),Y^{(q)}(t))n^{(q)}_{j}\ \text{for }-1<t<1\ (q=1,2,...,N),
\end{aligned}\tag{29}$$

and

$$\begin{aligned}
&\frac{1}{\pi}\,\mathrm{Re}\left\{\sum_{\alpha}\frac{L^{(q)}Q^{(2)}_{pkjl\alpha}n^{(q)}_{j}n^{(q)}_{l}}{\{(\gamma^{(q)}-\alpha^{(q)})+\tau^{(2)}_{\alpha}(\delta^{(q)}-\beta^{(q)})\}^{2}}\right\}\mathcal{H}\int_{-1}^{1}\frac{\Delta u^{(q)}_{p}(s)}{(t-s)^{2}}ds\\
&+\frac{1}{4\pi}\sum_{m=1}^{M+N}L^{(m)}\int_{-1}^{1}\Delta u^{(m)}_{p}(s)n^{(m)}_{l}n^{(q)}_{j}\\
&\times[\mathrm{Re}\{(1-\delta_{mq})\sum_{\alpha}Q^{(2)}_{pkjl\alpha}(\Xi^{(mq)}(s,t)+\tau^{(2)}_{\alpha}\Theta^{(mq)}(s,t))^{-2}\}\\
&+Z^{(2)}_{pkjl}(X^{(q)}(t),Y^{(q)}(t),X^{(m)}(s),Y^{(m)}(s))]ds\\
&=-\sigma^{(0)}_{kj}(X^{(q)}(t),Y^{(q)}(t))n^{(q)}_{j}\\
&\text{for }-1<t<1\ (q=N+1,N+2,...,N+M),
\end{aligned}\tag{30}$$

where $\mathcal{H}$ denotes that the integral is to be interpreted in the Hadamard finite-part sense, $\Delta u^{(q)}_{p}(s) = \Delta u_{p}(X^{(q)}(s), Y^{(q)}(s))$, $2X^{(q)}(s) = (\gamma^{(q)} + \alpha^{(q)}) + s(\gamma^{(q)} - \alpha^{(q)})$, $2Y^{(q)}(s) = (\delta^{(q)} + \beta^{(q)}) + s(\delta^{(q)} - \beta^{(q)})$, $n^{(q)}_{1} = (\delta^{(q)} - \beta^{(q)})/L^{(q)}$, $n^{(q)}_{2} = -(\gamma^{(q)} - \alpha^{(q)})/L^{(q)}$, $L^{(q)} = \sqrt{(\delta^{(q)} - \beta^{(q)})^{2} + (\gamma^{(q)} - \alpha^{(q)})^{2}}$, $\Xi^{(mq)}(s,t) = X^{(m)}(s) - X^{(q)}(t)$, and $\Theta^{(mq)}(s,t) = Y^{(m)}(s) - Y^{(q)}(t)$.

Equations (29) and (30) constitute a system of $3(N+M)$ hypersingular integral equations, from which we can solve for the $3(N+M)$ unknown functions $\Delta u_p^{(q)}(s)$ $(q = 1, 2, ..., N+M;\ p = 1, 2, 3)$. In general, these integral equations have to be solved numerically. An accurate and effective numerical technique for solving them is given in Kaya and Erdogan [6].

5 SPECIFIC EXAMPLES

We will now solve the hypersingular integral equations (29) and (30), numerically, for specific examples of the problem described in Section 2, and compute the crack tip stress intensity factors or the crack energy.

The elastic behaviour of a transversely isotropic material which has transverse planes parallel to the $0x_2x_3$ plane and which undergoes plane deformations is governed by the system

$$C\frac{\partial^2 u_1}{\partial x_1^2} + L\frac{\partial^2 u_1}{\partial x_2^2} + (F+L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} = 0,$$
$$A\frac{\partial^2 u_2}{\partial x_2^2} + L\frac{\partial^2 u_2}{\partial x_1^2} + (F+L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} = 0, \quad (31)$$

where A, F, C and L are the elastic coefficients of the materials. Notice that (31) is a special case of the more general system (1).

Equation (3) now becomes

$$AL\tau^4 - (F^2 + 2FL - AC)\tau^2 + CL = 0, \quad (32)$$

from which we obtain τ_1 and τ_2.

From (4), we find that the constants $A_{k\alpha}$ are given by

$$[A_{k\alpha}] = \begin{pmatrix} -\frac{i\tau_1(F+L)}{C+L\tau_1^2} & -\frac{i\tau_2(F+L)}{C+L\tau_2^2} \\ i & i \end{pmatrix}. \quad (33)$$

Other relevant constants like $L_{kj\alpha}$ and $N_{\alpha p}$ can be computed using (33). From now on, unless otherwise indicated, the latin and greek subscripts take the value of 1 and 2 only. For further details, refer to Clements [13].

To obtain some numerical results for the examples considered below, we will use the elastic constants for magnesium and titanium. For magnesium, these constants are given by $A = 5.96$, $F = 2.14$, $C = 6.14$ and $L = 1.64$;

for titanium, they are $A = 16.2$, $F = 6.9$, $C = 18.1$ and $L = 4.67$. If these constants are multiplied by 10^{11}, their units are in dynes per centimeter square.

5.1 Planar cracks normal to the interface in regions 1 and 2

Let us now consider the situation where each of regions 1 and 2 contains a planar crack lying on a plane which is perpendicular to the interface $x_2 = 0$. More specifically, we take $M = N = 1$, with $(\alpha^{(1)}, \beta^{(1)}) = (0, a)$, $(\gamma^{(1)}, \delta^{(1)}) = (0, b)$, $(\alpha^{(2)}, \beta^{(2)}) = (0, -a)$ and $(\gamma^{(2)}, \delta^{(2)}) = (0, -b)$, where a and b are positive constants such that $b > a$.

We assume that the internal stresses $\sigma_{kj}^{(0)}$ are such that $\sigma_{11}^{(0)} = P_0$ (P_0 is a given positive constant) and $\sigma_{12}^{(0)} = \sigma_{21}^{(0)} = 0$ on both cracks.

For this particular situation, we find that (29) and (30) reduce to:

$$\begin{aligned}
&\mathcal{H} \int_{-1}^{1} \frac{\chi^{(1)} \Delta u(s)}{(t-s)^2} ds + l^2 \int_{-1}^{1} \Delta u(s) Z_{1111}^{(1)}(0, k + lt, 0, k + ls) ds \\
&- l^2 \int_{-1}^{1} \frac{\chi^{(1)} \Delta v(s)}{(2k + l[s+t])^2} ds - l^2 \int_{-1}^{1} \Delta v(s) Z_{1111}^{(1)}(0, k + lt, 0, -k - ls) ds \\
&= -2\pi P_0 \text{ for } -1 < t < 1,
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
&- \mathcal{H} \int_{-1}^{1} \frac{\chi^{(2)} \Delta v(s)}{(t-s)^2} ds - l^2 \int_{-1}^{1} \Delta v(s) Z_{1111}^{(2)}(0, -k - lt, 0, -k - ls) ds \\
&+ l^2 \int_{-1}^{1} \frac{\chi^{(2)} \Delta u(s)}{(2k + l[s+t])^2} ds + l^2 \int_{-1}^{1} \Delta u(s) Z_{1111}^{(2)}(0, -k - lt, 0, k + ls) ds \\
&= -2\pi P_0 \text{ for } -1 < t < 1,
\end{aligned} \tag{35}$$

where $k = (a+b)/2$, $l = (b-a)/2$, $\chi^{(n)} = \text{Re}\{\sum_\alpha [\tau_\alpha^{(n)}]^{-2} Q_{1111\alpha}^{(n)}\}$, $\Delta u(s) = \Delta u_1^{(1)}(s)/l$ and $\Delta v(s) = \Delta u_1^{(2)}(s)/l$. Notice that, due to symmetry, $\Delta u_2^{(1)}(s) = \Delta u_2^{(2)}(s) = 0$.

To solve (34) and (35) numerically, we make the approximations

$$\Delta u(s) \simeq \sqrt{1-s^2}\sum_{j=1}^{J}\phi_j U_{j-1}(s),$$
$$\Delta v(s) \simeq \sqrt{1-s^2}\sum_{j=1}^{J}\psi_j U_{j-1}(s), \tag{36}$$

where ϕ_j and ψ_j are real constant coefficients to be determined and $U_j(x)$ is the j-th order Chebyshev polynomial of the second kind.

Substituting (36) into (34) and (35) leads to:

$$\begin{aligned}
&\sum_{j=1}^{J}\phi_j\left\{-\pi j U_{j-1}(t)\chi^{(1)} + l^2\int_{-1}^{1}\sqrt{1-s^2}U_{j-1}(s)\right.\\
&\qquad\qquad\left.\times Z_{1111}^{(1)}(0,k+lt,0,k+ls)ds\right\}\\
&- l^2\sum_{j=1}^{J}\psi_j\int_{-1}^{1}\sqrt{1-s^2}U_{j-1}(s)\\
&\qquad\times\left\{\frac{\chi^{(1)}}{(2k+l[s+t])^2} + Z_{1111}^{(1)}(0,k+lt,0,-k-ls)\right\}ds\\
&= -2\pi P_0 \text{ for } -1<t<1,
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\sum_{j=1}^{J}\psi_j\left\{\pi j U_{j-1}(t)\chi^{(2)} - l^2\int_{-1}^{1}\sqrt{1-s^2}U_{j-1}(s)\right.\\
&\qquad\qquad\left.\times Z_{1111}^{(2)}(0,-k-lt,0,-k-ls)ds\right\}\\
&+ l^2\sum_{j=1}^{N}\phi_j\int_{-1}^{1}\sqrt{1-s^2}U_{j-1}(s)\\
&\qquad\times\left\{\frac{\chi^{(2)}}{(2k+l[s+t])^2} + Z_{1111}^{(2)}(0,-k-lt,0,k+ls)\right\}ds\\
&= -2\pi P_0 \text{ for } -1<t<1.
\end{aligned} \tag{38}$$

For any given $t \in (-1,1)$, it is possible to compute the integrals in (37) and (38) accurately by using the quadrature formula (25.4.40) in Abramowitz and Stegun [16].

Now, there are $2J$ unknown constants ϕ_j and ψ_j in (37) and (38). We choose the free parameter t in (37) and (38) to be given in turn by

$$t = t_p = \cos([2p-1]\pi/[2J]) \text{ for } p = 1,\ 2,\ ...,\ J,$$

in order to generate a system of $2J$ linear algebraic equations in ϕ_j and ψ_j. The system thus generated is readily solved using standard computer packages.

With regions 1 and 2 occupied by titanium and magnesium respectively, we solve (34) and (35) numerically in order to compute the stress intensity factors at the tips $(0, \pm a)$ defined by

$$K_I^{(1)} = \lim_{y\to a^-} \sqrt{2(a-y)}\sigma_{11}^{(1)}(0,y) \text{ and } K_I^{(2)} = \lim_{y\to -a^+} \sqrt{2(y+a)}\sigma_{11}^{(2)}(0,y).$$

From (36), the stress intensity factors are approximately given by

$$K_I^{(1)} \simeq \frac{\chi^{(1)}}{2\sqrt{l}} \sum_{j=1}^{J} \phi_j U_{j-1}(-1) \text{ and } K_I^{(2)} \simeq \frac{\chi^{(2)}}{2\sqrt{l}} \sum_{j=1}^{J} \psi_j U_{j-1}(-1).$$

Hence, once ϕ_j and ψ_j are determined, numerical values of the stress intensity factors can be readily computed.

The numerical values of $K_I^{(n)}/(P_0\sqrt{l})$, thus obtained, are tabulated in Table 1 for various values of a/l. For a given a/l, we observe that $K_I^{(1)}/(P_0\sqrt{l}) > K_I^{(2)}/(P_0\sqrt{l})$, i.e. the state of stress around the crack in region 1 (titanium) is more severe than that around the crack in region 2 (magnesium). Notice that titanium is a more rigid material than magnesium. Moving the cracks closer to each other (i.e. decreasing a/l) has the effect of aggravating the stress around the crack tips. It is clear that as $a/l \to \infty$ both non-dimensionalised stress intensity factors $K_I^{(1)}/(P_0\sqrt{l})$ and $K_I^{(2)}/(P_0\sqrt{l})$ tend to unity.

In our calculations above, we *typically* use $J = 5$ in the approximation (36). When the calculations are repeated using $J = 10$, convergence to 3 or 4 significant figures is observed in the numerical results obtained. However, for cases where the cracks interact strongly with each other or the interface $x_2 = 0$, i.e. when a/l is extremely small, it may be necessary to use larger number of terms in (36) to achieve the same level accuracy in the computation.

Table 1. Stress intensity factors at the tips $(0, \pm a)$.

a/l	$K_I^{(1)}/(P_0\sqrt{l})$	$K_I^{(2)}/(P_0\sqrt{l})$
0.01	4.486	1.842
0.05	2.879	1.392
0.10	2.228	1.231
0.20	1.729	1.122
0.30	1.511	1.080
0.40	1.386	1.058
0.50	1.306	1.045
0.60	1.250	1.036
0.70	1.209	1.029
0.80	1.177	1.024
0.90	1.153	1.021
1.00	1.133	1.081
1.50	1.076	1.010
2.00	1.050	1.006
4.00	1.016	1.002
8.00	1.005	1.000

5.2 Parallel cracks normal to the interface in region 1

Let us consider the situation where, in region 1, there is a pair of parallel cracks lying on plane that are perpendicular to the $x_2 = 0$ interface. More precisely, we take $N = 2$ and $M = 0$, with $(\alpha^{(1)}, \beta^{(1)}) = (h, a)$, $(\gamma^{(1)}, \delta^{(1)}) = (h, b)$, $(\alpha^{(2)}, \beta^{(2)}) = (-h, a)$ and $(\gamma^{(2)}, \delta^{(2)}) = (-h, b)$, where a, b and h are positive constants such that $b > a$.

As in the previous example, we assume that that the internal stresses $\sigma_{kj}^{(0)}$ are such that $\sigma_{11}^{(0)} = P_0$ (P_0 is a given positive constant) and $\sigma_{12}^{(0)} = \sigma_{21}^{(0)} = 0$ on both cracks.

From the symmetry about the plane $x_1 = 0$, we know that $\Delta u_1^{(1)}(s) = \Delta u_1^{(2)}(s)$ and $\Delta u_2^{(1)}(s) = -\Delta u_2^{(2)}(s)$ for $-1 < s < 1$. If we define $\Delta u(s) =$

$\Delta u_1^{(1)}(s)/l$ and $\Delta v(s) = \Delta u_2^{(1)}(s)/l$, we find that (29) and (30) reduce to:

$$\mathcal{H}\int_{-1}^{1} \frac{\chi_{11}\Delta u(s)}{(t-s)^2}ds \; + l^2 \int_{-1}^{1} \Delta u(s)\,\mathrm{Re}\left\{\sum_\alpha \frac{Q^{(1)}_{1111\alpha}}{(-2h+\tau^{(1)}_\alpha l\{s-t\})^2}\right\}ds$$
$$+\, l^2 \int_{-1}^{1} \Delta u(s) Z_1(t,s)ds + l^2 \int_{-1}^{1} \Delta v(s) R_1(t,s)ds$$
$$\mathcal{H}\int_{-1}^{1} \frac{\chi_{21}\Delta v(s)}{(t-s)^2}ds \; + l^2 \int_{-1}^{1} \Delta v(s)\,\mathrm{Re}\left\{\sum_\alpha \frac{Q^{(1)}_{2111\alpha}}{(-2h+\tau^{(1)}_\alpha l\{s-t\})^2}\right\}ds$$
$$= -2\pi P_0 \quad \text{for } -1 < t < 1, \tag{39}$$

and

$$\mathcal{H}\int_{-1}^{1} \frac{\chi_{12}\Delta u(s)}{(t-s)^2}ds \; + l^2 \int_{-1}^{1} \Delta u(s)\,\mathrm{Re}\left\{\sum_\alpha \frac{Q^{(1)}_{1211\alpha}}{(-2h+\tau^{(1)}_\alpha l\{s-t\})^2}\right\}ds$$
$$+\, l^2 \int_{-1}^{1} \Delta u(s) Z_2(t,s)ds + l^2 \int_{-1}^{1} \Delta v(s) R_2(t,s)ds$$
$$\mathcal{H}\int_{-1}^{1} \frac{\chi_{22}\Delta v(s)}{(t-s)^2}ds \; + l^2 \int_{-1}^{1} \Delta v(s)\,\mathrm{Re}\left\{\sum_\alpha \frac{Q^{(1)}_{2211\alpha}}{(-2h+\tau^{(1)}_\alpha l\{s-t\})^2}\right\}ds$$
$$= 0 \quad \text{for } -1 < t < 1, \tag{40}$$

where $l = (b-a)/2$, $\chi_{pq} = \mathrm{Re}\{\sum_\alpha [\tau^{(1)}_\alpha]^{-2} Q^{(1)}_{pq11\alpha}\}$ and

$$Z_q(t,s) = Z^{(1)}_{1q11}(h, k+lt, h, k+ls) + Z^{(1)}_{1q11}(h, k+lt, -h, k+ls),$$
$$R_q(t,s) = Z^{(1)}_{2q11}(h, k+lt, h, k+ls) - Z^{(1)}_{2q11}(h, k+lt, -h, k+ls),$$

with $k = (a+b)/2$.

We proceed as before to solve (39) and (40) numerically. We approximate $\Delta u(s)$ and $\Delta v(s)$ using (36), substitute (36) into (39) and (40) and generate a system of linear algebraic equations from which we determine ϕ_j and ψ_j.

For the crack $x_1 = h$, $a < x_2 < b$, we define the crack energy E by

$$E = -\frac{P_0}{2}\int_a^b \Delta u_1(h, x_2)dx_2.$$

Uisng (36) and the orthogonality relation for Chebyshev polynomials, we obtain

$$E \simeq -\frac{\pi l P_0 \phi_1}{4}.$$

Hence, once ϕ_j and ψ_j are determined, the crack energy E can be readily computed.

Let us now work out the crack energy for the limiting case where a/l and h/l both tend to infinity. For this special case (which gives the corresponding problem of a single planar crack in an infinite homogeneous elastic space), we find that (39) and (40) reduces to a single equation

$$\mathcal{H}\int_{-1}^{1} \frac{\chi_{11}\Delta u(s)}{(t-s)^2} ds = -2\pi P_0 \quad \text{for } -1 < t < 1,$$

with solution given by

$$\Delta u(s) = \frac{2P_0}{\chi_{11}}\sqrt{1-s^2}.$$

Thus, the crack energy for the case where $a/l \to \infty$ and $h/l \to \infty$ is given by

$$E_0 = -\frac{\pi l^2 P_0^2}{2\chi_{11}}.$$

With regions 1 and 2 occupied by titanium and magnesium respectively, we solve (39) and (40) numerically and compute the non-dimensionalised crack energy E/E_0 for selected values of a/l and h/l in Table 2. It is clear that for a fixed h/l the crack energy decreases with increasing a/l, i.e. the cracks are more stable if they are located farther away from region 2. This is perhaps to be expected as region 2 is occupied by magnesium which is a less rigid material than titanium. We also observe that for a fixed a/l the crack energy decreases as h/l decreases.

Table 2. Non-dimensionalised crack energy E/E_0 ($E_0 \simeq 2.199952 P_0^2 l^2$).

h/l a/l	0.2500	0.5000	1.0000	2.0000	∞
0.2500	0.6374	0.7255	0.8718	0.9899	1.0874
0.5000	0.6175	0.7019	0.8471	0.9681	1.0545
1.0000	0.6011	0.6830	0.8232	0.9484	1.0281
2.0000	0.5906	0.6707	0.8076	0.9316	1.0119
∞	0.5828	0.6608	0.7947	0.9165	1.0000

Table 3. Non-dimensionalised crack energy E/E_0 ($E_0 \simeq 6.263115 P_0^2 l^2$).

h/l a/l	0.2500	0.5000	1.0000	2.0000	∞
0.2500	0.5416	0.6123	0.7360	0.8566	0.9304
0.5000	0.5555	0.6279	0.7522	0.8731	0.9542
1.0000	0.5684	0.6425	0.7695	0.8850	0.9754
2.0000	0.5774	0.6530	0.7824	0.9017	0.9894
∞	0.5845	0.6618	0.7938	0.9149	1.0000

We also compute the non-dimensionalised crack energy E/E_0 for the case where regions 1 and 2 occupied by magnesium and titanium respectively. The results for selected values of a/l and h/l are given in Table 3. In this case, for a fixed h/l, the crack energy increases as a/l increases, i.e. the cracks are more stable if they are nearer to region 2 which is occupied by more rigid material (titanium).

6 SUMMARY

In the present paper, we outlined an integral approach for solving numerically the problem of an arbitrary number of arbitrarily-located planar cracks in the interior of two dissimilar anisotropic half-spaces which adhere perfectly to each other. We expressed the displacements in terms of integrals taken over the crack faces, with the aid of a fundamental solution of elasticity which was modified in order to satisfy the continuity conditions on the interface separating the dissimilar half-spaces. The conditions on the cracks then give rise to a system of hypersingular integral equations, with the crack-opening displacements as unknown functions, to be solved. The hypersingular integral equations were solved numerically, using a collocation method, for some specific examples of the problem involving particular transversely-isotropic materials, in order to compute the crack tip stress intensity factors or the crack energy.

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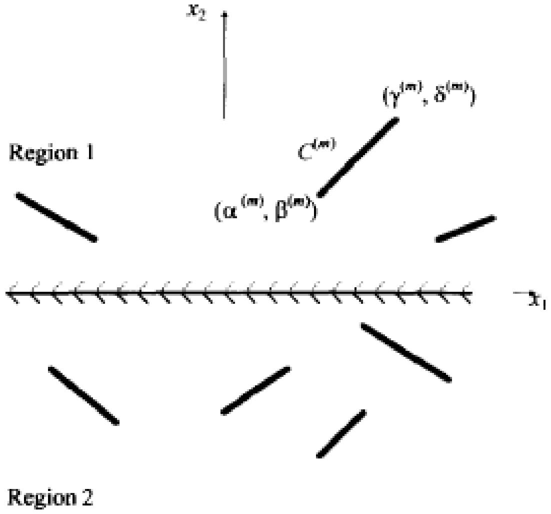

Figure 1

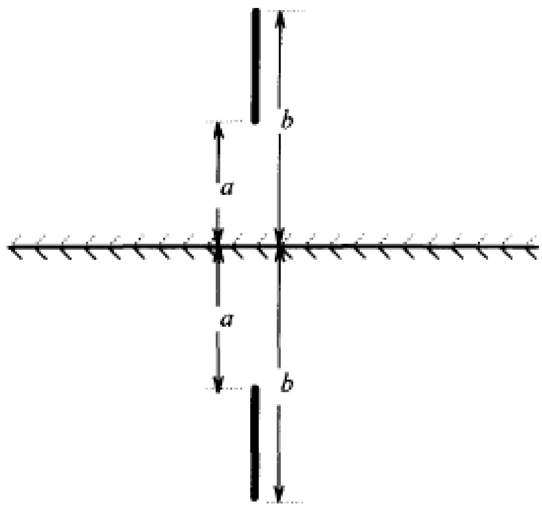

Figure 2

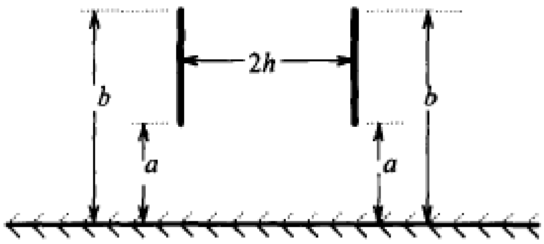

Figure 3