

A mode III crack in an inhomogeneous material

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Abstract

A mode III problem of a planar crack in an infinite elastic medium with shear modulus of the form $\mu = \mu_0 [1 + \epsilon\mu_1(x, y)]$, where μ_0 is a given positive constant, ϵ is a parameter of sufficiently small magnitude and μ_1 is a partially differentiable function of x and y , is examined. It is assumed that the problem has a series solution. The first two terms of the series are explicitly derived and used to obtain an approximate formula for the mode III crack tip stress intensity factor. The stress intensity factor is calculated for specific cases involving particular variations of the shear modulus.

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1 Introduction

The mathematical problem of determining the stress distribution around a crack in an inhomogeneous medium with elastic coefficients that vary continuously in space is inherently difficult to solve. Thus, it is usually considered only for special cases in which the elastic moduli assume certain specific elementary forms, such as linear or exponential variations (see, for example, Clements *et al*, 1997, 1978; Dhaliwal and Singh, 1978; and Gerasoulis and Srivasta, 1980).

The present paper examines a mode III problem which involves a planar crack in an infinite elastic medium with a slightly varying shear modulus of the rather general form

$$\mu = \mu_0 [1 + \epsilon \mu_1(x, y)] \quad (1)$$

where μ_0 is a given positive constant, ϵ is a positive real parameter such that $\epsilon \ll 1$ and μ_1 is any suitable function which is partially differentiable with respect to x and y in the domain of interest. It is assumed that the problem has a series solution expanded in terms of non-negative integer powers of ϵ . A Fourier integral transform technique is employed to obtain explicit expressions for the first two terms of the series. An approximate expression for the relevant stress intensity factor can then be derived.

Ang and Clements (1987) had used the solution approach mentioned above to solve the special case in which the planar crack lies on the plane $y = 0$ and μ_1 is a function of y alone, i.e. the shear modulus varies only in the direction perpendicular to the crack. The analysis was also extended to in-plane deformations by them, and then to a penny-shaped crack under torsion and normal extension by Ang (1987) and Ergüven and Gross (1999) respectively, also with the elastic coefficients varying only in the direction normal to the crack. The work in the present paper is a generalisation of the mode III crack problem considered by Ang and Clements (1987).

2 Basic equations

With reference to a Cartesian coordinate frame $Oxyz$, take an elastic medium undergoing an antiplane deformation in such a way that the only non-zero component of the displacement is the one in the z -direction and is given by the function $w(x, y)$. The only non-zero Cartesian stresses are then given by

$$\sigma_{xz} = \sigma_{zx} = \mu \frac{\partial w}{\partial x} \quad \text{and} \quad \sigma_{yz} = \sigma_{zy} = \mu \frac{\partial w}{\partial y}. \quad (2)$$

The antiplane deformation of the medium is governed by the partial differential equation

$$\frac{\partial}{\partial x} \left(\mu \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial w}{\partial y} \right) = 0. \quad (3)$$

If μ is given by Eq. (1), then Eq. (3) becomes

$$[1 + \epsilon\mu_1(x, y)] \nabla^2 w + \epsilon \left[\frac{\partial\mu_1}{\partial x} \cdot \frac{\partial w}{\partial x} + \frac{\partial\mu_1}{\partial y} \cdot \frac{\partial w}{\partial y} \right] = 0, \quad (4)$$

where ∇^2 denotes the Laplacian operator.

Following Ang and Clements (1987), we assume that w can be written in the series form

$$w = \sum_{n=0}^{\infty} \epsilon^n \phi_n(x, y), \quad (5)$$

where, from Eq. (4), ϕ_n are functions satisfying

$$\nabla^2 \phi_0 = 0, \quad (6)$$

and

$$\nabla^2 \phi_n = -\mu_1 \nabla^2 \phi_{n-1} - \frac{\partial\mu_1}{\partial x} \cdot \frac{\partial\phi_{n-1}}{\partial x} - \frac{\partial\mu_1}{\partial y} \cdot \frac{\partial\phi_{n-1}}{\partial y} \quad \text{for } n \geq 1. \quad (7)$$

Notice that Eqs. (6) and (7) are obtained by setting the coefficients of ϵ^n to zero after substituting Eq. (5) into Eq. (4).

From Eqs. (2) and (5), the stress σ_{yz} may be written as

$$\sigma_{yz} = \mu_0 \left[\frac{\partial\phi_0}{\partial y} + \sum_{n=0}^{\infty} \epsilon^{n+1} \left(\frac{\partial\phi_{n+1}}{\partial y} + \mu_1 \frac{\partial\phi_n}{\partial y} \right) \right]. \quad (8)$$

3 A mode III crack problem

Consider an an infinite elastic medium with shear modulus as given by Eq. (1), where μ_1 is such that $\mu_1(x, y) = \mu_1(-x, y) = \mu_1(x, -y)$. The medium contains a crack in the region $-a < x < a$, $y = 0$, $-\infty < z < \infty$, where a is a given positive constant. It is subject to a mode III deformation so that the basic equations given in the previous section are valid. An internal stress $\sigma_{yz} = -s_0(x)$ ($-a < x < a$), where s_0 is an even function of x , acts on the crack. The problem is to determine the displacement and stress fields throughout the medium.

From a mathematical standpoint, the problem is to solve Eq. (3) in the half plane region $y > 0$ subject to

$$w(x, 0) = 0 \text{ for } |x| > a, \quad (9)$$

$$\sigma_{yz}(x, 0) = -s_0(x) \text{ for } -a < x < a. \quad (10)$$

In addition, the stresses are required to vanish as $x^2 + y^2 \rightarrow \infty$ (within the half plane region $y > 0$).

If we assume that the problem under consideration has a series solution of the form Eq. (5) and if we are interested in only the first two terms of the series solution, then from Eqs. (5)-(8) the problem can be replaced by two consecutive boundary value problems as defined below.

Problem 1. For $y > 0$, solve Eq. (6) subject to

$$\phi_0(x, 0) = 0 \text{ for } |x| > a, \quad (11)$$

$$\sigma_{yz}^{(0)}(x, 0) = -s_0(x) \text{ for } -a < x < a, \quad (12)$$

where $\sigma_{yz}^{(0)} = \mu_0 \partial \phi_0 / \partial y$.

Problem 2. For $y > 0$, solve

$$\nabla^2 \phi_1 = -\frac{\partial \mu_1}{\partial x} \cdot \frac{\partial \phi_0}{\partial x} - \frac{\partial \mu_1}{\partial y} \cdot \frac{\partial \phi_0}{\partial y}, \quad (13)$$

subject to

$$\phi_1(x, 0) = 0 \text{ for } |x| > a, \quad (14)$$

$$\sigma_{yz}^{(1)}(x, 0) = 0 \text{ for } -a < x < a, \quad (15)$$

where

$$\sigma_{yz}^{(1)} = \mu_0 \left[\frac{\partial \phi_1}{\partial y} + \mu_1 \frac{\partial \phi_0}{\partial y} \right]. \quad (16)$$

4 Solution of problem 1

The solution of problem 1 is well documented in the literature for crack problems. In the quarter plane $x \geq 0, y \geq 0$, it is given by

$$\phi_0(x, y) = \int_0^\infty E(\xi) \exp(-\xi y) \cos(\xi x) d\xi, \quad (17)$$

where

$$\begin{aligned} E(\xi) &= \int_0^a r(t) J_0(\xi t) dt, \\ r(t) &= \frac{2t}{\pi \mu_0} \int_0^t \frac{s_0(u) du}{\sqrt{t^2 - u^2}} \quad \text{for } 0 < t < a, \end{aligned} \quad (18)$$

where J_0 is a Bessel function of order zero. Notice that because of the assumed symmetries in μ_1 and s_0 it is sufficient to state the solution of the problem in the region $x \geq 0, y \geq 0$ only.

On the plane $y = 0$, the stress $\sigma_{yz}^{(0)}$ is given by

$$\sigma_{yz}^{(0)}(x, 0) = -\mu_0 \frac{d}{dx} \int_0^{\min(x, a)} \frac{r(t) dt}{\sqrt{x^2 - t^2}} \quad \text{for } x \geq 0. \quad (19)$$

5 Solution of problem 2

To seek for the solution of problem 2 (in the quarter plane $x \geq 0, y \geq 0$), let

$$\phi_1 = \int_0^\infty [G(\xi, y) + F(\xi)] \exp(-\xi y) \cos(\xi x) d\xi, \quad (20)$$

where $G(\xi, y)$ and $F(\xi)$ are functions to be determined.

Direct substitution of Eqs. (17) and (20) into Eq. (13) yields

$$\begin{aligned} & \int_0^\infty \left[\frac{\partial^2 G}{\partial y^2} - 2\xi \frac{\partial G}{\partial y} \right] \exp(-\xi y) \cos(\xi x) d\xi \\ &= -\frac{\partial \mu_1}{\partial x} \cdot \frac{\partial \phi_0}{\partial x} - \frac{\partial \mu_1}{\partial y} \cdot \frac{\partial \phi_0}{\partial y}. \end{aligned} \quad (21)$$

Application of a Fourier inversion theorem on Eq. (21) gives

$$\begin{aligned} & \left[\frac{\partial^2 G}{\partial y^2} - 2\xi \frac{\partial G}{\partial y} \right] \exp(-\xi y) \\ &= -\frac{2}{\pi} \int_0^\infty \left[\frac{\partial \mu_1}{\partial x} \cdot \frac{\partial \phi_0}{\partial x} + \frac{\partial \mu_1}{\partial y} \cdot \frac{\partial \phi_0}{\partial y} \right] \cos(\xi x) dx. \end{aligned} \quad (22)$$

If we assume that μ_1 can be written in the form

$$\mu_1(x, y) = \int_0^\infty U(\xi, y) \cos(\xi x) d\xi, \quad (23)$$

then through the use of the convolution (Faltung) theorem (see Gradshteyn and Rhyszik, 1980) we find that Eq. (22) can be rewritten as

$$\frac{\partial^2 G}{\partial y^2} - 2\xi \frac{\partial G}{\partial y} = -\frac{\exp(\xi y)}{2} \Gamma(\xi, y), \quad (24)$$

where

$$\begin{aligned} \Gamma(\xi, y) = \int_0^\infty s E(s) \exp(-sy) [(s + \xi)U(s + \xi, y) + (s - \xi)U(s - \xi, y) \\ - U_y(s + \xi, y) - U_y(|s - \xi|, y)] ds, \end{aligned} \quad (25)$$

where U_y denotes the partial derivative of U with respect to y .

A solution of Eq. (24) is given by

$$G(\xi, y) = \frac{1}{4\xi} \int_0^y [\exp(\xi t) - \exp(\xi[2y - t])] \Gamma(\xi, t) dt. \quad (26)$$

From Eq. (20), condition Eq. (14) is satisfied if

$$F(\xi) + G(\xi, 0) = \int_0^a v(t) J_0(\xi t) dt, \quad (27)$$

where $v(t)$ is a function yet to be determined.

On the plane $y = 0$, $\sigma_{yz}^{(1)}$ is given by

$$\sigma_{yz}^{(1)}(x, 0) = \mu_0 \left\{ \int_0^\infty \frac{\partial G}{\partial y} \Big|_{y=0} \cos(\xi x) d\xi - \frac{d}{dx} \int_0^{\min(x, a)} \frac{[v(t) + \mu_1(x, 0)r(t)] dt}{\sqrt{x^2 - t^2}} \right\} \quad \text{for } x \geq 0. \quad (28)$$

Using Eq. (28), we find that condition Eq. (15) is satisfied if

$$\frac{d}{dx} \int_0^{\min(x, a)} \frac{v(t) dt}{\sqrt{x^2 - t^2}} = \int_0^\infty \frac{\partial G}{\partial y} \Big|_{y=0} \cos(\xi x) d\xi - \frac{\mu_1(x, 0) s_0(x)}{\mu_0} \quad \text{for } 0 < x < a. \quad (29)$$

Inversion of Eq. (29) as Abel's integral equation gives

$$v(t) = \frac{2t}{\pi} \int_0^t \frac{1}{\sqrt{t^2 - x^2}} \left[\int_0^\infty \frac{\partial G}{\partial y} \Big|_{y=0} \cos(\xi x) d\xi - \frac{\mu_1(x, 0) s_0(x)}{\mu_0} \right] dx$$

for $0 < t < a$. (30)

For the special case where $\mu_1(x, y) = f(y)$, from Eq. (23), it is obvious that we may write

$$U(\xi, y) = 2f(y)\delta(\xi - 0), \quad (31)$$

where δ denotes the Dirac-delta function. It follows that

$$\begin{aligned} \Gamma(\xi, y) &= -2\xi E(\xi) f'(y) \exp(-\xi y), \\ G(\xi, y) &= \xi E(\xi) \exp(2\xi y) \int_0^y \exp(-2\xi t) f(t) dt, \\ v(t) &= \frac{2t}{\pi} \int_0^t \frac{1}{\sqrt{t^2 - x^2}} \int_0^\infty \frac{\partial G}{\partial y} \Big|_{y=0} \cos(\xi x) d\xi dx - f(0)r(t) \text{ for } 0 < t < a. \end{aligned}$$

(32)

The results in Eq. (32) are in essential agreement with those given by Ang and Clements (1987) for the special case where $\mu_1(x, y) = f(y)$.

6 Stress intensity factor

At the crack tip $x = a, y = 0$, let us define the stress intensity factor

$$K = \lim_{x \rightarrow a^+} \sqrt{x - a} \cdot \sigma_{yz}(x, 0). \quad (33)$$

If we retain only the first two terms in the series solution, we obtain the approximation

$$K \simeq K^{(0)} + \epsilon K^{(1)}, \quad (34)$$

where

$$K^{(i)} = \lim_{x \rightarrow a^+} \sqrt{x - a} \cdot \sigma_{yz}^{(i)}(x, 0) \quad (i = 1, 2). \quad (35)$$

From Eqs. (19) and (28), we obtain

$$\begin{aligned}
K^{(0)} &= \frac{\mu_0 r(a)}{\sqrt{2a}}, \\
K^{(1)} &= \frac{\mu_0 [v(a) + \mu_1(a, 0)r(a)]}{\sqrt{2a}} \\
&\quad + \mu_0 \lim_{x \rightarrow a^+} \sqrt{x-a} \cdot \int_0^\infty \frac{\partial G}{\partial y} \Big|_{y=0} \cos(\xi x) d\xi. \tag{36}
\end{aligned}$$

7 Specific cases

Assume that the crack is acted upon by a uniform internal shear stress, i.e. $s_0(x) = \sigma_0$, where σ_0 is a given positive constant. For this particular case, $r(t) = \sigma_0 t / \mu_0$ and hence $K^{(0)} / (\sigma_0 \sqrt{2a}) = 1$. The function $E(\xi)$ in Eq. (18) is given by

$$\xi E(\xi) = \frac{a\sigma_0 J_1(a\xi)}{\mu_0}, \tag{37}$$

where J_1 is the Bessel function of order one.

For the variation of the shear modulus μ , let us firstly consider the specific case where μ_1 is given by

$$\mu_1(x, y) = \exp\left(-\frac{y}{h}\right) \cos\left(\frac{x}{r}\right), \tag{38}$$

where $h > 0$ and $r > 0$ are given constants.

From Eq. (23), it is clear that $U(\xi, y)$ may be written as

$$U(\xi, y) = \exp\left(-\frac{y}{h}\right) \delta\left(|\xi| - \frac{1}{r}\right). \tag{39}$$

If we use Eqs. (25), (26), (37) and (39) then after some manipulations we obtain (for $\xi \geq 0$)

$$\frac{\partial G}{\partial y} \Big|_{y=0} = \begin{cases} \frac{1}{2} a \sigma_0 \mu_0^{-1} [J_1(a[r^{-1} - \xi]) \\ \quad + (r+h)(2rh\xi + r+h)^{-1} J_1(a[r^{-1} + \xi])] & \text{if } \xi < 1/r, \\ \frac{1}{2} a \sigma_0 \mu_0^{-1} [(r+h)(2rh\xi + r+h)^{-1} J_1(a[r^{-1} + \xi]) \\ \quad + (r-h)(2rh\xi - h+r)^{-1} J_1(a[-r^{-1} + \xi])] & \text{if } \xi > 1/r. \end{cases} \tag{40}$$

Using Eqs. (30) and (36) together with (Watson, 1922)

$$\int_0^t \frac{\cos(\xi u) du}{\sqrt{t^2 - u^2}} = \frac{\pi}{2} J_0(\xi t), \quad (41)$$

we find that

$$\frac{K^{(1)}}{\sigma_0 \sqrt{2a}} = \frac{\mu_0}{2\sigma_0} \int_0^\infty \frac{\partial G}{\partial y} \Big|_{y=0} J_0(a\xi) d\xi - \frac{1}{2} J_0\left(\frac{a}{r}\right) + \frac{1}{2} \cos\left(\frac{a}{r}\right), \quad (42)$$

where the improper integral can be evaluated numerically using a suitable integration quadrature.

If we let $r \rightarrow \infty$ in Eqs. (42) and (40), we find that $K^{(1)}/(\sigma_0 \sqrt{2a})$ tends to the value obtained by using the results in Eq. (32), i.e. the corresponding value of $K^{(1)}/(\sigma_0 \sqrt{2a})$ for the variation $\mu_1 = \exp(-y/h)$ (as in one of the examples given in Ang and Clements, 1987).

We carry out the numerical computation the non-dimensionalized stress intensity factor $k = K^{(1)}/(\sigma_0 \sqrt{2a})$ using Eq. (42). The graphs of k against h/a (for $h/a \in [1/20, 5]$) for $r/a = 1, 2$ and 4 are given in Figure 1. For those values of r/a , it is obvious that k decreases with increasing h/a , i.e. if the exponential decay rate $1/h$ of μ_1 becomes lower the state of stress around the crack of fixed length $2a$ becomes less severe. Notice that for $r/a = 1, 2$ and 4 , the whole crack lies in the interior of the region $-\pi r/2 < x < \pi r/2$ where the shear modulus μ is greater than μ_0 (the shear modulus of the corresponding homogeneous material) and where for a fixed $y > 0$ the shear modulus μ increases with increasing h . From further calculations, we find that if part of the crack lies outside the region $-\pi r/2 < x < \pi r/2$, e.g. if $r/a = 1/2$, k does not necessarily decreases with h/a . If $r/a = 1/2$, the part of the crack that lies in the region having shear modulus greater than μ_0 is from the point $(-\pi a/4, 0)$ to $(\pi a/4, 0)$. The remaining part of the crack, including the crack tips, is in a region where the shear modulus μ is less than μ_0 and decreases with increasing h for a fixed $y > 0$. The graph of k against h/a (for $h/a \in [1/100, 5]$) for $r/a = 1/2$ is given in Figure 2. In this case, k clearly increases with increasing h/a .

Let us now consider the case where μ_1 varies according to

$$\mu_1(x, y) = \frac{a^2}{c^2 + x^2}, \quad (43)$$

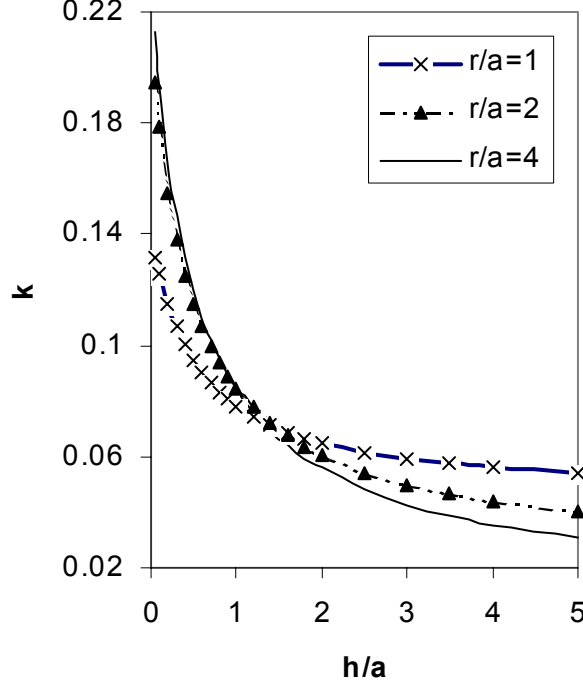


Figure 1: Graphs of k against h/a for selected values of r/a .

where $c > 0$ is a given constant. For this particular case, Eq. (23) gives

$$U(\xi, y) = \frac{a^2}{c} \exp(-c|\xi|). \quad (44)$$

It follows that

$$\left. \frac{\partial G}{\partial y} \right|_{y=0} = \frac{a^3 \sigma_0}{2c\mu_0} \int_0^\infty J_1(as) \left\{ \exp(-c|s + \xi|) + \frac{(s - \xi)}{(s + \xi)} \exp(-c|s - \xi|) \right\} ds, \quad (45)$$

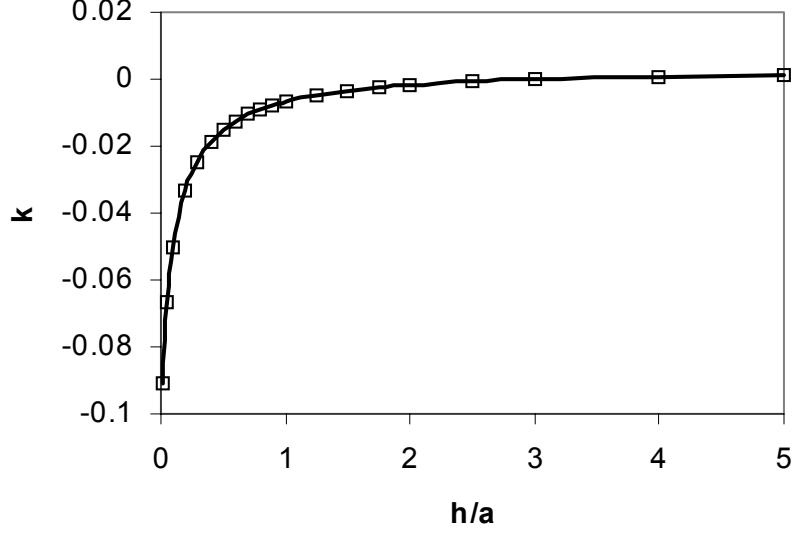


Figure 2: Graph of k against h/a for $r/a = 1/2$.

and

$$\begin{aligned} \frac{K^{(1)}}{\sigma_0\sqrt{2a}} &= \frac{\mu_0}{2\sigma_0} \int_0^\infty \left. \frac{\partial G}{\partial y} \right|_{y=0} J_0(a\xi) d\xi + \frac{a^2}{2(c^2 + a^2)} \\ &\quad - \frac{1}{\pi} \int_0^a \frac{a^2 dx}{(c^2 + x^2)\sqrt{a^2 - x^2}}. \end{aligned} \quad (46)$$

With some efforts, the improper integrals over $[0, \infty)$ in Eqs. (45) and (46) can be calculated numerically. The definite integral over the interval $[0, a]$ on the right hand side of Eq. (46) can be evaluated analytically using formula (3.3.50) in Abramowitz and Stegun (1970). Thus, the non-dimensionalized stress intensity factor $k = K^{(1)}/(\sigma_0\sqrt{2a})$ can be computed for various values of $c/a > 0$. A plot of k against c/a (for $c/a \in [1/20, 1/2]$) is given in Figure 3. For $c/a \in [1/20, 1/2]$, we find that $k < 0$ and k increases as c/a increases, i.e. for the range of c/a considered the state of stress around the crack becomes less severe as the positive parameter c/a becomes larger. Also, it appears that $k \rightarrow 0$ as $c/a \rightarrow \infty$. Notice that as c/a becomes larger the material surrounding the crack becomes softer.

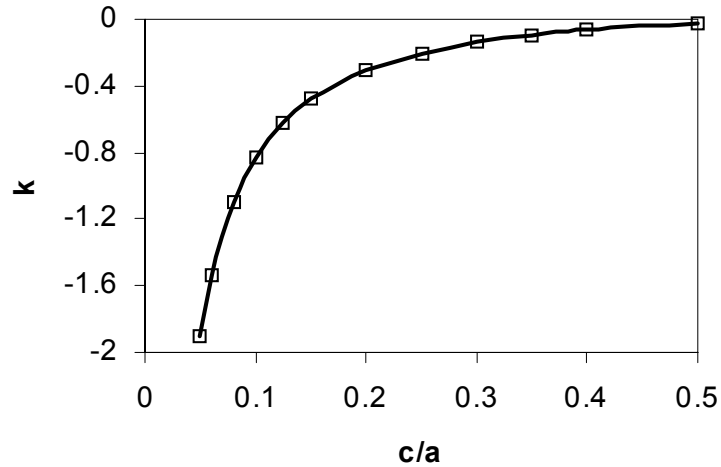


Figure 3: Graph of k against c/a .

8 Conclusion

A mode III problem concerning a planar crack in an elastic medium having shear modulus that varies slightly in space is analysed using a perturbation technique. An approximate expression for the relevant stress intensity factor is derived by using the first two terms of the series solution. For two particular variations of the shear modulus, we apply the analysis presented to compute the stress intensity factor numerically.

The analysis can be used to recover the special case considered in Ang and Clements (1987) where the shear modulus varies only in direction that is perpendicular to the crack. It may also be extended or modified to generalize the in-plane crack problem studied in Ang and Clements (1987) or possibly the penny-shaped crack problems in Ang (1987) and Ergüven and Gross (1999) to an even more general variation in the elastic coefficients.

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Keywords:

Mode III crack, inhomogeneous materials, stress intensity factor, perturbation technique.

Captions:

Figure 1: Graphs of k against h/a for selected values of r/a .

Figure 2: Graph of k against h/a for $r/a = 1/2$.

Figure 3: Graph of k against c/a .