A hypersingular boundary integral formulation for heat conduction across a curved imperfect interface

Whye-Teong Ang
Division of Engineering Mechanics
School of Mechanical and Aerospace Engineering
Nanyang Technological University
50 Nanyang Avenue, Singapore 639790
E-mail: mwtang@ntu.edu.sg
http://www.ntu.edu.sg/home/mwtang/

Abstract

The problem of determining the two-dimensional steady state temperature field in a bimaterial with a curved microscopically imperfect interface is considered. The temperature jump across the interface is proportional in magnitude to the interfacial heat flux. The conditions on the interface are formulated in terms of a boundary integral equation containing both Cauchy principal and Hadamard finite-part integrals. A numerical method based on this formulation is outlined for the numerical solution of the problem under consideration. It is applied to solve some specific problems.

Keywords: imperfect interface, heat conduction, hypersingular integral

This is a preprint of an article which would appear soon in Communications in Numerical Methods in Engineering published by Wiley Interscience. More details may be found at:
http://www3.interscience.wiley.com/cgi-bin/jhome/110541607

1 Introduction

Microscopic gaps inevitably exist along the common boundary (interface) between two bodies, no matter how well joined they may be. Because of this, there has recently been a surging interest in the investigation of microscopically imperfect interfaces (see e.g. Fan and Sze [1], Fan and Wang [2], Torquato and Rintoul [3], and other references therein). For heat conduction problems, a macroscopic model for taking into account the presence of microscopic imperfections allows for an interfacial temperature jump which is proportional in magnitude to the heat flux on the interface.

A Green's function satisfying the two-dimensional steady state thermal conditions on a straight homogeneously imperfect interface in a bimaterial is derived by Ang *et al.* [4]. With the Green's function, a boundary integral method which does not require the discretisation of the interface may be obtained for a class of steady state heat conduction problems involving bimaterials. The use of the Green's function gives accurate numerical values for the temperature, particularly at points very close to the interface. However, in general, a suitable Green's function may be difficult (if not impossible) to derive explicitly and analytically for an imperfect interface which is curved or inhomogeneous (or both).

The present paper considers a two-dimensional steady state heat conduction problem involving a thermally isotropic bimaterial with a curved inhomogeneously imperfect interface. A boundary integral formula is obtained for the temperature in the bimaterial. It is applied to express the interfacial conditions in terms of a boundary integral equation which contains both Cauchy principal and Hadamard finite-part integrals. The hypersingular boundary integral formulation is such that there is only one unknown function (given by the interfacial temperature jump) appearing in the integrand of the integrals over the interface. A numerical method based on this boundary integral formulation is outlined for the numerical solution of the heat conduction problem under consideration. It is applied to solve some specific problems.

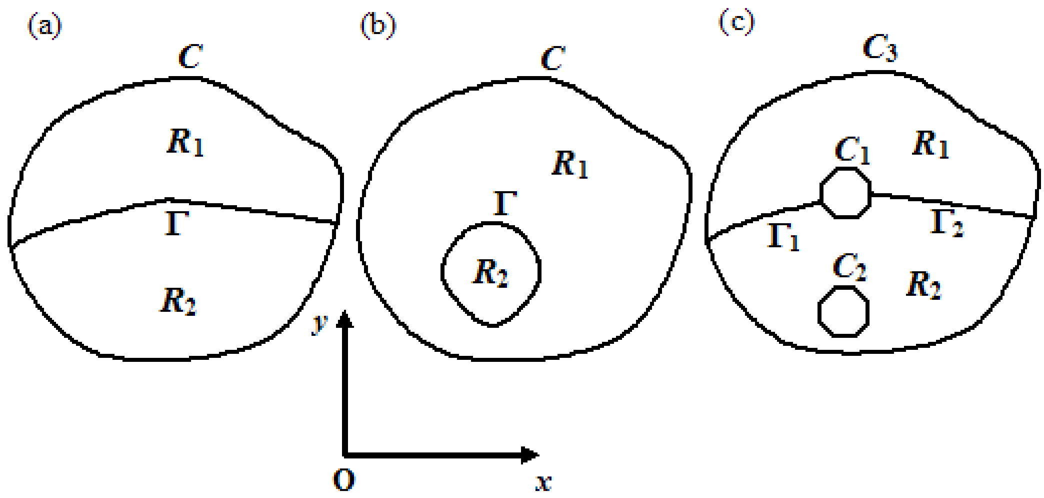

Figure 1: (a) The interface Γ is an open curve and C is a simple closed curve forming the exterior boundary of the bimaterial. (b) Both Γ and C are simple closed curves, with Γ lying in the interior of the region enclosed by C. (c) The interface Γ consists of two parts Γ_1 and Γ_2 and the boundary C comprises three parts C_1, C_2 and C_3, where C_1 and C_2 are the boundaries of the two holes and C_3 is the exterior boundary of the bimaterial.

2 Mathematical statement of the problem

With reference to a Cartesian coordinate frame $Oxyz$, a body has a geometry which does not vary along the z direction. The body comprises two possibly dissimilar materials joined along a curved boundary (interface) Γ. The regions occupied by the two dissimilar materials are denoted by R_1 and R_2. There may be several possible cases for the geometries of the interface Γ and the remaining boundary C of the bimaterial. Figure 1 (a) gives a sketch for the case in which Γ is an open curve and C is a simple closed curve. In Figure 1 (b), Γ and C are both simple closed curves, with R_1 being the region in between the curves Γ and C and R_2 the region enclosed by Γ. The bimaterial may also contain holes as shown in Figure 1 (c). The boundaries of the holes form parts of C.

The bond between the materials in R_1 and R_2 at the interface Γ is microscopically damaged such that the interfacial thermal conditions are given by

$$\begin{aligned} & k_1 \lim_{\epsilon \to 0} (n_1^{(\text{int})} \frac{\partial T}{\partial x} + n_2^{(\text{int})} \frac{\partial T}{\partial y})\Big|_{(x,y)=(\xi+|\epsilon| n_1^{(\text{int})}, \eta+|\epsilon| n_2^{(\text{int})})} \\ = \; & k_2 \lim_{\epsilon \to 0} (n_1^{(\text{int})} \frac{\partial T}{\partial x} + n_2^{(\text{int})} \frac{\partial T}{\partial y})\Big|_{(x,y)=(\xi-|\epsilon| n_1^{(\text{int})}, \eta-|\epsilon| n_2^{(\text{int})})} \quad \text{for } (\xi,\eta) \in \Gamma, \end{aligned} \tag{1}$$

and

$$\begin{aligned} & k_1 \lim_{\epsilon \to 0} (n_1^{(\text{int})} \frac{\partial T}{\partial x} + n_2^{(\text{int})} \frac{\partial T}{\partial y})\Big|_{(x,y)=(\xi+|\epsilon| n_1^{(\text{int})}, \eta+|\epsilon| n_2^{(\text{int})})} \\ = \; & \lambda \Delta T(\xi,\eta) \quad \text{for } (\xi,\eta) \in \Gamma, \end{aligned} \tag{2}$$

where $T(x, y)$ is the steady state temperature field in the bimaterial, k_1 and k_2 are the thermal conductivities of the materials in R_1 and R_2 respectively, $n_1^{(\text{int})}$ and $n_2^{(\text{int})}$ are respectively the x and y components of the unit normal vector to Γ pointing into R_1, λ is a given positive coefficient and $\Delta T(\xi, \eta)$ is the interfacial temperature jump defined by

$$\Delta T(\xi,\eta) = \lim_{\epsilon \to 0} [T(\xi + |\epsilon| n_1^{(\text{int})}, \eta + |\epsilon| n_2^{(\text{int})}) - T(\xi - |\epsilon| n_1^{(\text{int})}, \eta - |\epsilon| n_2^{(\text{int})})]. \tag{3}$$

For an inhomogeneously imperfect interface, λ is assumed to vary continuously from point to point on the interface Γ.

At each and every point on the boundary C, a linear combination of the temperature and the heat flux is specified, that is,

$$\alpha(x, y)T(x, y) + \beta(x, y)H(x, y) = g(x, y) \quad \text{for } (x, y) \in C, \tag{4}$$

where α and β are given functions such that $\alpha^2 + \beta^2 \neq 0$ at all points on C, g is a suitably given function and H is the heat flux across C. The heat flux

H is defined by

$$H(x,y) = -k(x,y)[n_1(x,y)\frac{\partial T}{\partial x} + n_2(x,y)\frac{\partial T}{\partial y}], \tag{5}$$

where $k(x,y)$ is the thermal conductivity and n_1 and n_2 are respectively the x and y components of the unit normal outward vector to C. The value of $k(x,y)$ is either k_1 or k_2 depending on whether the point (x,y) on C belongs to the material in R_1 or R_2.

The body is assumed to be thermally isotropic with thermal conductivities k_1 and k_2 being constants. According to the law of conservation of heat energy, the steady state temperature $T(x,y)$ is then required to satisfy the two-dimensional Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ for } (x,y) \in R_1 \cup R_2. \tag{6}$$

Mathematically, the problem of interest here is to solve the Laplace's equation (6) for $T(x,y)$ subject to the interfacial conditions (1) and (2) as well as the boundary condition (4).

3 Hypersingular boundary integral formulation

Using (6) together with (1) and (2), one may apply the reciprocal theorem and the fundamental solution for the two-dimensional Laplace's equation (Clements [5]) to obtain the following formula for the temperature at any point in the interior of R_1 or R_2:

$$\begin{aligned} T(\xi,\eta) &= \int_C [T(x,y)\Omega(x,y;\xi,\eta) + \Phi(x,y;\xi,\eta)H(x,y)]ds(x,y) \\ &+ \int_\Gamma \Delta T(x,y)[\lambda(x,y)\Delta\Phi(x,y;\xi,\eta) - \Omega^{(\text{int})}(x,y;\xi,\eta)]ds(x,y) \\ &\qquad \text{for } (\xi,\eta) \in R_1 \cup R_2, \end{aligned} \tag{7}$$

where

$$\begin{aligned}
\Phi(x,y;\xi,\eta) &= \frac{1}{4\pi k(x,y)}\ln([x-\xi]^2+[y-\eta]^2),\\
\Omega(x,y;\xi,\eta) &= \frac{n_1(x,y)(x-\xi)+n_2(x,y)(y-\eta)}{2\pi([x-\xi]^2+[y-\eta]^2)},\\
\Delta\Phi(x,y;\xi,\eta) &= \frac{k_2-k_1}{4\pi k_1 k_2}\ln([x-\xi]^2+[y-\eta]^2),\\
\Omega^{(\text{int})}(x,y;\xi,\eta) &= \frac{n_1^{(\text{int})}(x,y)(x-\xi)+n_2^{(\text{int})}(x,y)(y-\eta)}{2\pi([x-\xi]^2+[y-\eta]^2)}.
\end{aligned} \tag{8}$$

For the case in which (ξ,η) lies on a smooth part of the boundary C, the boundary integral equation (7) should be modified to become

$$\begin{aligned}
\frac{1}{2}T(\xi,\eta) &= \mathcal{C}\int\limits_C [T(x,y)\Omega(x,y;\xi,\eta)+\Phi(x,y;\xi,\eta)H(x,y)]ds(x,y)\\
&\quad+\int\limits_\Gamma \Delta T(x,y)[\lambda(x,y)\Delta\Phi(x,y;\xi,\eta)-\Omega^{(\text{int})}(x,y;\xi,\eta)]ds(x,y)\\
&\qquad\text{for } (\xi,\eta)\in C \text{ (smooth part)},
\end{aligned} \tag{9}$$

where $\mathcal{C}$ denotes that the integral is to be interpreted in the Cauchy principal sense.

Through the use of (7), condition (2) can be written as

$$\begin{aligned}
&(1-\frac{k_2-k_1}{2k_2})\lambda(\xi,\eta)\Delta T(\xi,\eta)\\
=&\int\limits_C [T(x,y)\Lambda(x,y;\xi,\eta)+\frac{k_1}{k(x,y)}\Omega^{(\text{int})}(\xi,\eta;x,y)H(x,y)]ds(x,y)\\
&+\frac{k_2-k_1}{k_2}\mathcal{C}\int\limits_\Gamma \Delta T(x,y)\lambda(x,y)\Omega^{(\text{int})}(\xi,\eta;x,y)ds(x,y)\\
&-\mathcal{H}\int\limits_\Gamma \Delta T(x,y)\Lambda^{(\text{int})}(x,y;\xi,\eta)ds(x,y) \text{ for } (\xi,\eta)\in\Gamma,
\end{aligned} \tag{10}$$

where $\mathcal{H}$ denotes that the integral is to be interpreted in the Hadamard finite-part sense and

$$\begin{aligned}\Lambda(x,y;\xi,\eta) &= \frac{k_1}{2\pi}(n_1^{(\mathrm{int})}(\xi,\eta)[n_1(x,y)([x-\xi]^2-[y-\eta]^2) \\ &\quad +2n_2(x,y)(x-\xi)(y-\eta)] \\ &\quad +n_2^{(\mathrm{int})}(\xi,\eta)[n_2(x,y)([y-\eta]^2-[x-\xi]^2) \\ &\quad +2n_1(x,y)(x-\xi)(y-\eta)])/([x-\xi]^2+[y-\eta]^2)^2,\end{aligned}$$

$$\begin{aligned}\Lambda^{(\mathrm{int})}(x,y;\xi,\eta) &= \frac{k_1}{2\pi}(n_1^{(\mathrm{int})}(\xi,\eta)[n_1^{(\mathrm{int})}(x,y)([x-\xi]^2-[y-\eta]^2) \\ &\quad +2n_2^{(\mathrm{int})}(x,y)(x-\xi)(y-\eta)] \\ &\quad +n_2^{(\mathrm{int})}(\xi,\eta)[n_2^{(\mathrm{int})}(x,y)([y-\eta]^2-[x-\xi]^2) \\ &\quad +2n_1^{(\mathrm{int})}(x,y)(x-\xi)(y-\eta)])/([x-\xi]^2+[y-\eta]^2)^2.\end{aligned} \tag{11}$$

4 Numerical procedure

A simple procedure based on (9) and (10) together with (4) for determining numerically T and H on C and the interfacial temperature jump ΔT is outlined here. Once T and H are completely known on C and ΔT is determined, the temperature T at any point (ξ,η) in the interior of $R_1 \cup R_2$ can be calculated by evaluating numerically the integrals on the right hand side of (7).

The boundary C is discretised into N straight line elements denoted by C_1, C_2, $\cdots$, C_{N-1} and C_N and the interface Γ into M straight line elements $\Gamma_1, \Gamma_2, \cdots, \Gamma_{M-1}$ and Γ_M. Over an element of C, T and H are approximated as constants. Similarly, ΔT is taken to be constant over an element of Γ. More specifically, one makes the approximation

$$\left.\begin{array}{r}T(x,y) \simeq T_n \\ H(x,y) \simeq H_n\end{array}\right\} \text{ for } (x,y) \in C_n \ (n=1,2,\cdots,N),$$
$$\Delta T(x,y) \simeq \Delta T_m \text{ for } (x,y) \in \Gamma_m \ (m=1,2,\cdots,M), \tag{12}$$

where T_n, H_n and ΔT_m are constants to be determined.

With (12), if one lets (ξ, η) in (9) to be the midpoint (ξ_p, η_p) of the element C_p, one obtains

$$\begin{aligned}
\frac{1}{2}T_p &= \frac{\delta_p[\ln(\frac{1}{2}\delta_p) - 1]}{2\pi k(\xi_p, \eta_p)} H_p \\
&+ \sum_{\substack{n=1 \\ n \neq p}}^{N} \{T_n \int\limits_{C_n} \Omega(x, y; \xi_p, \eta_p) ds(x, y) + H_n \int\limits_{C_n} \Phi(x, y; \xi_p, \eta_p) ds(x, y)\} \\
&+ \sum_{m=1}^{M} \Delta T_m \int\limits_{\Gamma_m} [\lambda_m \Delta\Phi(x, y; \xi_p, \eta_p) - \Omega^{(\text{int})}(x, y; \xi_p, \eta_p)] ds(x, y) \\
&\qquad \text{for } p = 1, 2, \cdots, N, \qquad (13)
\end{aligned}$$

where δ_p is the length of C_p, λ_m is the value of $\lambda(x, y)$ at $(x, y) = (\xi_{N+m}, \eta_{N+m})$ and (ξ_{N+m}, η_{N+m}) is the midpoint of Γ_m.

Similarly, letting (ξ, η) in (10) be the midpoint (ξ_{N+r}, η_{N+r}) of Γ_r, one finds that

$$\begin{aligned}
(1 - \frac{k_2 - k_1}{2k_2})\lambda_r \, \Delta T_r &= -\frac{2k_1}{\pi \delta_{N+r}} \Delta T_r \\
&+ \sum_{n=1}^{N} \{T_n \int\limits_{C_n} \Lambda(x, y; \xi_{N+r}, \eta_{N+r}) ds(x, y) \\
&+ H_n \int\limits_{C_n} \frac{k_1}{k(x, y)} \Omega^{(\text{int})}(\xi_{N+r}, \eta_{N+r}; x, y) ds(x, y)\} \\
&+ \sum_{\substack{m=1 \\ m \neq r}}^{M} \Delta T_m \int\limits_{\Gamma_m} [\frac{k_2 - k_1}{k_2} \lambda_m \Omega^{(\text{int})}(\xi_{N+r}, \eta_{N+r}; x, y) \\
&\qquad - \Lambda^{(\text{int})}(x, y; \xi_{N+r}, \eta_{N+r})] ds(x, y) \\
&\qquad \text{for } r = 1, 2, \cdots, M, \qquad (14)
\end{aligned}$$

where δ_{N+r} is the length of Γ_r.

Applying (4) at the midpoint of each element of the boundary C gives

$$\alpha(\xi_p, \eta_p) T_p + \beta(\xi_p, \eta_p) H_p = g(\xi_p, \eta_p) \text{ for } p = 1, 2, \cdots, N. \qquad (15)$$

Equations (13), (14) and (15) constitute a system of $2N + M$ linear algebraic equations in $2N + M$ unknowns given by T_p, H_p and ΔT_r for $p = 1, 2, \cdots, N$ and $r = 1, 2, \cdots, M$. Once the values of these unknowns are found, the temperature at any interior point in the bimaterial may be computed approximately by using the formula

$$\begin{aligned} T(\xi,\eta) \simeq{}& \sum_{n=1}^{N}\{T_n \int\limits_{C_n} \Omega(x,y;\xi,\eta)ds(x,y) + H_n \int\limits_{C_n} \Phi(x,y;\xi,\eta)ds(x,y)\} \\ &+ \sum_{m=1}^{M} \Delta T_m \int\limits_{\Gamma_m} [\lambda_m \Delta\Phi(x,y;\xi,\eta) - \Omega^{(\mathrm{int})}(x,y;\xi,\eta)]ds(x,y) \\ &\text{for } (\xi,\eta) \in R_1 \cup R_2. \end{aligned} \tag{16}$$

All the integrals over the straight line elements C_n or Γ_m in (13), (14) and (16) are proper. Analytical formulae can be obtained for the integrals as follows.

If the coordinates of points on either C_n or Γ_m are expressed in terms of linear functions of the parameter t for $0 \leq t \leq 1$, the proper integrals may be reduced to one of the following forms

$$\int\limits_0^1 \ln(At^2 + Bt + C)dt, \quad \int\limits_0^1 \frac{dt}{At^2 + Bt + C} \quad \text{and} \quad \int\limits_0^1 \frac{dt}{(At^2 + Bt + C)^2}, \tag{17}$$

where A, B and C are real coefficients which are independent of t and such that $4AC - B^2 > 0$.

For $4AC - B^2 > 0$, the definite integrals in (17) may be evaluated analytically by using

$$\begin{aligned} &\int \ln(At^2 + Bt + C)dt \\ ={}& t(\ln(A) - 2) + (t + \frac{B}{2A})\ln(t^2 + \frac{B}{A}t + \frac{C}{A}) \\ &+ \frac{1}{A}\sqrt{4AC - B^2}\arctan(\frac{2At + B}{\sqrt{4AC - B^2}}) + \text{constant}, \end{aligned}$$

$$\int \frac{dt}{At^2 + Bt + C}$$
$$= \frac{2}{\sqrt{4AC - B^2}} \arctan(\frac{2At + B}{\sqrt{4AC - B^2}}) + \text{constant},$$

$$\int \frac{dt}{(At^2 + Bt + C)^2}$$
$$= \frac{2At + B}{(4AC - B^2)(At^2 + Bt + C)}$$
$$+ \frac{4A}{(4AC - B^2)^{3/2}} \arctan \frac{2At + B}{\sqrt{(4AC - B^2)}} + \text{constant}.$$

For further details, one may refer to, for example, Ang [6].

Alternatively, if one finds the above analytical formulae cumbersome to use, one may choose to compute the proper integrals over the straight line elements C_n or Γ_m by using numerical integration.

5 Specific problems

To check its validity, the numerical procedure outlined in Section 4 is applied here to solve two specific problems.

Problem 1. The boundary C comprises two parts C_{inner} and C_{outer} as respectively given by $x^2 + y^2 = r_{\text{inner}}^2$ and $x^2 + y^2 = r_{\text{outer}}^2$, where $0 < r_{\text{inner}} < r_{\text{outer}}$. The interface Γ is given by $x^2 + y^2 = r_{\text{int}}^2$, where $r_{\text{inner}} < r_{\text{int}} < r_{\text{outer}}$. Thus, the regions R_1 and R_2 are given by $r_{\text{int}}^2 < x^2 + y^2 < r_{\text{outer}}^2$ and $r_{\text{inner}}^2 < x^2 + y^2 < r_{\text{int}}^2$ respectively.

Heat energy is added to or removed from the outer boundary C_{outer} by a convection process which is modelled by the boundary condition

$$H(x, y) = \gamma(T(x, y) - T_{\text{a}}) \text{ on } C_{\text{outer}}, \tag{18}$$

where γ and T_{a} are given constants. Note that T_{a} is the outside ambient temperature surrounding the body.

The inner boundary C_{inner} is maintained at a fixed constant temperature, that is,

$$T(x,y) = T_{\text{c}} \quad \text{on } C_{\text{inner}} \tag{19}$$

where T_{c} is a given constant.

It is assumed that (1) and (2) are applicable with λ being a constant, that is, the interface Γ is homogeneous.

The exact solution of this specific problem is

$$T(x,y) = \sigma_i + \frac{1}{2}\tau_i \ln(x^2+y^2) \ \text{ for } (x,y) \in R_i \ (i=1,2), \tag{20}$$

where

$$\begin{aligned}
\sigma_1 &= T_{\text{a}} - \tau_1\left[\frac{k_1}{\gamma r_{\text{outer}}} + \ln(r_{\text{outer}})\right],\\
\sigma_2 &= T_{\text{c}} - \tau_2 \ln(r_{\text{inner}}),\\
\tau_1 &= \frac{k_2}{k_1}\tau_2,\\
\tau_2 &= \frac{\lambda}{\chi}(T_{\text{a}} - T_{\text{c}})\\
\chi &= \frac{k_2}{r_{\text{int}}} - \lambda[\ln(r_{\text{inner}}) - \frac{k_2}{k_1}(\ln(r_{\text{outer}}) + \frac{k_1}{\gamma r_{\text{outer}}}) - (1-\frac{k_2}{k_1})\ln(r_{\text{int}})].
\end{aligned} \tag{21}$$

For the purpose of using the boundary integral method to solve the specific problem numerically, take $r_{\text{outer}} = 3/2$, $r_{\text{int}} = 1$, $r_{\text{inner}} = 1/2$, $k_1 = 1/2$, $k_2 = 3/4$, $\lambda = 10$, $\gamma = 1$, $T_{\text{a}} = 1$ and $T_{\text{c}} = 5$. The inner boundary C_{inner}, the interface Γ and the outer boundary C_{outer} are approximated as regular polygons with N_0, $2N_0$ and $3N_0$ sides respectively (so that $N = 4N_0$ and $M = 2N_0$).

It would be interesting to see if the boundary integral method could recover accurately the exact solution in (20). Equations (13), (14) and (15) are solved using $N_0 = 10$, 20 and 30 and the numerical values of T at various selected points in $R_1 \cup R_2$ as computed by using (16) are compared with the exact values in Table 1. The numerical values are in good agreement with the exact ones. It is also obvious that the accuracy of the numerical values improves significantly when N_0 is increased from 10 to 40.

Table 1. A comparison of the numerical values of T with the exact solution at various selected points.

Point	$N_0 = 10$	$N_0 = 20$	$N_0 = 40$	Exact
$(0.6000, 0.0000)$	4.5395	4.5916	4.6061	4.6113
$(0.3500, 0.6062)$	4.2130	4.2640	4.2776	4.2827
$(-0.6928, 0.4000)$	3.9337	3.9803	3.9931	3.9980
$(0.0000, 0.9000)$	3.6864	3.7301	3.7421	3.7470
$(0.9011, -0.6309)$	3.0024	3.0425	3.0534	3.0577
$(1.0392, 0.6000)$	2.7292	2.7658	2.7756	2.7794
$(-0.2257, 1.2803)$	2.4783	2.5113	2.5201	2.5235
$(-0.4788, -1.3156)$	2.2462	2.2756	2.2835	2.2865

For a given N_0, the numerical value of the temperature jump ΔT is found to have the same value on all the elements of the interface Γ, as expected. Furthermore, the percentage errors of the numerical values of ΔT are approximately 2%, 0.9% and 0.4% for N_0 given by 10, 20 and 40 respectively.

Problem 2. For another specific case, the boundary C is taken as comprising the four sides of the square with vertices $(0, 1/2)$, $(0, -1/2)$, $(1, -1/2)$ and $(1, 1/2)$. The interface Γ lies on part of the x axis from $(0, 0)$ to $(1, 0)$. The region R_1 is given by $0 < x < 1$, $0 < y < 1/2$, with $k_1 = 1$, and R_2 by $0 < x < 1$, $-1/2 < y < 0$, with $k_2 = 2$.

The interface Γ is inhomogeneous with

$$\lambda = \frac{2(1 + x - x^2)}{(1 + x^2)} \quad \text{for } 0 < x < 1. \tag{22}$$

The boundary conditions on the sides of the square are given by

$$T(x,y)=\begin{cases} x+11/6 & \text{for} \quad 0<x<1,\ y=1/2, \\ 1-y^2+2y^3/3+2y & \text{for} \quad x=0,\ 0<y<1/2, \\ 2-y^2+2y^3/3+2y & \text{for} \quad x=1,\ 0<y<1/2, \\ y^3/3+y & \text{for} \quad x=0,\ -1/2<y<0, \\ y^3/3+y & \text{for} \quad x=1,\ -1/2<y<0, \\ x^2/2-x/2-13/24 & \text{for} \quad 0<x<1,\ y=-1/2. \end{cases} \tag{23}$$

It may be verified that the exact solution of this specific problem is given by

$$T(x,y)=\begin{cases} 1+x^2-y^2-2x^2y+2y^3/3+2xy+2y & \text{for} \quad (x,y)\in R_1, \\ -x^2y+y^3/3+xy+y & \text{for} \quad (x,y)\in R_2. \end{cases} \tag{24}$$

To solve the specific problem numerically using the boundary integral method, the interface Γ is discretised into M elements and the exterior boundary C into $4M$. All the elements are of equal length $1/M$. Table 2 compares the numerical values of T at various selected points with the exact solution (24). The numerical values obtained using $M=18$ are more accurate that those from $M=6$.

Table 2. A comparison of the numerical values of T with the exact solution at various selected points.

Point	$M=6$	$M=18$	Exact
$(0.1000, 0.2000)$	1.4129	1.4114	1.4113
$(0.3000, 0.3000)$	1.7446	1.7741	1.7740
$(0.7000, 0.1000)$	1.7250	1.7229	1.7227
$(0.9000, 0.0500)$	1.9244	1.9172	1.9166
$(0.2000, -0.4000)$	-0.4854	-0.4853	-0.4853
$(0.4000, -0.2000)$	-0.2508	-0.2508	-0.2507
$(0.6000, -0.4900)$	-0.6474	-0.6470	-0.6468
$(0.8000, -0.0500)$	-0.0592	-0.0583	-0.0580

In Figure 2, the numerical and the exact interfacial temperature jump ΔT are plotted over the interval $0<x<1$. The numerical values of ΔT

in Figure 2 are obtained using $M = 18$. The two graphs are almost visually indistinguishable.

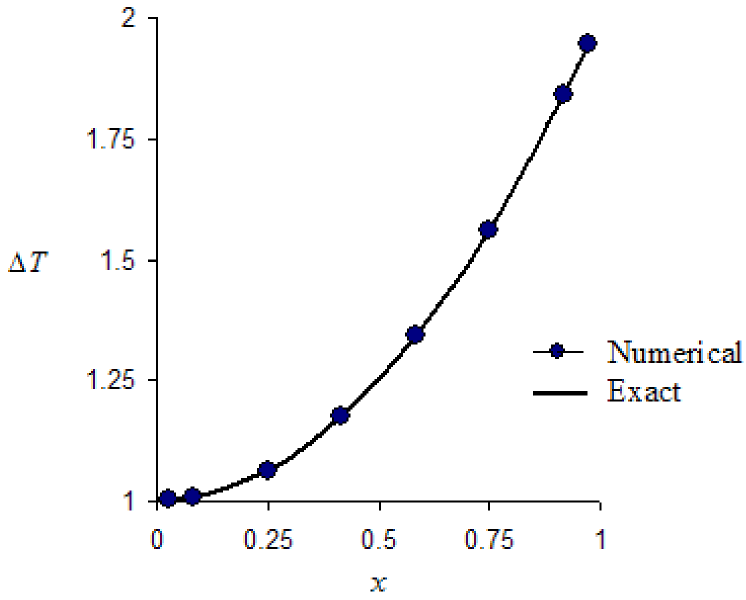

Figure 2: Plots of the numerical and the exact interfacial temperature jump ΔT over the interval $0 < x < 1$.

6 Conclusion

The problem of determining the two-dimensional steady state temperature distribution in a bimaterial with a curved inhomogeneously imperfect interface is formulated in terms of the boundary integral equations (9) and (10) which contain Cauchy principal and Hadamard finite-part integrals. A simple boundary integral method based on (9) and (10) is devised for solving the problem. The method reduces the problem under consideration to a system of linear algebraic equations. Two specific problems are solved using the boundary integral method. The numerical results obtained confirm the validity of the interfacial formulation (10) and the numerical procedure presented here.The numerical temperature at interior points in the bimaterial is observed to converge to the exact value when more boundary and interfacial elements are used in the calculation.

References

[1] Fan H, Sze KY. A micro-mechanics model for imperfect interface in dielectric materials. *Mechanics of Materials* 2001; **33**: 363-370.

[2] Fan H, Wang GF. Screw dislocation interacting with an imperfect interface. *Mechanics of Materials* 2003; **35:** 943-953.

[3] Torquato S, Rintoul MD. Effect of the interface on the properties of composite materials. *Physical Review Letters* 1995; **75:** 4067-4070.

[4] Ang WT, Choo KK. Fan H. A Green's function for steady-state two-dimensional isotropic heat conduction across a homogeneously imperfect interface. *Communications in Numerical Methods in Engineering* 2004; **20:** 391-399.

[5] Clements DL. *Boundary Value Problems Governed by Second Order Elliptic Systems*. Pitman, London, 1981.

[6] Ang WT. A times-stepping dual-reciprocity boundary element method for anisotropic heat diffusion subject to specification of energy. *Applied Mathematics and Computation* 2005; **162:** 661-678.