# Numerical solution of a linear elliptic partial differential equation with variable coefficients: a complex variable boundary element approach

**Whye-Teong Ang**

Division of Engineering Mechanics,
School of Mechanical and Aerospace Engineering,
Nanyang Technological University,
50 Nanyang Avenue, Singapore 639798
E-mail: mwtang@ntu.edu.sg
Tel: +65-790-5937, Fax: +65-6791-1859
http://www.ntu.edu.sg/home/mwtang/

**Abstract**

The present paper presents a complex variable boundary element method for the numerical solution of a second order elliptic partial differential equation with variable coefficients. To assess the validity and accuracy of the method, it is applied to solve some specific problems with known solutions.

# 1 Introduction

Of interest here is the numerical solution of the second order linear elliptic partial differential equation

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\gamma_{ij}\frac{\partial}{\partial x_i}(g(x_1,x_2)\frac{\partial}{\partial x_j}[\phi(x_1,x_2)])=0 \ \ \text{in } R, \tag{1}$$

subject to

$$\phi(x_1,x_2)=f_1(x_1,x_2) \ \text{ for } (x_1,x_2)\in D_1,$$
$$\psi(x_1,x_2)=f_2(x_1,x_2) \ \text{ for } (x_1,x_2)\in D_2, \tag{2}$$

where $(x_1,x_2)$ denotes a point on the $Ox_1x_2$ Cartesian plane, $R$ is a two-dimensional region bounded by a simple closed curve $C$ on the $Ox_1x_2$ plane, $\phi(x_1,x_2)$ is the unknown function to be determined, $\gamma_{ij}$ are given non-negative constants satisfying the symmetry property $\gamma_{ij}=\gamma_{ji}$ and the strict inequality $\gamma_{12}^2-\gamma_{11}\gamma_{22}<0$, $g(x_1,x_2)$ is a given function such that $g(x_1,x_2)$ is positive and at least twice partially differentiable with respect to the spatial variables $x_1$ and $x_2$ in $R$, $\psi(x_1,x_2)$ is defined by

$$\psi(x_1,x_2)=\sum_{i=1}^{2}\sum_{j=1}^{2}\gamma_{ij}g(x_1,x_2)n_i(x_1,x_2)\frac{\partial}{\partial x_j}[\phi(x_1,x_2)], \tag{3}$$

$[n_1(x_1,x_2),n_2(x_1,x_2)]$ is the unit normal vector to $C$ at the point $(x_1,x_2)$ pointing out of $R$, $D_1$ and $D_2$ are non-intersecting curves such that $C=D_1\cup D_2$, and $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are suitably prescribed functions for $(x_1,x_2)$ on $D_1$ and $D_2$ respectively.

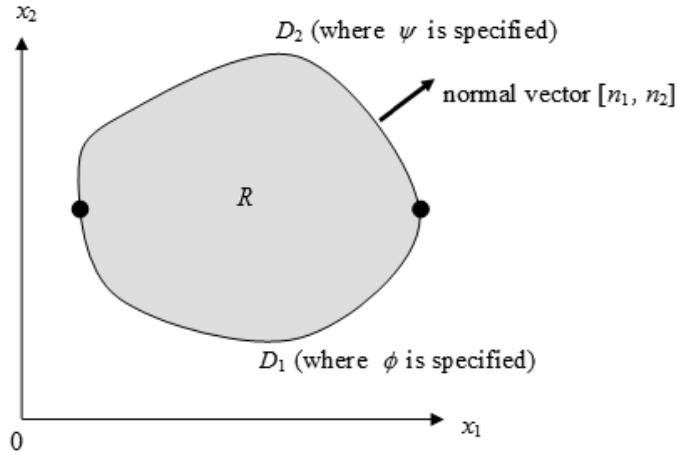A geometrical sketch of the problem is given in Figure 1.

**Figure 1**. A geometrical sketch of the problem.

For the special case in which $g(x_1, x_2) = 1$ (that is, the case in which the partial differential equation (1) has constant coefficients), it is possible to recast (1) into a boundary integral equation. The boundary element method for solving numerically the boundary value problem defined by (1) and (2) for such a case is well established (see Clements [7]). For the more general case in which $g(x_1, x_2)$ varies continuously in space, it may be mathematically difficult to derive a suitable fundamental solution for (1) in order to obtain a boundary integral formulation for the boundary value problem. If the fundamental solution for the special case $g(x_1, x_2) = 1$ is used for the general case, the resulting integral formulation does not contain only a boundary integral but also a domain integral. To deal with the domain integral in an effective manner or to obtain alternative formulations that do not require the solution domain to be discretised, various approaches have been proposed in the literature. For example, Rangogni [16] considered the case in which $\gamma_{ij} = \delta_{ij}$

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(Kronecker-delta) and $g(x_1, x_2) = 1 + \epsilon g_0(x_1, x_2)$ and employed the boundary element method together with the perturbation technique to solve the boundary value problem for small parameter $\epsilon$; Clements [6] and Ang, Kusuma and Clements [2] derived special fundamental solutions for the case in which $\gamma_{ij} = \delta_{ij}$ and $g(x_1, x_2) = X(x_1)Y(x_2)$; Kassab and Divo [11] introduced the idea of a generalised fundamental solution; Tanaka, Matsumoto and Suda [17] and Ang, Clements and Vahdati [1] applied the dual-reciprocity method proposed by Brebbia and Nardini [4] to approximate the domain integral as a boundary integral. Other related works of interest include Nerantzaki and Kandilas [13], Rangelov, Manolis and Dineva [15] and Katsikadelis [10].

For the special case in which $\gamma_{ij} = \delta_{ij}$ and $g(x_1, x_2) = X(x_1)Y(x_2)$, solutions of (1) can be expressed as a series containing an arbitrary complex function which is holomorphic in $R$ (Clements [6] and Ang, Kusuma and Clements [2]). In Park and Ang [14] and Ang, Park and Kang [3], the Cauchy integral formulae are employed to obtain a complex variable boundary element method for constructing the complex function which satisfies the boundary condition in (2). Such a complex variable boundary element approach is also used in Chen and Chen [5] and Hromadka and Lai [9] for solving potential problems governed by the two-dimensional Laplace's equation. The present paper outlines a complex variable boundary element method for solving (1) subject to (2) for more general coefficients $\gamma_{ij}$ and $g(x_1, x_2)$. The approach here is to use an appropriate substitution of variables together with a generalised radial basis function approximation of certain term in order to rewrite (1) as an elliptic partial differential equation whose solutions can be expressed in terms of an arbitrary holomorphic complex function.

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## 2 Generalised radial basis function approximation and complex variable formulation

Through the use of the substitution

$$\phi(x_1, x_2) = \frac{1}{\sqrt{g(x_1, x_2)}} w(x_1, x_2), \tag{4}$$

the governing equation in (1) can be rewritten as

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} [w(x_1, x_2)] = k(x_1, x_2) w(x_1, x_2), \tag{5}$$

where $k$ is given by

$$k(x_1, x_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{1}{\sqrt{g(x_1, x_2)}} \gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} [\sqrt{g(x_1, x_2)}]. \tag{6}$$

To approximate the right hand side of (5) using a meshfree method, we select $P$ well spaced out collocation points in $R \cup C$. The collocation points are denoted by $(\xi_1^{(1)}, \xi_2^{(1)})$, $(\xi_1^{(2)}, \xi_2^{(2)})$, $\cdots$, $(\xi_1^{(P-1)}, \xi_2^{(P-1)})$ and $(\xi_1^{(P)}, \xi_2^{(P)})$. Following closely Ang, Clements and Vahdati [1], we make the approximation

$$k(x_1, x_2) w(x_1, x_2) \simeq \sum_{p=1}^{P} \alpha^{(p)} \sigma^{(p)}(x_1, x_2), \tag{7}$$

where $\alpha^{(p)}$ is a constant coefficient and the generalised radial basis function $\sigma^{(p)}(x_1, x_2)$ centred about $(\xi_1^{(p)}, \xi_2^{(p)})$ is chosen here to be of the simple form

$$\sigma^{(p)}(x_1, x_2) = 1 + \left( [x_1 - \xi_1^{(p)} + \text{Re}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2 \right)^{1/2} \tag{8}$$

with

$$\tau = \frac{-\gamma_{12} + i\sqrt{\gamma_{11}\gamma_{22} - [\gamma_{12}]^2}}{\gamma_{22}} \quad (i = \sqrt{-1}). \tag{9}$$

Note that Re and Im respectively denote the real and the imaginary part of a complex number and $\tau$ is such that $\text{Im}\{\tau\} > 0$. For the special case in which $\gamma_{ij} = \delta_{ij}$, we find that $\tau = i$ and $\sigma^{(p)}(x_1, x_2)$ is a linear function of the distance between $(x_1, x_2)$ and $(\xi_1^{(p)}, \xi_2^{(p)})$.

We can let $(x_1, x_2)$ in (7) be given by $(\xi_1^{(m)}, \xi_2^{(m)})$ for $m = 1, 2, \cdots, P$, to set up a system of linear algebraic equations in $\alpha^{(p)}$. The system can be inverted to obtain

$$\alpha^{(p)} = \sum_{m=1}^{P} w^{(m)} k^{(m)} \chi^{(mp)}, \tag{10}$$

where $w^{(m)} = w(\xi_1^{(m)}, \xi_2^{(m)})$ and $k^{(m)} = k(\xi_1^{(m)}, \xi_2^{(m)})$ $(m = 1, 2, \cdots, P)$ and $\chi^{(mp)}$ are constants defined by

$$\sum_{m=1}^{P} \sigma^{(p)}(\xi_1^{(m)}, \xi_2^{(m)}) \chi^{(mr)} = \begin{cases} 1 & \text{if} \quad p = r, \\ 0 & \text{if} \quad p \neq r. \end{cases} \tag{11}$$

For the solution of (5), as in Dobroskok and Linkov [8], we write

$$w(x_1, x_2) = w_0(x_1, x_2) + w_1(x_1, x_2) \tag{12}$$

and choose $w_0(x_1, x_2)$ to satisfy

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} [w_0(x_1, x_2)] = k(x_1, x_2) w(x_1, x_2), \tag{13}$$

so that $w_1(x_1, x_2)$ is to be obtained by solving

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} [w_1(x_1, x_2)] = 0. \tag{14}$$

In view of (7), (8) and (10), an approximate solution of (13) may be given by

$$w_0(x_1, x_2) \simeq \sum_{m=1}^{P} B^{(m)}(x_1, x_2) w^{(m)} \tag{15}$$

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where

$$B^{(m)}(x_1, x_2) = k^{(m)} \sum_{p=1}^{P} \chi^{(mp)} \beta^{(p)}(x_1, x_2), \tag{16}$$

and

$$(\frac{\gamma_{11}\gamma_{22} - [\gamma_{12}]^2}{\gamma_{22}})\beta^{(p)}(x_1, x_2)$$
$$= \frac{1}{4}([x_1 - \xi_1^{(p)} + \text{Re}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2)$$
$$+ \frac{1}{9}([x_1 - \xi_1^{(p)} + \text{Re}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \xi_2^{(p)}\}]^2)^{3/2}. \tag{17}$$

In Clements [7], the general solution of (14) is given by

$$w_1(x_1, x_2) = \text{Re}\{F(x_1 + \tau x_2)\}, \tag{18}$$

where $F(x_1 + \tau x_2)$ is an arbitrary complex function which is holomorphic in $R$.

The boundary condition in (2) can be rewritten as

$$w(x_1, x_2) = p_1(x_1, x_2) \text{ for } (x_1, x_2) \in D_1,$$
$$\sum_{m=1}^{P} E^{(m)}(x_1, x_2)w^{(m)} + p_3(x_1, x_2) \text{Re}\{F(x_1 + \tau x_2)\}$$
$$+ \text{Re}\{\sum_{j=1}^{2} L_j F'(x_1 + \tau x_2)\}n_j(x_1, x_2)$$
$$= p_2(x_1, x_2) \text{ for } (x_1, x_2) \in D_2, \tag{19}$$

where the prime denotes differentiation with respect to the relevant argu-

ment, $L_j = \gamma_{j1} + \tau\gamma_{j2}$ and

$$
\begin{aligned}
p_1(x_1, x_2) &= \sqrt{g(x_1, x_2)} f_1(x_1, x_2), \\
p_2(x_1, x_2) &= \frac{f_2(x_1, x_2)}{\sqrt{g(x_1, x_2)}}, \\
p_3(x_1, x_2) &= -\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{2g(x_1, x_2)} \gamma_{ij} n_i(x_1, x_2) \frac{\partial}{\partial x_j}[g(x_1, x_2)], \\
E^{(m)}(x_1, x_2) &= \sum_{i=1}^{2}\sum_{j=1}^{2} \gamma_{ij} n_i(x_1, x_2) \frac{\partial}{\partial x_j}[B^{(m)}(x_1, x_2)] \\
&\quad + p_3(x_1, x_2) B^{(m)}(x_1, x_2).
\end{aligned}
\tag{20}
$$

If we can construct $F(x_1 + \tau x_2)$ which is holomorphic in $R$ and find the constants $w^{(1)}$, $w^{(2)}$, $\cdots$, $w^{(P-1)}$ and $w^{(P)}$ such that (19) is satisfied, then we have approximately solved the boundary value problem stated in Section 1 above. The required solution of the boundary value problem is then approximately given by

$$
\phi(x_1, x_2) \simeq \frac{1}{\sqrt{g(x_1, x_2)}} \left( \sum_{m=1}^{P} B^{(m)}(x_1, x_2) w^{(m)} + \mathrm{Re}\{F(x_1 + \tau x_2)\} \right). \tag{21}
$$

# 3  Numerical construction of complex function

According to the Cauchy integral formulae, for $(\xi_1, \xi_2) \in R$, we can write

$$
2\pi i F(\xi_1 + \tau\xi_2) = \oint_{(x_1, x_2) \in C} \frac{F(x_1 + \tau x_2)\, d(x_1 + \tau x_2)}{(x_1 - \xi_1 + \tau[x_2 - \xi_2])}, \tag{22}
$$

$$
2\pi i F'(\xi_1 + \tau\xi_2) = \oint_{(x_1, x_2) \in C} \frac{F(x_1 + \tau x_2)\, d(x_1 + \tau x_2)}{(x_1 - \xi_1 + \tau[x_2 - \xi_2])^2}, \tag{23}
$$

where $C$ is assigned an anticlockwise direction.

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We shall now apply (22) and (23) to devise a procedure for constructing numerically $F(x_1 + \tau x_2)$ and finding $w^{(1)}$, $w^{(2)}$, $\cdots$, $w^{(P-1)}$ and $w^{(P)}$ such that (19) is satisfied.

Put $M$ closely packed points $(x_1^{(1)}, x_2^{(1)})$, $(x_1^{(2)}, x_2^{(2)})$, $\cdots$, $(x_1^{(M-1)}, x_2^{(M-1)})$ and $(x_1^{(M)}, x_2^{(M)})$ on the curve $C$ following the anticlockwise direction. For $m = 1, 2, \cdots, M$, define $C^{(m)}$ to be the straight line segment from $(x_1^{(m)}, x_2^{(m)})$ to $(x_1^{(m+1)}, x_2^{(m+1)})$ (with $(x_1^{(M+1)}, x_2^{(M+1)}) = (x_1^{(1)}, x_2^{(1)})$). The first $M$ collocation points in (7) and (8) are chosen to be midpoints of $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(M-1)}$ and $C^{(M)}$, that is,

$$(\xi_1^{(m)}, \xi_2^{(m)}) = \frac{1}{2}(x_1^{(m)} + x_1^{(m+1)}, x_2^{(m)} + x_2^{(m+1)}) \text{ for } m = 1, 2, \cdots, M. \quad (24)$$

Another $N$ collocation points given by $(\xi_1^{(M+1)}, \xi_2^{(M+1)})$, $(\xi_1^{(M+2)}, \xi_2^{(M+2)})$, $\cdots$, $(\xi_1^{(M+N-1)}, \xi_2^{(M+N-1)})$ and $(\xi_1^{(M+N)}, \xi_2^{(M+N)})$ are chosen to lie in the interior of $R$. (Thus, the total number of collocation points is given by $P = M + N$.)

Following Park and Ang [14], we make the approximation $C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(M)}$ and discretise the Cauchy integral formula in (22) as

$$2\pi i F(z) = \sum_{k=1}^{M} (u^{(k)} + iv^{(k)}) \left[ \lambda(z^{(k)}, z^{(k+1)}, z) + i\theta(z^{(k)}, z^{(k+1)}, z) \right] \quad \text{for } z \in R, \quad (25)$$

where $z = x_1 + \tau x_2$, $z^{(m)} = x_1^{(m)} + \tau x_2^{(m)}$, $u^{(k)}$ and $v^{(k)}$ are real constants given by $u^{(k)} + iv^{(k)} = F(\xi_1^{(k)} + \tau \xi_2^{(k)})$ and $\lambda$ and $\theta$ are real parameters defined by

$$\lambda(a, b, c) = \ln|b - c| - \ln|a - c|$$

$$\theta(a, b, c) = \begin{cases} \Theta(a, b, c) & \text{if} \quad \Theta(a, b, c) \in (-\pi, \pi] \\ \Theta(a, b, c) + 2\pi & \text{if} \quad \Theta(a, b, c) \in [-2\pi, -\pi] \\ \Theta(a, b, c) - 2\pi & \text{if} \quad \Theta(a, b, c) \in (\pi, 2\pi] \end{cases}$$

$$\Theta(a, b, c) = \text{Arg}(b - c) - \text{Arg}(a - c). \quad (26)$$

Note that $\text{Arg}(z)$ denotes the principal argument of the complex number $z$. If the solution domain $R$ is convex in shape, $\theta(a, b, c)$ can be calculated directly from

$$\theta(a, b, c) = \cos^{-1}\left(\frac{|b - c|^2 + |a - c|^2 - |b - a|^2}{2\,|b - c|\,|a - c|}\right). \tag{27}$$

If we let $z \to \widehat{z}^{(k)} = \xi_1^{(k)} + \tau \xi_2^{(k)}$ (for each of the collocation points), the imaginary part of (25) gives

$$u^{(k)} = \frac{1}{2\pi} \sum_{m=1}^{M} \{u^{(m)}\theta(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)}) + v^{(m)}\lambda(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)})\}$$

$$\text{for } k = 1, 2, \cdots, M + N. \tag{28}$$

From (12), (15) and (18), we find that

$$u^{(k)} = \sum_{p=1}^{M+N} c^{(kp)}w^{(p)}, \tag{29}$$

where

$$c^{(kp)} = -B^{(p)}(\xi_1^{(k)}, \xi_2^{(k)}) + \begin{cases} 1 & \text{if } k = p \\ 0 & \text{if } k \neq p \end{cases}. \tag{30}$$

Hence, (28) can be written as

$$\sum_{p=1}^{M+N} d^{(kp)}w^{(p)} = \frac{1}{2\pi} \sum_{m=1}^{M} v^{(m)}\lambda(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)}) \text{ for } k = 1, 2, \cdots, M + N, \tag{31}$$

where

$$d^{(kp)} = c^{(kp)} - \frac{1}{2\pi} \sum_{m=1}^{M} c^{(mp)}\theta(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)}). \tag{32}$$

The first boundary condition in (19) can be written as

$$w^{(k)} = p_1(\xi_1^{(k)}, \xi_2^{(k)}) \text{ if } \phi \text{ is specified on } C^{(k)}. \tag{33}$$

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The Cauchy integral formula in (23) can be used to derive

$$\pi i F'(\widehat{z}^{(k)}) = \sum_{m=1}^{M} (u^{(m)} + iv^{(m)})[q(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)}) + ir(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)})]$$
$$\text{for } k = 1, 2, \cdots, M, \tag{34}$$

where $q$ and $r$ are real parameters defined by

$$q(a, b, c) + ir(a, b, c) = -\frac{1}{b - c} + \frac{1}{a - c}. \tag{35}$$

For further details, one may refer to Linkov and Mogilevskaya [12], Park and Ang [14] and Chen and Chen [5].

With (34), the second boundary condition in (19) can be written as

$$\sum_{p=1}^{M+N} T^{(kp)} w^{(p)} - \sum_{m=1}^{M} Q^{(km)} v^{(m)} = p_2(\xi_1^{(k)}, \xi_2^{(k)}) \text{ if } \psi \text{ is specified on } C^{(k)},$$
$$\tag{36}$$

where

$$T^{(kp)} = E^{(p)}(\xi_1^{(k)}, \xi_2^{(k)}) + p_3(\xi_1^{(k)}, \xi_2^{(k)}) c^{(kp)} + \sum_{m=1}^{M} c^{(mp)} R^{(km)}, \tag{37}$$

the real parameters $R^{(km)}$ and $Q^{(km)}$ are defined by

$$R^{(km)} + iQ^{(km)} = \frac{1}{\pi}(r(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)}) - iq(z^{(m)}, z^{(m+1)}, \widehat{z}^{(k)})) \sum_{j=1}^{2} L_j n_j^{(k)},$$
$$\tag{38}$$

and $[n_1^{(k)}, n_2^{(k)}]$ is the outward unit normal vector to $C^{(k)}$.

Note that (33) and (36) require the evaluation of the functions $p_1$ and $p_2$ at the midpoint $(\xi_1^{(k)}, \xi_2^{(k)})$ of the boundary element $C^{(k)}$. According to (20), $p_1$ and $p_2$ are expressed in terms of $f_1$ and $f_2$ given by the boundary conditions

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in (2). Now, depending on the geometry of $C$, the midpoint $(\xi_1^{(k)}, \xi_2^{(k)})$ may or may not lie on the actual physical boundary $C$. If $(\xi_1^{(k)}, \xi_2^{(k)})$ does not lie on $C$, then the value of $f_i$ ($i = 1, 2$) needed in the calculation of $p_i$ at $(\xi_1^{(k)}, \xi_2^{(k)})$ may be taken to be given by the average value of $f_i$ at the endpoints of $C^{(k)}$, as the endpoints of every boundary element are chosen to lie on $C$.
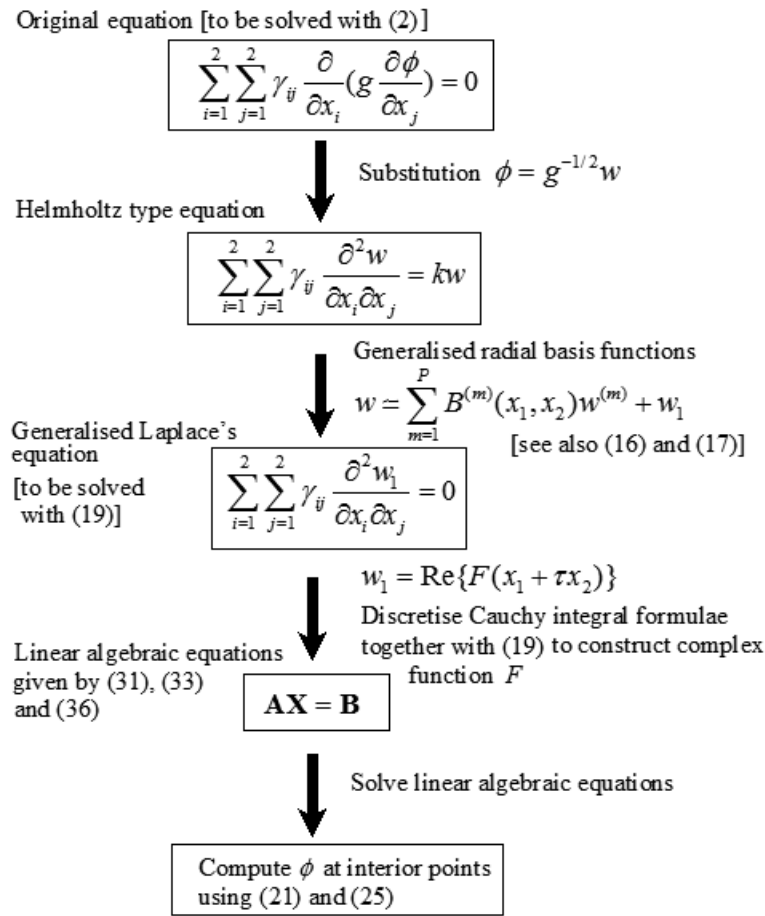
Original equation [to be solved with (2)]

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} \frac{\partial}{\partial x_i}\left(g\frac{\partial \phi}{\partial x_j}\right) = 0$$

Substitution $\phi = g^{-1/2}w$

Helmholtz type equation

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} = kw$$

Generalised radial basis functions

$$w = \sum_{m=1}^{P} B^{(m)}(x_1, x_2)w^{(m)} + w_1$$

[see also (16) and (17)]

Generalised Laplace's equation
[to be solved with (19)]

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} \frac{\partial^2 w_1}{\partial x_i \partial x_j} = 0$$

$w_1 = \mathrm{Re}\{F(x_1 + \tau x_2)\}$

Discretise Cauchy integral formulae together with (19) to construct complex function $F$

Linear algebraic equations given by (31), (33) and (36)

$$\mathbf{AX} = \mathbf{B}$$

Solve linear algebraic equations

Compute $\phi$ at interior points using (21) and (25)

**Figure 2**. A summary of the sequence of steps involved in the derivation and implementation of the numerical procedure.

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We may solve (31) for $k = 1, 2, \cdots, M + N$, together with (33), (36) and $v^{(M)} = 0$, as a system of $2M + N$ linear algebraic equations for the constants $w^{(p)}$ ($p = 1, 2, \cdots, M + N$) and $v^{(m)}$ ($m = 1, 2, \cdots, M - 1$). Note the $v^{(M)}$ is set to zero to ensure that the imaginary part of the complex function $F(x_1 + \tau x_2)$ is uniquely determined by the linear algebraic equations. Once the constants $w^{(p)}$ are found, $u^{(k)}$ can be computed from (29) and the required complex function $F(z)$ is given by (25). Note that the value of the solution $\phi$ at the collocation point $(\xi_1^{(p)}, \xi_2^{(p)})$ is given $w^{(p)}/\sqrt{g(\xi_1^{(p)}, \xi_2^{(p)})}$. The value of $\phi$ at any other point in the solution domain can be approximately calculated from (21) and (25).

For clarity, a summary of the sequence of steps involved in the derivation and implementation of the numerical procedure is given in Figure 2.

# 4   Test problems

In this section, specific test problems are solved using the complex variable boundary element method described above.

**Problem 1.** For a particular test problem, take the coefficients of the partial differential equation (1) to be given by $\gamma_{11} = 1$, $\gamma_{12} = \gamma_{21} = 1/2$, $\gamma_{22} = 1$ and $g(x_1, x_2) = 1 + x_1 + x_2$. The partial differential equation is to be solved in the square domain $0 < x_1 < 1$, $0 < x_2 < 1$ subject to the boundary condition

$$\phi(x_1, 0) = x_1 \text{ for } 0 < x_1 < 1,$$
$$\psi(0, x_2) = -\frac{1}{2}(1 + x_2) \text{ for } 0 < x_2 < 1,$$

$$\psi(1, x_2) = \frac{1}{2}(2 + x_2) \text{ for } 0 < x_2 < 1,$$

$$\psi(x_1, 1) = -\frac{1}{2}(2 + x_1) \text{ for } 0 < x_1 < 1.$$

**Table 1**. Numerical and exact values of $\phi$ at some interior points.

| $(x_1, x_2)$ | $h = 1/12$ | $h = 1/24$ | Exact |
|---|---|---|---|
| $(0.25, 0.25)$ | $0.0024$ | $0.0014$ | $0.0000$ |
| $(0.25, 0.50)$ | $-0.2485$ | $-0.2490$ | $-0.2500$ |
| $(0.25, 0.75)$ | $-0.4983$ | $-0.4989$ | $-0.5000$ |
| $(0.50, 0.25)$ | $0.2534$ | $0.2518$ | $0.2500$ |
| $(0.50, 0.50)$ | $0.0022$ | $0.0013$ | $0.0000$ |
| $(0.50, 0.75)$ | $-0.2491$ | $-0.2493$ | $-0.2500$ |
| $(0.75, 0.25)$ | $0.5034$ | $0.5020$ | $0.5000$ |
| $(0.75, 0.50)$ | $0.2535$ | $0.2519$ | $0.2500$ |
| $(0.75, 0.75)$ | $0.0024$ | $0.0010$ | $0.0000$ |
| Ave absolute error | $2.4 \times 10^{-3}$ | $1.4 \times 10^{-3}$ | $-$ |

For the purpose of obtaining some numerical results, each side of the square domain is discretised into $M_0$ elements of length $h = 1/M_0$ (so that $M = 4M_0$). The interior collocation points are chosen to be given by $(jh, kh)$ for $j = 1, 2, \cdots, M_0 - 1$ and $k = 1, 2, \cdots, M_0 - 1$ (so that $N = (M_0 - 1)^2$). Note that the distance separating any two consecutive interior collocation points lying on either a vertical or horizontal line is $h$. In Table 1, two sets of numerical values of $\phi$, as obtained using $h = 1/12$ and $h = 1/24$ (that is, using $M_0 = 12$ and $M_0 = 24$ respectively), are compared with the exact solution (given by $\phi(x_1, x_2) = x_1 - x_2$) at some interior points. For a given $h$, the average absolute error of the numerical values at the interior collocation points is given at the bottom of the table. It appears that the absolute error

is approximately halved when $h$ is decreased from $1/12$ to $1/24$. This seems to suggest that the absolute error of the numerical solution is $O(h)$, as one may expect (since the complex function in the complex variable formulation is approximated as a constant over a boundary element).

**Problem 2.** The coefficients $\gamma_{ij}$ are taken to be given by $\gamma_{11} = 1$, $\gamma_{12} = \gamma_{21} = 0$, $\gamma_{22} = 2$ and the function $g$ is given by

$$g(x_1, x_2) = (1 + 2x_2)^2 + \frac{1}{10}\sin(\pi x_2).$$

As in Problem 1 above, the solution domain is taken to be the square region $0 < x_1 < 1$, $0 < x_2 < 1$. The governing partial differential equation is to be solved in the square domain subject to the boundary conditions

$$\phi(x_1, 0) = 0 \text{ for } 0 < x_1 < 1,$$
$$\psi(0, x_2) = 0 \text{ for } 0 < x_2 < 1,$$
$$\psi(1, x_2) = 0 \text{ for } 0 < x_2 < 1,$$
$$\psi(x_1, 1) = 1 \text{ for } 0 < x_1 < 1.$$

The same test problem is solved in Ang, Clements and Vahdati [1] using a real variable dual-reciprocity boundary element method. In Figure 3, the numerical solution $\phi$ along the line $x_1 = 0.50$, as computed by using the complex variable boundary element approach (CVBEM) here with 160 equal length boundary elements and 361 interior collocation points, is plotted against $x_2$ (for $0.10 \le x_2 \le 0.90$). As shown graphically, the numerical values of $\phi(0.50, x_2)$ obtained here are in good agreement with those reported in [1].
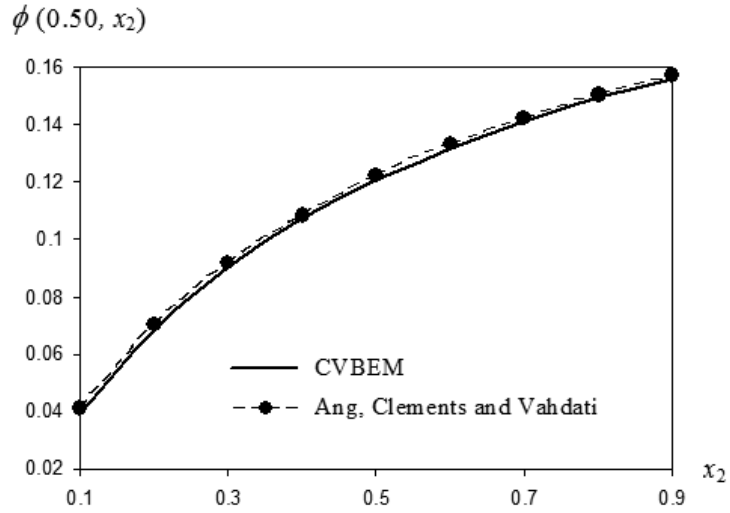
**Figure 3.** Plots of $\phi(0.50, x_2)$ against $x_2$ (for $0.10 \leq x_2 \leq 0.90$).

**Problem 3.** The coefficients $\gamma_{ij}$ and the function $g$ are taken to be given by $\gamma_{ij} = \delta_{ij}$ (kronecker-delta) and $g(x_1, x_2) = x_1^2 + x_2^2$. The solution domain is defined by $1 \leq x_1^2 + x_2^2 \leq 4$, $x_1 \geq 0$, $x_2 \geq 0$. The partial differential equation is to be solved subject to the boundary conditions

$$\psi = 0 \text{ on } x_1 = 0 \text{ for } 1 \leq x_2 \leq 2,$$

$$\psi = 0 \text{ on } x_2 = 0 \text{ for } 1 \leq x_1 \leq 2,$$

$$\phi = 0 \text{ on } x_1^2 + x_2^2 = 1 \text{ for } x_1 \geq 0 \text{ and } x_2 \geq 0,$$

$$\phi = 1 \text{ on } x_1^2 + x_2^2 = 4 \text{ for } x_1 \geq 0 \text{ and } x_2 \geq 0.$$

It is easy to verify that the exact solution for this test problem is given by

$$\phi(x_1, x_2) = \frac{4(x_1^2 + x_2^2 - 1)}{3(x_1^2 + x_2^2)}.$$

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The part of the boundary along the $x_1$ axis, that is, $1 \leq x_2 \leq 2$, $x_1 = 0$, is discretised into $J_0$ boundary elements, each of length $1/J_0$. So is the part $1 \leq x_1 \leq 2$, $x_2 = 0$. The part $x_1^2 + x_2^2 = 1$, $x_1 \geq 0$, $x_2 \geq 0$, is discretised into $2J_0$ equal length straight line elements, while $x_1^2 + x_2^2 = 4$, $x_1 \geq 0$, $x_2 \geq 0$, into $4J_0$ equal length elements. The total number of boundary elements is given by $M = 8J_0$. The interior collocations are chosen to be given by $([1 + k/(J_0 + 1)]\cos(p\pi/(2[J_0 + 1])), [1 + k/(J_0 + 1)]\sin(p\pi/(2[J_0 + 1])))$ for $k = 1, 2, \cdots, J_0$ and $p = 1, 2, \cdots, J_0$, that is, the number of interior collocation points is given by $N = J_0^2$.

**Table 2**. Numerical and exact values of $\phi$ at selected interior points.

| $(x_1, x_2)$ | $J_0 = 5$ | $J_0 = 11$ | $J_0 = 23$ | Exact |
|---|---|---|---|---|
| $(0.8250, 08250)$ | 0.3080 | 0.3424 | 0.3487 | 0.3537 |
| $(0.9428, 0.9428)$ | 0.5685 | 0.5773 | 0.5804 | 0.5833 |
| $(1.0607, 1.0607)$ | 0.7360 | 0.7386 | 0.7397 | 0.7407 |
| $(1.1785, 1.1785)$ | 0.8553 | 0.8542 | 0.8537 | 0.8533 |
| $(1.2964, 1.2964)$ | 0.9370 | 0.9396 | 0.9380 | 0.9366 |
| Ave absolute error | $1.4 \times 10^{-2}$ | $4.7 \times 10^{-3}$ | $2.2 \times 10^{-3}$ | − |

Table 2 compares the numerical values of $\phi$ obtained using $J_0 = 5$, $J_0 = 11$ and $J_0 = 23$ with the exact solution at five interior collocation points along the line $x_1 = x_2$. (For convenience and clarity, the numerical values of $\phi$ are given in Table 2 at only five interior collocation points.) For each $J_0$, the average absolute error of $\phi$ at those interior points is given in the last row of the table. There is an obvious reduction in the average absolute error when more boundary elements and interior collocation points are used in the calculation. Specifically, the average absolute error for $J_0 = 11$ is about

twice larger than that for $J_0 = 23$. The same observation applies too for average absolute error which is calculated using the numerical values at all the interior collocation points (instead of just the five points given in Table 2).

# 5   Summary

The numerical solution of a two-dimensional boundary value problem governed by a second order elliptic partial differential equation with variable coefficients is considered here. With an appropriate substitution and the use of generalised radial basis functions, the boundary value problem is reformulated as a problem which requires the construction of a holomorphic complex function. The Cauchy integral formulae are used to reduce the numerical construction of the holomorphic function to solving a system of linear algebraic equations. The numerical procedure does not require the solution domain to be divided into small elements. Only the boundary is discretised into straight line elements. To test its validity, the proposed complex variable boundary element procedure is applied to solve some specific test problems. The obtained numerical solutions agree favourably with known solutions. Convergence in the numerical values obtained is observed when the number of boundary elements and interior collocation points is increased.

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