A note on the CVBEM for the two-dimensional Helmholtz equation or its modified form

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Abstract

A complex variable boundary element method is outlined for the numerical solution of the two-dimensional Helmholtz equation or its modified form in a simply connected region subject to suitable boundary conditions. It is applied to solve a specific test problem.

Keywords: complex variable boundary element method, Helmholtz equation.

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1 INTRODUCTION

In the present note, we outline a complex variable boundary element method (CVBEM) for the numerical solution of the two-dimensional Helmholtz equation or its modified form as given by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \varkappa\phi = 0 \text{ in } R \tag{1}$$

subject to the boundary condition

$$\begin{aligned} \phi &= f(x,y) \text{ for } (x,y) \in C_1 \\ \frac{\partial\phi}{\partial n} &= g(x,y) \text{ for } (x,y) \in C_2 \end{aligned} \tag{2}$$

where f and g are suitably prescribed functions, $\varkappa$ is a given real non-zero constant, R is a two-dimensional region bounded by a simple closed curve C, C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$, $\partial\phi/\partial n = n_x\partial\phi/\partial x + n_y\partial\phi/\partial y$ and $[n_x, n_y]$ is the unit normal vector to C pointing away from R.

Hromadka and Lai [5] were among the earliest researchers to apply the Cauchy integral formula to derive a CVBEM for solving numerically potential problems governed by the two-dimensional Laplace equation. Recently, Ang and Park [1] proposed a version of the CVBEM for the numerical solution of a general system of elliptic partial differential equations in two-dimensional space. Park and Ang [7] extended the work to include an elliptic partial differential equation with variable coefficients. The approach in [1] and [7] differs from that in [5] in the treatment of the flux boundary conditions. The flux boundary conditions were treated using a differentiated form of the Cauchy integral formula in [1] and [7]. The same CVBEM approach was also independently introduced by Chen and Chen [3] for two-dimensional potential problems with or without degenerate boundaries.

A CVBEM formulation for the numerical solution of the boundary value problem (BVP) defined by (1)-(2) can be recovered from the analysis in Park and Ang [7]. It may be worthwhile to extract and write out explicitly this formulation as the Helmholtz equation or its modified form is an important partial differential equation in engineering science. To assess the validity of the formulation, it is applied to solve a specific test problem.

2 COMPLEX FORMULATION

Following the analysis in Clements [4], we obtain a general solution of (1) in terms of a complex function $\Phi(z)$ ($z = x + iy$, $i = \sqrt{-1}$) which is analytic in R as follows:

$$\phi(x,y) = \mathrm{Re}\left\{\Phi(z) + \sum_{m=1}^{\infty} m h_m(x) \int_a^z (z-t)^{m-1}\Phi(t)dt\right\}, \tag{3}$$

where $h_m(x)$ is the m-th order polynomial function of x defined by

$$\begin{aligned} h_m(x) &= \sum_{r=1}^{m} \beta_m^{(r)} x^r, \\ \beta_m^{(r)} &= -\frac{\varkappa}{2mr}\beta_{m-1}^{(r-1)} \text{ for } m \geq 1 \text{ and } r = 1, 2, \cdots, m, \end{aligned} \tag{4}$$

with $\beta_0^{(0)} = 1$ and $\beta_m^{(0)} = 0$ for $m \geq 1$.

From the recurrence relation for $\beta_m^{(r)}$ in (4), notice that the terms in the infinite series in (3) decay as the reciprocal of a factorial expression in m as m increases. Intuitively, this may seem to suggest that the series converges. A more rigorous investigation showing that the series converges uniformly may be carried out after the manner of Bergman [2]. This is beyond the scope of the present note, however.

With (3), the BVP defined by (1)-(2) can now reformulated as a problem which requires us to construct a complex function Φ which is analytic in $R \cup C$ and such that

$$\begin{aligned} &\mathrm{Re}\left\{\Phi(z) + \sum_{m=1}^{\infty} m h_m(x) \int_a^z (z-t)^{m-1}\Phi(t)dt\right\} \\ &= f(x,y) \text{ for } (x,y) \in C_1, \end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\mathrm{Re}\{(n_x + in_y)[\Phi'(z) + h_1(x)\Phi(z) \\
&+ \sum_{m=2}^{\infty} m(m-1)h_m(x)\int_a^z (z-t)^{m-2}\Phi(t)dt] \\
&+ n_x \sum_{m=1}^{\infty} m h_m'(x)\int_a^z (z-t)^{m-1}\Phi(t)dt\} \\
&= g(x,y) \text{ for } (x,y) \in C_2.
\end{aligned} \tag{6}$$

3 CVBEM

To construct approximately $\Phi(z)$ which is analytic in R and satisfy (5)-(6), we follow closely the analysis in Park and Ang [7].

For $c \in R$, the Cauchy integral formula and its differentiated form are given by

$$2\pi i\Phi(c) = \oint\limits_C \frac{\Phi(z)dz}{z-c} \text{ and } 2\pi i\Phi'(c) = \oint\limits_C \frac{\Phi(z)dz}{[z-c]^2}, \tag{7}$$

where C is assigned an anticlockwise direction.

To discretise the curve boundary C, M well-spaced out points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)}), \cdots, (x^{(M-1)}, y^{(M-1)})$ and $(x^{(M)}, y^{(M)})$ are placed on it in an anti-clockwise order. Denote the directed straight line segment from $(x^{(k)}, y^{(k)})$ to $(x^{(k+1)}, y^{(k+1)})$ by $C^{(k)}$ $(k = 1, 2, \cdots, M)$. [We define $(x^{(M+1)}, y^{(M+1)}) = (x^{(1)}, y^{(1)})$.] We make the approximation:

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(M-1)} \cup C^{(M)}. \tag{8}$$

Proceeding as in Park and Ang [7], we find that the real part of (7) approximately gives rise to:

$$\begin{aligned}
&\sum_{m=1}^{M} \big\{ U^{(m)}\gamma(z^{(m)}, z^{(m+1)}, \widehat{z}^{(p)}) \\
&-V^{(m)} \left[\theta(z^{(m)}, z^{(m+1)}, \widehat{z}^{(p)}) - 2\pi\delta_{pm}\right] \big\} = 0 \text{ for } p = 1, 2, \cdots, M,
\end{aligned} \tag{9}$$

where δ_{pm} is the Kronecker-delta, $z^{(m)} = x^{(m)} + iy^{(m)}$, $\widehat{z}^{(p)} = [z^{(p)} + z^{(p+1)}]/2$ is the midpoint of the line segment $C^{(p)}$, $U^{(p)}$ and $V^{(p)}$ are real constants such that $U^{(p)} + iV^{(p)} = \Phi(\widehat{z}^{(p)})$ and

$$\begin{aligned}
\gamma(a,b,c) &= \ln|b-c| - \ln|a-c| \\
\theta(a,b,c) &= \begin{cases} \Theta(a,b,c) & \text{if} \quad \Theta(a,b,c) \in [-\pi,\pi] \\ \Theta(a,b,c) + 2\pi & \text{if} \quad \Theta(a,b,c) \in [-2\pi,-\pi) \\ \Theta(a,b,c) - 2\pi & \text{if} \quad \Theta(a,b,c) \in (\pi, 2\pi] \end{cases} \\
\Theta(a,b,c) &= \operatorname{Arg}(b-c) - \operatorname{Arg}(a-c),
\end{aligned} \tag{10}$$

where $\operatorname{Arg}(z)$ denotes the principal argument of the complex number z.

In (9) is a system of M linear algebraic equations containing $2M$ unknowns $U^{(m)}$ and $V^{(m)}$ $(m = 1, 2, \cdots, M)$. Another M equations are needed to complete the system. These come from the boundary conditions (5)-(6) as:

$$\begin{aligned}
&U^{(p)} + \sum_{r=1}^{p} \left(\Gamma^{(pr)} U^{(r)} - \Psi^{(pr)} V^{(r)} \right) \\
&= f(\widehat{x}^{(p)}, \widehat{y}^{(p)}) \text{ if } \phi \text{ is specified on } C^{(p)},
\end{aligned} \tag{11}$$

or

$$\begin{aligned}
&\frac{1}{\pi} \sum_{r=1}^{M} \{ \, [w(z^{(r)}, z^{(r+1)}, \widehat{z}^{(p)}) n_x^{(p)} + q(z^{(r)}, z^{(r+1)}, \widehat{z}^{(p)}) n_y^{(p)}] U^{(r)} \\
&+ [q(z^{(r)}, z^{(r+1)}, \widehat{z}^{(p)}) n_x^{(p)} - w(z^{(r)}, z^{(r+1)}, \widehat{z}^{(p)}) n_y^{(p)}] V^{(r)} \, \} \\
&+ [n_x^{(p)} R_1^{(p)} - n_y^{(p)} S_1^{(p)}] U^{(p)} - [n_x^{(p)} S_1^{(p)} + n_y^{(p)} R_1^{(p)}] V^{(p)} \\
&+ \sum_{r=1}^{p} \left(\widetilde{\Gamma}^{(pr)} U^{(r)} - \widetilde{\Psi}^{(pr)} V^{(r)} \right) \\
&= g(\widehat{x}^{(p)}, \widehat{y}^{(p)}) \text{ if } \partial\phi/\partial n \text{ is specified on } C^{(p)},
\end{aligned} \tag{12}$$

where $\widehat{x}^{(p)} = (x^{(p)} + x^{(p+1)})/2$, $\widehat{y}^{(p)} = (y^{(p)} + y^{(p+1)})/2$, $R_m^{(p)}$, $S_m^{(p)}$, $\Gamma^{(pr)}$, $\Psi^{(pr)}$,

$\widetilde{\Gamma}^{(pr)}$, $\widetilde{\Psi}^{(pr)}$, $\widetilde{R}_m^{(p)}$, $\widetilde{S}_m^{(p)}$, q and w are real parameters defined by

$$\Gamma^{(pr)} + i\Psi^{(pr)} = \sum_{m=1}^{\infty} m(R_m^{(p)} + iS_m^{(p)})\Lambda_m^{(pr)},$$

$$\widetilde{\Gamma}^{(pr)} + i\widetilde{\Psi}^{(pr)} = (n_x^{(p)} + in_y^{(p)})\sum_{m=2}^{\infty} m(m-1)(R_m^{(p)} + iS_m^{(p)})\Lambda_{m-1}^{(pr)} + \sum_{m=1}^{\infty} mn_x^{(p)}(\widetilde{R}_m^{(p)} + i\widetilde{S}_m^{(p)})\Lambda_m^{(pr)},$$

$$\Lambda_m^{(pr)} = (1-\delta_{rp})\int_{z^{(r)}}^{z^{(r+1)}} [\widehat{z}^{(p)} - t]^{m-1}dt + \delta_{rp}\int_{z^{(r)}}^{\widehat{z}^{(r)}} [\widehat{z}^{(p)} - t]^{m-1}dt,$$

$$R_m^{(p)} + iS_m^{(p)} = h_m(\widehat{x}^{(p)}), \quad \widetilde{R}_m^{(p)} + i\widetilde{S}_m^{(p)} = h_m'(\widehat{x}^{(p)}),$$

$$q(a,b,c) + iw(a,b,c) = -\frac{1}{b-c} + \frac{1}{a-c}. \tag{13}$$

In deriving (12) from (6), we use the differentiated form of the Cauchy integral formula to express $\Phi'(\widehat{z}^{(p)})$ in terms of $U^{(r)}$ and $V^{(r)}$ ($r = 1, 2, \cdots, M$) as described in references [1], [3] and [7].

Also, in deriving (11) and (12), we have to deal with the integrals

$$\int_a^{\widehat{z}_p} (\widehat{z}_p - t)^{m-1}\Phi(t)dt \quad \text{for } m = 1, 2, \cdots. \tag{14}$$

As long as the path of integration lies in the region $R \cup C$, these integrals are path-independent. We are free to choose any path from a to $\widehat{z}_p$. We take $a = z^{(1)}$ (i.e. one of the endpoints of the line segment $C^{(1)}$) and integrate along the line segments $C^{(1)}$, $C^{(2)}, \cdots, C^{(p-2)}$, $C^{(p-1)}$ and $\widehat{C}^{(p)}$, where $\widehat{C}^{(p)}$ denotes the line segment from $(x^{(p)}, y^{(p)})$ to $(\widehat{x}^{(p)}, \widehat{y}^{(p)})$ (i.e. the first half of the line segment $C^{(p)}$). Furthermore, over $C^{(1)}$, $C^{(2)}, \cdots, C^{(p-2)}$, $C^{(p-1)}$ and $\widehat{C}^{(p)}$, we approximate $\Phi(t)$ as constants $U^{(1)} + iV^{(1)}$, $U^{(2)} + iV^{(2)}, \cdots$, $U^{(p-2)} + iV^{(p-2)}$, $U^{(p-1)} + iV^{(p-1)}$ and $U^{(p)} + iV^{(p)}$ respectively. Thus, in the derivation of (11) and (12), we use the approximation

$$\int_a^{\widehat{z}_p} (\widehat{z}_p - t)^{m-1} \Phi(t) dt$$
$$\simeq \sum_{r=1}^{p} \left(U^{(p)} + iV^{(p)}\right) \left\{ (1 - \delta_{rp}) \int_{z^{(r)}}^{z^{(r+1)}} [\widehat{z}^{(p)} - t]^{m-1} dt \right.$$
$$\left. + \delta_{rp} \int_{z^{(r)}}^{\widehat{z}^{(r)}} [\widehat{z}^{(p)} - t]^{m-1} dt \right\}. \tag{15}$$

Once the $2M$ unknown constants $U^{(r)}$ and $V^{(r)}$ $(r = 1, 2, \cdots, M)$ are determined from (9) together with (11) and/or (12), the solution of the BVP under consideration can be computed approximately at any point (ξ, η) in the interior of the solution domain using

$$\phi(\xi, \eta) = \frac{1}{2\pi i} \operatorname{Re} \left\{ \sum_{p=1}^{M} [U^{(p)} + iV^{(p)}] \right.$$
$$\times [\gamma(z^{(p)}, z^{(p+1)}, \xi + i\eta) + i\theta(z^{(p)}, z^{(p+1)}, \xi + i\eta)]$$
$$+ \sum_{m=1}^{\infty} m h_m(\xi) \sum_{p=1}^{M} [\gamma(z^{(p)}, z^{(p+1)}, \xi + i\eta) + i\theta(z^{(p)}, z^{(p+1)}, \xi + i\eta)]$$
$$\left. \times \sum_{k=1}^{p} (U^{(k)} + iV^{(k)}) \Lambda_m^{(pk)} \right\}. \tag{16}$$

4 A TEST PROBLEM

For a test problem, let us take R to be the region inside the circle $x^2 + y^2 = 1$.

If $\varkappa = -\omega^2$, a particular solution of (1) that is valid everywhere inside R including the circular boundary is

$$\phi(x, y) = K_0 \left(\omega \sqrt{[x - \frac{3}{2}]^2 + y^2} \right), \tag{17}$$

where K_0 is the modified Bessel function of the second kind and of order zero.

We apply the CVBEM to solve numerically the partial differential equation (1) with $\varkappa = -\omega^2$ in the circular region R subject to ϕ given on the upper half of the circle $x^2 + y^2 = 1$ and $\partial\phi/\partial n$ on the remaining circle. The values of ϕ on the upper half of the circle and $\partial\phi/\partial n$ on the remaining circle are generated using (17). If the CVBEM really works, it should recover the solution (17) approximately at any point (x, y) in the interior of R.

For $\omega = 1$, approximating the circular boundary by a M-sided regular polygon, we solve the resulting system of linear algebraic equations for the unknowns $U^{(k)}$ and $V^{(k)}$ $(k = 1, 2, \cdots, M)$ and use (16) to compute ϕ approximately at various selected interior points of R. The numerical values of ϕ obtained using $M = 20$, $M = 100$ and $M = 500$ are compared with the exact solution (17) in Table 1. Even for a relatively coarse discretization of the circular boundary using $M = 20$ (i.e. using boundary elements each of length 0.31 units), the numerical values of ϕ are reasonably accurate. From the table, it is obvious the numerical ϕ converges to the correct exact solution as M increases from 20 to 500.

Table 1. Numerical results for $\phi(x, y)$ at selected points (x, y) in the interior of the solution domain.

(x, y)	$M = 20$	$M = 100$	$M = 500$	Exact
$(0.50, 0.50)$	0.3437	0.3547	0.3562	0.3565
$(0.10, -0.60)$	0.2108	0.2084	0.2076	0.2075
$(0.98, 0.0)$	0.8146	0.8755	0.8888	0.8921
$(-0.60, 0.70)$	0.1066	0.09053	0.08829	0.08781
$(0.0, 0.80)$	0.1706	0.1662	0.1656	0.1655

Notice that in implementing the CVBEM we truncate the infinite series in (13) and (16) by replacing ∞ with a finite positive integer denoted by N_∞. One way of carrying out the truncation is simply to add up the well-ordered terms in the series one by one until there is no significant contribution to the infinite sum. The numerical results in Table 1 are obtained by using N_∞ not more than 10. We have experimented with solving the test problem for different values of ω. In general, as expected, we find that if ω has a higher magnitude it is necessary to use a larger number of boundary elements in the computation to achieve a given level of accuracy in the numerical results. The proposed CVBEM resembles the multiple reciprocity BEM (MRBEM) of Nowak [6] for solving the Helmholtz equation in two ways. Firstly, the MRBEM formulation also contains an infinite series which needs to be truncated for practical implementation. Secondly, the MRBEM also requires the discretisation of the boundary to be extremely fine when $\varkappa$ in (1) possesses a very high magnitude.

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