**A potential problem with a non-classical boundary condition**

Recently, Wu Hao, a researcher at the Institute for Solid State Physics in the University of Tokyo, told me he is interested in applying the BEM to solve a boundary value problem which goes something like this.

> "Solve
>
> $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \text{ in } R \text{ bounded by a curve } C, \tag{1}$$
>
> subject to
>
> $$\frac{\partial\phi}{\partial n}=G(x,y,\phi-\frac{\partial\phi}{\partial t}) \text{ for } (x,y)\in C, \tag{2}$$
>
> where $G$ is an expression containing $x$, $y$ and $\phi-\frac{\partial\phi}{\partial t}$, $\frac{\partial\phi}{\partial t}=\underline{\mathbf{t}}\cdot\nabla\phi$ and $\underline{\mathbf{t}}$ is the unit tangential vector on $C$ in the anticlockwise direction.
>
> In addition to (2), the value of $\phi-\frac{\partial\phi}{\partial t}$ can be computed at every point $(x,y)$ on $C$, that is,
>
> $$\phi-\frac{\partial\phi}{\partial t}=f(x,y) \text{ for } (x,y)\in C, \tag{3}$$
>
> where $f(x,y)$ is a suitably given function."

Actually, in the original problem stated by Hao Wu, $G(x,y,\phi-\frac{\partial\phi}{\partial t})$ is specifically given by something like $\sin(\phi-\frac{\partial\phi}{\partial t})$ but we generalise his problem here (so that later on we can easily create a test problem to check if the BEM procedure works).

One good way to proceed is to use the BEM to solve first (1) subject to

$$\frac{\partial\phi}{\partial n}=G(x,y,f(x,y)) \text{ for } (x,y)\in C. \tag{4}$$

The problem defined by (1) and (4) does not have a unique solution. If the BEM delivers us a particular solution denoted by $\phi_0(x,y)$, then $\phi_0(x,y)+B$ (with $B$ which is an arbitrary constant) is also a solution of (1) and (4). The constant $B$ for the problem defined by (1)-(3) can be determined in the BEM post-processing stage by using (3) as follows.

On the $i$-th element $C^{(i)}$, estimate the value of $\frac{\partial \phi_0}{\partial t}$ (from the BEM solution $\phi_0$) at the midpoint $(\overline{x}^{(i)}, \overline{y}^{(i)})$ of the element. The value of $B$ estimated from data on $C^{(i)}$ is given by

$$B^{(i)} = f(\overline{x}^{(i)}, \overline{y}^{(i)}) - \phi_0(\overline{x}^{(i)}, \overline{y}^{(i)}) + \left.\frac{\partial \phi_0}{\partial t}\right|_{(x,y)=(\overline{x}^{(i)}, \overline{y}^{(i)})}.$$

The constant $B$ can then be obtaining by averaging the values of $B^{(i)}$ over all the elements.

Now since we have to estimate the value of $\frac{\partial \phi_0}{\partial t}$ over an element, it may be better to use linear elements. If the discontinuous linear elements in Chapter 2 of the book "A Beginner's Course in Boundary Element Methods" are used then (in the notations of the book)

$$\begin{aligned}
f(\overline{x}^{(i)}, \overline{y}^{(i)}) &\simeq \frac{f(x^{(i)}, y^{(i)}) + f(^{(i+1)}, y^{(i+1)})}{2} \\
\phi_0(\overline{x}^{(i)}, \overline{y}^{(i)}) &\simeq \frac{\widehat{\phi}^{(N+i)} + \widehat{\phi}^{(i)}}{2} \\
\left.\frac{\partial \phi_0}{\partial t}\right|_{(x,y)=(\overline{x}^{(i)}, \overline{y}^{(i)})} &\simeq \frac{\widehat{\phi}^{(N+i)} - \widehat{\phi}^{(i)}}{\sqrt{(\xi^{(N+i)} - \xi^{(i)})^2 + (\eta^{(N+i)} - \eta^{(i)})^2}}.
\end{aligned}$$

A test problem

For a test problem, take $C$ to be the circle $x^2 + y^2 = 1$, $G(x, y, \phi - \frac{\partial \phi}{\partial t}) = 2(\phi - \frac{\partial \phi}{\partial t}) - 8xy$, and $f(x, y) = x^2 - y^2 + 4xy$. The exact solution for this problem is $\phi = x^2 - y^2$.

We use the subroutine `DLELAP1` (Chapter 2) to work out a particular solution for (1) and (4) and estimate the constant $B$ as explained above. Together with `DLELAP2`, the estimated value of $B$ is then used to compute the solution of (1)-(3) at selected points inside the solution domain $x^2 + y^2 < 1$.The complete main programme is in the file `haowu.txt`. Numerical results using $N = 100$ are compared with the exact solution in the table below at some selected interior points.

| $(x, y)$ | Numerical | Exact |
|---|---|---|
| $(0.1, 0.4)$ | $-0.1500$ | $-0.1500$ |
| $(0.9, 0.3)$ | $0.7186$ | $0.7200$ |
| $(-0.2, 0.7)$ | $-0.4492$ | $-0.4500$ |
| $(-0.6, -0.6)$ | $-0.0011$ | $0.0000$ |
| $(0.8, -0.2)$ | $0.6000$ | $0.6000$ |

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