A Green's function for steady-state two-dimensional isotropic heat conduction across a homogeneously imperfect interface

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Abstract

The problem of calculating the steady-state two-dimensional temperature field in a thermally isotropic bimaterial which has a homogeneously imperfect interface is considered. There is a temperature jump across the imperfect interface. To devise a boundary element method, which does not require the imperfect interface to be discretized, for the numerical solution of the problem, a special Green's function satisfying the appropriate interface condition is derived. A specific test problem is solved numerically using the boundary element method.

Key words: Green's function, imperfect interface, boundary integral method, bimaterial.

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1 Introduction

Many studies on layered media, such as those in Ang [1], Berger [2], Berger and Karageorghis [3] and Clements [4], assume that the dissimilar materials are perfectly bonded at their interfaces. From a microscopic standpoint, such perfect or ideal interfaces do not exist, however.

In recent years, there is a growing interest among researchers in the analyses of microscopically imperfect interfaces in layered and composite materials (see e.g. Benveniste [5], Fan and Sze [6], Fan and Wang [7] and other references therein). In the context of a heat conduction problem, a macroscopic model for studying such an imperfect interface allows for a temperature jump which is proportional in magnitude to the thermal heat flux at the interface. For the perfect interface, the temperature jump across the interface is taken to be zero.

In the present paper, we consider the two-dimensional problem of determining the steady-state temperature distribution in a thermally isotropic bimaterial with a homogeneously imperfect planar interface. The bimaterial is of a finite extent. Either the temperature or the heat flux (not both) is known at each and every point on the exterior boundary of the bimaterial.

A Green's function which satisfies the appropriate interface condition is derived. Using the Green's function, we obtain a solution for the governing partial differential equation, expressed in terms of an integral taken over only the exterior boundary of the bimaterial. The integral solution enables us to devise a boundary element method, which does not require the discretization of the imperfect interface, for the numerical solution of the problem under consideration.

To check the validity of the method, it is applied to solve a specific test problem with a known solution. From the numerical results obtained, it

appears that, with the use of the special Green's function, the temperature may be computed accurately at points very close to the interface, even if the discretization of the exterior boundary is relatively coarse. The advantages of using special Green's functions in boundary element formulations, such as an improved accuracy in the numerical results and a smaller number of boundary elements (hence unknowns) involved, are well known (see e.g. Berger [2], Clements [8], Clements and Haselgrove [9] and Cruse *et al.* [10]).

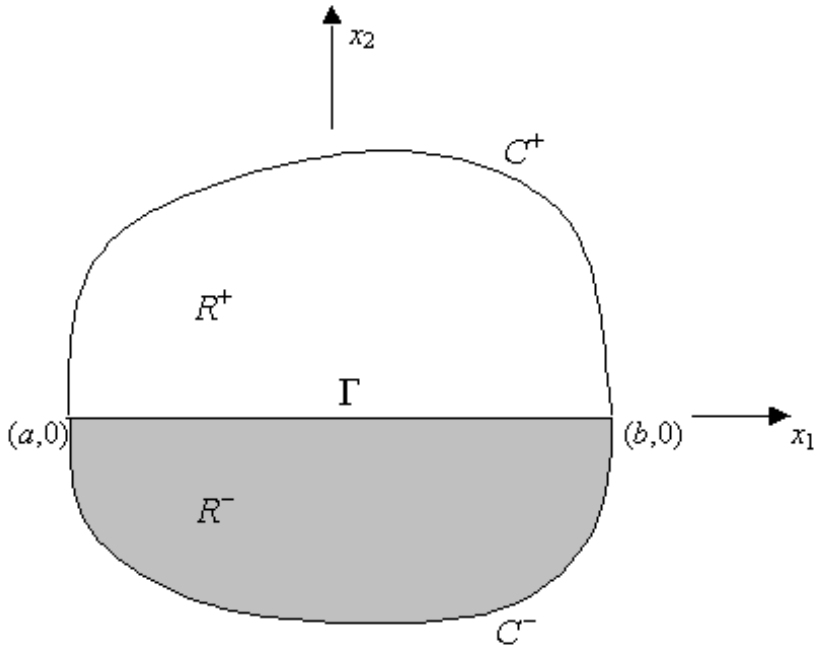

Figure 1. A sketch of the geometry of the problem.

2 The problem

With reference to an $0x_1x_2x_3$ Cartesian co-ordinate system, consider a finite body which is made up of two homogeneous materials having possibly different thermal properties. The geometry of the body does not vary along the

x_3-direction. On the $0x_1x_2$ plane, the materials are joined together along the straight line segment Γ which lies on part of the x_1-axis between the points $(a, 0)$ and $(b, 0)$ (where a and b are given real numbers such that $a < b$) and the exterior boundary of the body is the simple closed curve C. The curve C consists of two parts, namely C^+ which lies above the x_1-axis and C^- below the axis. A sketch of the geometry is given in Figure 1. The regions enclosed by $C^+ \cup \Gamma$ and $C^- \cup \Gamma$ are denoted by R^+ and R^- respectively.

If the steady-state temperature field in the body is independent of x_3 and given by $T(x_1, x_2)$, then together with the classical Fourier's law of heat conduction the energy equation gives rise to the two-dimensional Laplace's equation

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} = 0 \text{ in } R^{\pm} \cup C^{\pm} \cup \Gamma. \tag{1}$$

The bond between the materials in R^+ and R^- at the interface Γ is microscopically damaged. A macroscopic model for the heat conduction across the imperfect interface is given by

$$k^+ \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0^+} = k^- \left.\frac{\partial T}{\partial x_2}\right|_{x_2=0^-} = \lambda \Delta T(x_1) \quad \text{for } x_1 \in [a, b], \tag{2}$$

where k^+ and k^- are the (constant) thermal conductivities of the materials in R^+ and R^- respectively, λ is a given positive coefficient and $\Delta T(x_1) = T(x_1, 0^+) - T(x_1, 0^-)$. (Notice that according to (2) the x_2-component of the heat flux at the interface is given by $-\lambda \Delta T$. Thus, if $\Delta T < 0$, i.e. if $T(x_1, 0^+) < T(x_1, 0^-)$, one should find that $-\lambda \Delta T > 0$, since the direction of heat flow is always from high to low temperature. This implies that $\lambda > 0$. Similarly, one also finds that $\lambda > 0$ if $\Delta T > 0$.)

In the present paper, we assume that the coefficient λ is a constant, i.e. the microscopic imperfection or damage is uniformly distributed over the interface.

If we let $\lambda \to \infty$, the temperature jump given in (2) reduces to

$$\Delta T(x_1) = 0 \text{ for } x_1 \in [a, b], \tag{3}$$

that is, the condition for the perfect interface is recovered.

At each and every point on the exterior boundary $C = C^+ \cup C^-$, either the temperature T or its normal flux $-k\partial T/\partial n$ (but not both) is specified. (Note that k denotes the thermal conductivity and $\partial T/\partial n = \mathbf{n} \cdot \nabla T$, where $\mathbf{n}$ is the unit normal vector to C pointing out of the region R enclosed by C and $\nabla T = [\partial T/\partial x_1, \partial T/\partial x_2]$.) The problem is then to determine the temperature in the body by solving (1) subject to the boundary condition on C and the interface condition as given by (2).

3 Boundary integral formulation

A solution of (1), expressed in terms of a line integral over only the exterior boundary $C = C^+ \cup C^-$, can be obtained as follows.

For a general point (ξ_1, ξ_2) such that $\xi_2 \neq 0$, following the analysis in Clements [11], one can use (1) to derive the integral equations

$$\begin{aligned}
\gamma^+(\xi_1, \xi_2) T(\xi_1, \xi_2) = \int_{C^+} & [T(x_1, x_2) k^+ \frac{\partial}{\partial n} \Phi(x_1, x_2, \xi_1, \xi_2) \\
& - \Phi(x_1, x_2, \xi_1, \xi_2) k^+ \frac{\partial}{\partial n} T(x_1, x_2)] ds(x_1, x_2) \\
& + \int_a^b [-T(x_1, 0^+) k^+ \frac{\partial}{\partial x_2} \Phi(x_1, x_2, \xi_1, \xi_2) \Big|_{x_2 = 0^+} \\
& + \Phi(x_1, 0^+, \xi_1, \xi_2) k^+ \frac{\partial}{\partial x_2} T(x_1, x_2) \Big|_{x_2 = 0^+}] dx_1
\end{aligned} \tag{4}$$

and

$$\gamma^-(\xi_1,\xi_2)T(\xi_1,\xi_2) = \int\limits_{C^-} [T(x_1,x_2)k^- \frac{\partial}{\partial n}\Phi(x_1,x_2,\xi_1,\xi_2) - \Phi(x_1,x_2,\xi_1,\xi_2)k^- \frac{\partial}{\partial n}T(x_1,x_2)]ds(x_1,x_2) + \int\limits_a^b [T(x_1,0^-)k^- \frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\Big|_{x_2=0^-} - \Phi(x_1,0^-,\xi_1,\xi_2)k^- \frac{\partial}{\partial x_2}T(x_1,x_2)\Big|_{x_2=0^-}]dx_1, \quad (5)$$

where $\gamma^+(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R^+ \cup C^+$, $\gamma^+(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R^+$, $0 < \gamma^+(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C^+$, $\gamma^-(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R^- \cup C^-$, $\gamma^-(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R^-$, $0 < \gamma^-(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C^-$ and

$$\Phi(x_1,x_2,\xi_1,\xi_2) = \frac{1}{2\pi k^\pm}\,\mathrm{Re}\{\ln(z-c)\} + \Phi^\pm(x_1,x_2,\xi_1,\xi_2) \text{ for } \pm x_2 > 0, \quad (6)$$

with $z = x_1 + ix_2$, $c = \xi_1 + i\xi_2$, $i = \sqrt{-1}$ and $\Phi^\pm(x_1,x_2,\xi_1,\xi_2)$ being any arbitrary functions satisfying

$$\frac{\partial^2\Phi^\pm}{\partial x_1^2} + \frac{\partial^2\Phi^\pm}{\partial x_2^2} = 0 \text{ for } (x_1,x_2) \in R^\pm \cup C^\pm \cup \Gamma. \quad (7)$$

With the interface condition (2), if we choose the functions $\Phi^+(x_1,x_2,\xi_1,\xi_2)$ and $\Phi^-(x_1,x_2,\xi_1,\xi_2)$ in such a way that (7) is satisfied and

$$\begin{aligned} & k^+ \frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\Big|_{x_2=0^+} \\ =& k^- \frac{\partial}{\partial x_2}\Phi(x_1,x_2,\xi_1,\xi_2)\Big|_{x_2=0^-} \\ =& \lambda[\Phi(x_1,0^+,\xi_1,\xi_2) - \Phi(x_1,0^-,\xi_1,\xi_2)] \text{ for } -\infty < x_1 < \infty, \end{aligned} \quad (8)$$

then the use of (4) and (5) yields

$$\gamma(\xi_1,\xi_2)T(\xi_1,\xi_2) = \oint_C [T(x_1,x_2)k\frac{\partial}{\partial n}\Phi(x_1,x_2,\xi_1,\xi_2) - \Phi(x_1,x_2,\xi_1,\xi_2)k\frac{\partial}{\partial n}T(x_1,x_2)]ds(x_1,x_2), \tag{9}$$

where $\gamma(\xi_1,\xi_2) = \gamma^+(\xi_1,\xi_2) + \gamma^-(\xi_1,\xi_2)$, i.e. $\gamma(\xi_1,\xi_2) = 1$ if (ξ_1,ξ_2) lies inside R^+ or R^- and $0 < \gamma(\xi_1,\xi_2) < 1$ if (ξ_1,ξ_2) lies on C^+ or C^- [$\gamma(\xi_1,\xi_2) = 1/2$ if (ξ_1,ξ_2) lies on a smooth part of C^+ or C^-].

Equation (9) can be employed to obtain a boundary element method for solving the problem described in Section 2. With (9), only the exterior boundary C of the body has to be discretized into boundary elements. The interface condition (2) is automatically built into (9) through the use of a special Green's function $\Phi(x_1,x_2,\xi_1,\xi_2)$ satisfying (8). Thus, in using (9) for the numerical solution of the problem under consideration, no discretization of the imperfect interface is needed.

For details on how one may discretize the boundary C into boundary elements and then apply (9) to set up a system of linear algebraic equations for determining either the temperature or the heat flux (whichever is not known) on the boundary, one may refer to Clements [11]. Once the temperature and the heat flux are completely known on the boundary C, the temperature and its first order partial derivatives with respect to the spatial variables, at any point (ξ_1,ξ_2) in the interior of $R^+ \cup R^-$, can be computed approximately by evaluating approximately the line integral on the right hand side of (9).

4 A special Green's function

To find $\Phi^{\pm}(x_1, x_2, \xi_1, \xi_2)$ such that $\Phi(x_1, x_2, \xi_1, \xi_2)$ in (6) satisfies (8), let us write

$$\Phi^{\pm}(x_1, x_2, \xi_1, \xi_2) = \Phi^{\pm}_{\rm per}(x_1, x_2, \xi_1, \xi_2) + \Phi^{\pm}_{\rm imp}(x_1, x_2, \xi_1, \xi_2), \tag{10}$$

where $\Phi^{\pm}_{\rm per}(x_1, x_2, \xi_1, \xi_2)$ and $\Phi^{\pm}_{\rm imp}(x_1, x_2, \xi_1, \xi_2)$ satisfy (7).

The functions $\Phi^{\pm}_{\rm per}(x_1, x_2, \xi_1, \xi_2)$ are required to satisfy the conditions:

$$\begin{aligned}&[\Phi^{+}_{\rm per}(x_1, 0^+, \xi_1, \xi_2) - \Phi^{-}_{\rm per}(x_1, 0^-, \xi_1, \xi_2)] \\ &= -\frac{1}{2\pi}\left(\frac{1}{k^+} - \frac{1}{k^-}\right) \mathrm{Re}\{\ln(x_1 - \xi_1 - i\xi_2)\} \text{ for } -\infty < x_1 < \infty. \end{aligned} \tag{11}$$

and

$$\begin{aligned}&k^+ \frac{\partial}{\partial x_2}\Phi^{+}_{\rm per}(x_1, x_2, \xi_1, \xi_2)\bigg|_{x_2=0^+} \\ &= k^- \frac{\partial}{\partial x_2}\Phi^{-}_{\rm per}(x_1, x_2, \xi_1, \xi_2)\bigg|_{x_2=0^-} \text{ for } -\infty < x_1 < \infty. \end{aligned} \tag{12}$$

It is easy to verify by direct substitution that for $\xi_2 \neq 0$ a possible choice of functions $\Phi^{+}_{\rm per}(x_1, x_2, \xi_1, \xi_2)$ and $\Phi^{-}_{\rm per}(x_1, x_2, \xi_1, \xi_2)$ satisfying (7), (11) and (12) is given by

$$\begin{aligned}&\Phi^{+}_{\rm per}(x_1, x_2, \xi_1, \xi_2) \\ &= -\frac{k^- - k^+}{2\pi k^+(k^- + k^+)} \mathrm{Re}\{H(-\xi_2)\ln(z - c) + H(\xi_2)\ln(z - \overline{c})\}, \end{aligned} \tag{13}$$

and

$$\begin{aligned}&\Phi^{-}_{\rm per}(x_1, x_2, \xi_1, \xi_2) \\ &= \frac{k^- - k^+}{2\pi k^-(k^- + k^+)} \mathrm{Re}\{H(-\xi_2)\ln(\overline{z} - c) + H(\xi_2)\ln(\overline{z} - \overline{c})\}, \end{aligned} \tag{14}$$

where $H(x)$ is the Heaviside unit-step function and the bar denotes the complex conjugate of a complex number.

Notice that

$$\Phi(x_1, x_2, \xi_1, \xi_2) = \frac{1}{2\pi k^{\pm}} \operatorname{Re}\{\ln(z - c)\} + \Phi^{\pm}_{\text{per}}(x_1, x_2, \xi_1, \xi_2) \text{ for } \pm\, x_2 > 0 \tag{15}$$

is a Green's function for the perfect interface. It is a special case of the more general Green's function, given in Berger and Karageorghis [3], for the perfect interface between two thermally homogeneous anisotropic half-spaces.

With $\Phi^{\pm}_{\text{per}}(x_1, x_2, \xi_1, \xi_2)$ as given above, we require $\Phi^{\pm}_{\text{imp}}(x_1, x_2, \xi_1, \xi_2)$ satisfying (7) to be chosen in such a way that (8) holds. To find $\Phi^{\pm}_{\text{imp}}(x_1, x_2, \xi_1, \xi_2)$, let us write

$$\Phi^{\pm}_{\text{imp}}(x_1, x_2, \xi_1, \xi_2) = \frac{1}{2\pi} \operatorname{Re}\{\int_0^{\infty} G^{\pm}(t, \xi_1, \xi_2) \exp(\pm itz) dt\}, \tag{16}$$

where $G^{\pm}(t, \xi_1, \xi_2)$ are arbitrary functions to be determined.

To ensure that (8) is true, we require that

$$-k^{+} G^{+}(t, \xi_1, \xi_2) = k^{-} \overline{G}^{-}(t, \xi_1, \xi_2), \tag{17}$$

and

$$\begin{aligned} &\operatorname{Re}\{\int_0^{\infty} (\lambda[1 + \frac{k^{+}}{k^{-}}] + k^{+} t) G^{+}(t, \xi_1, \xi_2) \exp(itx_1) dt\} \\ &= \operatorname{Re}\{[1 - \frac{k^{-} - k^{+}}{k^{-} + k^{+}} H(-\xi_2)] \frac{i}{(x_1 - c)} \\ &\qquad - \frac{k^{-} - k^{+}}{k^{-} + k^{+}} H(\xi_2) \frac{i}{(x_1 - \overline{c})}\} \text{ for } -\infty < x_1 < \infty. \end{aligned} \tag{18}$$

Equation (18) can be inverted to obtain

$$\begin{aligned} &(\lambda[1 + \frac{k^{+}}{k^{-}}] + k^{+} t) G^{+}(t, \xi_1, \xi_2) \\ &= H(-\xi_2)[1 - \frac{k^{-} - k^{+}}{k^{-} + k^{+}}] \exp(-itc) - H(\xi_2)[1 + \frac{k^{-} - k^{+}}{k^{-} + k^{+}}] \exp(-it\overline{c}). \end{aligned} \tag{19}$$

In deriving (19), we make use of the result (which may be obtained from the table of exponential Fourier transforms compiled by Erdélyi *et al.* [12]):

$$\int_{-\infty}^{\infty} \frac{\exp(-itx)dx}{(\xi + i\eta - x)} = H(-\eta)2\pi i \exp(-it[\xi + i\eta]). \tag{20}$$

For $\xi_2 \neq 0$, the functions $G^+(t, \xi_1, \xi_2)$ and $G^-(t, \xi_1, \xi_2)$ as given by (17) and (19) decay exponentially to zero as t increases. Thus, $\Phi^{\pm}_{\rm imp}(x_1, x_2, \xi_1, \xi_2)$ and its partial derivatives with respect to x_1 or x_2 can be easily computed numerically from the integrals in (16) by using a suitable integration formula in Abramowitz and Stegun [13].

As $\lambda \to \infty$ (for the perfect interface), both $G^+(t, \xi_1, \xi_2)$ and $\overline{G}^-(t, \xi_1, \xi_2)$ [hence $\Phi^+_{\rm imp}(x_1, x_2, \xi_1, \xi_2)$ and $\Phi^-_{\rm imp}(x_1, x_2, \xi_1, \xi_2)$] tend to zero. This is as expected, since the corresponding Green's function for the perfect interface is given by (15).

5 Numerical results

To test the proposed Green's function and the associated boundary element method, let us take R^+ to be the region $0 < x_1 < 1$, $0 < x_2 < 1/2$, with $k^+ = 1/2$, and R^- to be $0 < x_1 < 1$, $-1/2 < x_2 < 0$, with $k^- = 1/3$. On the interface between R^+ and R^-, i.e $0 < x_1 < 1$, $x_2 = 0$, we impose the condition (2) with $\lambda = 1/2$.

A solution of (1) which satisfies the interface condition (2) with $k^+ = 1/2$, $k^- = 1/3$ and $\lambda = 1/2$ is given by

$$T(x_1, x_2) = \begin{cases} [3\cos(x_2) + 2\sin(x_2)]\exp(-x_1) & \text{for} \quad (x_1, x_2) \in R^+ \\ [\cos(x_2) + 3\sin(x_2)]\exp(-x_1) & \text{for} \quad (x_1, x_2) \in R^-. \end{cases} \tag{21}$$

To devise a test problem, let us use (21) to generate boundary values of the temperature T on the sides $x_2 = \pm 1/2$, $0 < x_1 < 1$, and boundary values

of the normal derivative of T on $x_1 = 0$, $-1/2 < x_2 < 1/2$ and also on $x_1 = 1$, $-1/2 < x_2 < 1/2$. The boundary element method is then applied to solve (1) subject to the boundary data thus generated and the interface condition (2). To implement the method, the sides of the bimaterial are divided into N equal length boundary elements. The temperature and the normal heat flux are assumed to be constant over the elements, i.e. constant elements are used. (To avoid ambiguity, we require a boundary element to be in either R^+ or R^- but not partly in both the regions. One of the endpoints of the element is allowed to be on Γ.) If the boundary element method really works, we should be able to recover the solution (21) approximately.

On the side $x_1 = 0$, $-1/2 < x_2 < 1/2$, the heat flux is specified. In Table 1, we compare the numerical values of the temperature at selected points on that part of the boundary, as obtained using $N = 24$ and $N = 72$, with the exact solution (21). Similarly, the numerical values of the outward normal derivative of T at selected points on $x_2 = 1/2$, $0 < x_1 < 1$ [i.e. $\partial T/\partial x_2$ on that part of the boundary] are compared with the exact ones in Table 2. In both tables, it is obvious that the numerical values improve significantly when the number of boundary elements is increased from 24 to 72.

Table 1. A comparison of the numerical and exact values of the temperature at selected points on $x_1 = 0$, $-1/2 < x_2 < 1/2$.

(x_1, x_2)	24 elements	72 elements	Exact
$(0, 0.4167)$	3.5286	3.5515	3.5528
$(0, 0.250)$	3.3983	3.4018	3.4015
$(0, 0.08333)$	3.1567	3.1565	3.1561
$(0, -0.08333)$	0.7361	0.7452	0.7468
$(0, -0.250)$	0.2178	0.2249	0.2267
$(0, -0.4167)$	-0.3070	-0.3025	-0.2997

Table 2. A comparison of the numerical and exact values of the outward normal derivative of the temperature at selected points on $x_2 = 1/2$, $0 < x_1 < 1$.

(x_1, x_2)	24 elements	72 elements	Exact
$(0.9167, 0.50)$	0.1134	0.1250	0.1267
$(0.750, 0.50)$	0.1451	0.1496	0.1497
$(0.5833, 0.50)$	0.1729	0.1762	0.1768
$(0.4167, 0.50)$	0.2040	0.2075	0.2089
$(0.250, 0.50)$	0.2363	0.2441	0.2468
$(0.08333, 0.50)$	0.3702	0.2828	0.2916

After using the boundary element method, the temperature and heat flux are completely known on the sides of the square region $0 < x_1 < 1$, $-1/2 < x_2 < 1/2$. We can now apply (9) (with $\gamma(\xi_1, \xi_2) = 1$) to compute the temperature at various points (ξ_1, ξ_2) in the interior of the square. In Table 3, we compare the numerical values of T at selected points, as obtained using $N = 24$ and $N = 72$, with the exact solution (21). On the whole, the numerical values of T at the selected interior points are in good agreement with the exact solution and improve in accuracy when a larger number of boundary elements is used.

Table 3. A comparison of the numerical and exact values of T with the exact ones at various points in the interior of the bimaterial.

(x_1, x_2)	24 elements	72 elements	Exact
$(0.50, 0.250)$	2.0640	2.0633	2.0631
$(0.20, -0.10)$	0.5577	0.5678	0.5694
$(0.80, 0.050)$	1.3931	1.3913	1.3912
$(0.30, -0.010)$	0.7106	0.7175	0.7186
$(0.10, 0.010)$	2.7428	2.7338	2.7325
$(0.990, 0.490)$	1.4991	1.3432	1.3333
$(0.010, -0.250)$	0.2031	0.2226	0.2244
$(0.750, 0.0010)$	1.4193	1.4181	1.4180

Notice that at the points $(0.990, 0.490)$ and $(0.010, -0.250)$ which are close to the exterior boundary of the solution domain, the calculation using 24 boundary elements yields relatively larger percentage errors than at other points shown in Table 3. This is consistent with the well known observation that the accuracy of the boundary element solution is usually poorer at points whose distance from the boundary is many times smaller than the length of the nearby boundary elements. However, the interface does not appear to have such an adverse boundary effect on the accuracy of the boundary element solution at points close to the interface, due to the use of the special Green's function. For example, at the point $(0.750, 0.0010)$ which is very close to the interface, the error of the numerical value of T is still low at well below 0.10% even when only 24 boundary elements (each of length about 0 .1667 units) are employed in the computation.

Numerical results obtained for a few other specific test problems (with known exact solutions) also exhibit the qualitative features reported above.

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