A COMPLEX VARIABLE BOUNDARY ELEMENT METHOD FOR AN ELLIPTIC PARTIAL DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

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Abstract

A boundary element method based on the Cauchy integral formulae is proposed for the numerical solution of a boundary value problem governed by a second order elliptic partial differential equation with variable coefficients. The boundary value problem has applications in engineering problems involving nonhomogeneous media. The method reduces the boundary value problem to the task of solving a system of linear algebraic equations. It can be easily implemented on the computer.

Keywords: complex variable boundary element method, elliptic partial differential equation with variable coefficients.

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1 INTRODUCTION

The second order elliptic partial differential equation (PDE)

$$\frac{\partial}{\partial x}\left(K(x)\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(K(x)\frac{\partial \phi}{\partial y}\right)=0, \quad \phi=\phi(x,y), \tag{1}$$

where $K(x)$ is a given function of the Cartesian coordinate x such that $K(x) > 0$ in the domain of interest, has extensive engineering applications in structural ceramics, modeling of mechanics of soils and other particulate materials (see e.g. Brennan [1] and Gibson [2]).

A boundary value problem (BVP) of interest is to solve (1) in a finite region R bounded by a simple closed curve C (on the $0xy$ plane) subject to the conditions

$$\phi = p(x,y) \quad \text{for } (x,y)\in D, \tag{2}$$

$$\frac{\partial \phi}{\partial n} = q(x,y) \quad \text{for } (x,y)\in E, \tag{3}$$

where p and q are suitably prescribed functions, D and E are non-intersecting curves such that $C = D\cup E$, $\partial\phi/\partial n = \mathbf{n}\cdot\nabla\phi$ and $\mathbf{n} = [n_1, n_2]$ is the unit normal outward vector to E.

Clements [3] had derived a general solution of (1) in the form

$$\phi(x,y)=\frac{1}{\sqrt{K(x)}}\,\mathrm{Re}\left\{\sum_{n=0}^{\infty} f_n(x)\Phi_n(z)\right\}, \tag{4}$$

where $i=\sqrt{-1}$ and $z = x+iy$. The functions f_n are particular solutions of the recurrence equations

$$2\frac{df_{n+1}}{dx} = -\frac{d^2 f_n}{dx^2} + \Lambda(x) f_n \ \text{ for } n = 0,1,2,3,\cdots, \tag{5}$$

with $f_0(x) = 1$ and

$$\Lambda(x)=\frac{d}{dx}\left[\left(\frac{dK}{dx}\right)^2/(4K)\right]/\left(\frac{dK}{dx}\right). \tag{6}$$

The complex functions $\Phi_n(z)$ are holomorphic in the domain of interest and satisfy

$$\frac{d\Phi_n}{dz} = \Phi_{n-1}(z) \quad \text{for } n = 1, 2, 3, \cdots . \tag{7}$$

In the present article, the solution (4) together with the Cauchy integral fromulae is applied to derive a numerical method for solving approximately the BVP defined by (1), (2) and (3). The method requires only the boundary C of the domain R to be discretized and is known as the complex variable boundary element method (CVBEM). It reduces the BVP to solving a system of linear algebraic equations. The coefficients of the linear algebraic equations are easy to compute. Hence the method can be easily implemented on the computer. The proposed method should offer itself as an interesting alternative to earlier BEM approaches (see e.g. Clements [3] and Rangogni [4]) for solving certain BVPs governed by (1).

The CVBEM was originally introduced by Hromadka and Lai [5] for BVPs governed by the two-dimensional Laplace equation, i.e. for the case where $K(x)$ is a constant function in (1). Recently, Ang and Park [6] developed a different version of the method for a more general system of second order elliptic PDEs with constant coefficients. The approach of [6] which differs from that of [5] in the approximation of the relevant complex functions and in the treatment of the boundary conditions is adopted here to solve the BVP under consideration.

2 CVBEM

Firstly, the boundary C is discretized as follows. Place M well-spaced out points $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(M-1)}, y^{(M-1)})$ and $(x^{(M)}, y^{(M)})$ in an anticlockwise order on C. Let $C^{(k)}$ be the straight line segment from $(x^{(k)}, y^{(k)})$ to $(x^{(k+1)}, y^{(k+1)})$ $(k = 1, 2, \cdots , M)$, where $(x^{(M+1)}, y^{(M+1)}) = (x^{(1)}, y^{(1)})$. We then make the approximation

$$C \approx C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(M-1)} \cup C^{(M)}. \tag{8}$$

Repeated application of the recurrence relation (7) gives

$$\Phi_n(z) = \frac{1}{(n-1)!}\int_a^z (z-t)^{n-1}\Phi_0(t)dt \ \text{ for } n = 1, 2, \cdots, \tag{9}$$

where a is a suitably chosen complex number.

Let $z^{(k)} = x^{(k)} + iy^{(k)}$ $(k = 1, 2, \cdots, M+1)$, $\widehat{z}^{(p)} = \xi^{(p)} + i\eta^{(p)}$, $\xi^{(p)} = (x^{(p)} + x^{(p+1)})/2$ and $\eta^{(p)} = (y^{(p)} + y^{(p+1)})/2$ $(p = 1, 2, \cdots, M)$. In (9), if we take $a = z^{(1)}$, $z = \widehat{z}^{(p)}$ and do the integration from a to z along the path $C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(p-1)} \cup \widehat{C}^{(p)}$ (where $\widehat{C}^{(p)}$is the straight line segment from $z^{(p)}$ to $\widehat{z}^{(p)}$), then

$$\begin{aligned}\Phi_n(\widehat{z}^{(p)}) = \frac{1}{(n-1)!}\Bigg\{&\sum_{k=1}^{p-1}\int_{z^{(k)}}^{z^{(k+1)}} (\widehat{z}^{(p)} - t)^{n-1}\Phi_0(t)dt \\ &+ \int_{z^{(p)}}^{\widehat{z}^{(p)}} (\widehat{z}^{(p)} - t)^{n-1}\Phi_0(t)dt\Bigg\} \\ &\text{for } p = 1, 2, \cdots, M \text{ and } n = 1, 2, \cdots,\end{aligned} \tag{10}$$

For $t \in C^{(k)}$, if we expand $\Phi_0(t)$ as a Taylor-Maclaurin series about $t = \widehat{z}^{(k)}$, i.e. if we write

$$\Phi_0(t) = \Phi_0(\widehat{z}^{(k)}) + (t - \widehat{z}^{(k)})\Phi_0'(\widehat{z}^{(k)}) + \frac{1}{2}(t - \widehat{z}^{(k)})^2\Phi_0''(\widehat{z}^{(k)}) + \cdots, \tag{11}$$

then, after ignoring terms whose magnitudes are $O(|z^{(k+1)} - z^{(k)}|^2)$ or higher order, we find that (10) gives (approximately)

$$\Phi_n(\widehat{z}^{(p)}) = \sum_{k=1}^{p}\Gamma_n^{(pk)}\Phi_0(\widehat{z}^{(k)}) \text{ for } p = 1, 2, \cdots, M \text{ and } n = 0, 1, 2, \cdots, \tag{12}$$

where $\Gamma_0^{(pk)} = \delta_{pk}$ and

$$\begin{aligned}\Gamma_n^{(pk)} = \frac{1}{(n-1)!}\left((1-\delta_{pk})\int_{z^{(k)}}^{z^{(k+1)}} (\widehat{z}^{(p)} - t)^{n-1}dt + \delta_{pk}\int_{z^{(p)}}^{\widehat{z}^{(p)}} (\widehat{z}^{(p)} - t)^{n-1}dt\right) \\ \text{for } n = 1, 2, 3, \cdots.\end{aligned} \tag{13}$$

Notice that $\Gamma_n^{(pk)}$can be easily evaluated.

If we write $\Phi_0(\widehat{z}^{(p)}) = u^{(p)} + iv^{(p)}$ where $u^{(p)}$ and $v^{(p)}$ are real constants then use of (12) together with (4) in the condition (2) yields

$$\sum_{k=1}^{p} \left\{ u^{(k)} \sum_{n=0}^{\infty} f_n(\xi^{(p)})\alpha_n^{(pk)} - v^{(k)} \sum_{n=0}^{\infty} f_n(\xi^{(p)})\beta_n^{(pk)} \right\}$$
$$= \sqrt{K(\xi^{(p)})}p(\xi^{(p)}, \eta^{(p)}) \text{ if } \phi \text{ is specified over } C^{(p)}, \tag{14}$$

where $\alpha_n^{(pk)}$ and $\beta_n^{(pk)}$ are real parameters given by $\Gamma_n^{(pk)} = \alpha_n^{(pk)} + i\beta_n^{(pk)}$.

Now, from (4) and (7), we find that

$$\frac{\partial}{\partial n}\left(\frac{1}{\sqrt{K(x)}}\text{Re}\left\{\sum_{n=0}^{\infty} f_n(x)\Phi_n(z)\right\}\right)$$
$$= \text{Re}\left\{\frac{(n_1 + in_2)}{\sqrt{K(x)}}\Phi_0'(z) + \sum_{n=0}^{\infty}\Phi_n(z)\left[n_1\left(\frac{f_{n+1}(x)}{\sqrt{K(x)}} + F_n(x)\right) + \frac{in_2 f_{n+1}(x)}{\sqrt{K(x)}}\right]\right\}, \tag{15}$$

where $F_n(x) = d\left(f_n(x)/\sqrt{K(x)}\right)/dx$ for $n = 0, 1, 2, \cdots$.

Thus, to deal with the condition in (3), we are required to evaluate $\Phi_0'(z)$ at z on the boundary. We shall apply the Cauchy integral formula

$$2\pi i\Phi_0'(z) = \sum_{k=1}^{M}\int_{C^{(k)}}\frac{\Phi_0(t)dt}{(t-z)^2} \text{ for } z \in R. \tag{16}$$

If we substitute (11) into (16) and ignore terms whose magnitudes are $O(|z^{(k+1)} - z^{(k)}|^2)$ or higher order, we obtain the approximation

$$2\pi i\Phi_0'(z) = \sum_{k=1}^{M}\Phi_0(\widehat{z}^{(k)})\left[q(z^{(k)}, z^{(k+1)}, z) + ir(z^{(k)}, z^{(k+1)}, z)\right] \text{ for a fixed } z \in R, \tag{17}$$

where q and r are real parameters defined by

$$q(a, b, c) + ir(a, b, c) = -\frac{1}{b-c} + \frac{1}{a-c}. \tag{18}$$

However, if we repeat the same exercise but with $z \to \widehat{z}^{(p)}$ (the midpoint of the p-th straight line segment), we find that

$$\pi i \Phi_0'(\widehat{z}^{(p)}) = \sum_{k=1}^{M} \Phi_0(\widehat{z}^{(k)}) \left[q(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)}) + ir(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)}) \right]. \tag{19}$$

Notice the difference in the factors that is multiplied to the derivative Φ_0' in (17) and (19).

With (12), (15) and (19), the condition in (4) gives

$$\begin{aligned}
&\sum_{k=1}^{M} u^{(k)} \left(\frac{n_2^{(p)} q(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)}) + n_1^{(p)} r(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)})}{\pi \sqrt{K(\xi^{(p)})}} \right) \\
&+ \sum_{k=1}^{p} u^{(k)} \sum_{n=0}^{\infty} \left(\alpha_n^{(pk)} \mu_n^{(p)} - \beta_n^{(pk)} \sigma_n^{(p)} \right) \\
&+ \sum_{k=1}^{M} v^{(k)} \left(\frac{n_1^{(p)} q(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)}) - n_2^{(p)} r(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)})}{\pi \sqrt{K(\xi^{(p)})}} \right) \\
&- \sum_{k=1}^{p} v^{(k)} \sum_{n=0}^{\infty} \left(\alpha_n^{(pk)} \sigma_n^{(p)} + \beta_n^{(pk)} \mu_n^{(p)} \right) \\
&= q(\xi^{(p)}, \eta^{(p)}) \text{ if } \frac{\partial \phi}{\partial n} \text{ is specified over } C^{(p)},
\end{aligned} \tag{20}$$

where $[n_1^{(p)}, n_2^{(p)}]$ is the unit normal (outward) vector to $C^{(p)}$and $\mu_n^{(p)}$and $\sigma_n^{(p)}$are real parameters defined by

$$\mu_n^{(p)} + i\sigma_n^{(p)} = \left[n_1^{(p)} \left(\frac{f_{n+1}(\xi^{(p)})}{\sqrt{K(\xi^{(p)})}} + F_n(\xi^{(p)}) \right) + \frac{i n_2^{(p)} f_{n+1}(\xi^{(p)})}{\sqrt{K(\xi^{(p)})}} \right]. \tag{21}$$

The equations (14) and (20) give M equations in $2M$ unknowns $u^{(k)}$ and $v^{(k)}$ $(k = 1, 2, \cdots, M)$. To obtain another M equations, we apply the Cauchy integral formula

$$2\pi i \Phi_0(z) = \sum_{k=1}^{M} \int_{C^{(k)}} \frac{\Phi_0(t)dt}{(t-z)} \text{ for } z \in R. \tag{22}$$

Proceeding as before, i.e. substituting (11) into (22) and ignoring terms whose magnitudes are $O(|z^{(k+1)} - z^{(k)}|^2)$ or higher order, we obtain

$$2\pi i\Phi_0(z) = \sum_{k=1}^{M}(u^{(k)} + iv^{(k)}) \left[\gamma(z^{(k)}, z^{(k+1)}, z) + i\theta(z^{(k)}, z^{(k+1)}, z)\right] \quad \text{for } z \in R, \tag{23}$$

where γ and θ are real parameters defined by

$$\begin{aligned} \gamma(a,b,c) &= \ln|b-c| - \ln|a-c| \\ \theta(a,b,c) &= \begin{cases} \Theta(a,b,c) & \text{if} \quad \Theta(a,b,c) \in [-\pi,\pi] \\ \Theta(a,b,c) + 2\pi & \text{if} \quad \Theta(a,b,c) \in [-2\pi,-\pi) \\ \Theta(a,b,c) - 2\pi & \text{if} \quad \Theta(a,b,c) \in (\pi,2\pi] \end{cases} \\ \Theta(a,b,c) &= \mathrm{Arg}(b-c) - \mathrm{Arg}(a-c), \end{aligned} \tag{24}$$

where $\mathrm{Arg}(z)$ denotes the principal argument of the complex number z. If the solution domain is convex in shape, $\theta(a,b,c)$ can be calculated directly from

$$\theta(a,b,c) = \cos^{-1}\left(\frac{|b-c|^2 + |a-c|^2 - |b-a|^2}{2\,|b-c|\,|a-c|}\right). \tag{25}$$

We may take the real part of equation (23) and let $z \to \widehat{z}^{(p)}$ to obtain

$$-2\pi v^{(p)} = \sum_{k=1}^{M}\left\{u^{(k)}\gamma(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)}) - v^{(k)}\theta(z^{(k)}, z^{(k+1)}, \widehat{z}^{(p)})\right\} \quad \text{for } p = 1,2,\cdots,M. \tag{26}$$

The equations (14), (20) and (26) constitute $2M$ equations from which the unknowns $u^{(k)}$ and $v^{(k)}$ $(k = 1,2,\cdots,M)$ can be determined. Once the unknowns are determined, we can calculate $\Phi_n(z)$ $(n = 1,2,\cdots)$ at the point $z = \widehat{z}^{(k)}$ $(k = 1,2,\cdots,M)$ using (12) and then at the point $z = c$ in the interior of R via

$$2\pi i\Phi_n(c) = \sum_{k=1}^{M}\Phi_n(\widehat{z}^{(k)}) \left[\gamma(z^{(k)}, z^{(k+1)}, c) + i\theta(z^{(k)}, z^{(k+1)}, c)\right]. \tag{27}$$

Notice that (27) is also valid for $n = 0$. Thus, all the complex functions needed for the solution of the BVP can be determined numerically at all points in $R \cup C$.

3 A TEST PROBLEM

For a test problem, we shall apply the CVBEM proposed above to solve

$$\frac{\partial}{\partial x}\left((x+2)^4 \frac{\partial \phi}{\partial x}\right) + (x+2)^4 \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ in } 0 < x < 1,\ 0 < y < \pi, \tag{28}$$

subject to

$$\left.\frac{\partial \phi}{\partial n}\right|_{y=0} = 0 \quad \text{for } 0 < x < 1, \tag{29}$$

$$\left.\frac{\partial \phi}{\partial n}\right|_{y=\pi} = 0 \quad \text{for } 0 < x < 1, \tag{30}$$

$$\phi(0, y) = 0 \quad \text{for } 0 < y < \pi, \tag{31}$$

$$\phi(1, y) = \cos(y) \quad \text{for } 0 < y < \pi. \tag{32}$$

The exact solution of the BVP can be obtained by the method of separation of variables. It is given by

$$\phi(x, y) = \frac{-27\left(-3x - 3 + x\exp(-2x) + 3\exp(-2x)\right)\exp(x+2)}{2e\left(3e^2 - 2\right)(x+2)^3} \cdot \cos(y). \tag{33}$$

For the PDE in (28),

$$K(x) = (x+2)^4, \quad f_0(x) = 1, \quad f_1(x) = -\frac{1}{(x+2)}$$
$$f_n(x) = 0 \quad \text{for } n \geq 2, \tag{34}$$

and

$$F_0(x) = -\frac{2}{(x+2)^3}, \quad F_1(x) = \frac{3}{(x+2)^4},$$
$$F_n(x) = 0 \quad \text{for } n \geq 2. \tag{35}$$

The CVBEM described in the previous section is applied to solve (28) subject to (29)-(32). The numerical values of ϕ are then calculated at selected interior points and compared with the exact values from (33). A comparison is made in Table 1 for numerical results obtained by putting 10 elements on each side of the rectangular boundary and also 40 elements per side (i.e. for $M = 40$ and $M = 120$) with the exact solution. For $M = 40$, the largest length of the elements is approximately 0.31 units, while for $M = 120$ the largest length is about 0.078 units. It is obvious from Table 1 that there is significant improvement in the accuracy of the numerical results when the number of elements is increased from 40 to 120. The accuracy is in general good except for interior points extremely close to the boundary. For interior points very close to the boundary, it is necessary to employ a larger number of elements for better accuracy. For example, at the point $(0.10, 3.10)$ which is extremely close to the boundary $y = \pi$, the percentage error in the calculation of ϕ is about 47% if the computation is carried out using $M = 40$. However, using $M = 120$, we are able to reduce the percentage error to 6.1%. Further calculation using $M = 400$ brings the percentage error down to well under 1%.

Table 1. A comparison of the CVBEM numerical results with the exact solution at selected interior points.

Interior point (x, y)	CVBEM $M = 40$	CVBEM $M = 120$	Exact
$(0.20.0.40)$	0.2477	0.2784	0.2797
$(0.40, 0.60)$	0.3903	0.4254	0.4331
$(0.80, 0.70)$	0.5951	0.6377	0.6537
$(0.50, 0.50)$	0.4840	0.5294	0.5413
$(0.99, 2.10)$	-0.4365	-0.5014	-0.5012
$(0.10, 3.10)$	-0.0878	-0.1552	-0.1653

4 SUMMARY

A CVBEM is proposed for the numerical solution of BVPs governed by a second order elliptic PDE with variable coefficients. A general solution of the PDE, expressible in terms of an arbitrary complex function that is analytic in the domain of interest, is available in the literature (Clements [3]). The proposed method makes use of the Cauchy integral formulae to construct approximately the relevant complex function that gives the required solution of the BVP. The task of constructing the complex function requires manipulation of the unknown data on only the boundary of the solution domain and can be eventually reduced to solving a system of linear algebraic equations. The method can be efficiently implemented on the computer, as only the boundary needs to be discretized and the coefficients of the algebraic equations are easy to compute. It is used to solve a test problem which has an explicit exact solution. The numerical results obtained indicates that the proposed CVBEM works. Convergence of the numerical solution to the exact one is also observed when the calculation is refined by increasing the number of boundary elements.

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