Nonlinear heat equation for nonhomogeneous anisotropic materials: a dual-reciprocity boundary element solution

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Abstract

A dual-reciprocity boundary element method is presented for the numerical solution of initial-boundary value problems governed by a nonlinear partial differential equation for heat conduction in nonhomogeneous anisotropic materials. To assess the validity and accuracy of the method, some specific problems are solved.

Keywords: Boundary element method; Heat conduction; Nonhomogeneous materials; Anisotropic materials; Nonlinear partial heat equation.

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1 Introduction

In recent years, many nonlinear problems in physical and engineering science have been solved numerically using various boundary element approaches (see, for example, Katsikadelis and Nerantzaki [1], Ang [2], Selvadurai [3], Chen, Hsiao, Chiu *et al* [4] and Dehghan and Mirzaei [5]). One such problem which is of considerable interest to many researchers deals with heat conduction in solids with temperature dependent thermal conductivity. In earlier works on boundary element techniques for solving the nonlinear heat conduction problem, such as Kikuta, Togoh and Tanaka [6] and Goto and Suzuki [7], the solids are assumed to be thermally isotropic with material properties (density, specific heat capacity and thermal conductivity) which are functions of temperature alone. Extensions to thermally anisotropic solids with material properties that vary with temperature and spatial coordinates are given by Clements and Budhi [8] and Azis and Clements [9].

In the present paper, we present a dual-reciprocity boundary element method for the problem of calculating the temperature in a nonhomogeneous anisotropic solid with density, specific heat capacity and thermal conductivity which are functions of temperature and spatial coordinates. Specifically, the thermal conductivity coefficients are taken to be of the form considered in Azis and Clements [9] (that is, as given in equation (2) in Section 2 below), but here (unlike in [9]) we do not require the function describing the spatial variation of the thermal conductivity (that is, $g(x_1, x_2)$) to satisfy a particular partial differential equation. In [9], the boundary element approach in Goto and Suzuki [7], which approximates the domain integral arising in the formulation by discretizing the solution domain into many small cells, is exended to solve the governing nonlinear heat equation.

The boundary element approach presented here follows closely the works in Ang, Clements and Vahdati [10] and Ang [11]. Firstly, guided by the analyses in Azis and Clements [9] and Ang, Clements and Vahdati [10], we convert the governing nonlinear heat equation into a suitable integro-differential equation. The integro-differential equation involves an unknown function $\psi(x_1, x_2, t)$ and its first order partial derivative with respect to time t. It does not contain any partial derivative of ψ with respect to the spatial coordinate x_i and can be accurately solved for numerical values of ψ by modifying the time-stepping dual-reciprocity boundary element procedure in Ang [11]. As in [11], $\partial\psi/\partial t$ is approximated in terms of a central difference formula, the integral over the boundary of the solution domain is discretized using discontinuous linear elements and the domain integral is approximated using the dual-reciprocity method. No meshing of the solution domain is required. The nonlinear term in the integrand of the integro-differential equation (that is, the term containing $\partial\psi/\partial t$ multiplied to a given function of x_1, x_2 and ψ) is treated using a predictor-corrector (iterative) approach similar to that described in Ang and Ang [12].

The task of finding numerical values of ψ at chosen points at consecutive time levels is eventually reduced to solving systems of linear algebraic equations. Once ψ is determined, the temperature can be recovered by solving an algebraic equation which expresses ψ as a function of the temperature and the spatial coordinates x_1 and x_2. To check its validity and to assess its accuracy, the numerical procedure presented here is applied to solve some specific problems with known solutions.

2 The problem

With reference to a Cartesian coordinate system $Ox_1x_2x_3$, consider a thermally nonhomogeneous anisotropic solid whose geometry does not vary along the x_3 axis. On the Ox_1x_2 plane, the body occupies the region R bounded by a simple closed curve C.

The temperature in the body is assumed to be independent of x_3. According to the classical theory of heat conduction, if there is no internal heat generation, the thermal behavior of the body is governed by the partial differential equation

$$\frac{\partial}{\partial x_i}(k_{ij}\frac{\partial T}{\partial x_j}) = \rho c \frac{\partial T}{\partial t}, \tag{1}$$

where k_{ij} are the thermal conductivity coefficients satisfying the symmetry relation $k_{ij} = k_{ji}$ and the strict inequality $k_{12}^2 - k_{11}k_{22} < 0$, $T(x_1, x_2, t)$ is the temperature at the point (x_1, x_2) at time $t \geq 0$ and ρ and c are respectively the density and the specific heat capacity of the body. The usual Einstenian convention of summing over a repeated index is assumed here for Latin subscripts running from 1 to 2.

The thermal conductivity coefficients k_{ij} for the thermally nonhomogeneous anisotropic solid are taken to be of the form

$$k_{ij} = \lambda_{ij} g(x_1, x_2) h(T), \tag{2}$$

where λ_{ij} are given positive constants such that $\lambda_{ij} = \lambda_{ji}$ and $\lambda_{12}^2 - \lambda_{11}\lambda_{22} < 0$, $g(x_1, x_2)$ is a given function such that $g(x_1, x_2)$ is positive and is at least twice partially differentiable with respect to x_1 and/or x_2 in $R \cup C$ and $h(T)$ is a given function which is integrable with respect to T. In general, the density ρ and the specific heat capacity c are suitably given functions which depend on x_1, x_2 and T.

The problem of interest here is to find the temperature $T(x_1, x_2, t)$ by solving (1) together with (2) in R subject to to the initial-boundary conditions

$$\begin{aligned} T(x_1, x_2, 0) &= p(x_1, x_2) \text{ for } (x_1, x_2) \in R, \\ T(x_1, x_2, t) &= u(x_1, x_2, t) \text{ for } (x_1, x_2) \in C_1 \text{ and } t > 0, \\ k_{ij} n_i \frac{\partial T}{\partial x_j} &= v(x_1, x_2, t) \text{ for } (x_1, x_2) \in C_2 \text{ and } t > 0, \end{aligned} \tag{3}$$

where p, u and v are suitably prescribed functions, C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$ and $[n_1(x_1, x_2), n_2(x_1, x_2)]$ is the unit normal outward vector to R at the point (x_1, x_2) on C.

3 Transformed equations

If we apply the Kirchhoff's transformation

$$\Theta(x_1, x_2, t) = \int h(T)dT \equiv K(T(x_1, x_2, t)) \tag{4}$$

and assume that it can be inverted to give the temperature as $T = M(\Theta)$, then equation (1) together with (2) can be rewritten as

$$\lambda_{ij}\frac{\partial}{\partial x_i}(g(x_1, x_2)\frac{\partial \Theta}{\partial x_j}) = S(x_1, x_2, \Theta)\frac{\partial \Theta}{\partial t}, \tag{5}$$

where

$$S(x_1, x_2, \Theta) = \frac{\rho(x_1, x_2, M(\Theta))c(x_1, x_2, M(\Theta))}{h(M(\Theta))}. \tag{6}$$

Furthermore, if we let

$$\Theta(x_1, x_2, t) = \frac{1}{\sqrt{g(x_1, x_2)}}\psi(x_1, x_2, t), \tag{7}$$

we find that (5) can be re-written as

$$\lambda_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j} = B(x_1, x_2)\psi + D(x_1, x_2, \psi)\frac{\partial \psi}{\partial t}, \tag{8}$$

where

$$\begin{aligned} B(x_1, x_2) &= \frac{1}{\sqrt{g(x_1, x_2)}}\lambda_{ij}\frac{\partial^2}{\partial x_i \partial x_j}[\sqrt{g(x_1, x_2)}], \\ D(x_1, x_2, \psi) &= \frac{1}{g(x_1, x_2)}S(x_1, x_2, \frac{1}{\sqrt{g(x_1, x_2)}}\psi). \end{aligned} \tag{9}$$

The initial-boundary conditions in (3) can be rewritten as

$$\begin{aligned} \psi(x_1, x_2, 0) &= \sqrt{g(x_1, x_2)}K(p(x_1, x_2)) \text{ for } (x_1, x_2) \in R, \\ \psi(x_1, x_2, t) &= \sqrt{g(x_1, x_2)}K(u(x_1, x_2, t)) \\ &\qquad \text{for } (x_1, x_2) \in C_1 \text{ and } t > 0, \\ q(x_1, x_2, t) &= f(x_1, x_2, t)\psi(x_1, x_2, t) + \frac{1}{\sqrt{g(x_1, x_2)}}v(x_1, x_2, t) \\ &\qquad \text{for } (x_1, x_2) \in C_2 \text{ and } t > 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} f(x_1,x_2) &= \frac{1}{2}\frac{1}{g(x_1,x_2)}n_i\lambda_{ij}\frac{\partial g}{\partial x_j} \\ q(x_1,x_2,t) &= n_i\lambda_{ij}\frac{\partial \psi}{\partial x_j}. \end{aligned} \tag{11}$$

Following closely the approach in Ang, Clements and Vahdati [10] and Ang [11], we recast (8) into an integro-differential equation given by

$$\begin{aligned} &\gamma(\xi_1,\xi_2)\psi(\xi_1,\xi_2,t) \\ &= \iint\limits_R \Phi(x_1,x_2,\xi_1,\xi_2)[B(x_1,x_2)\psi(x_1,x_2,t) \\ &\qquad + D(x_1,x_2,\psi(x_1,x_2,t))\frac{\partial}{\partial t}\{\psi(x_1,x_2,t)\}]dx_1dx_2 \\ &+ \oint\limits_C [\Gamma(x_1,x_2,\xi_1,\xi_2)\psi(x_1,x_2,t) - \Phi(x_1,x_2,\xi_1,\xi_2)q(x_1,x_2,t)]ds(x_1,x_2), \end{aligned} \tag{12}$$

where $\gamma(\xi_1,\xi_2) = 0$ if $(\xi_1,\xi_2) \notin R\cup C$, $\gamma(\xi_1,\xi_2) = 1$ if $(\xi_1,\xi_2) \in R$, $0 < \gamma(\xi_1,\xi_2) < 1$ if $(\xi_1,\xi_2) \in C$ $[\gamma(\xi_1,\xi_2) = 1/2$ if (ξ_1,ξ_2) lies on a smooth part of C and

$$\begin{aligned} \Phi(x_1,x_2,\xi_1,\xi_2) &= \frac{1}{2\pi\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}}\,\mathrm{Re}\{\ln(x_1-\xi_1+\tau[x_2-\xi_2])\}, \\ \Gamma(x_1,x_2,\xi_1,\xi_2) &= \frac{1}{2\pi\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}}\,\mathrm{Re}\left\{\frac{L(x_1,x_2)}{(x_1-\xi_1+\tau[x_2-\xi_2])}\right\}, \\ L(x_1,x_2) &= (\lambda_{11}+\tau\lambda_{12})n_1(x_1,x_2) + (\lambda_{21}+\tau\lambda_{22})n_2(x_1,x_2), \\ \tau &= \frac{-\lambda_{12}+i\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}}{\lambda_{22}} \quad (i=\sqrt{-1}). \end{aligned} \tag{13}$$

Note that Re denotes the real part of a complex number.

Section 4 describes a dual-reciprocity boundary element approach for finding $\psi(x_1,x_2,t)$ numerically from (12) together with (10). Once $\psi(x_1,x_2,t)$ is determined, the temperature $T(x_1,x_2,t)$ may be obtained by inverting the Kirchhoff's transformation in (4).

4 Dual-reciprocity boundary element method

For the dual-reciprocity boundary element method, we discretize the curve C_1 into N_1 straight line elements denoted by $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N_1-1)}$ and $C^{(N_1)}$ and C_2 into $C^{(N_1+1)}$, $C^{(N_1+2)}$, $\cdots$, $C^{(N_1+N_2-1)}$ and $C^{(N_1+N_2)}$. Note that C_1 and C_2 are parts of the boundary C (refer to the boundary conditions in (3) or (10)). Thus, the total number of boundary elements is given by $N = N_1+N_2$. The element $C^{(m)}$ ($m = 1, 2, \cdots, N$) has length $\ell^{(m)}$ and its starting and ending points are given by $(a_1^{(m)}, a_2^{(m)})$ and $(b_1^{(m)}, b_2^{(m)})$ respectively.

For an accurate approximation, ψ and q in (12) on the boundary are approximated using discontinuous linear boundary elements as outlined in París and Cañas [13]. For the discontinuous linear boundary elements, two points $(\eta_1^{(m)}, \eta_2^{(m)})$ and $(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ on $C^{(m)}$ are chosen as given by

$$\left.\begin{aligned} \eta_i^{(m)} &= a_i^{(m)} + r(b_i^{(m)} - a_i^{(m)}) \\ \eta_i^{(N+m)} &= b_i^{(m)} - r(b_i^{(m)} - a_i^{(m)}) \end{aligned}\right\} \text{ for a given } r \in (0, \frac{1}{2}). \tag{14}$$

Note that r is a selected constant between 0 and 1/2. It may typically be chosen to be 1/4.

If ψ is given by $\psi^{(m)}(t)$ and $\psi^{(N+m)}(t)$ at $(\eta_1^{(m)}, \eta_2^{(m)})$ and $(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ respectively then we make the approximation:

$$\begin{aligned} \psi(x_1, x_2, t) \simeq& [1 - d^{(m)}(x_1, x_2)]\psi^{(m)}(t) \\ &+ d^{(m)}(x_1, x_2)\psi^{(N+m)}(t) \text{ for } (x_1, x_2) \in C^{(m)}, \end{aligned} \tag{15}$$

where

$$d^{(m)}(x_1, x_2) = \frac{\sqrt{(x_1 - a_1^{(m)})^2 + (x_2 - a_2^{(m)})^2} - r\ell^{(m)}}{(1 - 2r)\ell^{(m)}}. \tag{16}$$

Similarly, for q, if it is given by $q^{(m)}(t)$ and $q^{(N+m)}(t)$ at $(\eta_1^{(m)}, \eta_2^{(m)})$ and $(\eta_1^{(N+m)}, \eta_2^{(N+m)})$ respectively, then

$$\begin{aligned} q(x_1, x_2, t) \simeq& [1 - d^{(m)}(x_1, x_2)]q^{(m)}(t) \\ &+ d^{(m)}(x_1, x_2)q^{(N+m)}(t) \text{ for } (x_1, x_2) \in C^{(m)}. \end{aligned} \tag{17}$$

With (15) and (17), the integro-differential equation in (12) can be approximately written as

$$
\begin{aligned}
&\gamma(\xi_1,\xi_2)\psi(\xi_1,\xi_2,t)\\
&=\iint\limits_R \Phi(x_1,x_2,\xi_1,\xi_2)[B(x_1,x_2)\psi(x_1,x_2,t)\\
&\qquad + D(x_1,x_2,\psi(x_1,x_2,t))\frac{\partial}{\partial t}\{\psi(x_1,x_2,t)\}]dx_1dx_2\\
&+\sum_{m=1}^{N}\left\{\psi^{(m)}(t)\Omega_1^{(m)}(\xi_1,\xi_2)+\psi^{(N+m)}(t)\Omega_2^{(m)}(\xi_1,\xi_2)\right\}\\
&-\sum_{m=1}^{N}\left\{q^{(m)}(t)\Omega_3^{(m)}(\xi_1,\xi_2)+q^{(N+m)}(t)\Omega_4^{(m)}(\xi_1,\xi_2)\right\}, \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_1^{(m)}(\xi_1,\xi_2) &= \int\limits_{C^{(m)}} [1-d^{(m)}(x_1,x_2)]\Gamma(x_1,x_2,\xi_1,\xi_2)ds(x_1,x_2),\\
\Omega_2^{(m)}(\xi_1,\xi_2) &= \int\limits_{C^{(m)}} d^{(m)}(x_1,x_2)\Gamma(x_1,x_2,\xi_1,\xi_2)ds(x_1,x_2),\\
\Omega_3^{(m)}(\xi_1,\xi_2) &= \int\limits_{C^{(m)}} [1-d^{(m)}(x_1,x_2)]\Phi(x_1,x_2,\xi_1,\xi_2)ds(x_1,x_2),\\
\Omega_4^{(m)}(\xi_1,\xi_2) &= \int\limits_{C^{(m)}} d^{(m)}(x_1,x_2)\Phi(x_1,x_2,\xi_1,\xi_2)ds(x_1,x_2). \qquad (19)
\end{aligned}
$$

Exact formulae for calculating the line integrals over $C^{(m)}$ in (19) are given in Ang [11].

To treat the domain integral in (18) using the dual-reciprocity method, we select P well spaced out points in the interior of the domain R. These interior points are denoted by $(\eta_1^{(2N+1)}, \eta_2^{(2N+1)})$, $(\eta_1^{(2N+2)}, \eta_2^{(2N+2)})$, $\cdots$, $(\eta_1^{(2N+P-1)}, \eta_2^{(2N+P-1)})$ and $(\eta_1^{(2N+P)}, \eta_2^{(2N+P)})$. We define $\psi^{(2N+j)}(t)=\psi(\eta_1^{(2N+1)},\eta_2^{(2N+1)},t)$ for $j=1, 2, \cdots, P$. Following the procedure detailed in Ang, Clements and

Vahdati [10] and Ang [11], we then approximate the domain integral using

$$\begin{aligned}
&\iint\limits_{R} \Phi(x_1, x_2, \xi_1, \xi_2)[B(x_1, x_2)\psi(x_1, x_2, t) \\
&\qquad + D(x_1, x_2, \psi(x_1, x_2, t))\frac{\partial}{\partial t}\{\psi(x_1, x_2, t)\}]dx_1 dx_2 \\
&\simeq \sum_{k=1}^{2N+P} [B(\eta_1^{(k)}, \eta_2^{(k)})\psi^{(k)}(t) \\
&+ D(\eta_1^{(k)}, \eta_2^{(k)}, \psi^{(k)}(t))\frac{d}{dt}\{\psi^{(k)}(t)\}] \sum_{j=1}^{2N+P} \chi^{(kj)}\Psi^{(j)}(\xi_1, \xi_2),
\end{aligned} \tag{20}$$

where

$$\sum_{k=1}^{2N+P} \sigma^{(j)}(\eta_1^{(k)}, \eta_2^{(k)})\chi^{(km)} = \begin{cases} 1 & \text{if} \quad j = m \\ 0 & \text{if} \quad j \neq m \end{cases}$$
$$\text{for } j, m = 1, 2, \cdots, 2N + P,$$

$$\begin{aligned}
\sigma^{(j)}(x_1, x_2) = 1 &+ \left([x_1 - \eta_1^{(j)} + \text{Re}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2\right) \\
&+ \left([x_1 - \eta_1^{(j)} + \text{Re}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2 + [\text{Im}\{\tau\}\{x_2 - \eta_2^{(j)}\}]^2\right)^{3/2} \\
&\text{for } j = 1, 2, \cdots, 2N + P,
\end{aligned}$$

$$\begin{aligned}
&\Psi^{(j)}(\xi_1, \xi_2) \\
&= \gamma(\xi_1, \xi_2)\theta^{(j)}(\xi_1, \xi_2) - \int\limits_{C} \Phi(x_1, x_2, \xi_1, \xi_2)\beta^{(j)}(x_1, x_2)ds(x_1, x_2) \\
&- \int\limits_{C} \theta^{(j)}(x_1, x_2)\Gamma(x_1, x_2, \xi_1, \xi_2)ds(x_1, x_2) \\
&\quad \text{for } j = 1, 2, \cdots, 2N + P,
\end{aligned}$$

$$
\begin{aligned}
&\left(\frac{\lambda_{11}\lambda_{22}-\lambda_{12}^2}{\lambda_{22}}\right)\theta^{(j)}(x_1,x_2)\\
&=\frac{1}{4}\left([x_1-\eta_1^{(j)}+\mathrm{Re}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2\right)\\
&+\frac{1}{16}\left([x_1-\eta_1^{(j)}+\mathrm{Re}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2\right)^2\\
&+\frac{1}{25}\left([x_1-\eta_1^{(j)}+\mathrm{Re}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2+[\mathrm{Im}\{\tau\}\{x_2-\eta_2^{(j)}\}]^2\right)^{5/2},
\end{aligned}
$$

$$
\beta^{(j)}(x_1,x_2)=-\sum_{i=1}^{2}\sum_{k=1}^{2}\lambda_{ik}n_i(x_1,x_2)\frac{\partial\theta^{(j)}}{\partial x_k}. \tag{21}
$$

Note that $\sigma^{(j)}(x_1,x_2)$ are the radial basis functions used in Ang, Clements and Vahdati [10] for anisotropic media. For $\lambda_{ij}=\delta_{ij}$ (Kronecker-delta), we find that $\tau=i$ and $\sigma^{(j)}(x_1,x_2)$ given above reduces to give the local interpolating functions suggested by Zhang and Zhu [14].

The functions $\Psi^{(j)}(\xi_1,\xi_2)$ in (21) are expressed in terms of line integrals over the boundary C and can easily be computed approximately using

$$
\begin{aligned}
\Psi^{(j)}(\xi_1,\xi_2)\simeq{}&\gamma(\xi_1,\xi_2)\theta^{(j)}(\xi_1,\xi_2)\\
&-\sum_{m=1}^{N}\{\beta^{(j)}(\eta_1^{(m)},\eta_2^{(m)})\Omega_3^{(m)}(\xi_1,\xi_2)\\
&-\beta^{(j)}(\eta_1^{(N+m)},\eta_2^{(N+m)})\Omega_4^{(m)}(\xi_1,\xi_2)\\
&-\theta^{(j)}(\eta_1^{(m)},\eta_2^{(m)})\Omega_1^{(m)}(\xi_1,\xi_2)\\
&-\theta^{(j)}(\eta_1^{(N+m)},\eta_2^{(N+m)})\Omega_2^{(m)}(\xi_1,\xi_2)\}.
\end{aligned}\tag{22}
$$

No discretization of the solution domain into elements is therefore needed in computing the domain integral in (18).

If we make the approximation

$$
\begin{aligned}
\psi^{(k)}(t)&\simeq\frac{1}{2}[\psi^{(k)}(t+\frac{1}{2}\Delta t)+\psi^{(k)}(t-\frac{1}{2}\Delta t)],\\
\frac{d}{dt}[\psi^{(k)}(t)]&\simeq\frac{\psi^{(k)}(t+\frac{1}{2}\Delta t)-\psi^{(k)}(t-\frac{1}{2}\Delta t)}{\Delta t},
\end{aligned}\tag{23}
$$

let (ξ_1, ξ_2) in (18) be given in turn by $(\eta_1^{(p)}, \eta_2^{(p)})$ for $p = 1,\ 2,\ \cdots,\ 2N + P$, and use the boundary conditions in (10), we obtain

$$
\begin{aligned}
&(\frac{1}{2}\alpha^{(p)}[\psi^{(p)}(t+\frac{1}{2}\Delta t)+\psi^{(p)}(t-\frac{1}{2}\Delta t)]+(1-\alpha^{(p)})R(\eta_1^{(p)},\eta_2^{(p)},t))\gamma(\eta_1^{(p)},\eta_2^{(p)})\\
&\quad=\sum_{k=1}^{2N+P}[\frac{1}{2}B(\eta_1^{(k)},\eta_2^{(k)})\{\psi^{(k)}(t+\frac{1}{2}\Delta t)+\psi^{(k)}(t-\frac{1}{2}\Delta t)\}\\
&\qquad\quad+\frac{1}{\Delta t}E^{(k)}(t)\{\psi^{(k)}(t+\frac{1}{2}\Delta t)-\psi^{(k)}(t-\frac{1}{2}\Delta t)\}]\alpha^{(k)}\mu^{(kp)}\\
&\quad+\sum_{k=1}^{2N}(1-\alpha^{(k)})\mu^{(kp)}(B(\eta_1^{(k)},\eta_2^{(k)})R(\eta_1^{(k)},\eta_2^{(k)},t)+E^{(k)}(t)\frac{d}{dt}\{R(\eta_1^{(k)},\eta_2^{(k)},t)\})\\
&\quad+\sum_{m=1}^{N_1}\{\Omega_1^{(m)}(\eta_1^{(p)},\eta_2^{(p)})R(\eta_1^{(m)},\eta_2^{(m)},t)+\Omega_2^{(m)}(\eta_1^{(p)},\eta_2^{(p)})R(\eta_1^{(N+m)},\eta_2^{(N+m)},t)\}\\
&\quad+\sum_{m=N_1+1}^{N}\{\frac{1}{2}(\Omega_1^{(m)}(\eta_1^{(p)},\eta_2^{(p)})-\Omega_3^{(m)}(\eta_1^{(p)},\eta_2^{(p)})f(\eta_1^{(m)},\eta_2^{(m)}))\\
&\qquad\qquad\times[\psi^{(m)}(t+\frac{1}{2}\Delta t)+\psi^{(m)}(t-\frac{1}{2}\Delta t)]\\
&\qquad\quad+\frac{1}{2}(\Omega_2^{(m)}(\eta_1^{(p)},\eta_2^{(p)})-\Omega_4^{(m)}(\eta_1^{(p)},\eta_2^{(p)})f(\eta_1^{(N+m)},\eta_2^{(N+m)}))\\
&\qquad\qquad\times[\psi^{(N+m)}(t+\frac{1}{2}\Delta t)+\psi^{(N+m)}(t-\frac{1}{2}\Delta t)]\}\\
&\quad-\sum_{m=1}^{N_1}\{q^{(m)}(t)\Omega_3^{(m)}(\eta_1^{(p)},\eta_2^{(p)})+q^{(N+m)}(t)\Omega_4^{(m)}(\eta_1^{(p)},\eta_2^{(p)})\}\\
&\quad-\sum_{m=N_1+1}^{N}\{\Omega_3^{(m)}(\eta_1^{(p)},\eta_2^{(p)})\frac{1}{\sqrt{g(\eta_1^{(m)},\eta_2^{(m)})}}v(\eta_1^{(m)},\eta_2^{(m)},t)]\\
&\qquad\qquad+\Omega_4^{(m)}(\eta_1^{(p)},\eta_2^{(p)})\frac{1}{\sqrt{g(\eta_1^{(N+m)},\eta_2^{(N+m)})}}v(\eta_1^{(N+m)},\eta_2^{(N+m)},t)\}\\
&\qquad\qquad\text{for } p=1,2,\cdots,2N+P, \qquad (24)
\end{aligned}
$$

where

$$
\begin{aligned}
E^{(k)}(t) &= D(\eta_1^{(k)}, \eta_2^{(k)}, \psi^{(k)}(t)), \\
\mu^{(kp)} &= \sum_{j=1}^{2N+P} \chi^{(kj)} \Psi^{(j)}(\eta_1^{(p)}, \eta_2^{(p)}), \\
\alpha^{(k)} &= \begin{cases} 0 & \text{for } 1 \leq k \leq N_1 \text{ or } N+1 \leq k \leq N+N_1, \\ 1 & \text{otherwise,} \end{cases} \\
R(\eta_1, \eta_2, t) &= \sqrt{g(\eta_1, \eta_2)} K(u(\eta_1, \eta_2, t)).
\end{aligned} \tag{25}
$$

If $E^{(n)}(t)$ and $\psi^{(n)}(t - \frac{1}{2}\Delta t)$ are assumed known for $n = 1, 2, \cdots, 2N+P$ then (24) constitutes a system of $2N+P$ linear algebraic equations containing $2N+P$ unknowns given by $q^{(m)}(t)$ and $q^{(N+m)}(t)$ for $m = 1, 2, \cdots, N_1$, $\psi^{(m)}(t+\frac{1}{2}\Delta t)$ and $\psi^{(N+m)}(t+\frac{1}{2}\Delta t)$ for $m = N_1+1, N_1+2, \cdots, N_1+N_2$, and $\psi^{(p)}(t+\frac{1}{2}\Delta t)$ for $p = 2N+1, 2N+2, \cdots, 2N+P$. (Note that $N_1+N_2 = N$.) The unknowns can be determined numerically by repeating the steps below until the numerical values of ψ at the selected points are obtained at the desired time level.

1. From the initial condition given in (10), compute the values of $\psi^{(n)}(0)$ for $n = 1, 2, \cdots, 2N+P$. Choose a small positive time-step Δt. Set the integer $J = 0$. Go to Step 2.

2. Estimate the values of $E^{(n)}(J\Delta t)$ using the latest known values of $\psi^{(n)}(J\Delta t)$, that is, $E^{(n)}(J\Delta t) \simeq D(\eta_1^{(n)}, \eta_2^{(n)}, \psi^{(n)}(J\Delta t))$. Go to Step 3.

3. Using the latest known values of $E^{(n)}(J\Delta t)$ and $\psi^{(n)}(J\Delta t)$, let $t = (J+\frac{1}{2})\Delta t$ in (24) to set up a system of linear algebraic equations and solve for the unknowns $q^{(m)}((J+\frac{1}{2})\Delta t)$ and $q^{(N+m)}((J+\frac{1}{2})\Delta t)$ for $m = 1, 2, \cdots, N_1$, $\psi^{(j)}((J+1)\Delta t)$ and $\psi^{(N+j)}((J+1)\Delta t)$ for $j = N_1+1$,

$N_1+2, \cdots, N_1+N_2$, and $\psi^{(p)}((J+1)\Delta t)$ for $p = 2N+1, 2N+2, \cdots, 2N+P$. Go to Step 4.

4. Use the latest known values of $\psi^{(n)}((J+1)\Delta t)$ obtained in Step 3 above to compute $\psi^{(n)}((J+\frac{1}{2})\Delta t) = \frac{1}{2}[\psi^{(n)}((J+1)\Delta t) + \psi^{(n)}(J\Delta t)]$ for $n = 1, 2, \cdots, 2N+P$. Re-calculate $E^{(n)}(J\Delta t)$ using $E^{(n)}(J\Delta t) \simeq D(\eta_1^{(n)}, \eta_2^{(n)}, \psi^{(n)}((J+\frac{1}{2})\Delta t))$. Check whether the newly obtained values of $E^{(n)}(J\Delta t)$ agree with the previous values to within a specified number of significant figures. If the required convergence is not achieved, go to Step 3. Otherwise, increase the current value of J by 1 and go to Step 2.

5 Specific numerical examples

To assess the validity and accuracy of the dual-reciprocity boundary element procedure outlined above, it is applied to solve some specific problems here.

Problem 1. The thermal conductivity, density and specific heat capacity of the material are taken to be given by $k_{ij} = \delta_{ij}(1+T)$, $\rho = 1$ and $c = 1+\frac{1}{2}T$ respectively. Note that δ_{ij} is the Kronecker-delta. The solution domain R is taken to be rectangular in shape, defined by $0 < x_1 < 1$, $0 < x_2 < 1/5$. Refer to Figure 1. The initial-boundary conditions are given by

$$\begin{aligned}
T(x_1, x_2, 0) &= 0 \text{ for } (x_1, x_2) \in R, \\
T(0, x_2, t) &= 1 \text{ for } 0 < x_2 < \frac{1}{5} \text{ and } t > 0, \\
T(1, x_2, t) &= 0 \text{ for } 0 < x_2 < \frac{1}{5} \text{ and } t > 0, \\
k_{ij} n_i \frac{\partial T}{\partial x_j}\bigg|_{x_2=0} &= 0 \text{ for } 0 < x_1 < 1 \text{ and } t > 0, \\
k_{ij} n_i \frac{\partial T}{\partial x_j}\bigg|_{x_2=1/5} &= 0 \text{ for } 0 < x_1 < 1 \text{ and } t > 0.
\end{aligned}$$

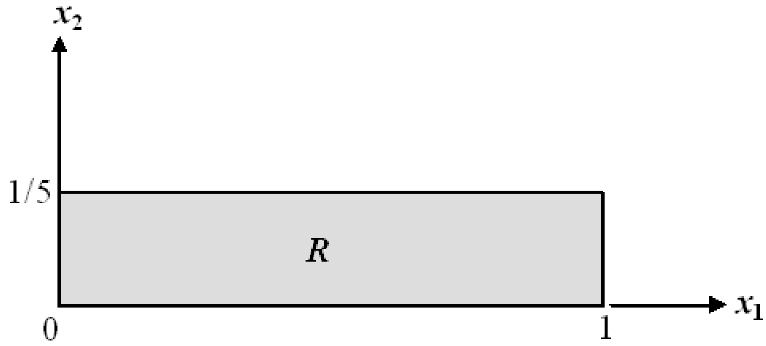

Figure 1. A sketch of the solution domain for Problem 1.

For this specific problem, $\lambda_{ij} = \delta_{ij}$, $h(T) = 1+T$ and $g(x_1, x_2) = 1$. Hence, from (4) and (7), we may write $\psi/\sqrt{g} = T + \frac{1}{2}T^2$. Solving for T and taking the positive square root of the determinant yields $T = -1 + \sqrt{2\psi/\sqrt{g} + 1}$. It follows that the partial differential equation to solve is given by

$$\frac{\partial^2\psi}{\partial x_i \partial x_i} = \frac{1 + \sqrt{\left(2\psi/\sqrt{g} + 1\right)}}{2\sqrt{\left(2\psi/\sqrt{g} + 1\right)}} \frac{\partial\psi}{\partial t},$$

and the initial-boundary conditions are

$$\begin{aligned}
\psi(x_1, x_2, 0) &= 0 \text{ for } (x_1, x_2) \in R, \\
\psi(0, x_2, t) &= \frac{3}{2} \text{ for } 0 < x_2 < \frac{1}{5} \text{ and } t > 0, \\
\psi(1, x_2, t) &= 0 \text{ for } 0 < x_2 < \frac{1}{5} \text{ and } t > 0, \\
q(x_1, 0, t) &= 0 \text{ for } 0 < x_1 < 1 \text{ and } t > 0, \\
q(x_1, \frac{1}{5}, t) &= 0 \text{ for } 0 < x_1 < 1 \text{ and } t > 0.
\end{aligned}$$

The sides of the rectangular domain are discretized into N equal length boundary elements. The interior collocation points are given by $(m/(M_1 + 1), n/[5(M_2+1)])$ for $m = 1, 2, \cdots, M_1$ and $n = 1, 2, \cdots, M_2$. The parameter r in (14) is chosen to be $1/4$ and the time-step Δt by $1/(J_0 + \frac{1}{2})$, where J_0

is a selected positive integer. The time-stepping dual-reciprocity boundary element method (DRBEM) is applied to compute $\psi(x_1, x_2, t)$ numerically. Specifically, in following the steps outlined in Section 4, we take $J = 0$, 1, 2, $\cdots$, J_0 (consecutively) to compute numerically the values of ψ at the chosen collocation points at consecutive time levels until we obtain $\psi^{(n)}(1) = \frac{1}{2}[\psi^{(n)}((J_0+1)/(J_0+\frac{1}{2})) + \psi^{(n)}(J_0/(J_0+\frac{1}{2}))]$ for $n = 1, 2, \cdots, 2N + M_1M_2$. For each value of J, that is, at each time level, the final numerical values of $\psi^{(n)}((J+\frac{1}{2})/(J_0+\frac{1}{2}))$ are obtained after iterating 3 times to and fro the last two steps in Section 4. The numerical values of $\psi^{(n)}(1)$ obtained using $N = 96$ (each element of length 0.125 units), $M_1 = 9$, $M_2 = 7$ (63 interior collocation points) and $J_0 = 10$ ($\Delta t = 2/21 \simeq 0.0952$) are used to compute the temperature at the collocation points $(0.20, 0.10)$, $(0.40, 0.10)$, $(0.60, 0.10)$ and $(0.80, 0.10)$ at time $t = 1$. The numerical values of the temperature are compared with those given in Azis and Clements [9] and Goto and Suzuki [7] in Table 1. (In references [7] and [9], the temperature is calculated using a different boundary element approach.) The three sets of numerical values appear to be reasonably close to one another, but our results here seem to be closer to those given by Goto and Suzuki [7] than Azis and Clements [9].

Table 1. Numerical values of T at time $t = 1$ at selected interior points.

(x_1, x_2)	Azis and Clements	Goto and Suzuki	Present DRBEM
$(0.20, 0.10)$	0.8398	0.8439	0.8448
$(0.40, 0.10)$	0.6664	0.6733	0.6738
$(0.60, 0.10)$	0.4754	0.4832	0.4826
$(0.80, 0.10)$	0.2589	0.2649	0.2649

Problem 2. In a particular problem considered in Azis and Clements [9], the thermal conductivity, density and specific heat capacity are taken to be such that

$$[k_{ij}] = (1 + \frac{1}{10}x_1)^2 \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \text{ and } \rho c = \frac{9}{2T}(1 + \frac{1}{10}x_1).$$

Here $\lambda_{11} = 3$, $\lambda_{12} = \lambda_{21} = 1$, $\lambda_{22} = 4$, $g(x_1, x_2) = (1 + \frac{1}{10}x_1)^2$ and $h(T) = 1$.

With k_{ij} and ρc as given above, it may be verified that a particular solution of (1) is given by

$$T(x_1, x_2, t) = \frac{1 - \frac{1}{4}(x_1 + x_2)^2}{(1 + t)(1 + \frac{1}{10}x_1)}.$$

Proceeding as in Azis and Clements [9], we use the above particular solution to generate boundary data for T and $k_{ij}n_i\partial T/\partial x_j$ respectively on the horizontal and vertical sides of the square domain $0 < x_1 < 1$, $0 < x_2 < 1$. The solution is also used to generate initial data for T at $t = 0$.

For this particular problem, $\psi(x_1, x_2, t) = (1+\frac{1}{10}x_1)T(x_1, x_2, t)$. We apply the DRBEM to solve for ψ in the square domain subject to the generated initial-boundary data. The boundary of the square domain is divided into N elements of equal length and the interior collocation points are chosen as $(m/(M + 1), n/(M + 1))$ for $m = 1, 2, \cdots, M$ and $n = 1, 2, \cdots, M$. As in the first problem above, we take the parameter r in (14) to be 1/4.

Two sets of numerical values of T at the point $(0.50, 0.50)$ at selected time instants are compared with the exact solution in Table 2. Set A is obtained by using $(N, M) = (60, 7)$ (49 interior collocation points) and $\Delta t = 0.20$, while Set B by $(N, M) = (160, 15)$ (225 interior collocation points) and $\Delta t = 2/30 \simeq 0.06667$. At a given time level, both sets of numerical values are obtained after iterating 5 times to and fro the last two steps in Section 4. Set B gives more accurate numerical values of T than Set A, that is, the numerical solution shows convergence towards the exact one when the DRBEM calculation is refined. Numerical values of T obtained by Azis and Clements [9] by using a very small time-step of 0.001, discretizing the boundary into 160 boundary elements and subdividing the solution domain into 1600 cells (for treating the domain integral in their formulation) are also shown in Table 2 for $t = 0.10$, 0.30 and 0.50. (No numerical value of T is given in [9] for $t = 0.70$ and $t = 0.90$.) Even though a relatively large time-step is used here in Set A, the numerical values obtained appear to be quite comparable in accuracy with those of [9].

Table 2. Numerical and exact values of T at $(0.50, 0.50)$ at selected time instants

Time t	Set A	Set B	Azis and Clements	Exact
0.10	0.65327	0.64973	0.64907	0.64935
0.30	0.54941	0.54946	0.54914	0.54945
0.50	0.47550	0.47614	0.47587	0.47619
0.70	0.41969	0.42010	–	0.42017
0.90	0.37542	0.37585	–	0.37594

Problem 3. In the two particular problems above, the spatial function g in (2) is such that the coefficient B in the partial differential equation in (8) is zero. For a problem in which B does not vanish in the solution domain, here we take

$$[k_{ij}] = T\exp(x_1)\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } \rho c = T\exp(x_1),$$

so that $\lambda_{11} = 1$, $\lambda_{12} = \lambda_{21} = 1$, $\lambda_{22} = 3$, $g(x_1, x_2) = \exp(x_1)$ and $h(T) = T$.

The solution domain is as in Problem 1 (Figure 1), that is, $0 < x_1 < 1$, $0 < x_2 < 1/5$. The initial-boundary conditions are taken to be

$$\begin{aligned}
T(x_1, x_2, 0) &= \exp(-\frac{1}{4}x_1) \text{ for } (x_1, x_2) \in R, \\
T(0, x_2, t) &= \exp(-\frac{1}{8}t) \text{ for } 0 < x_2 < \frac{1}{5} \text{ and } t > 0, \\
T(1, x_2, t) &= \exp(-\frac{1}{8}t - \frac{1}{4}) \text{ for } 0 < x_2 < \frac{1}{5} \text{ and } t > 0, \\
k_{ij}n_i\frac{\partial T}{\partial x_j}\bigg|_{x_2=0} &= 0 \text{ for } 0 < x_1 < 1 \text{ and } t > 0, \\
k_{ij}n_i\frac{\partial T}{\partial x_j}\bigg|_{x_2=1/5} &= 0 \text{ for } 0 < x_1 < 1 \text{ and } t > 0.
\end{aligned}$$

To apply the DRBEM to solve for $\psi = \frac{1}{2}\exp(\frac{1}{2}x_1)T^2$, we discretize the sides of the rectangular region into 180 equal length boundary elements, employ 105 evenly spaced out collocation points in the interior of the solution domain and choose r in (14) and Δt to be 1/4 and 2/41 respectively.

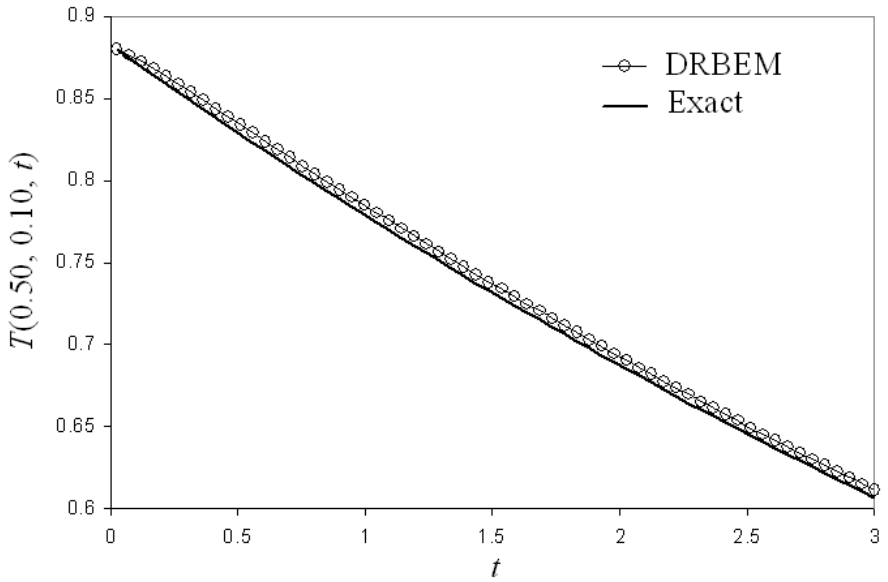

Figure 2. A graphical comparison of the numerical and exact T .

In Figure 2, over the time interval $0 \leq t \leq 3$, we make a graphical comparison of the numerical values of $T(0.50, 0.10, t)$ with the exact solution given by $T = \exp(-\frac{1}{8}t - \frac{1}{4}x_1)$. The two graphs are in reasonably good agreement with each other. The percentage errors for all the numerical values of $T(0.50, 0.10, t)$ over $0 \leq t \leq 3$ are less than 0.8%.

6 Conclusion

The task of solving a class of two-dimensional initial-boundary value problems governed by a generalized nonlinear heat equation for nonhomogeneous anisotropic media is considered. With the aid of the Kirchhoff's transformation and an appropriate substitution of variables, the partial differential equation is recast in a form that allows the initial-boundary value problem to be formulated in terms of an integro-differential equation suitable for the development of a dual-reciprocity boundary element method. A time-stepping dual-reciprocity boundary element method is presented for the numerical solution of the initial-boundary value problem. To assess the validity and accuracy of the dual-reciprocity boundary element method, it is applied to solve a few specific problems with known solutions. The numerical results obtained agree favorably with the known solutions indicating that the method can be used to provide reliable and accurate numerical solutions for the nonlinear heat equation.

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