Hypersingular integral equations for periodic arrays of planar cracks in a periodically layered anisotropic elastic space under antiplane shear stress

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Abstract

The problem of determining the antiplane shear stress around periodic arrays of planar cracks in a periodically-layered anisotropic elastic space is considered. With a suitable integral expression for the displacement, it is eventually reduced to a system of simultaneous hypersingular integral equations containing the 'crack-opening displacements' as unknown functions. Once the integral equations are solved, crack parameters of interest, such as the crack tip stress intensity factors, may be readily computed. For some specific examples of the problem, the integral equations are solved numerically by using a collocation technique, in order to compute the relevant stress intensity factors.

Keywords: Periodically-located cracks, periodically-layered anisotropic material, antiplane shear stress, hypersingular integral equations.

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1 Introduction

Composites which are made up of two or more layers of dissimilar materials are playing an increasingly important role in modern technology. For example, media comprising a large number of very fine layers are employed in optical recording, and synthetic materials, such as plywood and fabric laminates, are widely used in the design and construction of modern aircrafts. The importance of assessing the reliability and integrity of such composites has prompted many researchers to carry out studies of cracks in multilayered materials, e.g. Ryvkin [1], Lahiri et al. [2], Ang and Clements [3], Ang [4] and Clements [5].

In the present paper, we consider the problem of analysing the stress distribution around planar cracks in an elastic medium comprising infinitely many layers that are alternately occupied by two dissimilar anisotropic materials. The cracks are acted upon by an internal antiplane shear stress. The geometries of the layers and the cracks as well as the internal stress acting on the cracks are periodic along the direction that is normal to the interfaces separating the two dissimilar materials.

Problems concerning periodic arrays of cracks in elastic media may be of useful relevance to certain physical situations in which multiple cracking occurs. They have been studied by various authors, e.g. Ang and Park [6], Ang and Clements [7], Tweed and Melrose [8] and Benthem and Koiter [9].

To solve the problem presently under consideration, we first derive a suitable integral expressions for the displacement, satisfying the relevant continuity conditions on the interfaces of the layers. The conditions on the cracks then lead to a system of hypersingular integral equations with the 'crack-opening displacements' as unknown functions. The integral equations can be solved numerically. Once they are solved, crack parameters of interest, such as the crack tip stress intensity factors, can be easily computed. For specific examples of the problem, we solve the integral equations numerically in order to calculate the relevant stress intensity factors.

2 Statement of the problem

With reference to a Cartesian coordinate system given by $0x_1x_2x_3$, consider an infinite elastic space which comprises infinitely many layers $S^{(2m)} = \{(x_1, x_2, x_3) : \ 2m(h+d) < x_2 < 2m(h+d)+2h\}$ and $S^{(2m+1)} = \{(x_1, x_2, x_3) : 2m(h+d)+2h < x_2 < 2(m+1)(h+d)\}$, where $m = 0, \pm 1, \pm 2, \cdots$, and $2h$ and $2d$ are the thicknesses of the layers $S^{(2m)}$ and $S^{(2m+1)}$ respectively. The layers $S^{(2m)}$ and $S^{(2m+1)}$ are alternately occupied by two dissimilar *anisotropic* materials. The materials are assumed to adhere perfectly to each other along the plane interfaces $x_2 = 2m(h+d) + 2h$ and $x_2 = 2(m+1)(h+d)$ $(m = 0, \pm 1, \pm 2, \cdots)$.

The interior of each of the layers $S^{(2m)}$ $(m = 0, \pm 1, \pm 2, \cdots)$ contains N arbitrarily-located planar cracks denoted by $C_m^{(1)}, C_m^{(2)}, \cdots, C_m^{(N-1)}$ and $C_m^{(N)}$. We assume that the geometries of the cracks do not vary along the x_3-axis and the cracks do not intersect with one another or the interfaces separating the layers. On the $0x_1x_2$ plane, the tips of a typical k-th crack in the layer $S^{(2m)}$, i.e. $C_m^{(k)}$, are given by $(a^{(k)}, b^{(k)} + 2m[h+d])$ and $(c^{(k)}, d^{(k)} + 2m[h+d])$, where $0 < b^{(k)} < 2h$ and $0 < d^{(k)} < 2h$. Thus, for a fixed integer k $(1 \leq k \leq N)$, $C_0^{(k)}$, $C_{\pm 1}^{(k)}$, $C_{\pm 2}^{(k)}$, $C_{\pm 3}^{(k)}, \cdots$ constitute an array of periodically-located planar cracks in the periodically-layered elastic space. The other layers $S^{(\pm 1)}$, $S^{(\pm 3)}$, $S^{(\pm 5)}, \cdots$ are *flawless*, devoid of any cracks. A sketch of the problem for $N = 2$ is given in Figure 1.

The cracks are opened up by the action of a suitably prescribed internal traction which is independent of time and the x_3 coordinate and which are periodic functions of x_2 with period $2[h+d]$. We further assume that the cracks are deformed in such a way that the only non-zero component of the Cartesian displacement is the one along the x_3-direction. The stress is required to vanish as $|x_1| \to \infty$. The problem is then to determine the Cartesian displacement and stress in the layers, particularly near the cracks.

3 Hypersingular integral equations

For the problem described in Section 2, if the x_3-component (the only non-zero component) of the displacement is given by $u_3^{(m)} = u_3^{(m)}(x_1, x_2)$ for $(x_1, x_2) \in S^{(2m)} \cup S^{(2m+1)}$, then the only non-zero components of the stress (in $S^{(2m)} \cup S^{(2m+1)}$) are

$$\sigma_{k3}^{(m)} = \lambda_{kj} \frac{\partial u_3^{(m)}}{\partial x_j} \text{ for } k = 1, 2, \tag{1}$$

where the elastic moduli $\lambda_{kj} = \lambda_{jk}$ $(k, j = 1, 2)$ are piecewise constant functions of x_2 given by

$$\lambda_{kj} = \begin{cases} \lambda_{kj}^{(0)} & \text{if} \quad 2m(h+d) < x_2 < 2m(h+d) + 2h, \\ \lambda_{kj}^{(1)} & \text{if} \quad 2m(h+d) + 2h < x_2 < 2(m+1)(h+d). \end{cases} \tag{2}$$

Physical constraint requires that the *constants* $\lambda_{kj}^{(n)}$ $(n = 0, 1)$ satisfy the strict inequality $(\lambda_{12}^{(n)})^2 - \lambda_{11}^{(n)}\lambda_{22}^{(n)} < 0$. Also, notice that in (1) and throughout the present paper, the Einsteinian convention of summing over a repeated index is assumed to hold for only latin subscripts which run from 1 to 2.

Let us define

$$\left.\begin{array}{r} w^{(m)}(y_1,y_2) \equiv u_3^{(m)}(y_1,y_2+2m[h+d]) \\ s_k^{(m)}(y_1,y_2) \equiv \sigma_{k3}^{(m)}(y_1,y_2+2m[h+d]) \end{array}\right\} \text{ for } y_2 \in (0,2h)\cup(2h,2[h+d]). \tag{3}$$

Since the materials occupying the layered space adhere perfectly to each other, we have to impose the continuity conditions

$$\begin{array}{c} w^{(m)}(y_1,[2h]^-) - w^{(m)}(y_1,[2h]^+) = 0 \\ s_2^{(m)}(y_1,[2h]^-) - s_2^{(m)}(y_1,[2h]^+) = 0 \\ w^{(m)}(y_1,[2(h+d)]^-) - w^{(m)}(y_1,0^+) = 0 \\ s_2^{(m)}(y_1,[2(h+d)]^-) - s_2^{(m)}(y_1,0^+) = 0 \end{array} \tag{4}$$

for $-\infty < y_1 < \infty$. Notice that, due to the periodicity in the geometries and boundary conditions along the x_2 direction, we have $w^{(0)}(\underline{\mathbf{y}}) = w^{(m)}(\underline{\mathbf{y}})$ and $s_k^{(0)}(\underline{\mathbf{y}}) = s_k^{(m)}(\underline{\mathbf{y}})$ ($\underline{\mathbf{y}} = (y_1,y_2)$) for $-\infty < y_1 < \infty$, $y_2 \in (0,2h)\cup(2h,2[h+d])$, and $m = \pm1,\pm2,\pm3,\cdots$.

We choose $w^{(m)}(\underline{\mathbf{y}})$ (for $m = 0,\pm1,\pm2,\cdots$) to be given by

$$w^{(m)}(\underline{\mathbf{y}}) = \sum_{p=1}^{N} \int_{D^{(p)}} r(\underline{\boldsymbol{\xi}}) U(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) dS(\underline{\boldsymbol{\xi}}), \tag{5}$$

where $\underline{\boldsymbol{\xi}} = (\xi_1,\xi_2)$ (with $-\infty < \xi_1 < \infty$ and $0 < \xi_2 < 2h$), $D^{(p)}$ denotes the directed straight line from $(a^{(p)},b^{(p)})$ to $(c^{(p)},d^{(p)})$, $r(\underline{\boldsymbol{\xi}})$ is an unknown function yet to be determined, $\underline{\mathbf{n}}^{(p)} = ([d^{(p)}-b^{(p)}]/\ell^{(p)},[a^{(p)}-c^{(p)}]/\ell^{(p)}) \equiv (n_1^{(p)},n_2^{(p)})$, $\ell^{(p)} = \sqrt{(a^{(p)}-c^{(p)})^2+(d^{(p)}-b^{(p)})^2}$ is the length of the cracks $C_m^{(p)}$, and

$$\begin{aligned} & U(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) \\ = \; & \frac{1}{2\pi}[H(y_2)-H(y_2-2h)]\,\mathrm{Re}\left\{\frac{\Theta^{(p)}}{(\xi_1-y_1)+\tau^{(0)}(\xi_2-y_2)}\right\} \\ & +G(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) \;\text{ for } y_2 \in (0,2h)\cup(2h,2[h+d]), \end{aligned} \tag{6}$$

where $L_k^{(p)} = \lambda_{k1}^{(p)} + \tau^{(p)}\lambda_{k2}^{(p)}$, $\tau^{(p)} = \left(-\lambda_{12}^{(p)} + i\sqrt{\lambda_{11}^{(p)}\lambda_{22}^{(p)} - (\lambda_{12}^{(p)})^2}\right)/\lambda_{22}^{(p)}$ ($p = 0,1$), $i = \sqrt{-1}$, $\Theta^{(p)} = L_k^{(0)} n_k^{(p)}$, $H(y)$ is the usual Heaviside unit-step function, $G(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ is a function yet to be determined and Re denotes the real part of a complex number.

For the displacement $w^{(m)}(\underline{\mathbf{y}})$ in (5) to satisfy the equilirium equations of elasticity, the function $G(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ in (6) must satisfy

$$\begin{aligned} \lambda_{kj}^{(0)} \frac{\partial^2 G}{\partial y_k \partial y_j} &= 0 \text{ for } 0 < y_2 < 2h, \\ \lambda_{kj}^{(1)} \frac{\partial^2 G}{\partial y_k \partial y_j} &= 0 \text{ for } 2h < y_2 < 2[h+d]. \end{aligned} \tag{7}$$

For (4) to be satisfied, the function G must be chosen in such a way that, for $-\infty < y_1 < \infty$,

$$\begin{aligned}
U(y_1,[2h]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - U(y_1,[2h]^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) &= 0,\\
\Xi(y_1,[2h]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - \Xi(y_1,[2h]^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) &= 0,\\
U(y_1,[2(h+d)]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - U(y_1,0^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) &= 0,\\
\Xi(y_1,[2(h+d)]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - \Xi(y_1,0^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) &= 0,
\end{aligned} \tag{8}$$

where

$$\Xi(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) = \begin{cases} \lambda_{2j}^{(0)}\partial U/\partial y_j & \text{if} \quad 0 < y_2 < 2h, \\ \lambda_{2j}^{(1)}\partial U/\partial y_j & \text{if} \quad 2h < y_2 < 2[h+d]. \end{cases}$$

In addition, the partial derivatives of $U(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ with respect to y_j are required to vanish as $|y_1| \to \infty$.

We can rewrite conditions (8) as:

$$\begin{aligned}
\int_{-\infty}^{\infty}\left[U(y_1,[2h]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - U(y_1,[2h]^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\right]\exp(-i\gamma y_1)dy_1 &= 0,\\
\int_{-\infty}^{\infty}\left[\Xi(y_1,[2h]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - \Xi(y_1,[2h]^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\right]\exp(-i\gamma y_1)dy_1 &= 0,\\
\int_{-\infty}^{\infty}\left[U(y_1,[2(h+d)]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - U(y_1,0^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\right]\exp(-i\gamma y_1)dy_1 &= 0,\\
\int_{-\infty}^{\infty}\left[\Xi(y_1,[2(h+d)]^-;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) - \Xi(y_1,0^+;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\right]\exp(-i\gamma y_1)dy_1 &= 0,
\end{aligned} \tag{9}$$

where $\gamma > 0$ is a constant parameter.

For the solution of (7) subject to (9), we choose

$$\begin{aligned}
& G(\underline{\mathbf{y}};\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\\
= \; & \frac{1}{2\pi}\left[H(y_2) - H(y_2-2h)\right]\\
& \times \operatorname{Re}\int_0^\infty \Big[E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(iuz^{(0)})\\
& \qquad + \overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(iu\overline{z}^{(0)})du + \Lambda(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\Big]\,du\\
& + \frac{1}{2\pi}\left[H(y_2-2h) - H(y_2-2[h+d])\right]\\
& \times \operatorname{Re}\int_0^\infty \Big[E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(iuz^{(1)})\\
& \qquad + \overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(iu\overline{z}^{(1)})du + \Lambda(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\Big]\,du\\
& \text{for } y_2 \in (0,2h) \cup (2h, 2[h+d]),
\end{aligned} \tag{10}$$

where $z^{(p)} = y_1 + \tau^{(p)} y_2$ $(p = 0,1)$, $E^{(m)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$, $\overline{F}^{(m)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ $(m = 0,1)$ and $\Lambda(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ are functions yet to be determined, and the overhead bar denotes the complex conjugate of a complex number.

Now, the first two conditions in (9) give rise to

$$\begin{aligned}
& E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(1)}h)+\overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(1)}h) \\
& -E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(0)}h)-\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(0)}h) \\
= \;& i\Theta^{(p)}\exp(-iu[\xi_1+\tau^{(0)}\{\xi_2-2h\}]), \\
& L_2^{(1)}E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(1)}h)+\overline{L}_2^{(1)}\overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(1)}h) \\
& -L_2^{(0)}E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(0)}h)-\overline{L}_2^{(0)}\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(0)}h) \\
= \;& iL_2^{(0)}\Theta^{(p)}\exp(-iu[\xi_1+\tau^{(0)}\{\xi_2-2h\}]), \qquad (11)
\end{aligned}$$

and the last two conditions yield

$$\begin{aligned}
& E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(1)}[h+d]) \\
& +\overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(1)}[h+d]) \\
& -E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})-\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) \\
= \;& i\overline{\Theta}^{(p)}\exp(-iu[\xi_1+\overline{\tau}^{(0)}\xi_2]), \\
& L_2^{(1)}E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(1)}[h+d]) \\
& +\overline{L}_2^{(1)}\overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(1)}[h+d]) \\
& -L_2^{(0)}E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})-\overline{L}_2^{(0)}\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) \\
= \;& i\overline{L}_2^{(0)}\overline{\Theta}^{(p)}\exp(-iu[\xi_1+\overline{\tau}^{(0)}\xi_2]). \qquad (12)
\end{aligned}$$

Notice that, in our derivation of (11) and (12), we make use of the inequality $0<\xi_2<2h$ and the results (taken from Erdélyi *et al.* [10])

$$\begin{aligned}
\int_{-\infty}^{\infty}(a-ix)^{-1}\exp(-ixy)dx &= H(y)2\pi\exp(-ay), \\
\int_{-\infty}^{\infty}(a+ix)^{-1}\exp(-ixy)dx &= -H(-y)2\pi\exp(ay), \\
\int_{-\infty}^{\infty}\exp(ip[u-x])dp &= 2\pi\delta(u-x),
\end{aligned}$$

where a is a constant such that $\mathrm{Re}\{a\}>0$ and $\delta(x)$ is the Dirac-delta function.

From (11), we obtain

$$\begin{aligned}
& E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(1)}h) \\
= \;& -\Psi\left\{i\Omega\Theta^{(p)}\exp(-iu[\xi_1+\tau^{(0)}\{\xi_2-2h\}])\right.
\end{aligned}$$

$$
\begin{aligned}
&+\Omega E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(0)}h)+\overline{\Gamma}\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(0)}h)\Big\},\\
&\overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(1)}h)\\
=\ &\Psi\Big\{i\Gamma\Theta^{(p)}\exp(-iu[\xi_1+\tau^{(0)}\{\xi_2-2h\}])\\
&+\Gamma E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\tau^{(0)}h)+\overline{\Omega}\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\exp(2iu\overline{\tau}^{(0)}h)\Big\},
\end{aligned}
\tag{13}
$$

where $\Psi=1/(L_2^{(1)}-\overline{L}_2^{(1)})$, $\Omega=\overline{L}_2^{(1)}-L_2^{(0)}$ and $\Gamma=L_2^{(1)}-L_2^{(0)}$.

Substitution of (13) into (12) yields

$$
\begin{aligned}
&\left[\Upsilon_1(u)\exp(2iu\tau^{(0)}h)-1\right]E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\\
&+\left[\overline{\Upsilon}_1(-u)\exp(2iu\overline{\tau}^{(0)}h)-1\right]\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\\
=\ &i\left[\overline{\Theta}^{(p)}\exp(-iu\overline{\tau}^{(0)}\xi_2)-\Theta^{(p)}\Upsilon_1(u)\exp(-iu\tau^{(0)}\{\xi_2-2h\})\right]\exp(-iu\xi_1),\\
&\left[\Upsilon_2(u)\exp(2iu\tau^{(0)}h)-L_2^{(0)}\right]E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\\
&+\left[\overline{\Upsilon}_2(-u)\exp(2iu\overline{\tau}^{(0)}h)-\overline{L}_2^{(0)}\right]\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\\
=\ &i\Big[\overline{L}_2^{(0)}\overline{\Theta}^{(p)}\exp(-iu\overline{\tau}^{(0)}\xi_2)\\
&-\Theta^{(p)}\Upsilon_2(u)\exp(-iu\tau^{(0)}\{\xi_2-2h\})\Big]\exp(-iu\xi_1),
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
\Upsilon_1(u)&=\Psi\left[\Gamma\exp(2iu\overline{\tau}^{(1)}d)-\Omega\exp(2iu\tau^{(1)}d)\right],\\
\Upsilon_2(u)&=\Psi\left[\overline{L}_2^{(1)}\Gamma\exp(2iu\overline{\tau}^{(1)}d)-L_2^{(1)}\Omega\exp(2iu\tau^{(1)}d)\right].
\end{aligned}
\tag{15}
$$

Equations (14) can be readily inverted to obtain $E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$. The functions $E^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(1)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ can then be determined from (13). From (14), it is clear that $E^{(m)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(m)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ $(m=0,1)$ are singular at $u=0$. Specifically, they behave as $\mathbf{O}(1/u)$ for u close to 0. The improper integrals over $[0,\infty)$ in (10) will be divergent if we do not choose $\Lambda(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ in a careful manner to cancel out the singularities in the integrands. To substract away these singularities, we select

$$
\Lambda(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})=-\frac{1}{u}\kappa(\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}),
\tag{16}
$$

where

$$
\kappa(\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})=\lim_{u\to0^+}u\left[E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})+\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})\right].
\tag{17}
$$

Now that we can determine $E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ from (14), we are ready to use (5) together with (6) and (10) to deal with the conditions on the crack faces. These conditions can be written as

$$s_k^{(0)}(\underline{\mathbf{y}})n_k^{(q)} \to -s_k^{(\mathrm{int})}(\underline{\boldsymbol{\xi}})n_k^{(q)} \text{ as } \underline{\mathbf{y}} \to \underline{\boldsymbol{\xi}} \in D^{(q)} \ (q=1,2,\cdots,N), \qquad (18)$$

where $s_k^{(\mathrm{int})}(\underline{\boldsymbol{\xi}}) = \sigma_{k3}^{(\mathrm{int})}(\underline{\boldsymbol{\xi}})$ and $\sigma_{k3}^{(\mathrm{int})}(x_1,x_2)$ which give the non-zero components of the internal stress acting on the cracks are assumed to be periodic functions of x_2 with period $2[h+d]$. Notice that in (18) $\underline{\boldsymbol{\xi}} = (\xi_1,\xi_2)$, with $0<\xi_2<2h$.

Using (5) together with (6) and (10), we find that the conditions in (18) give rise to:

$$\begin{aligned} & \chi^{(q)}\mathcal{H}\int_{-1}^{1}\frac{R^{(q)}(t)dt}{(t-s)^2} + \sum_{p=1}^{N}\int_{-1}^{1}K^{(qp)}(t,s)R^{(p)}(t)dt \\ = \ & -s_k^{(\mathrm{int})}(X_1^{(q)}(s),X_2^{(q)}(s))n_k^{(q)} \ \text{ for } -1<s<1 \quad (q=1,2,\cdots,N), \end{aligned} \qquad (19)$$

where $\mathcal{H}$ denotes that the integral over the appropriate interval is to be interpreted in the Hadamard finite-part sense, $2X_1^{(q)}(s) = (a^{(q)}+c^{(q)})+(c^{(q)}-a^{(q)})s$, $2X_2^{(q)}(s) = (b^{(q)}+d^{(q)})+(d^{(q)}-b^{(q)})s$, $R^{(q)}(t) = r(X_1^{(q)}(t),X_2^{(q)}(t))$ and

$$\begin{aligned} \chi^{(q)} &= \frac{1}{\pi}\mathrm{Re}\left\{\frac{\left[\Theta^{(q)}\right]^2\ell^{(q)}}{\left[(c^{(q)}-a^{(q)})+\tau^{(0)}(d^{(q)}-b^{(q)})\right]^2}\right\}, \\ K^{(qp)}(t,s) &= \frac{\ell^{(p)}}{4\pi}\mathrm{Re}\{\frac{\Theta^{(q)}\Theta^{(p)}(1-\delta_{qp})}{\left[(X_1^{(p)}(t)-X_1^{(q)}(s))+\tau^{(0)}(X_2^{(p)}(t)-X_2^{(q)}(s))\right]^2} \\ & \quad +\int_0^\infty iu\left[\Theta^{(q)}E^{(0)}(u;X_1^{(p)}(t),X_2^{(p)}(t);\underline{\mathbf{n}}^{(p)})\right. \\ & \qquad \times\exp(iu\tau^{(0)}X_2^{(q)}(s)) \\ & \quad \left.+\overline{\Theta}^{(q)}\overline{F}^{(0)}(u;X_1^{(p)}(t),X_2^{(p)}(t);\underline{\mathbf{n}}^{(p)})\exp(iu\overline{\tau}^{(0)}X_2^{(q)}(s))\right] \\ & \qquad \times\exp[iuX_1^{(q)}(s)]du\}. \end{aligned} \qquad (20)$$

Equations (19) constitute a system of N hypersingular integral equations from which we can solve for the unknown functions $R^{(q)}(t)$ $(q=1,2,...,N)$ (and hence $r(\underline{\mathbf{y}})$ for $\underline{\mathbf{y}}\in D^{(q)}$). Once the unknown functions are determined, the displacement can be computed by using (5), and other physical quantities of interest, such as the crack tip stress intensity factors, can also be readily calculated. Numerical techniques for solving the hypersingular integral equations are available.

Using (5) together with the limit

$$\lim_{\varepsilon\to 0^{\pm}} \varepsilon \int_{-1}^{1} \frac{r(t)dt}{[t-s]^2+\varepsilon^2} = \pm\pi r(s) \text{ for } -1<s<1,$$

the functions $R^{(q)}(s)$ can be shown to be directly related to the difference between the displacement on the "upper" face of the crack $C_m^{(q)}$ and that on the "lower" face, or the so called "crack-opening displacement." As we shall see in the next section, once $R^{(p)}(u)$ are determined, physical quantities of interest, such as the crack tip stress intensity factors, can be readily computed.

4 Special cases

If we let either $h\to\infty$ or $d\to\infty$ (not both), the periodically-layered elastic space degenerates into an elastic layer of finite width sandwiched between two similar elastic half-spaces. For the case where $h\to\infty$, the elastic layer is flawless (devoid of any cracks) and planar cracks are present in the half-space(s). On the other hand, for $d\to\infty$, planar cracks exist in the elastic layer but not in the half-space. (The case $d\to\infty$ corresponds geometrically to the problem considered in Ang and Clements [3].)

If both h and d tend to infinity, the periodically-layered elastic space reduces to two dissimilar elastic half-spaces which adhere perfectly to each other. Planar cracks are found in only one of the elastic half-spaces.

For these limiting cases, the calculation of $E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ becomes somewhat simpler.

From (14), it can be shown that, for the limiting case where $h\to\infty$, $E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ are given by

$$\begin{aligned} & E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) \\ = \; & \frac{i\,\overline{\Theta}^{(p)}\left[\overline{\Upsilon}_2(-u)-\overline{L}_2^{(0)}\overline{\Upsilon}_1(-u)\right]\exp(-iu\left[\xi_1+\overline{\tau}^{(0)}\xi_2\right])}{L_2^{(0)}\overline{\Upsilon}_1(-u)-\overline{\Upsilon}_2(-u)}, \\ & \overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) = 0. \end{aligned} \tag{21}$$

Similarly, for $d\to\infty$, we find that $E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ are given by

$$\begin{aligned} E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) &= -\frac{i\,\overline{\Gamma}\exp(-iu\xi_1)}{\Delta(u)}\Big\{\Theta^{(p)}\Psi\Gamma\exp(-iu\tau^{(0)}\{\xi_2-2h\}) \\ &\qquad + \overline{\Theta^{(p)}\Psi\Omega}\exp(-iu\overline{\tau}^{(0)}\xi_2)\exp(2iu\overline{\tau}^{(0)}h)\Big\}, \\ \overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)}) &= \frac{i\Psi\Gamma\exp(-iu\xi_1)}{\Delta(u)}\Big\{\Theta^{(p)}\Omega\exp(-iu\tau^{(0)}\{\xi_2-2h\}) \end{aligned}$$

$$-\overline{\Theta^{(p)}\Gamma}\exp(-iu\overline{\tau}^{(0)}\xi_2)\exp(2iu\tau^{(0)}h)\Big\},\tag{22}$$

where $\Delta(u)=\Psi\Gamma\overline{\Gamma}\exp(2iu\tau^{(0)}h)+\overline{\Psi\Omega}\Omega\exp(2iu\overline{\tau}^{(0)}h)$.

Lastly, for $h\to\infty$ and $d\to\infty$ together, from either (21) or (22), we find that $E^{(0)}(u;\boldsymbol{\xi};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\boldsymbol{\xi};\underline{\mathbf{n}}^{(p)})$ reduce to

$$\begin{aligned}E^{(0)}(u;\boldsymbol{\xi};\underline{\mathbf{n}}^{(p)}) &= -\frac{i\,\overline{\Theta^{(p)}\Gamma}}{\Omega}\exp(-iu\{\xi_1+\overline{\tau}^{(0)}\xi_2\}),\\ \overline{F}^{(0)}(u;\boldsymbol{\xi};\underline{\mathbf{n}}^{(p)}) &= 0.\end{aligned}\tag{23}$$

For isotropic materials (i.e. $\lambda_{ij}^{(n)}=\mu^{(n)}\delta_{ij}$ $[n=0,1]$), it can be verified that (22) and (23) are essentially in agreement with the analysis of Ang and Clements [3].

5 Numerical calculations for specific examples

In this section, we shall solve the hypersingular integral equations (19) for some specific examples where the elastic constants $\lambda_{ij}^{(m)}$ $(m=0,1)$ are such that $\lambda_{12}^{(m)}=\lambda_{21}^{(m)}=0$.

5.1 A periodic array of cracks perpendicular to the interfaces

The interior of each of the layers $S^{(2m)}$ $(m=0,\pm1,\pm2,\cdots)$ contains a single planar crack of length 2ℓ on a plane that is perpendicular to the interfaces separating the layers. More specifically, the tips of the crack are given by $(a^{(1)},b^{(1)})=(0,b)$ and $(c^{(1)},d^{(1)})=(0,b+2\ell)$, where b and ℓ are *positive* constants such that $b+2\ell<2h$. The internal stress is such that $s_1^{(\mathrm{int})}=\sigma_0$ (a constant) on the crack.

For this particular case, the system of hypersingular integral equations in (19) reduces to a single equation given by

$$\frac{1}{2\ell\pi}\chi\mathcal{H}\int_{-1}^{1}\frac{R(t)dt}{(t-s)^2}+\int_{-1}^{1}K(t,s)R(t)dt=-\sigma_0\ \text{ for }\ -1<s<1,\tag{24}$$

where $R(t)=R^{(1)}(t)$, $\chi=\mathrm{Re}\{[L_1^{(0)}/\tau^{(0)}]^2\}$ and

$$\begin{aligned}K(t,s) &= \frac{\ell}{2\pi}\mathrm{Re}\{\int_0^\infty iu\left[L_1^{(0)}E^{(0)}(u;0,Y(t);1,0)\exp(iu\tau^{(0)}Y(s))\right.\\ &\qquad\left.+\overline{L}_1^{(0)}\overline{F}^{(0)}(u;0,Y(t);1,0)\exp(iu\overline{\tau}^{(0)}Y(s))\right]du,\end{aligned}\tag{25}$$

with $Y(t)=b+(t+1)\ell$.

For the solution of (19), we make the approximation (as in, e.g. [3])

$$R(t) \approx \sqrt{1-t^2} \sum_{j=1}^{J} A_j U_{j-1}(t), \tag{26}$$

where A_j are real constant coefficients yet to be determined and $U_k(x)$ is the k-th order Chebyshev polynomial of the second kind.

Substitution of (26) into (19) yields

$$\sum_{j=1}^{J} A_j M_j(s) = -\sigma_0 \quad \text{for } -1 < s < 1, \tag{27}$$

where

$$M_j(s) = -\frac{1}{2\ell} \chi j U_{j-1}(s) + \int_{-1}^{1} \sqrt{1-t^2} U_{j-1}(t) K(t,s) dt. \tag{28}$$

The integral in (28) can be accurately calculated by using the numerical quadrature (25.4.40) in Abramowitz and Stegun [11].

There are J unknowns A_j $(j = 1, 2, \cdots, J)$ in (27). As many as J equations are needed to solve for these unknowns. To generate J equations out of (27), we choose the value of the parameter s to be given, in turn, by

$$s = s_p \equiv \cos\left[\frac{(2p-1)\pi}{2J}\right] \quad \text{for } p = 1, 2, \cdots, J. \tag{29}$$

The resulting system of linear algebraic equations can be easily solved by using standard computer packages.

We define the stress intensity factor

$$K^- = \lim_{\varepsilon \to 0^+} \sqrt{2\varepsilon}\sigma_{13}(0, b-\varepsilon) \text{ and } K^+ = \lim_{\varepsilon \to 0^+} \sqrt{2\varepsilon}\sigma_{13}(0, b+2\ell+\varepsilon)$$

which can be approximately computed using

$$K^{\pm} \approx \frac{1}{2\sqrt{\ell}} \chi \sum_{j=1}^{J} A_j U_{j-1}(\pm 1), \tag{30}$$

once we have determined A_j.

For the purpose of using (30) to obtain numerical values of the stress intensity factor, we consider three separate cases for the elastic moduli of the materials in the layers:

(a) the layers $S^{(2m)}$ occupied by magnesium and the layers $S^{(2m+1)}$ by titanium, i.e. $\lambda_{11}^{(0)} = 0.164$, $\lambda_{22}^{(0)} = 0.170$, $\lambda_{11}^{(1)} = 0.467$ and $\lambda_{22}^{(1)} = 0.350$ (in gram per centimeter per microsecond square),

(b) all the layers are occupied by the materials having the same elastic moduli, i.e. $\lambda_{11}^{(0)} = \lambda_{11}^{(1)}$ and $\lambda_{22}^{(0)} = \lambda_{22}^{(1)}$, and

(c) the layers $S^{(2m)}$ occupied by titanium and the layers $S^{(2m+1)}$ by magnesium, i.e. $\lambda_{11}^{(0)} = 0.467$, $\lambda_{22}^{(0)} = 0.350$, $\lambda_{11}^{(1)} = 0.164$ and $\lambda_{22}^{(1)} = 0.170$.

Notice that titanium is a more rigid material than magnesium and case (b) gives the corresponding problem involving a periodic array of collinear cracks of equal length in an infinite homogeneous elastic space.

For $b = h - \ell$ (i.e. the crack is centrally located in the interior of each of the layers $S^{(2m)}$) and for selected values $h/\ell = d/\ell = \delta/\ell > 1$ (i.e. layers $S^{(2m)}$ and $S^{(2m+1)}$ have equal width given by δ), numerical results for the non-dimensionalised stress intensity factor $K^+/(\sigma_0\sqrt{\ell})$ or $K^-/(\sigma_0\sqrt{\ell})$ ($K^+/(\sigma_0\sqrt{\ell}) = K^-/(\sigma_0\sqrt{\ell})$ for the centrally-located crack) obtained are given in Table 1 for cases (a), (b) and (c) described above. From the table, it is obvious that, for any fixed δ/ℓ, the state of stress around the cracks in case (b) is more severe than that in case (a) but is less severe than that in case (c). In case (a), for the values of δ/ℓ considered, it appears that $K^+/(\sigma_0\sqrt{\ell})$ increases with increasing δ/ℓ, while in cases (b) and (c) $K^+/(\sigma_0\sqrt{\ell})$ decreases with increasing δ/ℓ. In all the three cases, $K^+/(\sigma_0\sqrt{\ell}) \to 1$ as $\delta/\ell \to \infty$, as expected. (Notice $K^+/(\sigma_0\sqrt{\ell}) = 1$ for the corresponding problem involving a single planar crack in a homogeneous elastic space.)

Table 1. Numerical values of $K^+/(\sigma_0\sqrt{\ell})$ ($= K^-/(\sigma_0\sqrt{\ell})$).

δ/ℓ	case (a)	case (b)	case (c)
1.100	0.8728	1.102	1.444
1.250	0.9359	1.075	1.257
1.500	0.9682	1.050	1.146
1.750	0.9805	1.036	1.098
2.000	0.9866	1.027	1.071
2.500	0.9924	1.017	1.043
3.000	0.9951	1.012	1.029
4.000	0.9974	1.007	1.016
5.000	0.9984	1.004	1.010
6.000	0.9988	1.003	1.007

For the degenerate geometry $h \to \infty$, a sketch of the problem is given in Figure 2. From the figure, it is clear that case (b) (where $\lambda_{11}^{(0)} = \lambda_{11}^{(1)}$ and $\lambda_{22}^{(0)} = \lambda_{22}^{(1)}$) corresponds to the problem of a single planar crack in an infinite homogeneous elastic space. Hence, $K^\pm/(\sigma_0\sqrt{\ell})$ is unity for case (b), no matter what value we give to b/ℓ. For $h \to \infty$, $d/\ell = 1.000$ and selected values of b/ℓ, we calculate the non-dimensionalised stress intensity factor $K^{\pm}/(\sigma_0\sqrt{\ell})$ for cases (a) and (c). (We solve (24) using $E^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ and $\overline{F}^{(0)}(u;\underline{\boldsymbol{\xi}};\underline{\mathbf{n}}^{(p)})$ as given by (21) in (25) in order to compute $K^{\pm}/(\sigma_0\sqrt{\ell})$.) The numerical results of $K^{\pm}/(\sigma_0\sqrt{\ell})$ are tabulated in Tables 2 and 3 for cases (a) and (c) respectively.

From Table 2, for case (a), we observe that the magnitudes of $K^{\pm}/(\sigma_0\sqrt{\ell})$ is less than unity, i.e. the state of stress around the crack tips are less severe than that around a single crack in an infinite homogeneous elastic space. It appears that the presence of the titanium strip has the effect of reducing the stress around the crack which is embedded in magnesium. This is perhaps to be expected since titanium is a more rigid material than magnesium. For case (a), it is interesting to observe that $K^{+}/(\sigma_0\sqrt{\ell}) > K^{-}/(\sigma_0\sqrt{\ell})$, i.e. the state of stress around the crack tip $(0,b)$ is less severe than that around the tip $(0,b+2\ell)$. The tip $(0,b)$ is located nearer to the titanium strip than the other tip. It is also clear that moving the crack farther away from the titanium strip aggravates the stress around both crack tips.

The results for case (c) in Table 3 indicate that the presence of the magnesium strip worsens the stress around the crack in titanium, the state of stress around the crack tip $(0,b)$ is more severe than that around the other tip and locating the crack farther away from the magnesium strip reduces the stress distribution around the crack tips.

Table 2. Numerical values of $K^{\pm}/(\sigma_0\sqrt{\ell})$ for case (a).

b/ℓ	$K^{-}/(\sigma_0\sqrt{\ell})$	$K^{+}/(\sigma_0\sqrt{\ell})$
0.1000	0.8596	0.9656
0.2000	0.9109	0.9723
0.4000	0.9513	0.9806
0.6000	0.9686	0.9856
0.8000	0.9781	0.9888
1.000	0.9838	0.9912
2.000	0.9946	0.9964
4.000	0.9986	0.9989
8.000	0.9997	0.9998

Table 3. Numerical values of $K^{\pm}/(\sigma_0\sqrt{\ell})$ for case (c).

b/ℓ	$K^{-}/(\sigma_0\sqrt{\ell})$	$K^{+}/(\sigma_0\sqrt{\ell})$
0.1000	1.1650	1.0383
0.2000	1.0975	1.0292
0.4000	1.0499	1.0193
0.6000	1.0310	1.0139
0.8000	1.0212	1.0105
1.000	1.0153	1.0082
2.000	1.0048	1.0032
4.000	1.0012	1.0009
8.000	1.0002	1.0002

5.2 A periodic array of cracks parallel to the interfaces

The interior of each of the layers $S^{(2m)}$ $(m = 0, \pm 1, \pm 2, \cdots)$ contains a pair of cracks of equal length lying on planes parallel to the interfaces separating the layers. More precisely, the tips of the cracks are given by $(a^{(1)}, b^{(1)}) = (\ell, 3h/2)$, $(c^{(1)}, d^{(1)}) = (-\ell, 3h/2)$, $(a^{(2)}, b^{(2)}) = (\ell, h/2)$ and $(c^{(2)}, d^{(2)}) = (-\ell, h/2)$, where ℓ is a *positive* constant. The internal stress is such that $s_2^{(\mathrm{int})} = \sigma_0$ (a constant) on the cracks.

For this case, the system of hypersingular integral equations in (19) also reduces to (24) but with $R(t) = R^{(1)}(t) = R^{(2)}(t)$, $\chi = \mathrm{Re}\{[L_2^{(0)}]^2\}$ and

$$
\begin{aligned}
K(t,s) &= \frac{\ell}{2\pi}\mathrm{Re}\{\frac{[L_2^{(0)}]^2}{[(X(t)-X(s))-\tau^{(0)}h]^2} \\
&\quad + \int_0^\infty iu\left[L_2^{(0)}E(u;t,s) + \overline{L}_2^{(0)}\overline{F}(u;t,s)\right]\exp[iuX(s)]du\},
\end{aligned} \tag{31}
$$

where $X(t) = -t\ell$ and

$$
\begin{aligned}
E(u;t,s) &= \left[E^{(0)}(u;X(t),3h/2;0,1) + E^{(0)}(u;X(t),h/2;0,1)\right] \\
&\quad \times \exp(3ihu\tau^{(0)}/2), \\
\overline{F}(u;t,s) &= \left[\overline{F}^{(0)}(u;X(t),3h/2;0,1) + \overline{F}^{(0)}(u;X(t),h/2;0,1)\right] \\
&\quad \times \exp(3ihu\overline{\tau}^{(0)}/2).
\end{aligned} \tag{32}
$$

The stress intensity factor defined by

$$
K = \lim_{\varepsilon \to 0^+} \sqrt{2\varepsilon}\,\sigma_{23}(\ell+\varepsilon, 3h/2)
$$

can be computed approximately using (30).

As in the previous example, to compute the non-dimensionalised stress intensity factor $K/(\sigma_0\sqrt{\ell})$ using (30), for the elastic moduli of the layers, we consider cases (a), (b) and (c) as described above.

Numerical results of $K/(\sigma_0\sqrt{\ell})$ for selected values of $h/\ell = d/\ell = \delta/\ell > 0$ are given in Table 4. In all the three cases, $K/(\sigma_0\sqrt{\ell})$ increases with increasing δ/ℓ, i.e. it seems that in each case the cracks are more stable when they are closer to one another. It is also clear that in all the cases $K/(\sigma_0\sqrt{\ell})$ approaches unity from below as δ/ℓ increases. The non-dimensionalised stress intensity factor $K/(\sigma_0\sqrt{\ell})$ for case (b) is greater in magnitude than that for case (a) but less than that for case (c).

Table 4. Numerical values of $K/(\sigma_0\sqrt{\ell})$.

δ/ℓ	case (a)	case (b)	case (c)
0.1250	0.2305	0.2892	0.3536
0.2500	0.3394	0.4291	0.5298
0.5000	0.4811	0.5969	0.7175
0.7500	0.5798	0.7018	0.8128
1.000	0.6554	0.7718	0.8659
1.500	0.7613	0.8565	0.9210
2.000	0.8292	0.9035	0.9483
3.000	0.9042	0.9494	0.9735
5.000	0.9600	0.9798	0.9896
10.00	0.9892	0.9947	0.9973

In the calculations above for the examples, we use $J = 10$ in the approximation (26). When the calculations are repeated using $J = 20$, convergence to at least 3 significant figures is observed in the numerical results.

6 Closing remarks

To solve the problem under consideration, we have constructed a suitable singular solution (of the equilibrium equation of elasticity) that satisfies the relevant continuity conditions on the interfaces separating the dissimilar materials. The solution is given by (6) together with (10) and (13)-(17). It is used to form an integral expression for the displacement as in (5). The integral expression contains unknown functions which are determined from the system of hypersingular integral equations in (19). We have also shown that the analysis presented can also be used to recover special cases of the problem which involve bimaterials and trilayered materials.

For specific examples of the problem involving particular transversely-isotropic materials, we have solved the hypersingular integral equations numerically in order to compute the relevant stress intensity factors.

In principle, it is possible to extend the analysis in the present paper to include plane deformations. We may, however, expect the algebraic manipulations required in the extension to be more tedious.

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