STRESSES AROUND A PERIODIC ARRAY OF PLANAR CRACKS IN AN ANISOTROPIC BIMATERIAL

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Abstract

The problem of calculating the stress distribution around a periodic array of planar cracks in an anisotropic elastic half-space which adhere perfectly to another anisotropic half-space is considered. It is formulated in terms of a system of hypersingular integral equations with the crack-opening displacements as unknown functions. For a specific case involving transversely-isotropic materials, the integral equations are solved numerically through the use of a collocation technique and numerical values of useful quantities, such as the crack tip stress intensity factors, are computed.

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1 INTRODUCTION

There is definitely a need to assess the reliability and integrity of anisotropic and composite structures that are presently playing an ever increasing role in modern technology, such as in engineering construction and manufacturing and in orthopaedic surgery (e.g. artificial implants in bones). It is not surprising then that the task of analysing the stress distribution around cracks in anisotropic layered materials has been given considerable attention in the literature (see e.g. Willis [1], Clements [2], and Ang [3],[4]).

In the present paper, we consider the problem of a periodic array of planar cracks in an anisotropic elastic half-space which adheres perfectly to another anisotropic half-space. The cracks are assumed to open up under the action of suitably prescribed internal tractions. Problems involving periodic arrays of cracks in elastic media may be of useful relevance to some practical situations in which multiple cracking occurs, and have been examined by various researchers (e.g. Benthem and Koiter [5], Nied [6], Ang [7] and, Tweed and Melrose [8]).

With the displacements and stresses written in terms of suitable integral expressions, the problem under consideration is reduced to solving a system of hypersingular (Hadamard finite-part) integral equations. The unknown functions are the crack-opening displacements. The integral equations apply for the most general anisotropic materials, i.e. the materials are not required to possess any symmetries in their anisotropy. However, the equations are solved numerically, using a collocation technique, for a specific case involving particular transversely isotropic materials, in order to compute the crack tip stress intensity factors.

2 THE PROBLEM

By referring to an $0x_1x_2x_3$ Cartesian coordinate system, consider an infinite elastic medium which consists of two regions: $x_2 > h$ (region 1) and $x_2 < h$ (region 2) (h is a positive constant). The regions are occupied by dissimilar anisotropic materials which adhere perfectly to each other along the interface $x_2 = h$.

Region 2 contains an infinite number of periodically-located planar cracks with geometries that do not vary with x_3. Region 1 is flawless, devoid of any cracks. Specifically, the cracks (in region 2) are given by $C_0, C_{\pm 1}, C_{\pm 2}, ...$, where $C_{2n} = \{(x_1, x_2, x_3) : x_1 = nd + sa\cos\theta,\ x_2 = sa\sin\theta,\ -\infty < x_3 < \infty,$ $-1 < s < 1\}$ and $C_{2n+1} = \{(x_1, x_2, x_3) : x_1 = c + nd + sb\cos\phi,\ x_2 = sb\sin\phi,$ $-\infty < x_3 < \infty,\ -1 < s < 1\}$ for $n = 0, \pm 1, \pm 2, ...$. The constants c and d obey the inequality $0 < c < d$; and $a > 0$, $b > 0$, θ and ϕ are such that the cracks do *not* intersect with one another and the interface $x_2 = h$.

The cracks open up and become traction-free under the action of suitably prescribed internal stresses which are periodic along the $0x_1$ direction, with period d, and are independent of time and the x_3 coordinate. The displacements and stresses (generated by the presence of the cracks) are required to vanish, as $|x_2| \to \infty$. The problem is to determine the displacement and stress fields throughout the elastic medium.

3 SOME BASIC EQUATIONS

For a two-dimensional anisotropic elastic material occupying a region R bounded by a closed curve D, on the $0x_1x_2$ plane, it can be shown that (see Clements [9])

$$u_k(\underset{\sim}{x}) = \int_D [u_r(\underset{\sim}{\xi})\Gamma_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) - p_r(\underset{\sim}{\xi})\Phi_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi})]dS(\underset{\sim}{\xi}) \text{ for } \underset{\sim}{x} \in R, \tag{1}$$

where u_k are the Cartesian displacements, $\underset{\sim}{x} = (x_1, x_2)$, $\underset{\sim}{\xi} = (\xi_1, \xi_2)$, p_r are the tractions acting across D and

$$\Phi_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \operatorname{Re}\{\sum_\alpha A_{r\alpha} N_{\alpha j} \ln(c_\alpha - z_\alpha)\} d_{jk},$$
$$\Gamma_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \operatorname{Re}\{\sum_\alpha L_{rj\alpha} N_{\alpha p}(c_\alpha - z_\alpha)^{-1}\} n_j(\underset{\sim}{\xi}) d_{pk}, \tag{2}$$

where $\sum_\alpha$ denotes the summation over the greek subscript α from 1 to 3, $z_\alpha = x_1 + \tau_\alpha x_2$, $c_\alpha = \xi_1 + \tau_\alpha \xi_2$, $n_j(\underset{\sim}{\xi})$ are the components of the unit normal outward vector to D at $\underset{\sim}{\xi}$, and $[d_{jk}]$ is defined by the relation

$$-\frac{i}{2} \sum_\alpha \{L_{j2\alpha} N_{\alpha p} - \bar{L}_{j2\alpha} \bar{N}_{\alpha p}\} d_{pk} = \delta_{jk},$$

where $i = \sqrt{-1}$, $\bar{z}$ denotes the complex conjugate of z and δ_{jk} is the kronecker-delta. The constants $A_{r\alpha}$, $L_{rj\alpha}$ and $N_{\alpha p}$ are related to the elastic moduli c_{ijkl} of the anisotropic material, as explained in Clements [9]. Throughout the present paper, unless otherwise mentioned, the Einstein convention of summing over a repeated index is adopted for only latin subscripts which run from 1 to 3.

Now, if R covers the entire $0x_1x_2$ plane and contains several straight cuts or cracks (of finite lengths), denoted by L_1,L_2, ...,L_{N-1} and L_N, in its interior, and if the displacements $u_k(\underset{\sim}{x})$ behave as $O(|\underset{\sim}{x}|^{-s})$ $(s>0)$, as $|\underset{\sim}{x}| \to \infty$, then, from (1), the displacements $u_k(\underset{\sim}{x})$ can be written as:

$$u_k(\underset{\sim}{x}) = \sum_{m=1}^{N} \int_{L_m^+} \Delta u_r(\underset{\sim}{\xi}) \Gamma_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi}) dS(\underset{\sim}{\xi}), \tag{3}$$

where L_m^- and L_m^+ denote respectively the "lower" and "upper" faces of the cut L_m and $\Delta u_r(\underset{\sim}{\xi}) = [u_r(\underset{\sim}{\xi})]^+ - [u_r(\underset{\sim}{\xi})]^-$, with $[u_r(\underset{\sim}{\xi})]^\pm$ denoting the value of $u_r(\underset{\sim}{\xi})$ for $\underset{\sim}{\xi} \in L_m^\pm$. In the derivation of (3), we assume that the stresses are continuous across the cuts and note that $\Phi_{rk}(\underset{\sim}{x}, \underset{\sim}{\xi})$ are single-valued functions for all $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ on the $0x_1x_2$ plane (provided that $\underset{\sim}{\xi} \neq \underset{\sim}{x}$).

4 SOLUTION OF THE PROBLEM

For the solution of the problem under consideration, guided by (3), we choose the displacements in region n to be given by

$$u_k^{(n)}(\underset{\sim}{x}) = \sum_{m=-\infty}^{\infty} \int_{C_m^+} \Delta u_p(\underset{\sim}{\xi}) U_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi}) dS(\underset{\sim}{\xi}), \tag{4}$$

where C_m^+ denotes the "upper" face of the crack C_m, Δu_p give the crack-opening displacements and

$$U_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \mathrm{Re}\{\delta_{n2} \sum_{\alpha} L_{pj\alpha}^{(n)} \, N_{\alpha r}^{(n)} (c_\alpha^{(n)} - z_\alpha^{(n)})^{-1}\} n_j(\underset{\sim}{\xi}) d_{rk}^{(n)} + G_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi}), \tag{5}$$

where $c_\alpha^{(n)} = \xi_1 + \tau_\alpha^{(n)} \xi_2$ and $z_\alpha^{(n)} = x_1 + \tau_\alpha^{(n)} x_2$. The superscript (n) indicates that constants such as $L_{pj\alpha}$, $N_{\alpha r}$, d_{rk} and τ_α are to be computed using the elastic moduli $c_{ijkl}^{(n)}$ of the material in region n.

For (4) to satisfy the governing equations of anisotropic elasticity, the functions $G_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi})$ in (5) must satisfy

$$c_{rskq}^{(n)} \frac{\partial^2 G_{pk}^{(n)}}{\partial x_s \partial x_q} = 0 \text{ for all } \underset{\sim}{x} \text{ in region } n. \tag{6}$$

The different materials making up the composite adhere perfectly to each other along the $x_2 = h$ interface. Thus, the functions $G_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi})$ must be chosen in such a way that, for $-\infty < x_1 < \infty$,

$$\lim_{\varepsilon \to 0^+} [U_{pk}^{(1)}(x_1, h + \varepsilon, \underset{\sim}{\xi}) - U_{pk}^{(2)}(x_1, h - \varepsilon, \underset{\sim}{\xi})] = 0,$$
$$\lim_{\varepsilon \to 0^+} [S_{pk2}^{(1)}(x_1, h + \varepsilon, \underset{\sim}{\xi}) - S_{pk2}^{(2)}(x_1, h - \varepsilon, \underset{\sim}{\xi})] = 0, \tag{7}$$

where $S_{pkj}^{(n)} = c_{kjrs}^{(n)} \partial U_{pr}^{(n)} / \partial x_s$. In addition, $G_{pk}^{(n)}(\underset{\sim}{x}, \underset{\sim}{\xi})$ are required to vanish as $|x_2| \to \infty$ (in region n).

For the solution of (6) subject to (7), we choose

$$G_{pk}^{(1)}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \mathrm{Re}\{\sum_{\alpha} A_{k\alpha}^{(1)} \int_0^\infty E_{p\alpha}(u, \underset{\sim}{\xi}) \exp(iu[z_\alpha^{(1)} - \tau_\alpha^{(1)} h]) du\},$$
$$G_{pk}^{(2)}(\underset{\sim}{x}, \underset{\sim}{\xi}) = \frac{1}{2\pi} \mathrm{Re}\{\sum_{\alpha} A_{k\alpha}^{(2)} \int_0^\infty F_{p\alpha}(u, \underset{\sim}{\xi}) \exp(-iu[z_\alpha^{(2)} - \tau_\alpha^{(2)} h]) du\}, \tag{8}$$

where $E_{p\alpha}(u, \underset{\sim}{\xi})$ and $F_{p\alpha}(u, \underset{\sim}{\xi})$ are functions yet to be determined. The system (6) is satisfied by (8).

From a Fourier inversion theorem, we know that conditions (7) can be rewritten as:

$$\int_{-\infty}^{\infty} \lim_{\varepsilon \to 0^+} [U_{pk}^{(1)}(x_1, h + \varepsilon, \underset{\sim}{\xi}) - U_{pk}^{(2)}(x_1, h - \varepsilon, \underset{\sim}{\xi})] \times \exp(-i\gamma x_1) dx_1 = 0, \tag{9}$$

$$\int_{-\infty}^{\infty} \lim_{\varepsilon \to 0^+} [S_{pk2}^{(1)}(x_1, h + \varepsilon, \underset{\sim}{\xi}) - S_{pk2}^{(2)}(x_1, h - \varepsilon, \underset{\sim}{\xi})] \times \exp(-i\gamma x_1) dx_1 = 0, \tag{10}$$

where $\gamma > 0$ is a real parameter.

Using the results (Erdélyi *et al.* [10])

$$\int_{-\infty}^{\infty} (a - ix)^{-1} \exp(-ixy) dx = H(y) 2\pi \exp(-ay),$$
$$\int_{-\infty}^{\infty} (a + ix)^{-1} \exp(-ixy) dx = -H(-y) 2\pi \exp(ay), \tag{11}$$

where a is a constant such that $\mathrm{Re}\{a\} > 0$ and $H(x)$ is the Heaviside unit-step function, from (9), we find that (10) becomes

$$\sum_{\alpha}\{A^{(1)}_{k\alpha}E_{p\alpha}(u,\underset{\sim}{\xi}) - \bar{A}^{(2)}_{k\alpha}\bar{F}_{p\alpha}(u,\underset{\sim}{\xi})\} \\ = \sum_{\alpha} in_j(\underset{\sim}{\xi})T^{(2)}_{pj\alpha k}\exp(-iu[\xi_1 + \tau^{(2)}_{\alpha}\{\xi_2 - h\}]) \text{ for } \xi_2 < h, \tag{12}$$

where $T^{(2)}_{pj\alpha k} = L^{(2)}_{pj\alpha}N^{(2)}_{\alpha r}d^{(2)}_{rk}$.

In similar manner, (10) gives rise to

$$\sum_{\alpha}\{L^{(1)}_{k2\alpha}E_{p\alpha}(u,\underset{\sim}{\xi}) - \bar{L}^{(2)}_{k2\alpha}\bar{F}_{p\alpha}(u,\underset{\sim}{\xi})\} \\ = \sum_{\alpha} in_l(\underset{\sim}{\xi})Q^{(2)}_{pk2l\alpha}\exp(-iu[\xi_1 + \tau^{(2)}_{\alpha}\{\xi_2 - h\}]) \text{ for } \xi_2 < h, \tag{13}$$

where $Q^{(2)}_{pkjl\alpha} = (c^{(2)}_{kjr1} + \tau^{(2)}_{\alpha}c^{(2)}_{kjr2})T^{(2)}_{pl\alpha r}$.

Solving (12) and (13) for $E_{p\alpha}(u,\underset{\sim}{\xi})$ and $F_{p\alpha}(u,\underset{\sim}{\xi})$, we obtain

$$E_{p\alpha}(u,\underset{\sim}{\xi}) = \sum_{\beta,\gamma} Z_{\alpha\beta}(\bar{N}^{(2)}_{\beta k}T^{(2)}_{pl\gamma k} - \bar{M}^{(2)}_{\beta k}Q^{(2)}_{pk2l\gamma}) \\ \times \exp(-iu[\xi_1 + \tau^{(2)}_{\gamma}\{\xi_2 - h\}])in_l(\underset{\sim}{\xi}) \text{ for } \xi_2 < h, \tag{14}$$

and

$$\bar{F}_{p\alpha}(u,\underset{\sim}{\xi}) = \sum_{\beta,\gamma} W_{\alpha\beta}(N^{(1)}_{\beta k}T^{(2)}_{pl\gamma k} - M^{(1)}_{\beta k}Q^{(2)}_{pk2l\gamma}) \\ \times \exp(-iu[\xi_1 + \tau^{(2)}_{\gamma}\{\xi_2 - h\}])in_l(\underset{\sim}{\xi}) \text{ for } \xi_2 < h, \tag{15}$$

where $[Z_{\alpha\beta}]$ and $[W_{\alpha\beta}]$ are obtained from the relations

$$\sum_{\beta} Z_{\alpha\beta}[\bar{N}^{(2)}_{\beta k}A^{(1)}_{k\gamma} - \bar{M}^{(2)}_{\beta k}L^{(1)}_{k2\gamma}] = \delta_{\alpha\gamma}, \\ \sum_{\beta} W_{\alpha\beta}[M^{(1)}_{\beta k}\bar{L}^{(2)}_{k2\gamma} - N^{(1)}_{\beta k}\bar{A}^{(2)}_{k\gamma}] = \delta_{\alpha\gamma}. \tag{16}$$

Substituting (14) and (15) into (8), we obtain

$$G_{pk}^{(1)}(\underset{\sim}{x},\underset{\sim}{\xi}) = -\frac{1}{2\pi}\,\mathrm{Re}\{\sum_{\alpha,\beta,\gamma} A_{k\alpha}^{(1)} Z_{\alpha\beta}(\bar{N}_{\beta q}^{(2)} T_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(2)} Q_{pq2l\gamma}^{(2)})$$
$$\times (z_{\alpha}^{(1)} - [\xi_1 + \tau_{\gamma}^{(2)}\{\xi_2 - h\}])^{-1}\} \text{ for } \xi_2 < h, \tag{17}$$

and

$$G_{pk}^{(2)}(\underset{\sim}{x},\underset{\sim}{\xi}) = -\frac{1}{2\pi}\,\mathrm{Re}\{\sum_{\alpha,\beta,\gamma} A_{k\alpha}^{(2)} \bar{W}_{\alpha\beta}(\bar{N}_{\beta q}^{(1)} \bar{T}_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(1)} \bar{Q}_{pq2l\gamma}^{(2)})$$
$$\times (z_{\alpha}^{(2)} - [\xi_1 + \bar{\tau}_{\gamma}^{(2)}\{\xi_2 - h\}])^{-1}\} \text{ for } \xi_2 < h. \tag{18}$$

The displacements as given by (4) together with (5), (17) and (18), satisfy the governing equations of elasticity, as well as the continuity conditions on the interface $x_2 = h$. The remaining conditions to be satisfied are those on the crack faces, specifically given by

$$\sigma_{kj}^{(2)}(\underset{\sim}{x}) n_j(\underset{\sim}{x}) \to -\sigma_{kj}^{(0)}(\underset{\sim}{y}) n_j(\underset{\sim}{y}) \text{ as } \underset{\sim}{x} \to \underset{\sim}{y} \in C_m^+ \ (m = 0, \pm 1, \pm 2, ...), \tag{19}$$

where $\sigma_{kj}^{(2)}$ are the Cartesian stresses in region 2, and $\sigma_{kj}^{(0)}$ are the internal stresses acting on the cracks. The stresses $\sigma_{kj}^{(0)}$ are assumed to be periodic along the $0x_1$ direction, with period d.

Now, from (4), (5), (17) and (18) together with

$$\sum_{n=-\infty}^{\infty} \frac{1}{p - ns} = \frac{i\pi}{s} \coth\left(\frac{i\pi p}{s}\right),$$

we obtain

$$\sigma_{kj}^{(2)}(\underset{\sim}{x}) = -\frac{\pi a}{2d^2} \int_{-1}^{1} \Delta U_p(s)\,\mathrm{Re}\{\sum_{\alpha} Q_{pkjl\alpha}^{(2)}$$
$$\times \operatorname{csch}^2(\frac{i\pi}{d}\{z_{\alpha}^{(2)} - sa[\cos\theta + \tau_{\alpha}^{(2)}\sin\theta]\})$$
$$+ \sum_{\alpha,\beta,\gamma} L_{kj\alpha}^{(2)} \bar{W}_{\alpha\beta}(\bar{N}_{\beta q}^{(1)} \bar{T}_{pl\gamma q}^{(2)} - \bar{M}_{\beta q}^{(1)} \bar{Q}_{pq2l\gamma}^{(2)})$$
$$\times \operatorname{csch}^2(\frac{i\pi}{d}\{z_{\alpha}^{(2)} - sa[\cos\theta + \bar{\tau}_{\gamma}^{(2)}\sin\theta] + \bar{\tau}_{\gamma}^{(2)} h\})\} N_l ds$$

$$
\begin{aligned}
&-\frac{\pi b}{2d^2}\int_{-1}^{1}\Delta V_p(s)\,\mathrm{Re}\{\sum_{\alpha} Q^{(2)}_{pkjl\alpha} \\
&\times \operatorname{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha}-c-sb[\cos\phi+\tau^{(2)}_{\alpha}\sin\phi]\}) \\
&+\sum_{\alpha,\beta,\gamma} L^{(2)}_{kj\alpha}\bar{W}_{\alpha\beta}(\bar{N}^{(1)}_{\beta q}\bar{T}^{(2)}_{pl\gamma q}-\bar{M}^{(1)}_{\beta q}\bar{Q}^{(2)}_{pq2l\gamma}) \\
&\times \operatorname{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha}-c-sb[\cos\phi+\bar{\tau}^{(2)}_{\gamma}\sin\phi]+\bar{\tau}^{(2)}_{\gamma}h\})\}N^*_l ds,
\end{aligned}
\tag{20}
$$

where $\Delta U_p(s) = \Delta u_p(md + sa\cos\theta, sa\sin\theta)$ and $\Delta V_p(s) = \Delta u_p(c + md + sb\cos\phi, sb\sin\phi)$ $(m = 0, \pm 1, \pm 2, ...,)$, $N_1 = \sin\theta$, $N_2 = -\cos\theta$, $N^*_1 = \sin\phi$, and $N^*_2 = -\cos\phi$.

For convenience, we may think of the crack C_m as being a closed curve which is assigned a clockwise direction and which encloses an elliptical region having an area that vanishes to zero. In the derivation of (20), we take the "upper" face of the crack C_{2m} to be the part of the ellipse assigned the direction from the tip $(md - a\cos\theta, -a\sin\theta)$ to $(md + a\cos\theta, a\sin\theta)$, and the "upper" face of C_{2m+1} to be the part of the ellipse from $(c + md - b\cos\phi, -b\sin\phi)$ to $(c + md + b\cos\phi, b\sin\phi)$. Also, in the derivation of (20), we use the fact that, for the problem under consideration, the displacements are periodic along the $0x_1$ direction, with period d.

Use of (20) in (19) yields the system of hypersingular integral equations given by

$$
\begin{aligned}
&\frac{1}{2\pi a}\mathrm{Re}\{\sum_{\alpha}\frac{Q^{(2)}_{pkjl\alpha}N_jN_l}{(\cos\theta+\tau^{(2)}_{\alpha}\sin\theta)^2}\}\mathcal{H}\int_{-1}^{1}\frac{\Delta U_p(s)}{(t-s)^2}ds \\
&+\int_{-1}^{1}\Delta U_p(s)\Omega^{(A)}_{pk}(s, ta\cos\theta, ta\sin\theta)ds \\
&+\int_{-1}^{1}\Delta V_p(s)\Omega^{(B)}_{pk}(s, ta\cos\theta, ta\sin\theta)ds \\
&= -\sigma^{(0)}_{kj}(ta\cos\theta, ta\sin\theta)N_j \text{ for } -1 < t < 1,
\end{aligned}
\tag{21}
$$

and

$$\frac{1}{2\pi b}\,\mathrm{Re}\{\sum_{\alpha}\frac{Q^{(2)}_{pkjl\alpha}N^*_jN^*_l}{(\cos\phi+\tau^{(2)}_{\alpha}\sin\phi)^2}\}\mathcal{H}\int_{-1}^{1}\frac{\Delta V_p(s)}{(t-s)^2}ds$$
$$+\int_{-1}^{1}\Delta V_p(s)\Omega^{(C)}_{pk}(s,c+tb\cos\phi,tb\sin\phi)ds$$
$$+\int_{-1}^{1}\Delta U_p(s)\Omega^{(D)}_{pk}(s,c+tb\cos\phi,tb\sin\phi)ds$$
$$=-\sigma^{(0)}_{kj}(c+tb\cos\phi,tb\sin\phi)N^*_j\ \text{for}\ -1<t<1, \tag{22}$$

where $\mathcal{H}$ denotes that the integral is to be interpreted in the Hadamard finite-part sense and

$$\Omega^{(A)}_{pk}(s,x_1,x_2)=-\frac{\pi a}{2d^2}N_lN_j\,\mathrm{Re}\{\sum_{\alpha}Q^{(2)}_{pkjl\alpha}$$
$$\times[\mathrm{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha}-sa[\cos\theta+\tau^{(2)}_{\alpha}\sin\theta]\})$$
$$+\frac{d^2}{\pi^2(z^{(2)}_{\alpha}-sa[\cos\theta+\tau^{(2)}_{\alpha}\sin\theta])^2}]$$
$$+\sum_{\alpha,\beta,\gamma}L^{(2)}_{kj\alpha}\bar{W}_{\alpha\beta}(\bar{N}^{(1)}_{\beta q}\bar{T}^{(2)}_{pl\gamma q}-\bar{M}^{(1)}_{\beta q}\bar{Q}^{(2)}_{pq2l\gamma})$$
$$\times\,\mathrm{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha}-sa[\cos\theta+\bar{\tau}^{(2)}_{\gamma}\sin\theta]+\bar{\tau}^{(2)}_{\gamma}h\})\},$$

$$\Omega^{(B)}_{pk}(s,x_1,x_2)=-\frac{\pi b}{2d^2}N^*_lN_j\,\mathrm{Re}\{\sum_{\alpha}Q^{(2)}_{pkjl\alpha}$$
$$\times\,\mathrm{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha}-c-sb[\cos\phi+\tau^{(2)}_{\alpha}\sin\phi]\})$$
$$+\sum_{\alpha,\beta,\gamma}L^{(2)}_{kj\alpha}\bar{W}_{\alpha\beta}(\bar{N}^{(1)}_{\beta q}\bar{T}^{(2)}_{pl\gamma q}-\bar{M}^{(1)}_{\beta q}\bar{Q}^{(2)}_{pq2l\gamma})$$
$$\times\,\mathrm{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha}-c-sb[\cos\phi+\bar{\tau}^{(2)}_{\gamma}\sin\phi]+\bar{\tau}^{(2)}_{\gamma}h\})\},$$

$$
\begin{aligned}
\Omega^{(C)}_{pk}(s,x_1,x_2) = &-\frac{\pi b}{2d^2} N_l^* N_j^* \operatorname{Re}\{ \sum_{\alpha} Q^{(2)}_{pkjl\alpha} \\
&\times [\operatorname{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha} - c - sb[\cos\phi + \tau^{(2)}_{\alpha}\sin\phi]\}) \\
&+ \frac{d^2}{\pi^2(z^{(2)}_{\alpha} - c - sb[\cos\phi + \tau^{(2)}_{\alpha}\sin\phi])^2}] \\
&+ \sum_{\alpha,\beta,\gamma} L^{(2)}_{kj\alpha} \bar{W}_{\alpha\beta} (\bar{N}^{(1)}_{\beta q} \bar{T}^{(2)}_{pl\gamma q} - \bar{M}^{(1)}_{\beta q} \bar{Q}^{(2)}_{pq2l\gamma}) \\
&\times \operatorname{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha} - c - sb[\cos\phi + \bar{\tau}^{(2)}_{\gamma}\sin\phi] + \bar{\tau}^{(2)}_{\gamma} h\})\},
\end{aligned}
$$

$$
\begin{aligned}
\Omega^{(D)}_{pk}(s,x_1,x_2) = &-\frac{\pi a}{2d^2} N_l N_j^* \operatorname{Re}\{ \sum_{\alpha} Q^{(2)}_{pkjl\alpha} \\
&\times \operatorname{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha} - sa[\cos\theta + \tau^{(2)}_{\alpha}\sin\theta]\}) \\
&+ \sum_{\alpha,\beta,\gamma} L^{(2)}_{kj\alpha} \bar{W}_{\alpha\beta} (\bar{N}^{(1)}_{\beta q} \bar{T}^{(2)}_{pl\gamma q} - \bar{M}^{(1)}_{\beta q} \bar{Q}^{(2)}_{pq2l\gamma}) \\
&\times \operatorname{csch}^2(\frac{i\pi}{d}\{z^{(2)}_{\alpha} - sa[\cos\theta + \bar{\tau}^{(2)}_{\gamma}\sin\theta] + \bar{\tau}^{(2)}_{\gamma} h\})\}.
\end{aligned}
\tag{23}
$$

Equations (21) and (22) constitute a system of hypersingular integral equations from which the unknown functions $\Delta U_p(s)$ and $\Delta V_p(s)$ can be solved for. Once the functions are determined, the displacements and stresses, and other useful quantities such as the crack tip stress intensity factors, can be calculated using (4). An accurate and effective numerical technique for solving the hypersingular integral equations is described by Kaya and Erdogan [11].

5 NUMERICAL RESULTS FOR A SPECIFIC CASE

In the present section, for the purpose of illustration, we will solve equations (21) and (22) numerically for a specific case involving particular transversely-isotropic materials, and compute the crack tip stress intensity factors. The stress intensity factors are important quantities for forming criteria for crack extension.

The elastic behaviour of a transversely isotropic material which has transverse planes parallel to the $0x_2x_3$ plane and which undergoes *plane* deformations is governed by the system

$$
\begin{aligned}
C\frac{\partial^2 u_1}{\partial x_1^2} + L\frac{\partial^2 u_1}{\partial x_2^2} + (F+L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} &= 0, \\
A\frac{\partial^2 u_2}{\partial x_2^2} + L\frac{\partial^2 u_2}{\partial x_1^2} + (F+L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} &= 0,
\end{aligned} \tag{24}
$$

where A, F, C and L are the elastic coefficients of the material. Some details on the computation of constants like $A_{k\alpha}$, τ_α and $L_{kj\alpha}$ which correspond to the system (24) are given in Clements [9] or Ang [4].

We take the elastic behaviour of the material in region n to be governed by (24), with the elastic coefficients $A = A^{(n)}$, $F = F^{(n)}$, $C = C^{(n)}$ and $L = L^{(n)}$.

Let us now consider the specific case where region 2 contains a periodic array of equal length and evenly spaced out cracks which lie on a plane parallel to the interface $x_2 = h > 0$ and are subject to a plane deformation. Specifically, in accordance with the notations used in Section 2, we take $\theta = \phi = 0$, $a = b$ and $c = d/2$, with $d > 4a$, and assume that the cracks are acted upon by an internal stress which is such that $\sigma_{22}^{(0)} = P_0$ (P_0 is a positive constant) and $\Delta U_1(s) = \Delta U_3(s) = \Delta V_1(s) = \Delta V_3(s) = 0$.

For this particular case, the system of hypersingular integral equations given by (21) and (22) gives

$$
\chi\mathcal{H}\int_{-1}^{1} \frac{\Delta u(s) ds}{(t-s)^2} + 2\pi a \int_{-1}^{1} \Delta u(s)\Omega(t,s) ds = 2\pi P_0 \text{ for } -1 < t < 1, \tag{25}
$$

where $\Delta u(s) = \Delta U_2(s)/a = \Delta V_2(s)/a$, $\chi = \mathrm{Re}\{\sum_\alpha Q_{2222\alpha}^{(2)}\}$ and $\Omega(t,s) = \Omega_{22}^{(A)}(s, ta, 0) + \Omega_{22}^{(B)}(s, ta, 0)$.

For the numerical solution of (25), we make the approximation:

$$
\Delta u(s) \simeq \sqrt{1-s^2} \sum_{j=1}^{J} \varphi_j U_{j-1}(s), \tag{26}
$$

where φ_j are real constant coefficients to be determined and $U_j(x)$ is the j-th order Chebyshev polynomial of the second kind.

Substituting (26) into (25), and using some results in Kaya and Erdogan [11], we obtain

$$\sum_{j=1}^{J} K_j(t)\varphi_j = 2\pi P_0 \text{ for } -1 < t < 1, \tag{27}$$

where

$$K_j(t) = -\pi j U_{j-1}(t)\chi + 2\pi a \int_{-1}^{1} \sqrt{1-s^2}U_{j-1}(s)\Omega(t,s)ds. \tag{28}$$

It is possible to compute the integral in (28) accurately by using the quadrature formula (25.4.40) in Abramowitz and Stegun [12].

Equation (27) contains J unknowns, namely φ_1, φ_2, ..., φ_J. We choose the free parameter t in (27) to be given in turn by

$$t = t_p = \cos([2p-1]\pi/[2J]) \text{ for } p = 1,\ 2,\ ...,\ J,$$

in order to generate a system of J linear algebraic equations in φ_j. The system thus generated is readily solved using standard computer packages.

For this particular case, the mode I stress intensity factor which is of practical interest has the same value at all the crack tips. We define the factor by

$$K_I = \lim_{x\to a^+} \sqrt{2(x-a)}\sigma_{22}^{(2)}(x,0),$$

and it can be computed approximately via

$$K_I \simeq -\frac{\chi}{2\sqrt{a}}\sum_{j=1}^{J}\varphi_j U_{j-1}(1), \tag{29}$$

once φ_j are determined.

To obtain some numerical results, we will use the elastic constants for magnesium and titanium. For magnesium, these constants are given by $A = 5.96$, $N = 2.57$, $F = 2.14$, $C = 6.14$ and $L = 1.64$; for titanium, they are $A = 16.2$, $F = 6.9$, $C = 18.1$ and $L = 4.67$. If these constants are multiplied by 10^{11}, their units are in dynes per centimeter square. Notice that the magnitudes of the elastic constants for magnesium are lower than the corresponding ones for titanium, i.e. magnesium is a "softer" material than titanium.

Let us now solve (25), with regions 1 and 2 being occupied by titanium and magnesium respectively, and then compute the stress intensity factor $K_I/(P_0\sqrt{a})$ according to (29). From our calculation, we observe that, for a fixed $d/a > 4$, the stress intensity factor increases in magnitude as h/a increases, i.e. the cracks are more stable if they are closer to region 1. This is as expected, since region 1 is occupied by a material which is "harder" than that in region 2. Also, for a fixed $h/a > 0$, $K_I/(P_0\sqrt{a})$ is larger in magnitude for smaller d/a, i.e. the cracks are less stable when the crack spacing is smaller. In Table 1, we present the numerical values of $K_I/(P_0\sqrt{a})$ for some selected values of h/a and d/a.

We repeat the calculation of the stress intensity factor, with regions 1 and 2 occupied by magnesium and titanium respectively. As in the earlier calculation above, we observe that, for a fixed h/a, the stress intensity factor $K_I/(P_0\sqrt{a})$ decreases with increasing d/a. However, for a fixed d/a, it is observed that the factor decreases in magnitude as h/a increases. This is not surprising as, in the present case, the material in region 1 is "softer" than that in region 2 and the cracks may be expected to be less stable when they are closer to the "softer" material. Numerical values of $K_I/(P_0\sqrt{a})$ for some selected values of h/a and d/a are given in Table 2.

Lastly, we carry out the computation of the stress intensity factor for the case where region 2 is occupied by magnesium and region 1 by a material with elastic constants $A^{(1)} = \xi \geq 16.2$, $F^{(1)} = 6.9$, $L^{(1)} = 4.67$ and $C^{(1)} = 18.1$. As $\xi \to \infty$, the material in region 1 becomes inextensible along the $0x_2$ direction. Since fiber-reinforced materials may be reasonably modelled using inextensible anisotropic materials (see, e.g. Clements [13]), it may be of some practical relevance for us to examine the effect of increasing the parameter ξ on the stress intensity factor. In Table 3, for $h/a = 1.00$ and $d/a = 5.00$, we present the stress intensity factor $K_I/(P_0\sqrt{a})$ for various values of ξ. It is clear from the table that the stress intensity factor $K_I/(P_0\sqrt{a})$ decreases as ξ increases, i.e. the cracks are more stable as the material in region 1 becomes more and more inextensible along the $0x_2$ direction. Also, $K_I/(P_0\sqrt{a})$ converges slowly (from above) to the particular value 1.495 (correct to 4 significant figures) as $\xi \to \infty$.

In our calculation above, we *typically* use $J = 5$ in the approximation (26). When the calculation is repeated using $J = 10$, convergence to 3 or 4 significant figures is observed in the numerical results. However, for situations involving extreme parameters, i.e. for very small h/a or d/a very close to 4, it is necessary to use a larger number of terms in (26) to achieve the same level of accuracy in the computation.

6 SUMMARY

We have considered the task of determining the displacement and stress fields around a periodic array of planar cracks in an anisotropic elastic half-space which adheres perfectly to another anisotropic half-space. A singular solution which satisfies the continuity conditions on the interface separating the two half-spaces is constructed and used to form a suitable integral expression for the displacements. The conditions on the crack faces then give rise to a system of hypersingular integral equations with the crack-opening displacements as unknown functions. The integral equations can be readily solved using a numerical technique. For a specific case involving a periodic array of planar cracks and particular transversely-isotropic materials, we have carried out the task of solving the integral equations numerically and compute the relevant crack tip stress intensity factor.

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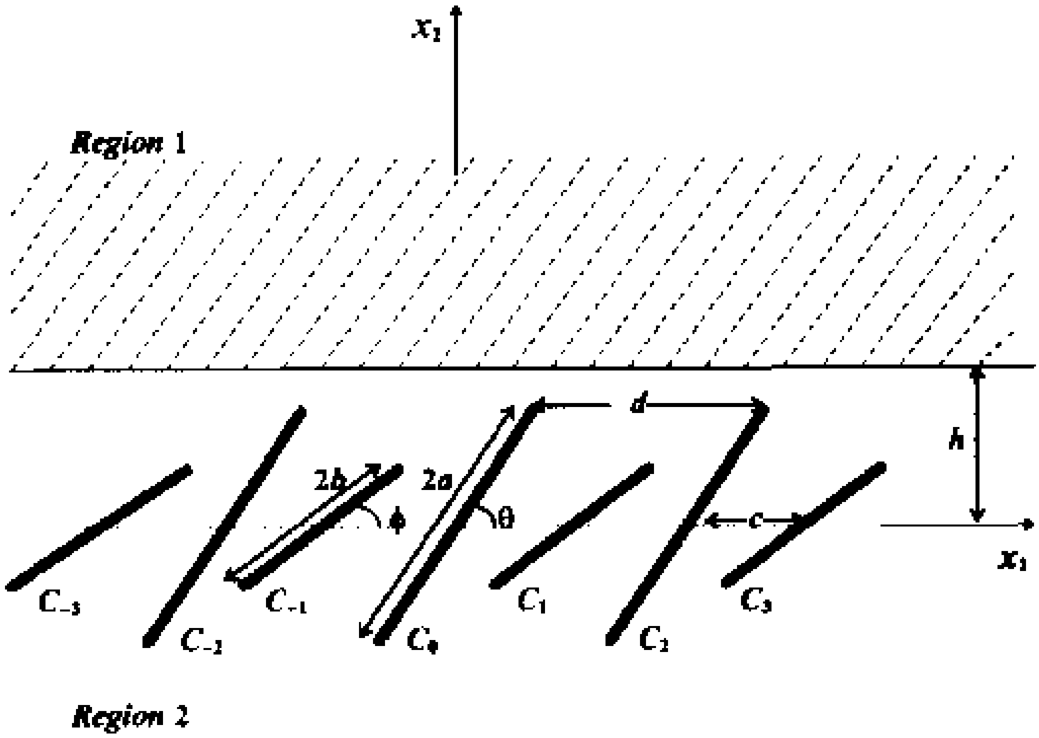

Figure 1

d/a h/a	4.20	4.40	4.60	4.80	5.00
0.20	2.514	1.867	1.577	1.412	1.304
0.40	2.889	2.126	1.778	1.574	1.440
0.60	2.940	2.174	1.829	1.625	1.488
0.80	2.949	2.189	1.849	1.649	1.515
1.00	2.952	2.195	1.860	1.663	1.531

Table 1

h/a \ d/a	4.20	4.40	4.60	4.80	5.00
0.20	4.582	3.313	2.794	2.494	2.295
0.40	3.041	2.315	2.024	1.867	1.767
0.60	2.973	2.243	1.940	1.776	1.675
0.80	2.962	2.223	1.910	1.738	1.631
1.00	2.959	2.215	1.896	1.718	1.606

Table 2

ξ	$K_I(P_0 a^{1/2})$
16.2	1.532
50	1.516
100	1.509
500	1.501
1000	1.499
5000	1.497
10,000	1.496
50,000	1.496
100,000	1.496

Table 3