

Dynamic interaction of multiple arbitrarily oriented planar cracks in a piezoelectric space: a semi-analytic solution*

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Abstract

The problem of determining the electro-elastic fields around arbitrarily oriented planar cracks in an infinite piezoelectric space is considered. The cracks which are acted upon by a transient load are either electrically impermeable or permeable. A semi-analytic method based on the theory of exponential Fourier transformation is proposed for solving the problem in the Laplace transform domain. The Laplace transforms of the jumps in the displacements and electric potential across opposite crack faces are determined by solving a system of hypersingular integral equations. Once these displacement and electric potential jumps are obtained, the displacements and electric potential and other physical quantities of interest, such as the crack tip stress and electric displacement intensity factors, can be computed with the help of a suitable algorithm for inverting Laplace transforms. The stress and electric displacement intensity factors are computed for some specific cases of the problem.

Keywords: multiple cracks, transient loads, piezoelectric materials, hypersingular integral equations

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1 Introduction

In recent years, the problem of determining the electro-elastic fields around cracks in piezoelectric materials has been a subject of considerable interest among many researchers. It appears that the majority of analyses on piezoelectric crack problems (see, for example, Athanasius, Ang and Sridhar [4], Li [20], Shindo, Watanabe and Narita [27] and Wang and Mai [32]) are concerned with electro-elastostatic deformations. According to Kuna [18], there are comparatively fewer works on piezoelectric cracks that are acted upon by time dependent loads.

The governing partial differential equations for piezoelectric materials undergoing dynamic antiplane deformations are relatively simpler in form, being reducible to a pair of equations comprising the two-dimensional Helmholtz and Laplace's equations. Thus, most papers giving semi-analytic solutions for dynamic piezoelectric crack problems assume that the cracks undergo antiplane deformations. For example, the dynamic response of a single electrically impermeable planar crack in an infinite transversely isotropic piezoelectric material under pure electric load and undergoing an antiplane deformation is investigated by Chen [7]. Chen and Karihaloo [8], Chen and Meguid [9], Li and Fan [21] and Li and Tang [24] have solved problems involving a single planar crack in an infinitely long piezoelectric strip under antiplane deformations. Coplanar cracks undergoing antiplane deformations in piezoelectric materials are examined in Chen and Worswick [10] and Meguid and Chen [25]. Kwon and Lee [19] and Li and Lee [22]-[23] have investigated the antiplane deformation of edge cracks in piezoelectric materials.

There are apparently few papers in which semi-analytic solutions for cracks undergoing dynamic inplane deformations in piezoelectric materials may be found. One of them is Shindo [26]. In [26], the problem of a single planar crack in a piezoelectric ceramic under normal impact is formulated in

terms of a pair of dual integral equations by representing the displacement and electric potential in the Laplace transform domain by suitable Fourier sine and cosine transform representations. The dual integral equations are solved as explained in Sneddon and Lowengrub [30], by reducing them to Fredholm integral equations of the second kind. The dynamic piezoelectric problem can also be formulated in terms of hypersingular integral equations using the approach in a recent paper by García-Sánchez, Zhang, Sládek and Sládek [13]. In [13], the kernels of the hypersingular integral formulation contain second order spatial derivatives of a suitable dynamic Green's function for piezoelectric solids. The Green's function is derived using Radon transform. Its evaluation is a rather involved exercise, requiring the computation of a line integral over a unit circle with integrand that is expressed in terms of exponential integrals (Wang and Zhang [33]).

The present paper derives a semi-analytic solution for an electro-elastic problem involving an arbitrary number of arbitrarily oriented planar cracks in an infinite piezoelectric space. The cracks are acted upon by internal stresses that are time dependent and are either electrically impermeable or permeable. The displacement and electric potential in the Laplace transform domain are expressed as a linear combination of suitably constructed exponential Fourier transform representations. The integrands of the Fourier integrals contain unknown functions that are directly related to the jumps in the Laplace transforms of the displacement and electrical potential across opposite crack faces. The unknown functions are to be determined by solving numerically a system of hypersingular integral equations. Once they are determined, the displacements and electric potential and other physical quantities of interest, such as the crack tip stress and electric displacement intensity factors, can be computed with the help of a suitable algorithm for inverting Laplace transforms. The analysis presented here is general in that

it covers both inplane and antiplane deformations. The crack tip stress and electric displacement intensity factors are computed for some specific cases of the problem. For the case involving a single planar crack, the values of the stress and electric displacement intensity factors computed are compared with those published in the literature.

The solution presented here is applicable to cracks that are either electrically impermeable or permeable. The electrically semi-permeable cracks proposed by Hao and Shen [14] may be of interest in engineering analysis too. The solution here may be used together with an iterative procedure (such as the one described in Ang and Athanasius [3]) to analyze the dynamic interaction of electrically semi-permeable cracks.

Numerical methods may be applied to solve dynamic piezoelectric problems involving cracks in solids of finite extent. Examples of such numerical works include: Enderlein, Ricoeur and Kuna [11] (finite element method); García-Sánchez, Zhang and Sáez [12], Kögl and Gaul [17] and Wünsche, García-Sánchez, Sáez and Zhang [34] (boundary element method); and Sladek, Sladek, Zhang, García-Sánchez and Wünsche [28] (meshless method) . The semi-analytic solution given here may be used to check the validity of those numerical techniques for solving dynamic piezoelectric crack problems.

2 The problem

Referring to an $Ox_1x_2x_3$ Cartesian coordinate system, consider an infinite piezoelectric space that contains N arbitrarily oriented non-intersecting planar cracks whose geometries do not change along the x_3 axis. The cracks are denoted by $\Gamma^{(1)}$, $\Gamma^{(2)}$, \dots , $\Gamma^{(N-1)}$ and $\Gamma^{(N)}$. The n -th planar crack $\Gamma^{(n)}$ lies in the region

$$-\ell^{(n)} < a_{j1}^{(n)}(x_j - c_j^{(n)}) < \ell^{(n)}, \quad a_{j2}^{(n)}(x_j - c_j^{(n)}) = 0, \quad -\infty < x_3 < \infty, \quad (1)$$

where

$$[a_{ij}^{(n)}] = \begin{pmatrix} \sin(\theta^{(n)}) & \cos(\theta^{(n)}) & 0 \\ -\cos(\theta^{(n)}) & \sin(\theta^{(n)}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

On the Ox_1x_2 plane, the crack $\Gamma^{(n)}$ is a straight line cut, $2\ell^{(n)}$ is the length of the crack, $(c_1^{(n)}, c_2^{(n)})$ is the midpoint of the crack and $\theta^{(n)}$ is the angle between the crack and the vertical line passing through $(c_1^{(n)}, c_2^{(n)})$ (such that $0 \leq \theta^{(n)} \leq \pi$), as shown in Figure 1. Note that the usual convention of summing over a repeated subscript is adopted in (1) for lower case Latin subscripts that run from 1 to 3.

It will be assumed that here the electroelastic deformation of the cracked piezoelectric space does not vary along the x_3 direction. The problem is to determine the displacements $u_k(x_1, x_2, t)$ and electric potential $\phi(x_1, x_2, t)$ in the piezoelectric space for time $t > 0$ such that suitably prescribed boundary conditions on the cracks are satisfied.

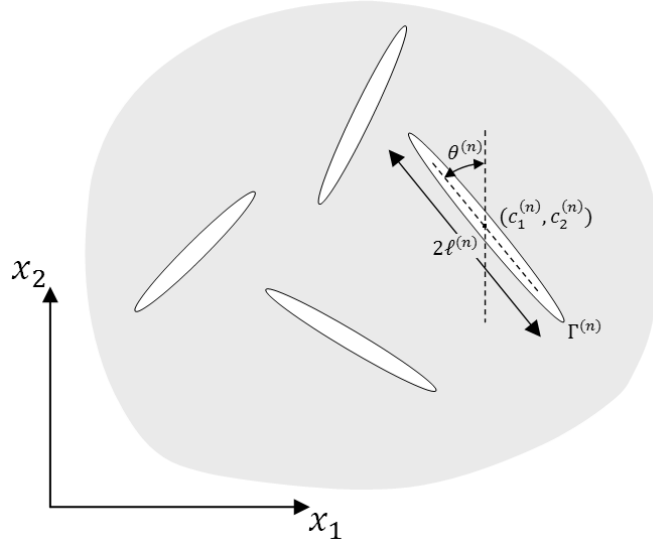


Figure 1. A geometrical sketch of the problem.

More specifically, the conditions the cracks are given by

$$\begin{aligned} \sigma_{kj}(x_1, x_2, t)m_j^{(n)} &\rightarrow -P_k^{(n)}(\xi_1, \xi_2, t) \quad (k = 1, 2, 3) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N), \end{aligned} \quad (3)$$

and either

$$\begin{aligned} d_j(x_1, x_2, t)m_j^{(n)} &\rightarrow -P_4^{(n)}(\xi_1, \xi_2, t) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ \text{if the cracks are electrically impermeable,} \end{aligned} \quad (4)$$

or

$$\begin{aligned} \Delta\phi(x_1, x_2, t) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ \text{if the cracks are electrically permeable,} \end{aligned} \quad (5)$$

where σ_{ij} and d_i are respectively the stresses and electric displacements, $P_1^{(n)}(\xi_1, \xi_2, t)$, $P_2^{(n)}(\xi_1, \xi_2, t)$, $P_3^{(n)}(\xi_1, \xi_2, t)$ and $P_4^{(n)}(\xi_1, \xi_2, t)$ are suitably prescribed functions for $(\xi_1, \xi_2) \in \Gamma^{(n)}$, $m_i^{(n)} = -a_{i2}^{(n)}$ are the components of a unit magnitude normal vector to the crack $\Gamma^{(n)}$ and $\Delta\phi(x_1, x_2)$ denotes the jump in the electrical potential ϕ across the crack $\Gamma^{(n)}$ as defined by

$$\begin{aligned} \Delta\phi(x_1, x_2, t) &= \lim_{\varepsilon \rightarrow 0} [\phi(x_1 - |\varepsilon|m_1^{(n)}, x_2 - |\varepsilon|m_2^{(n)}, t) \\ &\quad - \phi(x_1 + |\varepsilon|m_1^{(n)}, x_2 + |\varepsilon|m_2^{(n)}, t)] \\ &\text{for } (x_1, x_2) \in \Gamma^{(n)}. \end{aligned} \quad (6)$$

Furthermore, it is assumed here that the displacements u_k and its partial derivative with respect to time (that is, $\partial u_k / \partial t$) are both zero at time $t = 0$ and the stresses $\sigma_{kj}(x_1, x_2, t)$ and electric displacement $d_k(x_1, x_2, t)$ generated by the cracks vanish as $x_1^2 + x_2^2 \rightarrow \infty$.

3 Basic equations of electroelasticity

The governing equations for the displacements u_k and electric potential ϕ in a homogeneous piezoelectric material are given by

$$\begin{aligned} c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} + e_{lij} \frac{\partial^2 \phi}{\partial x_j \partial x_\ell} &= \rho \frac{\partial^2 u_i}{\partial t^2}, \\ e_{jkl} \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} - \kappa_{j\ell} \frac{\partial^2 \phi}{\partial x_j \partial x_\ell} &= 0, \end{aligned} \quad (7)$$

where c_{ijkl} , e_{lij} and $\kappa_{i\ell}$ are the constant elastic moduli, piezoelectric coefficients and dielectric coefficients respectively and ρ is the density.

The constitutive equations relating (σ_{ij}, d_j) and (u_k, ϕ) are

$$\begin{aligned} \sigma_{ij} &= c_{ijkl} \frac{\partial u_k}{\partial x_\ell} + e_{lij} \frac{\partial \phi}{\partial x_\ell}, \\ d_j &= e_{jkl} \frac{\partial u_k}{\partial x_\ell} - \kappa_{jp} \frac{\partial \phi}{\partial x_p}. \end{aligned} \quad (8)$$

Following closely the approach of Barnett and Lothe [5], we define

$$\begin{aligned} U_J &= \begin{cases} u_j & \text{for } J = j = 1, 2, 3, \\ \phi & \text{for } J = 4, \end{cases} \\ S_{Ij} &= \begin{cases} \sigma_{ij} & \text{for } I = i = 1, 2, 3, \\ d_j & \text{for } I = 4, \end{cases} \\ C_{IjK\ell} &= \begin{cases} c_{ijkl} & \text{for } I = i = 1, 2, 3 \text{ and } K = k = 1, 2, 3, \\ e_{lij} & \text{for } I = i = 1, 2, 3 \text{ and } K = 4, \\ e_{jkl} & \text{for } I = 4 \text{ and } K = k = 1, 2, 3, \\ -\kappa_{j\ell} & \text{for } I = 4 \text{ and } K = 4, \end{cases} \end{aligned} \quad (9)$$

so that (7) and (8) may be respectively written more compactly as

$$C_{IjK\ell} \frac{\partial^2 U_K}{\partial x_j \partial x_\ell} = B_{IK} \frac{\partial^2 U_K}{\partial t^2} \quad (I = 1, 2, 3, 4) \quad (10)$$

and

$$S_{Ij} = C_{IjK\ell} \frac{\partial U_K}{\partial x_\ell} \quad (I = 1, 2, 3, 4; j = 1, 2, 3) \quad (11)$$

where

$$B_{IK} = \begin{cases} \rho & \text{if } I = K \text{ and } I \neq 4, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

Note that uppercase Latin subscripts have values 1, 2, 3 and 4. Summation is also implied for repeated uppercase Latin subscripts.

It follows that the problem stated in Section 2 requires solving (10) within the piezoelectric space for time $t > 0$ subject to the initial-boundary conditions

$$U_K = 0 \text{ and } \frac{\partial U_K}{\partial t} = 0 \text{ at } t = 0 \quad (K = 1, 2, 3), \quad (13)$$

$$\begin{aligned} S_{Ij}(x_1, x_2, t) m_j^{(n)} &\rightarrow -P_I^{(n)}(\xi_1, \xi_2, t) \quad (I = 1, 2, 3) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N), \end{aligned} \quad (14)$$

and either

$$\begin{aligned} S_{4j}(x_1, x_2, t) m_j^{(n)} &\rightarrow -P_4^{(n)}(\xi_1, \xi_2, t) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ &\text{if the cracks are electrically impermeable,} \end{aligned} \quad (15)$$

or

$$\begin{aligned} \Delta U_4(x_1, x_2, t) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ &\text{if the cracks are electrically permeable,} \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Delta U_I(x_1, x_2, t) &= \lim_{\varepsilon \rightarrow 0} [U_I(x_1 - |\varepsilon| m_1^{(n)}, x_2 - |\varepsilon| m_2^{(n)}) \\ &\quad - U_I(x_1 + |\varepsilon| m_1^{(n)}, x_2 + |\varepsilon| m_2^{(n)})] \\ &\text{for } (x_1, x_2) \in \Gamma^{(n)}. \end{aligned} \quad (17)$$

In addition, it is required that $S_{Ij}(x_1, x_2, t) \rightarrow 0$ as $x_1^2 + x_2^2 \rightarrow \infty$.

4 Formulation in Laplace transform domain

We denote the Laplace transformation of $F(x_1, x_2, t)$ over time $t \geq 0$ by $\widehat{F}(x_1, x_2, s)$, that is, we define

$$\widehat{F}(x_1, x_2, s) = \int_0^{\infty} F(x_1, x_2, t) \exp(-st) dt, \quad (18)$$

where s is the Laplace transformation parameter.

Application of the Laplace transformation on both sides of (10) together with the initial conditions (13) yields

$$C_{IjK\ell} \frac{\partial^2 \widehat{U}_K}{\partial x_j \partial x_\ell} - s^2 B_{IK} \widehat{U}_K = 0 \quad (I = 1, 2, 3, 4). \quad (19)$$

In the Laplace transform domain, the problem is to solve (19) subject to

$$\begin{aligned} \widehat{S}_{Ij}(x_1, x_2, s) m_j^{(n)} &\rightarrow -\widehat{P}_I^{(n)}(\xi_1, \xi_2, s) \quad (I = 1, 2, 3) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N), \end{aligned} \quad (20)$$

and either

$$\begin{aligned} \widehat{S}_{4j}(x_1, x_2, s) m_j^{(n)} &\rightarrow -\widehat{P}_4^{(n)}(\xi_1, \xi_2, s) \\ \text{as } (x_1, x_2) &\rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ &\text{if the cracks are electrically impermeable,} \end{aligned} \quad (21)$$

or

$$\begin{aligned} \Delta \widehat{U}_4(x_1, x_2, s) &\rightarrow 0 \text{ as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)} \quad (n = 1, 2, \dots, N) \\ &\text{if the cracks are electrically permeable.} \end{aligned} \quad (22)$$

It is also required that $\widehat{S}_{Ij}(x_1, x_2, s) \rightarrow 0$ as $x_1^2 + x_2^2 \rightarrow \infty$.

5 Method of solution

In this section, a method based on the theory of exponential Fourier transformation is proposed for solving (19) subject to (20)-(22).

5.1 Solution in Fourier integral form

For the solution of the piezoelectric crack problem, let

$$\begin{aligned} \widehat{U}_K(x_1, x_2, s) = & \sum_{n=1}^N \operatorname{Re} \left\{ \sum_{\alpha=1}^4 \int_0^{\infty} A_{K\alpha}^{(n)}(\xi, s) [H(a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{1\alpha}^{(n)}(\xi, s) \right. \\ & \times \exp(i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)})(x_j - c_j^{(n)})) \\ & + H(-a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{2\alpha}^{(n)}(\xi, s) \\ & \left. \times \exp(-i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)})(x_j - c_j^{(n)})) \right] d\xi \}, \end{aligned} \quad (23)$$

where $H(x)$ is the unit-step Heaviside function, $F_{1\alpha}^{(n)}(\xi, s)$ and $F_{2\alpha}^{(n)}(\xi, s)$ are functions yet to be determined, $\tau_{\alpha}^{(n)}(\xi, s)$ ($n = 1, 2, \dots, N$) are roots, with positive imaginary parts, of the 8-th order polynomial equation (in τ) given by

$$\begin{aligned} \det \left[\frac{s^2}{\xi^2} B_{IK} + (a_{11}^{(n)} + \tau a_{12}^{(n)})^2 C_{I1K1} \right. \\ \left. + (a_{21}^{(n)} + \tau a_{22}^{(n)})(a_{11}^{(n)} + \tau a_{12}^{(n)})(C_{I1K2} + C_{I2K1}) \right. \\ \left. + (a_{21}^{(n)} + \tau a_{22}^{(n)})^2 C_{I2K2} \right] = 0, \end{aligned} \quad (24)$$

$A_{K\alpha}^{(n)}(\xi, s)$ ($n = 1, 2, \dots, N$) are non-trivial solutions of the system

$$\begin{aligned} \left[\frac{s^2}{\xi^2} B_{IK} + (a_{11}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{12}^{(n)})^2 C_{I1K1} \right. \\ \left. + (a_{21}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{22}^{(n)})(a_{11}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{12}^{(n)})(C_{I1K2} + C_{I2K1}) \right. \\ \left. + (a_{21}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{22}^{(n)})^2 C_{I2K2} \right] A_{K\alpha}^{(n)} = 0. \end{aligned} \quad (25)$$

From (11) and (23), we obtain

$$\begin{aligned}\widehat{S}_{Ij}(x_1, x_2, s) &= \sum_{n=1}^N \operatorname{Re}\left\{ \sum_{\alpha=1}^4 \int_0^{\infty} i\xi L_{Ij\alpha}^{(n)}(\xi, s) [H(a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{1\alpha}^{(n)}(\xi, s) \right. \\ &\quad \times \exp(i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)})(x_j - c_j^{(n)})) \\ &\quad - H(-a_{r2}^{(n)}(x_r - c_r^{(n)})) F_{2\alpha}^{(n)}(\xi, s) \\ &\quad \left. \times \exp(-i\xi(a_{j1}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{j2}^{(n)})(x_j - c_j^{(n)})) \right\} d\xi, \quad (26)\end{aligned}$$

$L_{Ij\alpha}^{(n)}(\xi, s)$ ($n = 1, 2, \dots, N$) are given by

$$\begin{aligned}L_{Ij\alpha}^{(n)}(\xi, s) &= [(a_{11}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{12}^{(n)})C_{IjK1} \\ &\quad + (a_{21}^{(n)} + \tau_{\alpha}^{(n)}(\xi, s)a_{22}^{(n)})C_{IjK2}]A_{K\alpha}^{(n)}.\end{aligned} \quad (27)$$

The integral representation for $\widehat{U}_K(x_1, x_2, s)$ in (23) is obtained by generalizing the analyses in Ang [1]-[2] and Clements [6].

Note that $\widehat{U}_K(x_1, x_2, s)$ and $\widehat{S}_{Ij}(x_1, x_2, s)$ in (23) and (26) respectively are represented by different integral expressions in different parts of the piezoelectric space. To ensure that $\widehat{S}_{Ij}(x_1, x_2, s)m_j^{(n)}$ are continuous on $a_{r2}^{(n)}(x_r - c_r^{(n)}) = 0$, the functions $F_{1\alpha}^{(n)}(\xi, s)$ and $F_{2\alpha}^{(n)}(\xi, s)$ are chosen to be given by

$$F_{1\alpha}^{(n)}(\xi, s) = M_{\alpha P}^{(n)}(\xi, s)\psi_P^{(n)}(\xi, s) \text{ and } F_{2\alpha}^{(n)}(\xi, s) = M_{\alpha P}^{(n)}(\xi, s)\overline{\psi_P^{(n)}}(\xi, s), \quad (28)$$

where $\psi_P^{(n)}(\xi, s)$ are functions to be determined and the overhead bar denotes the complex conjugate of a complex number and $M_{\alpha P}^{(n)}(\xi, s)$ are defined by

$$\sum_{\alpha=1}^4 m_j^{(n)} L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) = \delta_{IP} \quad (n = 1, 2, \dots, N). \quad (29)$$

The functions $\widehat{U}_K(x_1, x_2, s)$ are continuous on the plane $a_{r2}^{(n)}(x_r - c_r^{(n)}) = 0$ at points not on any of the cracks if $\psi_P^{(n)}(\xi, s)$ are chosen to be

$$\psi_P^{(n)}(\xi, s) = iT_{PJ}^{(n)}(\xi, s) \int_{-\ell^{(n)}}^{\ell^{(n)}} r_J^{(n)}(u, s) \exp(-i\xi u) du, \quad (30)$$

where $i = \sqrt{-1}$, $r_j^{(n)}(u, s)$ are real functions yet to be determined and $T_{PJ}^{(n)}(\xi, s)$ are real functions defined by

$$i \sum_{\alpha=1}^4 [A_{K\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) - \overline{A}_{K\alpha}^{(n)}(\xi, s) \overline{M}_{\alpha P}^{(n)}(\xi, s)] T_{PJ}^{(n)}(\xi, s) = \delta_{KJ}. \quad (31)$$

Use of (31) in (23) together with

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \int_{-\ell}^{\ell} \frac{\psi(u)}{\epsilon^2 + (v-u)^2} du = \pi \psi(v) \text{ for } -\ell < v < \ell, \quad (32)$$

gives

$$r_K^{(n)}(a_{j1}^{(n)}(x_j - c_j^{(n)}), s) = \frac{1}{\pi} \Delta \widehat{U}_K(x_1, x_2, s) \\ \text{for } -\ell^{(n)} < a_{j1}^{(n)}(x_j - c_j^{(n)}) < \ell^{(n)}, \quad a_{j2}^{(n)}(x_j - c_j^{(n)}) = 0, \quad (33)$$

where $\Delta \widehat{U}_K(x_1, x_2, s)$ are the Laplace transform of the generalized crack opening displacements as defined in (17).

The functions $\widehat{S}_{Ij}(x_1, x_2, s)$ in (26) can now be written as

$$\widehat{S}_{Ij}(x_1, x_2, s) = - \sum_{n=1}^N \int_{-\ell^{(n)}}^{\ell^{(n)}} r_K^{(n)}(u, s) \operatorname{Re} \left\{ \sum_{\alpha=1}^4 \int_0^{\infty} \xi [H(a_{r2}^{(n)}(x_r - c_r^{(n)})) \right. \\ \times L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) \exp(i\xi \tau_{\alpha}^{(n)}(\xi, s) a_{k2}^{(n)}(x_k - c_k^{(n)})) \\ + H(-a_{r2}^{(n)}(x_r - c_r^{(n)})) \overline{L}_{Ij\alpha}^{(n)}(\xi, s) \overline{M}_{\alpha P}^{(n)}(\xi, s) \\ \times \exp(i\xi \overline{\tau}_{\alpha}^{(n)}(\xi, s) a_{k2}^{(n)}(x_k - c_k^{(n)}))] \\ \left. \times T_{PK}^{(n)}(\xi, s) \exp(i\xi [a_{k1}^{(n)}(x_k - c_k^{(n)}) - u]) d\xi \right\} du. \quad (34)$$

5.2 Electrically impermeable cracks

From (34), conditions (20) and (21) for electrically impermeable cracks give the hypersingular integral equations

$$\begin{aligned}
& \frac{1}{\ell^{(q)}} \mathcal{H} \int_{-1}^1 \frac{D_{IK}^{(q)} \Delta \widehat{U}_K^{(q)}(u, s)}{(v-u)^2} du + \ell^{(q)} \int_{-1}^1 \Delta \widehat{U}_K^{(q)}(u, s) \Omega_{IK}^{(q)}(u, v, s) du \\
& + \ell^{(q)} \mathcal{C} \int_{-1}^1 s^2 G_{IK}^{(q)} \Delta \widehat{U}_K^{(q)}(u, s) \cosh(\ell^{(q)} \eta |v-u|) \ln(\ell^{(q)} \eta |v-u|) du \\
& + \sum_{\substack{n=1 \\ n \neq q}}^N \ell^{(n)} \int_{-1}^1 \Delta \widehat{U}_K^{(n)}(u, s) \Theta_{IK}^{(nq)}(u, v, s) du \\
& = -\pi \widehat{P}_I^{(q)}(X_1^{(q)}(v), X_2^{(q)}(v), s) \quad (I = 1, 2, 3, 4) \\
& \quad \text{for } -1 < v < 1 \quad (q = 1, 2, \dots, N), \tag{35}
\end{aligned}$$

where $\Delta \widehat{U}_K^{(q)}(u, s) = \pi r_K^{(q)}(\ell^{(q)} u, s)$, $X_1^{(q)}(v) = c_1^{(q)} + \ell^{(q)} v \sin(\theta^{(q)})$, $X_2^{(q)}(v) = c_2^{(q)} - \ell^{(q)} v \cos(\theta^{(q)})$, \mathcal{C} denotes that the integral is to be interpreted in the Cauchy principal sense and \mathcal{H} denotes that the integral is to be interpreted in the Hadamard finite-part sense, $D_{IK}^{(q)}$, $G_{IK}^{(q)}$ and $W_{IK}^{(q)}(\xi, s)$ are given by

$$\begin{aligned}
D_{IK}^{(q)} &= \lim_{(\xi/s) \rightarrow \infty} T_{IK}^{(q)}(\xi, s), \\
G_{IK}^{(q)} &= \lim_{(\xi/s) \rightarrow \infty} \left(\frac{\xi}{s}\right)^2 [T_{IK}^{(q)}(\xi, s) - D_{IK}^{(q)}], \\
W_{IK}^{(q)}(\xi, s) &= T_{IK}^{(q)}(\xi, s) - D_{IK}^{(q)} - \frac{s^2 G_{IK}^{(q)}}{\xi^2 + \eta^2} \quad (\eta > 0), \tag{36}
\end{aligned}$$

and $\Omega_{IK}^{(q)}(u, v, s)$ and $\Theta_{IK}^{(nq)}(u, v, s)$ are respectively defined by

$$\begin{aligned}\Omega_{IK}^{(q)}(u, v, s) = & - \int_0^{\infty} \xi W_{IK}^{(q)}(\xi, s) \cos(\ell^{(q)} \xi [v - u]) d\xi \\ & - s^2 G_{IK}^{(q)} [\text{Shi}(\ell^{(q)} \eta |v - u|) \sinh(\ell^{(q)} \eta |v - u|) \\ & - \frac{1}{2} \cosh(\ell^{(q)} \eta |v - u|) (\text{Ei}(\ell^{(q)} \eta |v - u|) - E_1(\ell^{(q)} \eta |v - u|)) \\ & + \cosh(\ell^{(q)} \eta |v - u|) \ln(\ell^{(q)} \eta |v - u|)],\end{aligned}\quad (37)$$

and

$$\begin{aligned}\Theta_{IK}^{(nq)}(u, v, s) = & - \text{Re} \left\{ \sum_{\alpha=1}^4 \int_0^{\infty} \xi m_j^{(q)} [H(Y_2^{(nq)}(u, v)) \right. \\ & \times L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) \exp(i\xi \tau_{\alpha}^{(n)}(\xi, s) Y_2^{(nq)}(u, v)) \\ & + H(-Y_2^{(nq)}(u, v)) \bar{L}_{Ij\alpha}^{(n)}(\xi, s) \bar{M}_{\alpha P}^{(n)}(\xi, s) \\ & \times \exp(i\xi \bar{\tau}_{\alpha}^{(n)}(\xi, s) Y_2^{(nq)}(u, v))] \\ & \left. \times T_{PK}^{(n)}(\xi, s) \exp(i\xi Y_1^{(nq)}(u, v)) d\xi \right\} \\ & \text{if it is assumed that } Y_2^{(nq)}(u, v) \neq 0,\end{aligned}\quad (38)$$

with $Y_p^{(nq)}(u, v) = a_{kp}^{(n)}(X_k^{(q)}(v) - c_k^{(n)}) - \ell^{(n)} \delta_{p1} u$ and

$$\begin{aligned}\text{Shi}(u) &= \int_0^u \frac{\sinh(x)}{x} dx, \\ \text{Ei}(u) &= -\mathcal{C} \int_{-u}^{\infty} \frac{\exp(-x)}{x} dx, \\ E_1(u) &= \int_u^{\infty} \frac{\exp(-x)}{x} dx.\end{aligned}\quad (39)$$

The functions $W_{IK}^{(q)}(\xi, s)$ behave as $O(s^4/\xi^4)$ for very large ξ . Thus, the improper integral over $[0, \infty)$ that appears in the definition of $\Omega_{IK}^{(q)}(u, v, s)$ in (37) is well defined.

Note that the expressions for $\Theta_{IK}^{(nq)}(u, v, s)$ as given in (38) are valid for $Y_2^{(nq)}(u, v) \neq 0$. If $Y_2^{(nq)}(u, v) = 0$ then (38) has to be modified accordingly. The modification gives

$$\begin{aligned}
\Theta_{IK}^{(nq)}(u, v, s) = & \operatorname{Re}\left\{ \frac{\tilde{D}_{IK}^{(nq)}}{[Y_1^{(nq)}(u, v)]^2} - \int_0^\infty \xi \tilde{W}_{IK}^{(nq)}(\xi, s) \exp(i\xi Y_1^{(nq)}(u, v)) d\xi \right. \\
& - s^2 \tilde{G}_{IK}^{(nq)} [\operatorname{Shi}(\eta|Y_1^{(nq)}(u, v)|) \sinh(\eta|Y_1^{(nq)}(u, v)|) \\
& - \frac{1}{2} \cosh(\eta|Y_1^{(nq)}(u, v)|) \\
& \times (\operatorname{Ei}(\eta|Y_1^{(nq)}(u, v)|) - E_1(\eta|Y_1^{(nq)}(u, v)|)) \\
& + i\frac{\pi}{2} \operatorname{sgn}(Y_1^{(nq)}(u, v)) \\
& \left. \times (\cosh(\eta|Y_1^{(nq)}(u, v)|) - \sinh(\eta|Y_1^{(nq)}(u, v)|)) \right\} \\
& \text{if } Y_2^{(nq)}(u, v) = 0, \tag{40}
\end{aligned}$$

where $\operatorname{sgn}(x)$ denotes the sign of x and

$$\begin{aligned}
\tilde{D}_{IK}^{(nq)} &= \lim_{(\xi/s) \rightarrow \infty} \tilde{T}_{IK}^{(nq)}(\xi, s), \\
\tilde{T}_{IK}^{(nq)}(\xi, s) &= \sum_{\alpha=1}^4 m_j^{(q)} L_{Ij\alpha}^{(n)}(\xi, s) M_{\alpha P}^{(n)}(\xi, s) T_{IK}^{(n)}(\xi, s), \\
\tilde{G}_{IK}^{(nq)} &= \lim_{(\xi/s) \rightarrow \infty} \left(\frac{\xi}{s}\right)^2 [\tilde{T}_{IK}^{(nq)}(\xi, s) - \tilde{D}_{IK}^{(nq)}] \\
\tilde{W}_{IK}^{(nq)}(\xi, s) &= \tilde{T}_{IK}^{(nq)}(\xi, s) - \tilde{D}_{IK}^{(nq)} - \frac{s^2 \tilde{G}_{IK}^{(nq)}}{\xi^2 + \eta^2} \quad (\eta > 0). \tag{41}
\end{aligned}$$

If the cracks $\Gamma^{(n)}$ and $\Gamma^{(q)}$ are coplanar then $m_j^{(q)} = m_j^{(n)}$, hence $\tilde{T}_{IK}^{(nq)}(\xi, s) = T_{IK}^{(n)}(\xi, s)$, $\tilde{D}_{IK}^{(nq)} = D_{IK}^{(n)}$, $\tilde{G}_{IK}^{(nq)} = G_{IK}^{(n)}$ and $\tilde{W}_{IK}^{(nq)}(\xi, s) = W_{IK}^{(n)}(\xi, s)$. Note that $Y_2^{(nq)}(u, v) = 0$ for all u and v if $\Gamma^{(n)}$ and $\Gamma^{(q)}$ are coplanar.

The derivations of (35), (37) and (40) make use of the following results:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_{-1}^1 \frac{(\epsilon^2 - (v-u)^2)\psi(u)}{(\epsilon^2 + (v-u)^2)^2} du &= -\mathcal{H} \int_{-1}^1 \frac{\psi(u)}{(v-u)^2} du \text{ for } -1 < v < 1, \\ \int_0^\infty \frac{\xi}{\xi^2 + \eta^2} \cos(a\xi) d\xi &= -\frac{1}{2} \cosh(a\eta) (\text{Ei}(a\eta) - E_1(a\eta)) \\ &\quad + \text{Shi}(a\eta) \sinh(a\eta) \quad (a\eta > 0), \\ \int_0^\infty \frac{\xi}{\xi^2 + \eta^2} \sin(a\xi) d\xi &= \frac{\pi}{2} \text{sgn}(a) [\cosh(|a\eta|) - \sinh(|a\eta|)]. \end{aligned} \quad (42)$$

Note that $\text{Ei}(x) - E_1(x)$ tend to $2 \ln(x)$ as $x \rightarrow 0^+$. This explains the presence of the Cauchy principal integral in (35).

If the cracks are electrically impermeable, the functions $\Delta \widehat{U}_K^{(q)}(u, s)$ ($q = 1, 2, \dots, N$) in (30) are to be determined by solving the hypersingular integral equations in (35). If we make the approximation (as in Kaya and Erdogan [16])

$$\Delta \widehat{U}_K^{(q)}(u, s) \simeq \sqrt{1-u^2} \sum_{j=1}^J \omega_K^{(qj)}(s) U^{(j-1)}(u), \quad (43)$$

where $U^{(j)}(x) = \sin([j+1] \arccos(x)) / \sin(\arccos(x))$ is the j^{th} order Chebyshev polynomial of the second kind and $\omega_P^{(nj)}(s)$ are unknown coefficients, then (35) can be used to set up a system of linear algebraic equations to determine $\omega_P^{(nj)}(s)$ for any fixed value of s (Athanasius, Ang and Sridhar [4]).

5.3 Electrically permeable cracks

From (16) and (33), $\Delta \widehat{U}_4^{(q)}(u, s) = 0$ for $-1 < u < 1$ and $q = 1, 2, \dots, N$, if the cracks are electrically permeable. According to (14), the unknown functions $\Delta \widehat{U}_1^{(q)}(u, s)$, $\Delta \widehat{U}_2^{(q)}(u, s)$ and $\Delta \widehat{U}_3^{(q)}(u, s)$ that can be approximated as above by (43) are governed by (35) (with $\Delta \widehat{U}_4^{(q)}(u, s) = 0$) for $I = 1, 2, 3$ (instead of $I = 1, 2, 3, 4$).

6 Stress and electric displacement intensity factors

The dynamic stress and electric displacement intensity factors at the tips $(X_1^{(n)}(-1), X_2^{(n)}(-1))$ and $(X_1^{(n)}(1), X_2^{(n)}(1))$ of the n -th crack $\Gamma^{(n)}$ are defined as follows:

$$\begin{aligned} & K_I(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\ &= \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)} (S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_1^{(n)} \\ &+ S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_2^{(n)}) m_j^{(n)}, \end{aligned}$$

$$\begin{aligned} & K_{II}(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\ &= \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)} (S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_2^{(n)} \\ &- S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_1^{(n)}) m_j^{(n)}, \end{aligned}$$

$$\begin{aligned} & K_{III}(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\ &= \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)} S_{3j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_j^{(n)}, \end{aligned}$$

$$\begin{aligned} & K_{IV}(X_1^{(n)}(-1), X_2^{(n)}(-1), t) \\ &= \lim_{u \rightarrow -1^-} \sqrt{-2\ell^{(n)}(u+1)} S_{4j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_j^{(n)}, \end{aligned}$$

$$\begin{aligned} & K_I(X_1^{(n)}(1), X_2^{(n)}(1), t) \\ &= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} (S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_1^{(n)} \\ &+ S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_2^{(n)}) m_j^{(n)}, \end{aligned}$$

$$\begin{aligned} & K_{II}(X_1^{(n)}(1), X_2^{(n)}(1), t) \\ &= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} (S_{1j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_2^{(n)} \\ &- S_{2j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_1^{(n)}) m_j^{(n)}, \end{aligned}$$

$$\begin{aligned}
& K_{III}(X_1^{(n)}(1), X_2^{(n)}(1), t) \\
&= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} S_{3j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_j^{(n)}, \\
& K_{IV}(X_1^{(n)}(1), X_2^{(n)}(1), t) \\
&= \lim_{u \rightarrow 1^+} \sqrt{2\ell^{(n)}(u-1)} S_{4j}(X_1^{(n)}(u), X_2^{(n)}(u), t) m_j^{(n)}. \tag{44}
\end{aligned}$$

Once the coefficients $\omega_P^{(nj)}(s)$ in (43) are determined, the above intensity factors can be approximately calculated in the Laplace transform domain using

$$\begin{aligned}
\widehat{K}_I(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)} m_1^{(n)} + D_{P2}^{(n)} m_2^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_{II}(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)} m_2^{(n)} - D_{P2}^{(n)} m_1^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_{III}(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P3}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_{IV}(X_1^{(n)}(-1), X_2^{(n)}(-1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P4}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(-1), \\
\widehat{K}_I(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)} m_1^{(n)} + D_{P2}^{(n)} m_2^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1), \\
\widehat{K}_{II}(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq \frac{1}{\sqrt{\ell^{(n)}}} (D_{P1}^{(n)} m_2^{(n)} - D_{P2}^{(n)} m_1^{(n)}) \\
&\quad \times \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1),
\end{aligned}$$

$$\begin{aligned}\widehat{K}_{III}(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P3}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1), \\ \widehat{K}_{IV}(X_1^{(n)}(1), X_2^{(n)}(1), s) &\simeq -\frac{1}{\sqrt{\ell^{(n)}}} D_{P4}^{(n)} \sum_{j=1}^J \omega_P^{(nj)}(s) U^{(j-1)}(+1).\end{aligned}\quad (45)$$

The dynamics stress and electric displacement intensity factors at any time t may be recovered by using the numerical Laplace transform algorithm in Stehfest [31], that is, by using the formula

$$f(t) \simeq \frac{\ln(2)}{t} \sum_{n=1}^{2M} V_n \widehat{f}\left(\frac{n \ln(2)}{t}\right), \quad (46)$$

where $\widehat{f}(s)$ denotes the Laplace transform of $f(t)$, M is a positive integer and

$$V_n = (-1)^{n+M} \sum_{m=\lceil (n+1)/2 \rceil}^{\min(n, M)} \frac{m^M (2m)!}{(M-m)! m! (m-1)! (n-m)! (2m-n)!}, \quad (47)$$

with $[r]$ denoting the integer part of the real number r .

Note that the Stehfest's algorithm requires the problem under consideration to be solved for only real Laplace transform parameter s . It has been widely used by researchers for the numerical inversion of Laplace transforms in solving many problems in engineering (see, for example, Ang [1], Hemker [15] and Smith, Edwards and Beselli [29]).

7 Specific cases

In this section, the dynamics crack tip stress and electric displacement intensity factors are computed for some specific cases of the problem.

Problem 1. Consider the case of a single electrically impermeable crack in an infinite piezoelectric space. The crack lies in the region $-a < x_1 < a$,

$x_2 = 0$, and the only non-zero uniform load acting on it is given by $S_{22} = -H(t)\sigma_0$, where $H(t)$ is the unit-step Heaviside function. (Note that S_{12} , S_{32} and S_{42} are taken to be zero on the crack.)

The electrical poling direction is taken to be along the x_2 direction so that

$$\begin{aligned}
C_{1111} &= C_{3333} = A, C_{1133} = C_{3311} = N, C_{2222} = C, \\
C_{1122} &= C_{2211} = C_{2233} = C_{3322} = F, \\
C_{1212} &= C_{2112} = C_{2121} = C_{1221} = C_{2323} = C_{3223} = C_{3232} = C_{2332} = L, \\
C_{1313} &= C_{3113} = C_{3131} = C_{1331} = \frac{1}{2}(A - N), \\
C_{2141} &= C_{1241} = C_{3243} = C_{2343} = C_{4121} = C_{4112} = C_{4332} = C_{4323} = e_1, \\
C_{1142} &= C_{3342} = C_{4211} = C_{4233} = e_2, \\
C_{2242} &= C_{4222} = e_3, C_{4141} = C_{4343} = -\epsilon_1, C_{4242} = -\epsilon_2,
\end{aligned} \tag{48}$$

where A , N , F , C , L , e_1 , e_2 , e_3 , ϵ_1 and ϵ_2 are independent constants.

The piezoelectric material is PZT-BaTiO₃ with material constants A , N , F , C , L , e_1 , e_2 , e_3 , ϵ_1 and ϵ_2 and density ρ given by

$$\begin{aligned}
A &= 15.0 \times 10^{10}, N = 7.78 \times 10^{10}, F = 6.6 \times 10^{10}, \\
C &= 14.6 \times 10^{10}, L = 4.4 \times 10^{10}, e_1 = 11.4, \\
e_2 &= -4.35, e_3 = 17.5, \epsilon_1 = 98.7 \times 10^{-10}, \\
\epsilon_2 &= 112 \times 10^{-10}, \rho = 5800.
\end{aligned} \tag{49}$$

The values of A , N , F , C and L above are in N/m², e_1 , e_2 and e_3 are in C/m², ϵ_1 and ϵ_2 are in C/(Vm) and ρ in kg/m³.

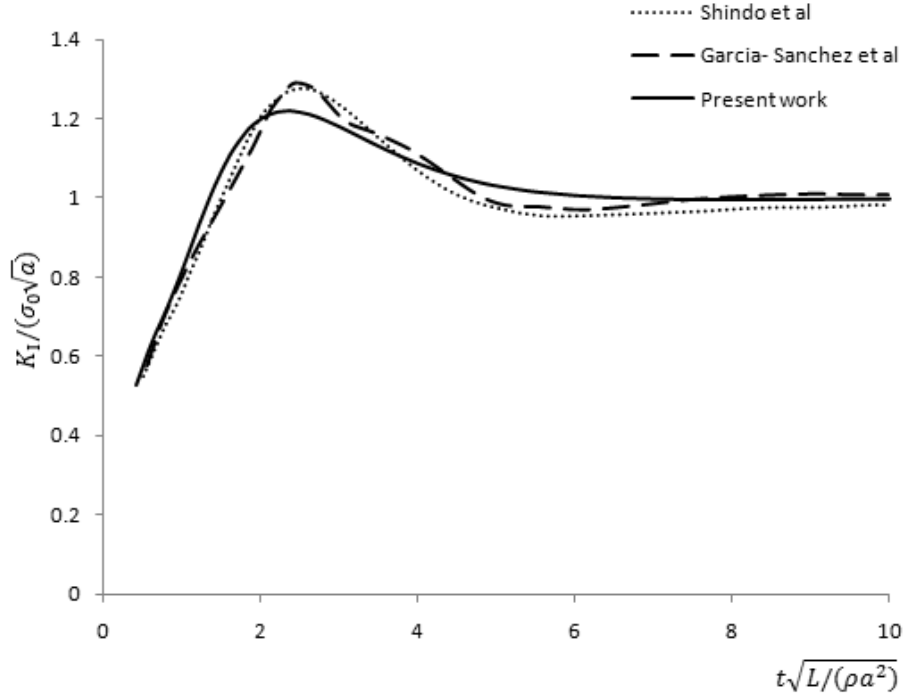


Figure 2. Plots of $K_I/(\sigma_0\sqrt{a})$ against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$.

The numerical stress intensity factor $K_I/(\sigma_0\sqrt{a})$ and electric displacement intensity factor $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at the crack tip $(a, 0)$ are plotted against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ in Figures 2 and 3 respectively. The results are obtained by using $J = 10$ in the numerical solution of the hypersingular integral equations and $M = 4$ (8 terms) in the Stehfest's formula for inverting Laplace transform. In Figures 2 and 3, the numerical values of $K_I/(\sigma_0\sqrt{a})$ and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ are also compared with those extracted from Shindo [26] and García-Sánchez, Zhang, Sládek and Sládek [13]. (Numerical values of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ are not given in Shindo [26].) All the plots of $K_I/(\sigma_0\sqrt{a})$ in Figure 2 are quite close to one another, exhibiting the

same general trends and reaching peak values at about the same time. So are the plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ in Figure 3. As pointed out earlier on, 8 terms ($M = 4$) are used in the Stehfest's formula for obtaining the plots in Figure 2. Convergence is observed in the numerical values of the intensity factors when the number of terms in the formula is increased to 10. The numerical inversion of Laplace transform becomes unstable when the number of terms is increased beyond 10. To use more terms, it is necessary to refine the calculation to obtain more a more accurate solution of the hypersingular integral equations. A higher arithmetic precision in the computing machine is needed too.

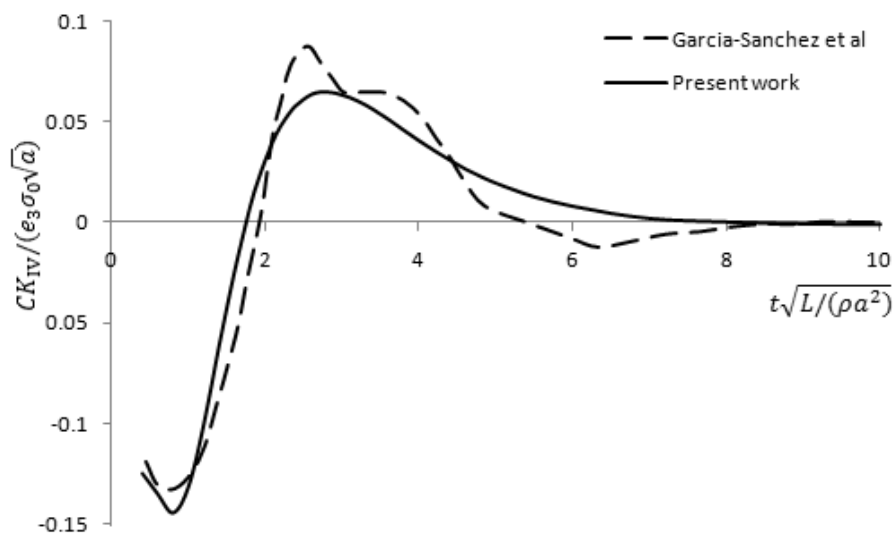


Figure 3. Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the non-dimensionalized time $t\sqrt{L}/(\rho a^2)$.

Problem 2. Consider a pair of coplanar cracks, each of length $2a$, as shown in Figure 4. The distance between the inner tips of the cracks is $2d$. The uniform tractions acting on the crack faces are given by $S_{22} = -H(t)\sigma_0$. Here

electrically impermeable and permeable cracks will be examined separately. For the case in which the coplanar cracks are electrically impermeable, the condition $S_{42} = -H(t)D_0$, where D_0 is a constant, applies.

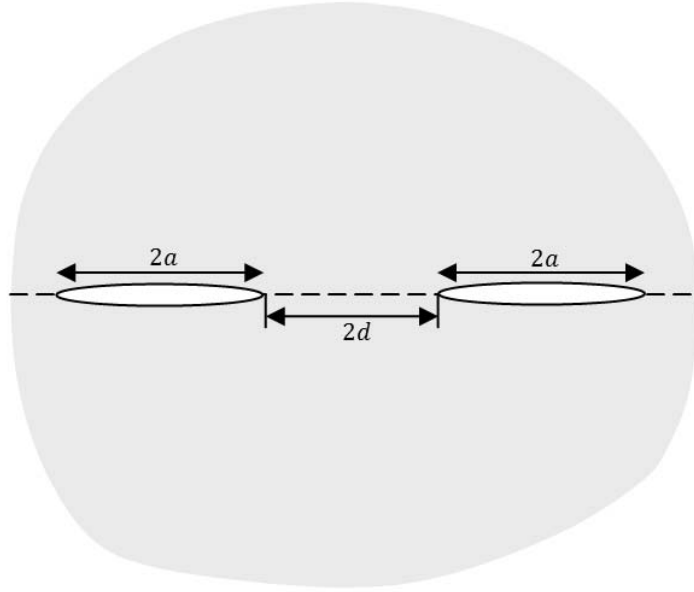


Figure 4. A pair of coplanar cracks.

The electrical poling is along the x_2 direction. For the purpose of obtaining some numerical results for the dynamic stress and electric displacement intensity factors at the inner and outer tips of the coplanar cracks, the material constants of PZT- BaTiO₃ (as in Problem 1) are used and the load ratio σ_0/D_0 for electrically impermeable cracks is taken to be 10^{10} NC^{-1} .

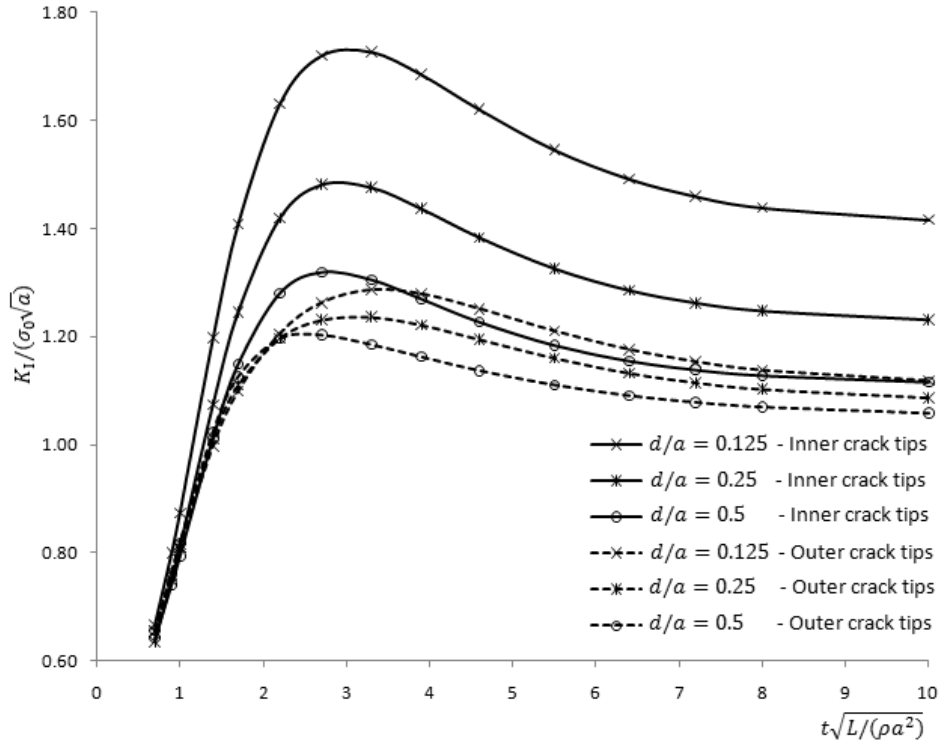


Figure 5. Plots of $K_I/(\sigma_0\sqrt{a})$ against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ at inner and outer crack tips of electrically impermeable cracks for selected values of d/a .

In Figure 5, the non-dimensionalized stress intensity factor $K_I/(\sigma_0\sqrt{a})$ at the inner and outer crack tips are plotted against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ for $d/a = 0.125, 0.50$ and 0.25 . The plots of $K_I/(\sigma_0\sqrt{a})$ for electrically permeable and impermeable cracks are almost indistinguishable. Thus, plots of $K_I/(\sigma_0\sqrt{a})$ are given in Figure 5 for only electrically impermeable cracks. In each of the plots, $K_I/(\sigma_0\sqrt{a})$ increases rapidly to a peak value before settling down to the corresponding value of the static stress intensity factor. For each of the values of d/a in Figure 5, the peak values of $K_I/(\sigma_0\sqrt{a})$ at the inner and the outer crack tips are significantly different

and the peak value at the inner tips is larger than that at the outer tips. As may be expected, the peak value of $K_I/(\sigma_0\sqrt{a})$ at each crack tip is larger if the cracks are closer to each other. Further calculations show that the plot of $K_I/(\sigma_0\sqrt{a})$ at the inner tips is almost identical as that at the outer tips for $d/a > 3$.

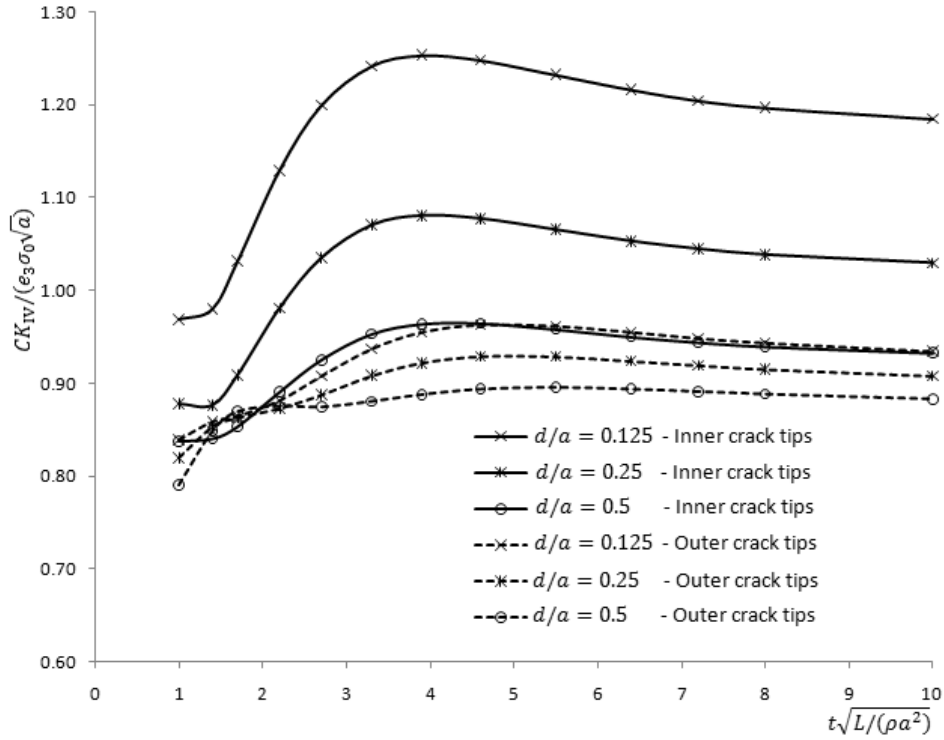


Figure 6. Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the non-dimensionalized time $t\sqrt{L}/(\rho a^2)$ at inner and outer crack tips of electrically impermeable cracks for selected values of d/a .

Plots of the non-dimensionalized electric displacement intensity factors $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at the inner and the outer crack tips against $t\sqrt{L}/(\rho a^2)$ for

$d/a = 0.125, 0.50$ and 0.25 are given in Figures 6 and 7 for electrically impermeable and permeable cracks respectively. The plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ for electrically impermeable cracks are distinct from those for electrically permeable cracks. For a fixed d/a , the peak value of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at each crack tip is apparently higher for the electrically permeable cracks than that for the impermeable cracks.

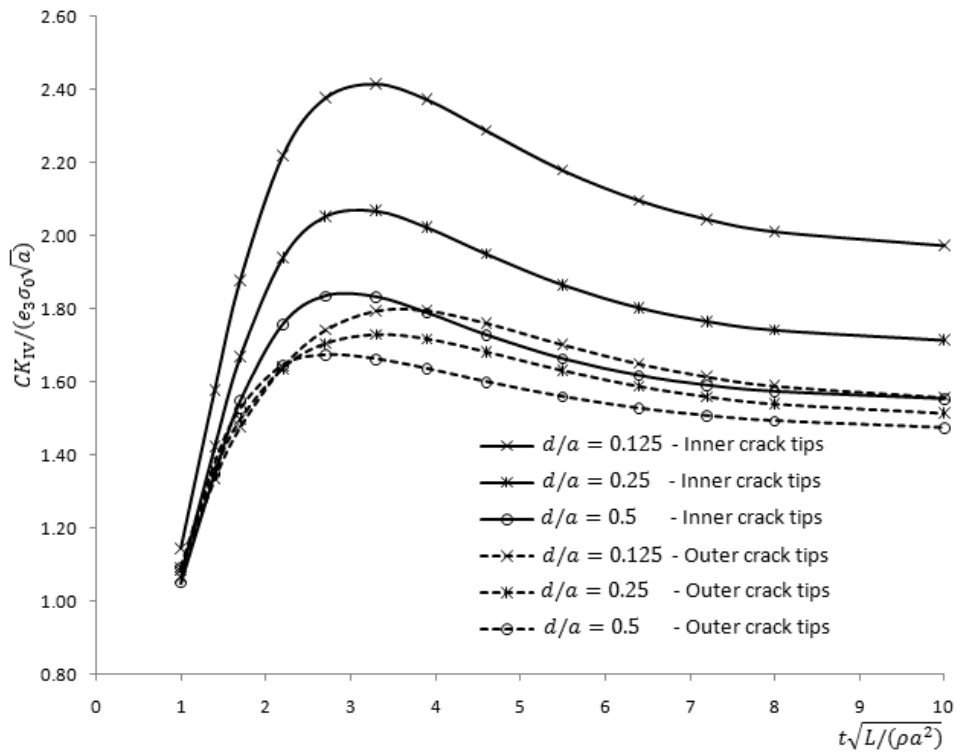


Figure 7. Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ at inner and outer crack tips of electrically permeable cracks for selected values of d/a .

Problem 3. Consider two equal length parallel electrically impermeable cracks as sketched in Figure 8. The half length of each crack is given by

a. The centers of the cracks lie on a vertical line and are separated by a distance denoted by d . The non-zero constant loads acting on the crack faces are given by $S_{22} = -H(t)\sigma_0$ and $S_{42} = -H(t)D_0$, with $\sigma_0/D_0 = 10^{10} \text{ NC}^{-1}$.

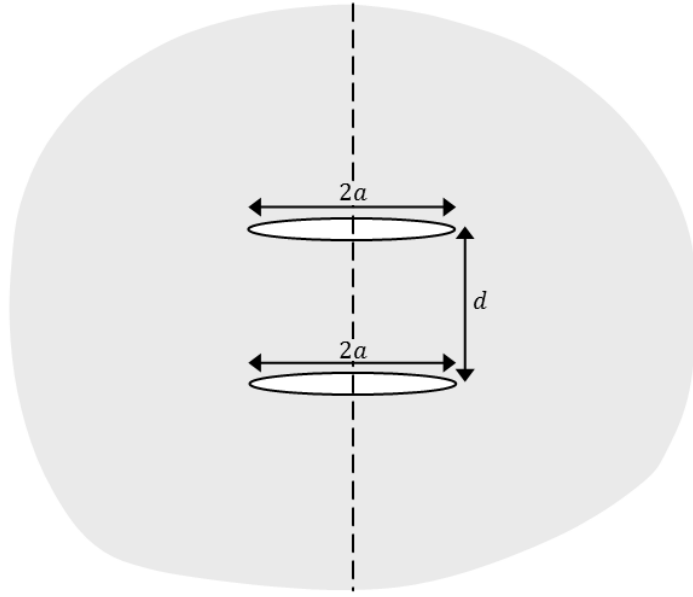


Figure 8. Two parallel cracks.

The electrical poling is along the x_2 direction. Using the material constants of PZT- BaTiO₃ (as in Problem 1), for selected values of d/a , we plot the non-dimensionalized crack tip stress intensity factors $K_I/(\sigma_0\sqrt{a})$ and $CK_{II}/(F\sigma_0\sqrt{a})$ and the non-dimensionalized crack tip electric displacement intensity factor $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ in Figures 9, 10 and 11 respectively.

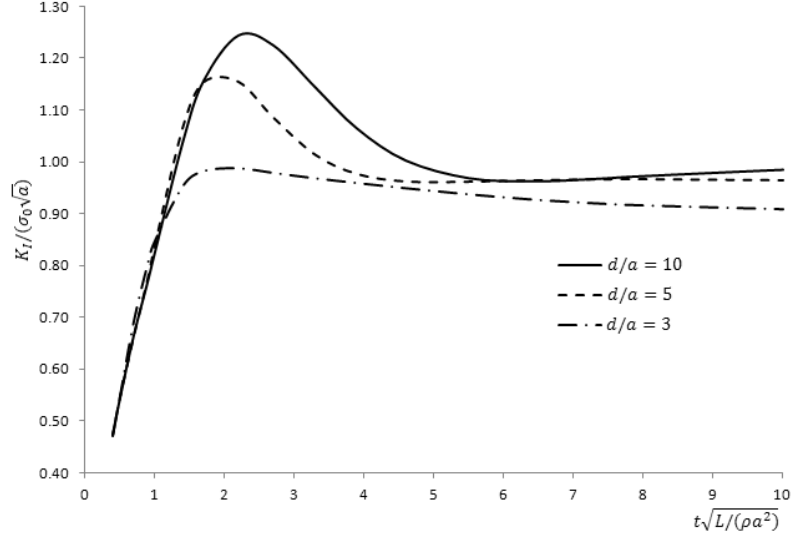


Figure 9. Plots of $K_I/(\sigma_0\sqrt{a})$ against the non-dimensionalized time for selected values of d/a .

In Figure 9, for a given d/a , the non-dimensionalized stress intensity factor $K_I/(\sigma_0\sqrt{a})$ rises to a peak (maximum) value and then drops to a trough (minimum) value before gradually settling down to approach its static value. Both the trough and the peak values decrease in magnitude as d/a decreases. A similar observation may be made of the non-dimensionalized electric displacement intensity factor $CK_{IV}/(e_3\sigma_0\sqrt{a})$ in Figure 11.

As d/a tends to infinity, the non-dimensionalized stress intensity factor $CK_{II}/(F\sigma_0\sqrt{a})$ vanishes. Nevertheless, when the cracks come close to each other, there is an increase in the magnitude $CK_{II}/(F\sigma_0\sqrt{a})$ due to larger differences in the stress distribution on opposite crack faces. This is shown in Figure 10. For $d/a = 10$, the magnitude of $CK_{II}/(F\sigma_0\sqrt{a})$ is very small at all time. Note that the fact that K_{II} is not zero for the parallel cracks in Figure 10 may be explained by the well known phenomenon called Poisson effect. Due to Poisson effect, compressive stresses are generated on opposite crack faces. For the parallel cracks, they are unequal, thereby giving rise to

shear stresses on each of the cracks.

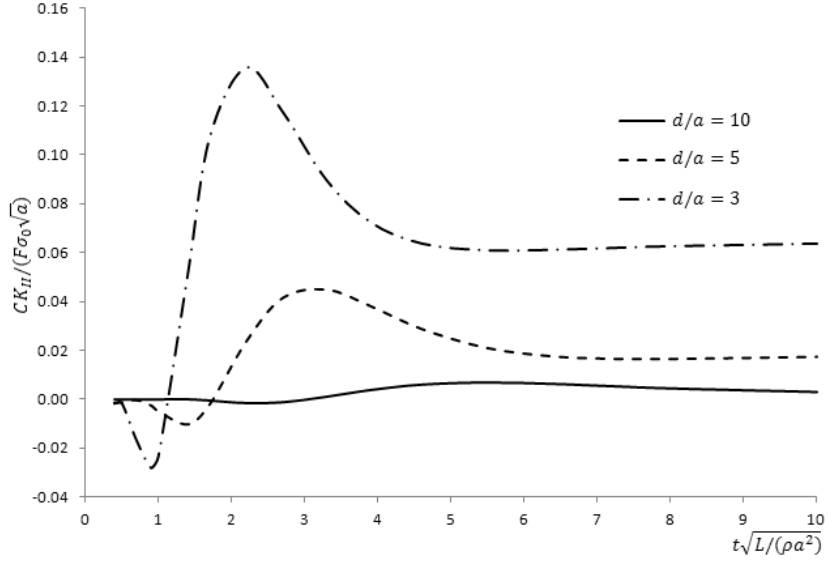


Figure 10. Plots of $CK_{II}/(F\sigma_0\sqrt{a})$ against the non-dimensionalized time for selected values of d/a .

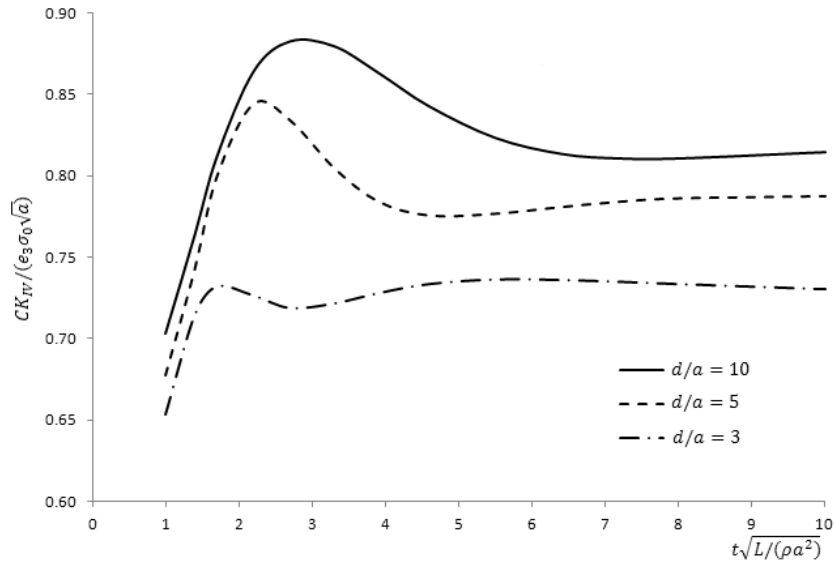


Figure 11. Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against the non-dimensionalized time for selected values of d/a .

Problem 4. Consider now the two pairs of electrically impermeable cracks, which are of equal length $2a$, in the piezoelectric space, as sketched in Figure 12. The faces of the horizontal cracks are subject to internal uniform loads given by $S_{12} = S_{22} = 0$, $S_{32} = -H(t)\tau_0$ and $S_{42} = -H(t)D_0$, such that $\tau_0/D_0 = 10^{10} \text{ NC}^{-1}$. The internal loads on the faces of the vertical cracks are given by $S_{K1} = 0$ (for $K = 1, 2, 3, 4$).

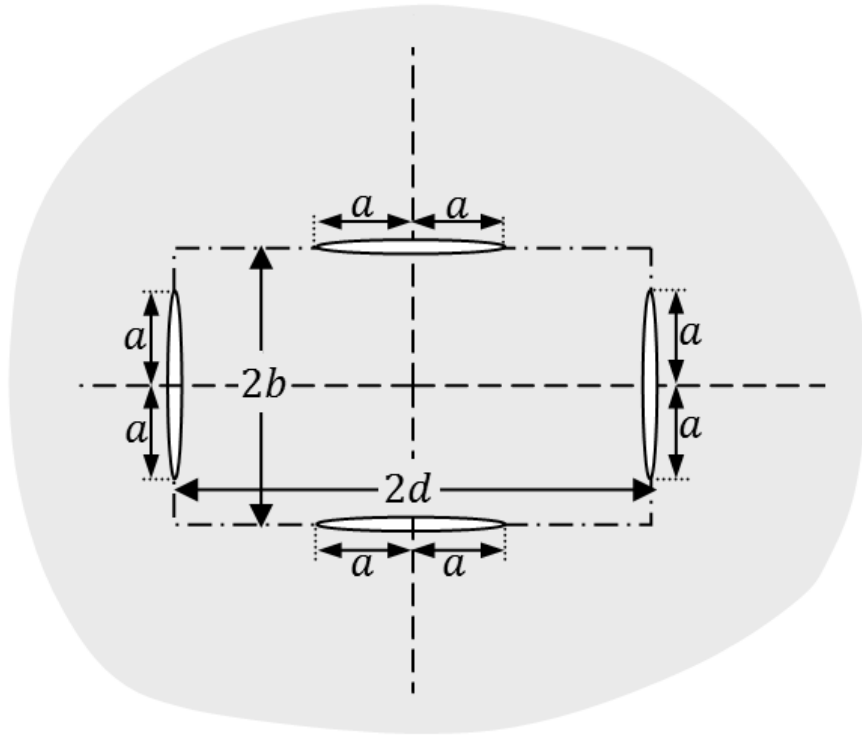


Figure 12. Two pairs of cracks.

The electrical poling is in the x_3 direction, so that C_{IjKl} are given in terms of the independent constants $A, N, F, C, L, e_1, e_2, e_3, \epsilon_1$ and ϵ_2 by

$$\begin{aligned}
C_{1111} &= C_{2222} = A, \quad C_{1122} = C_{2211} = N, \quad C_{3333} = C, \\
C_{1133} &= C_{3311} = C_{2233} = C_{3322} = F, \\
C_{1313} &= C_{3113} = C_{3131} = C_{1331} = C_{2323} = C_{3223} = C_{3232} = C_{2332} = L, \\
C_{1212} &= C_{2112} = C_{2121} = C_{1221} = \frac{1}{2}(A - N), \\
C_{3141} &= C_{1341} = C_{2342} = C_{3242} = C_{4131} = C_{4113} = C_{4223} = C_{4232} = e_1, \\
C_{1143} &= C_{2243} = C_{4311} = C_{4322} = e_2, \\
C_{3343} &= C_{4333} = e_3, \quad C_{4141} = C_{4242} = -\epsilon_1, \quad C_{4343} = -\epsilon_2.
\end{aligned} \tag{50}$$

As in the problems above, the material occupying the piezoelectric space is taken to be PZT-BaTiO₃ with material constants as given in (49).

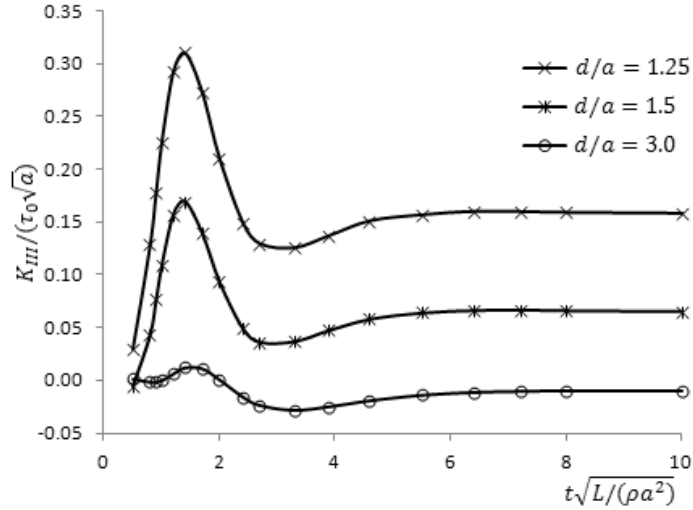


Figure 13. Plots of $K_{III}/(\tau_0\sqrt{a})$ at the upper tip of the left vertical crack against the non-dimensionalized time for $b/a = 1.25$ and selected values of d/a .

The deformation of the cracks is antiplane. The non-dimensionalized crack tip stress and electrical displacement intensity factors $K_{III}/(\tau_0\sqrt{a})$ and $K_{IV}/(D_0\sqrt{a})$ are of interest here. For $b/a = 1.25$ and a few selected values of d/a , these intensity factors at the lower tip of the left vertical crack are plotted against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ in Figures 13 and 14. For the same values of b/a and d/a , plots of the intensity factors at the left tip of the upper horizontal crack are given in Figures 15 and 16. As shown in Figures 14 and 16, $K_{IV}/(D_0\sqrt{a})$ at both crack tips does not vary with time. In Figures 13 and 15, for a fixed b/a and d/a , $K_{III}/(\tau_0\sqrt{a})$ rises quickly to a peak value, drops to a trough value and gradually approaches the corresponding static value. As expected, as d/a increases, both $K_{III}/(\tau_0\sqrt{a})$ and $K_{IV}/(D_0\sqrt{a})$ for the vertical cracks decrease in magnitude, becoming closer to zero.

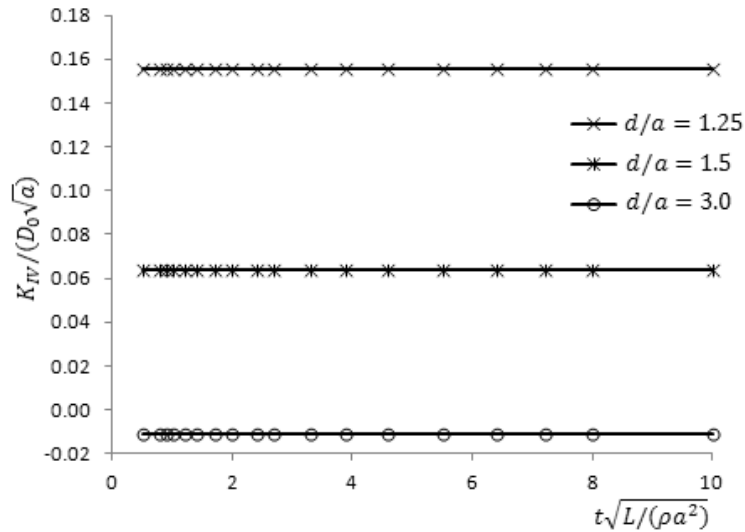


Figure 14. Plots of $K_{IV}/(D_0\sqrt{a})$ at the upper tip of the left vertical crack against the non-dimensionalized time for $b/a = 1.25$ and selected values of d/a .

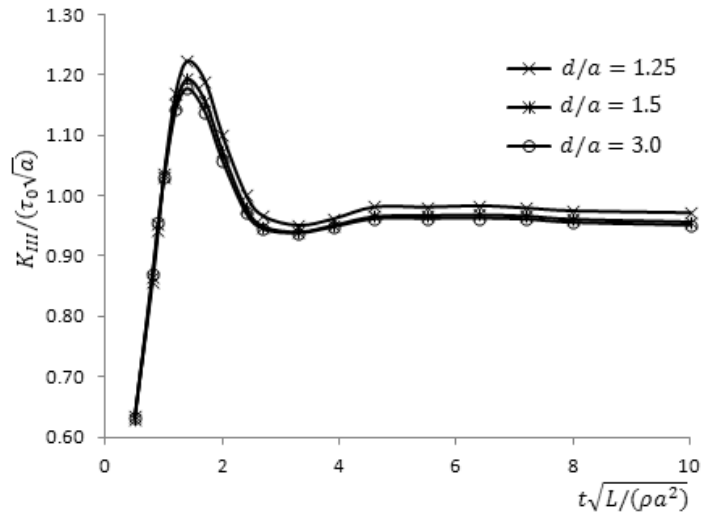


Figure 15. Plots of $K_{III}/(\tau_0\sqrt{a})$ at the left tip of the upper horizontal crack against the non-dimensionalized time for $b/a = 1.25$ and selected values of d/a .

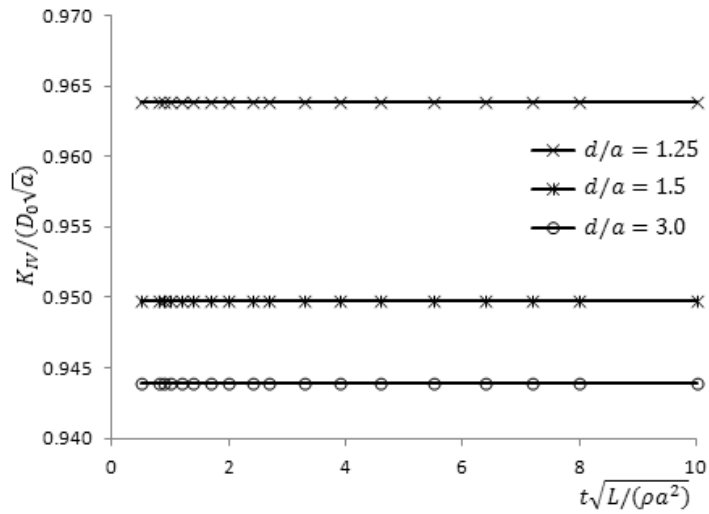


Figure 16. Plots of $K_{IV}/(D_0\sqrt{a})$ at the left tip of the upper horizontal crack against the non-dimensionalized time for $b/a = 1.25$ and selected values of d/a .

8 Summary

Through the use of Laplace and exponential Fourier transforms, a semi-analytic solution is derived for an electroelastodynamic problem involving an arbitrary number of arbitrarily located planar cracks in a piezoelectric space. The problem is eventually reduced to solving a system of hypersingular integral equations. The unknown functions in the hypersingular integral equations are the Laplace transforms of the jumps in the displacements and electric potential across opposite crack faces. Once they are determined, the crack tip stress and electric displacement intensity factors can be easily computed in the Laplace transform domain. A numerical technique for inverting Laplace transforms is employed to recover the intensity factors in the physical domain.

The solution is applied to study some specific cases of the problem. For the case of a single crack under impact loadings, the computed crack tip stress and electric displacement intensity factors are found to be in reasonably good agreement with those published in the literature. Numerical results are also obtained for other cases which include one which involves four interacting planar cracks under antiplane deformations.

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