Dynamic response of planar cracks in an infinitely long piezoelectric strip

W. T. Ang* and L. Athanasius
Division of Engineering Mechanics
School of Mechanical and Aerospace Engineering
Nanyang Technological University
Singapore 639798

Abstract

The problem of determining the electro-elastic fields around an array of parallel planar cracks in an infinitely long piezoelectric strip is considered. The cracks, acted upon by dynamic loads, are either electrically impermeable or permeable. A semi-analytic method based on the theory of exponential Fourier transformation is presented for solving the problem in the Laplace transform domain. The Laplace transforms of the jumps in the displacements and electric potential across opposite crack faces are determined by solving a system of hypersingular integral equations. Once these displacement and electric potential jumps are obtained, the displacements and electric potential and other physical quantities of interest, such as the crack tip stress and electric displacement intensity factors, can be recovered with the help of a suitable algorithm for inverting Laplace transforms. The stress and electric displacement intensity factors are computed for some specific cases of the problem.

This is a preprint of an article accepted for publication in *Applied Mathematics and Computation*. When the article is in press, it can be accessed at: http://dx.doi.org/10.1016/j.amc.2013.01.059.

* Author for correspondence (W. T. Ang, mwtang@ntu.edu.sg).

1 Introduction

During the last few decades, the analysis of cracks in piezoelectric materials has been investigated by many researchers. The vast majority of works on piezoelectric crack problems are, however, concerned with electro-elastostatic deformations (see, for example, Athanasius, Ang and Sridhar [3], Li [16], Shindo, Watanabe and Narita [24] and Wang and Mai [27]). In a recent paper, Kuna [14] pointed out that there are comparatively fewer works on piezoelectric cracks that are acted upon by time dependent loads.

Most papers presenting semi-analytic solutions for dynamic piezoelectric crack problems assume that the cracks undergo out of plane or antiplane deformations. For example, Chen [6] studied the dynamic response of a single electrically impermeable planar crack in an infinite transversely isotropic piezoelectric material under pure electric load and undergoing an antiplane deformation; Chen and Karihaloo [7], Chen and Meguid [8], Li and Fan [17] and Li and Tang [20] solved problems involving a single planar crack in an infinitely long piezoelectric strip under antiplane deformations; Chen and Worswick [9] and Meguid and Chen [22] examined the behaviours of coplanar cracks undergoing antiplane deformations in piezoelectric materials; and Kwon and Lee [15] and Li and Lee [18]-[19] investigated the antiplane deformation of edge cracks in piezoelectric materials.

Apparently fewer papers giving semi-analytic solutions for cracks undergoing dynamic inplane deformations in piezoelectric materials may be found in the literature. Shindo [23] formulated the problem of a single planar crack in a piezoelectric ceramic under normal in terms of a pair of integral equations by representing the displacement and electric potential in the Laplace transform domain by suitable Fourier sine and cosine transform representations. The integral equations were reduced to Fredholm integral equations of the second kind solved as explained in Sneddon and Lowengrub [25]. Using the method of dislocations, Wang and Yu [28] reduced the two-dimensional analysis of a mode I planar crack in an infinitely long piezoelectric strip to solving Cauchy singular integral equations. The approach in [23] and [28] was extended by Liu and Zhong [21] to analyze the transient response of a pair of collinear cracks in a piezoelectric space of infinite extent.

Through the use of boundary integral equations for piezoelectricity, the dynamic piezoelectric crack problems may also be formulated in terms of hypersingular integral equations using the approach in a recent paper by García-Sánchez, Zhang, Sládek and Sládek [10]. In [10], the kernels of the hypersingular integral formulation contain second order spatial derivatives of a suitable dynamic Green's function for piezoelectric solids. Such an approach has been successfully used for solving elastostatic crack problems (see Chen and Hong [5] and Hong and Chen [12]). Nevertheless, the evaluation of the dynamic Green's function (unlike the static one) is a rather involved exercise, requiring the computation of a line integral over a unit circle with integrand that is expressed in terms of exponential integrals (see, for example, Wang and Zhang [29]).

Recently, Ang and Athanasius [2] derived a semi-analytic solution for an electro-elastic problem involving an arbitrary number of arbitrarily oriented planar cracks in an infinite piezoelectric space. The displacement and electric potential in the Laplace transform domain are expressed as a linear combination of suitably constructed exponential Fourier transform representations. The integrands of the Fourier integrals contain unknown functions that are directly related to the jumps in the Laplace transforms of the displacement and electrical potential across opposite crack faces. The task of determining the unknown functions is eventually reduced to solving numerically a system of hypersingular integral equations.

In the present paper, we extend the analysis in [2] to the case of an array of planar cracks in an infinitely long piezoelectric strip. The cracks are parallel to the edges of the strip and are arbitrarily located relative to one another. As in [2], once the Laplace transforms of the jumps in displacement and electrical potential across opposite crack faces are determined, the displacements and electric potential and other physical quantities of interest, such as the crack tip stress and electric displacement intensity factors, can be extracted with the aid of a suitable algorithm for inverting Laplace transforms. The crack tip stress and electric displacement intensity factors are calculated for some specific cases of the problem. To check the validity of the analysis presented here, values of the stress and electric displacement intensity factors are computed for the special case of a single crack in a strip and compared with those published in the literature.

2 The problem

Referring to an $Ox_1x_2x_3$ Cartesian coordinate system, consider an infinitely long piezoelectric strip $-\infty < x_1 < \infty$, $0 < x_2 < h$, $-\infty < x_3 < \infty$, where h is a given positive constant. The interior of the strip contains N arbitrarily oriented non-intersecting planar cracks that are parallel to its edges $x_2 = 0$ and $x_2 = h$. The geometries of the cracks do not change along the x_3 axis. The cracks are denoted by $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\cdots$, $\Gamma^{(N-1)}$ and $\Gamma^{(N)}$. The n-th planar crack $\Gamma^{(n)}$ lies in the region $-\ell^{(n)} + c_1^{(n)} < x_1 < \ell^{(n)} + c_1^{(n)}$, $x_2 = c_2^{(n)}$, $-\infty < x_3 < \infty$. On the Ox_1x_2 plane, the crack $\Gamma^{(n)}$ is a straight line cut, $2\ell^{(n)}$ is the length of the crack and $(c_1^{(n)}, c_2^{(n)})$ is the midpoint of the crack.

It will be assumed that here the electroelastic deformation of the cracked piezoelectric space does not vary along the x_3 direction. The problem is to determine the displacements $u_k(x_1, x_2, t)$ and electric potential $\phi(x_1, x_2, t)$ in the piezoelectric strip for time $t > 0$ such that suitably prescribed boundary conditions on the cracks and the edges of the strip are satisfied.

More specifically, the conditions the cracks are given by

$$\begin{aligned} \sigma_{k2}(x_1, x_2, t) &\rightarrow -P_k^{(n)}(\xi_1, \xi_2, t) \ (k = 1, 2, 3) \\ &\quad \text{as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)}(n = 1, 2, \cdots, N), \end{aligned} \tag{1}$$

and either

$$\begin{aligned} d_2(x_1, x_2, t) &\rightarrow -P_4^{(n)}(\xi_1, \xi_2, t) \\ &\quad \text{as } (x_1, x_2) \rightarrow (\xi_1, \xi_2) \in \Gamma^{(n)}(n = 1, 2, \cdots, N) \\ &\quad \text{if the cracks are electrically impermeable,} \end{aligned} \tag{2}$$

or

$$\begin{aligned} &\Delta\phi^{(n)}(x_1, t) = 0 \text{ for } -\ell^{(n)} + c_1^{(n)} < x_1 < \ell^{(n)} + c_1^{(n)} \ (n = 1, 2, \cdots, N) \\ &\qquad \text{if the cracks are electrically permeable,} \end{aligned} \tag{3}$$

where σ_{ij} and d_i are respectively the stresses and electric displacements, $P_1^{(n)}(\xi_1, \xi_2, t)$, $P_2^{(n)}(\xi_1, \xi_2, t)$, $P_3^{(n)}(\xi_1, \xi_2, t)$ and $P_4^{(n)}(\xi_1, \xi_2, t)$ are suitably prescribed functions for $(\xi_1, \xi_2) \in \Gamma^{(n)}$ giving the internal dynamic loads on the cracks and $\Delta\phi^{(n)}(x_1, t)$ denotes the jump in the electrical potential ϕ across the crack $\Gamma^{(n)}$ as defined by

$$\begin{aligned} \Delta\phi^{(n)}(x_1, t) &= \lim_{\varepsilon \rightarrow 0}[\phi(x_1, c_2^{(n)} + |\varepsilon|, t) - \phi(x_1, c_2^{(n)} - |\varepsilon|, t)] \\ &\quad \text{for } -\ell^{(n)} + c_1^{(n)} < x_1 < \ell^{(n)} + c_1^{(n)}. \end{aligned} \tag{4}$$

The conditions on the edges of the strip are given by

$$\left.\begin{aligned} \sigma_{i2}(x_1, 0, t) = 0 \\ d_2(x_1, 0, t) = 0 \\ \sigma_{i2}(x_1, h, t) = 0 \\ d_2(x_1, h, t) = 0 \end{aligned}\right\} \text{ for } -\infty < x_1 < \infty. \tag{5}$$

Furthermore, it is assumed here that the displacements u_k and and its partial derivative with respect time (that is, $\partial u_k/\partial t$) are both zero at time $t=0$ and the stresses $\sigma_{k1}(x_1,x_2,t)$ and electric displacement $d_1(x_1,x_2,t)$ generated by the cracks vanish as $|x_1|\to\infty$.

3 Basic equations of electroelasticity

The governing equations for the displacements u_k and electric potential ϕ in a homogeneous piezoelectric material are given by

$$\begin{aligned} c_{ijk\ell}\frac{\partial^2 u_k}{\partial x_j \partial x_\ell}+e_{\ell ij}\frac{\partial^2 \phi}{\partial x_j \partial x_\ell} &= \rho\frac{\partial^2 u_i}{\partial t^2}, \\ e_{jk\ell}\frac{\partial^2 u_k}{\partial x_j \partial x_\ell}-\kappa_{j\ell}\frac{\partial^2 \phi}{\partial x_j \partial x_\ell} &= 0, \end{aligned} \tag{6}$$

where $c_{ijk\ell}$, $e_{\ell ij}$ and $\kappa_{i\ell}$ are the constant elastic moduli, piezoelectric coefficients and dielectric coefficients respectively and ρ is the density. The Einstenian convention of summing over a repeated index applies here for lowercase Latin subscripts.

The constitutive equations relating (σ_{ij},d_j) and (u_k,ϕ) are

$$\begin{aligned} \sigma_{ij} &= c_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}+e_{\ell ij}\frac{\partial \phi}{\partial x_\ell}, \\ d_j &= e_{jk\ell}\frac{\partial u_k}{\partial x_\ell}-\kappa_{jp}\frac{\partial \phi}{\partial x_\ell}. \end{aligned} \tag{7}$$

Following closely the approach of Barnett and Lothe [4], we define

$$\begin{aligned} U_J &= \begin{cases} u_j & \text{for } J=j=1,2,3, \\ \phi & \text{for } J=4, \end{cases} \\ S_{Ij} &= \begin{cases} \sigma_{ij} & \text{for } I=i=1,2,3, \\ d_j & \text{for } I=4, \end{cases} \\ C_{IjK\ell} &= \begin{cases} c_{ijk\ell} & \text{for } I=i=1,2,3 \text{ and } K=k=1,2,3, \\ e_{\ell ij} & \text{for } I=i=1,2,3 \text{ and } K=4, \\ e_{jk\ell} & \text{for } I=4 \text{ and } K=k=1,2,3, \\ -\kappa_{j\ell} & \text{for } I=4 \text{ and } K=4, \end{cases} \end{aligned} \tag{8}$$

so that (6) and (7) may be respectively written more compactly as

$$C_{IjK\ell}\frac{\partial^2 U_K}{\partial x_j \partial x_\ell} = B_{IK}\frac{\partial^2 U_K}{\partial t^2}\ (I = 1, 2, 3, 4) \tag{9}$$

and

$$S_{Ij} = C_{IjK\ell}\frac{\partial U_K}{\partial x_\ell}\ (I = 1, 2, 3, 4;\ j = 1, 2, 3) \tag{10}$$

where

$$B_{IK} = \begin{cases} \rho & \text{if } I = K \text{ and } I \neq 4, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Note that uppercase Latin subscripts have values 1, 2, 3 and 4. Summation is also implied for repeated uppercase Latin subscripts.

It follows that the problem stated in Section 2 requires solving (9) within the piezoelectric strip for time $t > 0$ subject to initial-boundary conditions stated as follows.

The initial conditions are

$$U_K = 0 \text{ and } \frac{\partial U_K}{\partial t} = 0 \text{ at } t = 0\ (K = 1, 2, 3). \tag{12}$$

The conditions on the cracks are given by

$$\begin{aligned} S_{I2}(x_1, x_2, t) &\to -P_I^{(n)}(\xi_1, \xi_2, t)\ (I = 1, 2, 3) \\ \text{as } (x_1, x_2) &\to (\xi_1, \xi_2) \in \Gamma^{(n)}\ (n = 1, 2, \cdots, N), \end{aligned} \tag{13}$$

and either

$$\begin{aligned} S_{42}(x_1, x_2, t) &\to -P_4^{(n)}(\xi_1, \xi_2, t) \\ \text{as } (x_1, x_2) &\to (\xi_1, \xi_2) \in \Gamma^{(n)}\ (n = 1, 2, \cdots, N) \\ &\qquad \text{if the cracks are electrically impermeable,} \end{aligned} \tag{14}$$

or

$$\begin{aligned} &\Delta U_4^{(n)}(x_1, t) \to 0 \text{ for } -\ell^{(n)} + c_1^{(n)} < x_1 < \ell^{(n)} + c_1^{(n)}\ (n = 1, 2, \cdots, N) \\ &\qquad \text{if the cracks are electrically permeable,} \end{aligned} \tag{15}$$

where

$$\Delta U_I^{(n)}(x_1,t) = \lim_{\varepsilon\to 0}[U_I(x_1,c_2^{(n)}+|\varepsilon|) - U_I(x_1,c_2^{(n)}-|\varepsilon|)] \quad \text{for } -\ell^{(n)}+c_1^{(n)} < x_1 < \ell^{(n)}+c_1^{(n)}. \tag{16}$$

From (5), the conditions on the edges of the strip are given by

$$\left.\begin{array}{l} S_{I2}(x_1,0,t)=0 \\ S_{I2}(x_1,h,t)=0 \end{array}\right\} \text{ for } -\infty < x_1 < \infty \ (I=1,2,3,4). \tag{17}$$

In addition, it is required that $S_{I1}(x_1,x_2,t) \to 0$ as $|x_1| \to \infty$.

4 Formulation in Laplace transform domain

We denote the Laplace transformation of $F(x_1,x_2,t)$ over time $t \geq 0$ by $\widehat{F}(x_1,x_2,s)$, that is, we define

$$\widehat{F}(x_1,x_2,s) = \int_0^\infty F(x_1,x_2,t)\exp(-st)dt, \tag{18}$$

where s is the Laplace transformation parameter.

Application of the Laplace transformation on both sides of (9) together with the initial conditions (12) yields

$$C_{IjK\ell}\frac{\partial^2 \widehat{U}_K}{\partial x_j \partial x_\ell} - s^2 B_{IK}\widehat{U}_K = 0 \ (I=1,2,3,4). \tag{19}$$

In the Laplace transform domain, the problem is to solve (19) subject to boundary conditions stated as follows.

On the cracks, the boundary conditions are given by

$$\begin{aligned} \widehat{S}_{I2}(x_1,x_2,s) &\to -\widehat{P}_I^{(n)}(\xi_1,\xi_2,s) \ (I=1,2,3) \\ \text{as } (x_1,x_2) &\to (\xi_1,\xi_2) \in \Gamma^{(n)} \ (n=1,2,\cdots,N), \end{aligned} \tag{20}$$

and by either

$$
\begin{aligned}
\widehat{S}_{42}(x_1,x_2,s) &\to -\widehat{P}_4^{(n)}(\xi_1,\xi_2,s)\\
\text{as } (x_1,x_2) &\to (\xi_1,\xi_2)\in\Gamma^{(n)}\ (n=1,2,\cdots,N)\\
&\quad\text{if the cracks are electrically impermeable,}
\end{aligned}
\tag{21}
$$

or

$$
\Delta\widehat{U}_4^{(n)}(x_1,s)\to 0 \text{ for } -\ell^{(n)}+c_1^{(n)}<x_1<\ell^{(n)}+c_1^{(n)}\ (n=1,2,\cdots,N)
$$

$$
\text{if the cracks are electrically permeable.} \tag{22}
$$

On the edges of the strip, the boundary conditions are given by either

$$
\left.\begin{array}{l}
\widehat{S}_{I2}(x_1,0,s)=0\\
\widehat{S}_{I2}(x_1,h,s)=0
\end{array}\right\}\ \text{for } -\infty<x_1<\infty\ (I=1,2,3,4). \tag{23}
$$

It is also required that $\widehat{S}_{I1}(x_1,x_2,s)\to 0$ as $|x_1|\to\infty$.

5 Method of solution

In this section, a semi-analytical method is proposed for solving (19) subject to (20)-(23). As in Ang and Athanasius [2], the generalized displacements and stresses in the Laplace transform domain are expressed explicitly in terms of Fourier integral representations. As we shall see, the integrands of the Fourier integrals contain unknown functions which are directly related to the Laplace transform of the generalized displacement jumps across opposite crack faces. The method of solution here is regarded as semi-analytical as the unknown functions are to be determined by solving numerically a system of hypersingular integral equations. Moreover, a numerical technique for inverting Laplace transformation will be employed to recover the required physical quantities in the real time domain.

5.1 Explicit solution in Fourier integral form

For the solution of the piezoelectric crack problem in the Laplace transform space, let

$$
\begin{aligned}
\widehat{U}_K(x_1,x_2,s) \quad = \quad & \sum_{\alpha=1}^{4} \mathrm{Re}\{\int_0^\infty A_{K\alpha}(\xi,s)[E_{1\alpha}(\xi,s)\exp(i\xi(x_1+\tau_\alpha x_2)\\
& +E_{2\alpha}(\xi,s)\exp(-i\xi(x_1+\tau_\alpha x_2))]d\xi\}\\
& +\sum_{n=1}^{N}\mathrm{Re}\{\sum_{\alpha=1}^{4}\int_0^\infty A_{K\alpha}(\xi,s)[H(x_2-c_2^{(n)}))F_{1\alpha}^{(n)}(\xi,s)\\
& \times\exp(i\xi(x_1-c_1^{(n)}+\tau_\alpha(x_2-c_2^{(n)})))\\
& +H(-x_2+c_2^{(n)})F_{2\alpha}^{(n)}(\xi,s)\\
& \times\exp(-i\xi(x_1-c_1^{(n)}+\tau_\alpha(x_2-c_2^{(n)})))]d\xi\},
\end{aligned} \tag{24}
$$

where the overhead bar denotes the complex conjugate of a complex number, $H(x)$ is the unit-step Heaviside function, $E_{1\alpha}^{(n)}(\xi,s)$, $E_{2\alpha}^{(n)}(\xi,s)$, $F_{1\alpha}^{(n)}(\xi,s)$ and $F_{2\alpha}^{(n)}(\xi,s)$ are functions yet to be determined, $\tau_\alpha(\xi,s)$ are roots, with positive imaginary parts, of the 8-th order polynomial equation (in τ) given by

$$
\det[\frac{s^2}{\xi^2}B_{IK}+C_{I1K1}+\tau(C_{I1K2}+C_{I2K1})+\tau^2C_{I2K2}]=0, \tag{25}
$$

$A_{K\alpha}(\xi,s)$ are non-trivial solutions of the system

$$
[\frac{s^2}{\xi^2}B_{IK}+C_{I1K1}+\tau_\alpha(\xi,s)(C_{I1K2}+C_{I2K1})+(\tau_\alpha(\xi,s))^2C_{I2K2}]A_{K\alpha}=0. \tag{26}
$$

From (10) and (24), we obtain

$$
\begin{aligned}
\widehat{S}_{Ij}(x_1,x_2,s) \;=\; & \mathrm{Re}\{\sum_{\alpha=1}^{4}\int_{0}^{\infty} i\xi L_{Ij\alpha}(\xi,s)[E_{1\alpha}(\xi,s)\exp(i\xi(x_1+\tau_\alpha x_2) \\
& -E_{2\alpha}(\xi,s)\exp(-i\xi(x_1+\tau_\alpha x_2))]d\xi\} \\
& +\sum_{n=1}^{N}\mathrm{Re}\{\sum_{\alpha=1}^{4}\int_{0}^{\infty} i\xi L_{Ij\alpha}(\xi,s)[H(x_2-c_2^{(n)})F_{1\alpha}^{(n)}(\xi,s) \\
& \times\exp(i\xi(x_1-c_1^{(n)}+\tau_\alpha(x_2-c_2^{(n)}))) \\
& -H(-x_2+c_2^{(n)})F_{2\alpha}^{(n)}(\xi,s) \\
& \times\exp(-i\xi(x_1-c_1^{(n)}+\tau_\alpha(x_2-c_2^{(n)})))]d\xi\}, \qquad (27)
\end{aligned}
$$

$L_{Ij\alpha}(\xi,s)$ are given by

$$L_{Ij\alpha}(\xi,s)=[C_{IjK1}+\tau_\alpha(\xi,s)C_{IjK2}]A_{K\alpha}. \qquad (28)$$

Note that $\widehat{U}_K(x_1,x_2,s)$ and $\widehat{S}_{Ij}(x_1,x_2,s)$ in (24) and (27) respectively are represented by different integral expressions in different parts of the piezoelectric strip. Thus, to ensure that $\widehat{S}_{I2}(x_1,x_2,s)$ are continuous on $x_2=c_2^{(n)}$, the functions $F_{1\alpha}^{(n)}(\xi,s)$ and $F_{2\alpha}^{(n)}(\xi,s)$ are chosen to be given by

$$F_{1\alpha}^{(n)}(\xi,s)=M_{\alpha P}(\xi,s)\psi_P^{(n)}(\xi,s) \text{ and } F_{2\alpha}^{(n)}(\xi,s)=M_{\alpha P}(\xi,s)\overline{\psi}_P^{(n)}(\xi,s)\ , \qquad (29)$$

where $\psi_P^{(n)}(\xi,s)$ are functions to be determined and $M_{\alpha P}(\xi,s)$ are defined by

$$\sum_{\alpha=1}^{4} L_{I2\alpha}(\xi,s)M_{\alpha P}(\xi,s)=\delta_{IP}, \qquad (30)$$

where δ_{IP} is the kronecker-delta.

The functions $\widehat{U}_K(x_1,x_2,s)$ are continuous on the plane $x_2=c_2^{(n)}$ at points not on any of the cracks if $\psi_P^{(n)}(\xi,s)$ are chosen to be

$$\psi_P^{(n)}(\xi,s)=iT_{PJ}(\xi,s)\int_{-\ell^{(n)}}^{\ell^{(n)}} r_J^{(n)}(u,s)\exp(-i\xi u)du, \qquad (31)$$

where $i = \sqrt{-1}$, $r_J^{(n)}(u,s)$ are real functions yet to be determined and $T_{PJ}(\xi,s)$ are real functions defined by

$$i\sum_{\alpha=1}^{4}[A_{K\alpha}(\xi,s)M_{\alpha P}(\xi,s) - \overline{A}_{K\alpha}(\xi,s)\overline{M}_{\alpha P}(\xi,s)]T_{PJ}(\xi,s) = \delta_{KJ}. \qquad (32)$$

Use of (32) in (24) together with

$$\lim_{\epsilon\to 0^+} \epsilon \int_{-\ell}^{\ell} \frac{\psi(u)}{\epsilon^2 + (v-u)^2} du = \pi\psi(v) \text{ for } -\ell < v < \ell, \qquad (33)$$

gives

$$r_K^{(n)}(x_1 - c_1^{(n)}, s) = \frac{1}{\pi}\Delta\widehat{U}_K^{(n)}(x_1,s) \text{ for } -\ell^{(n)} < x_1 - c_1^{(n)} < \ell^{(n)}, \qquad (34)$$

where $\Delta\widehat{U}_K(x_1,s)$ are the Laplace transform of the generalized crack opening displacements as defined in (16).

The functions $\widehat{S}_{Ij}(x_1,x_2,s)$ in (27) can now be written as

$$\begin{aligned}
\widehat{S}_{Ij}(x_1,x_2,s) \;=\; & \sum_{\alpha=1}^{4}\mathrm{Re}\{\int_{0}^{\infty} i\xi L_{Ij\alpha}(\xi,s)[E_{1\alpha}(\xi,s)\exp(i\xi(x_1+\tau_\alpha x_2) \\
& -E_{2\alpha}(\xi,s)\exp(-i\xi(x_1+\tau_\alpha x_2))]d\xi\} \\
& -\sum_{n=1}^{N}\int_{-\ell^{(n)}}^{\ell^{(n)}} r_K^{(n)}(u,s)\,\mathrm{Re}\{\sum_{\alpha=1}^{4}\int_{0}^{\infty}\xi[H(x_2-c_2^{(n)}) \\
& \times L_{Ij\alpha}(\xi,s)M_{\alpha P}(\xi,s)\exp(i\xi\tau_\alpha(\xi,s)(x_2-c_2^{(n)})) \\
& +H((-x_2+c_2^{(n)}))\overline{L}_{Ij\alpha}(\xi,s)\overline{M}_{\alpha P}(\xi,s) \\
& \times\exp(i\xi\overline{\tau}_\alpha(\xi,s)(x_2-c_2^{(n)}))] \\
& \times T_{PK}(\xi,s)\exp(i\xi[(x_1-c_1^{(n)})-u])d\xi\}du. \qquad (35)
\end{aligned}$$

If we let

$$\begin{aligned} E_{1\alpha}(\xi,s) &= M_{\alpha P}(\xi,s)\sum_{n=1}^{N} Z^{(n)}_{PK1}(\xi,s)\int_{-\ell^{(n)}}^{\ell^{(n)}} r^{(n)}_K(u,s)\exp(-i\xi u)du, \\ E_{2\alpha}(\xi,s) &= M_{\alpha P}(\xi,s)\sum_{n=1}^{N} \overline{Z}^{(n)}_{PK2}(\xi,s)\int_{-\ell^{(n)}}^{\ell^{(n)}} r^{(n)}_K(u,s)\exp(i\xi u)du. \end{aligned} \tag{36}$$

then the boundary conditions on the edges of the strip given by (23) give

$$\begin{aligned} & Z^{(n)}_{IK1}(\xi,s)+Z^{(n)}_{IK2}(\xi,s) \\ = & -i\sum_{\alpha=1}^{4}\overline{L}_{I2\alpha}(\xi,s)\overline{M}_{\alpha P}(\xi,s)T_{PK}(\xi,s) \\ & \times\exp(-i\xi[c^{(n)}_1+\overline{\tau}_\alpha(\xi,s)c^{(n)}_2]) \text{ for } I=1,2,3,4, \end{aligned} \tag{37}$$

and

$$\begin{aligned} & \sum_{\alpha=1}^{4}\{L_{I2\alpha}(\xi,s)M_{\alpha P}(\xi,s)\exp(i\xi\tau_\alpha h)Z^{(n)}_{PK1}(\xi,s) \\ & +\overline{L}_{I2\alpha}(\xi,s)\overline{M}_{\alpha P}(\xi,s)\exp(i\xi\overline{\tau}_\alpha h)Z^{(n)}_{PK2}(\xi,s)\} \\ = & -i\sum_{\alpha=1}^{4}L_{I2\alpha}(\xi,s)M_{\alpha P}(\xi,s)T_{PK}(\xi,s)\exp(i\xi[-c^{(n)}_1+\tau_\alpha(\xi,s)(h-c^{(n)}_2)]) \\ & \text{for } I=1,2,3,4. \end{aligned} \tag{38}$$

5.2 Electrically impermeable cracks

From (35), conditions (20) and (21) for electrically impermeable cracks give the hypersingular integral equations

$$\begin{aligned}
& \frac{1}{\ell^{(q)}} \int_{-1}^{1} \frac{D_{IK} \Delta V_K^{(q)}(u,s)}{(v-u)^2} du + \ell^{(q)} \int_{-1}^{1} \Delta V_K^{(q)}(u,s) \Omega_{IK}^{(q)}(u,v,s) du \\
& + \ \ell^{(q)} \int_{-1}^{1} s^2 G_{IK} \Delta V_K^{(q)}(u,s) \cosh(\ell^{(q)} \eta |v-u|) \ln(\ell^{(q)} \eta |v-u|) du \\
& + \sum_{\substack{n=1 \\ n \neq q}}^{N} \ell^{(n)} \int_{-1}^{1} \Delta V_K^{(n)}(u,s) \Theta_{IK}^{(nq)}(u,v,s) du \\
& + \sum_{n=1}^{N} \ell^{(n)} \int_{-1}^{1} \Delta V_K^{(n)}(u,s) \Lambda_{IK}^{(nq)}(u,v,s) du \\
= & \ -\pi \widehat{P}_I^{(q)}(c_1^{(q)} + \ell^{(q)} v, c_2^{(q)}, s) \ (I = 1,2,3,4) \\
& \qquad \text{for } -1 < v < 1 \ (q = 1, 2, \cdots, N),
\end{aligned} \tag{39}$$

where $\Delta V_K^{(q)}(u,s) = \Delta \widehat{U}_K^{(q)}(c_1^{(q)} + \ell^{(q)} u, s) = \pi r_K^{(q)}(\ell^{(q)} u, s)$, $\fint$ denotes that the integral is to be interpreted in the Cauchy principal sense and $⨎$ denotes that the integral is to be interpreted in the Hadamard finite-part sense, D_{IK}, G_{IK} and $W_{IK}(\xi, s)$ are given by

$$\begin{aligned}
D_{IK} &= \lim_{(\xi/s) \to \infty} T_{IK}(\xi, s), \\
G_{IK} &= \lim_{(\xi/s) \to \infty} (\frac{\xi}{s})^2 [T_{IK}(\xi, s) - D_{IK}] \ \ (\eta > 0), \\
W_{IK}(\xi, s) &= T_{IK}(\xi, s) - D_{IK} - \frac{s^2 G_{IK}}{\xi^2 + \eta^2},
\end{aligned} \tag{40}$$

and $\Omega^{(q)}_{IK}(u,v,s)$ and $\Theta^{(nq)}_{IK}(u,v,s)$ are respectively defined by

$$\begin{aligned}
\Omega^{(q)}_{IK}(u,v,s) &= -\int_0^\infty \xi W_{IK}(\xi,s)\cos(\ell^{(q)}\xi[v-u])d\xi \\
&-s^2 G_{IK}[\mathrm{Shi}(\ell^{(q)}\eta|v-u|)\sinh(\ell^{(q)}\eta|v-u|) \\
&-\frac{1}{2}\cosh(\ell^{(q)}\eta|v-u|)(\mathrm{Ei}(\ell^{(q)}\eta|v-u|)-E_1(\ell^{(q)}\eta|v-u|)) \\
&+\cosh(\ell^{(q)}\eta|v-u|)\ln(\ell^{(q)}\eta|v-u|)], \qquad (41)
\end{aligned}$$

$$\begin{aligned}
\Theta^{(nq)}_{IK}(u,v,s) &= -\mathrm{Re}\{\sum_{\alpha=1}^{4}\int_0^\infty \xi[H(Y^{(nq)}_2(u,v)) \\
&\times L_{I2\alpha}(\xi,s)M_{\alpha P}(\xi,s)\exp(i\xi\tau_\alpha(\xi,s)Y^{(nq)}_2(u,v)) \\
&+H(-Y^{(nq)}_2(u,v))\overline{L}_{I2\alpha}(\xi,s)\overline{M}_{\alpha P}(\xi,s) \\
&\times\exp(i\xi\overline{\tau}_\alpha(\xi,s)Y^{(nq)}_2(u,v))] \\
&\times T_{PK}(\xi,s)\exp(i\xi Y^{(nq)}_1(u,v))d\xi\} \\
&\qquad \text{if } Y^{(nq)}_2(u,v)\neq 0, \qquad (42)
\end{aligned}$$

$$\begin{aligned}
\Theta^{(nq)}_{IK}(u,v,s) &= \frac{D_{IK}}{[Y^{(nq)}_1(u,v)]^2}-\int_0^\infty \xi W_{IK}(\xi,s)\cos(\xi Y^{(nq)}_1(u,v))d\xi \\
&-s^2 G_{IK}[\mathrm{Shi}(\eta|Y^{(nq)}_1(u,v)|)\sinh(\eta|Y^{(nq)}_1(u,v)|) \\
&-\frac{1}{2}\cosh(\eta|Y^{(nq)}_1(u,v)|) \\
&\times(\mathrm{Ei}(\eta|Y^{(nq)}_1(u,v)|)-E_1(\eta|Y^{(nq)}_1(u,v)|))] \\
&\qquad \text{if } Y^{(nq)}_2(u,v)=0, \qquad (43)
\end{aligned}$$

and

$$\begin{aligned}
\Lambda^{(nq)}_{IK}(u,v,s) &= \mathrm{Re}\{\sum_{\alpha=1}^{4}\int_0^\infty i\xi[L_{I2\alpha}(\xi,s)M_{\alpha P}(\xi,s)Z^{(n)}_{PK1}(\xi,s)\exp(i\xi\tau_\alpha c^{(q)}_2) \\
&+\overline{L}_{I2\alpha}(\xi,s)\overline{M}_{\alpha P}(\xi,s)Z^{(n)}_{PK2}(\xi,s)\exp(i\xi\overline{\tau}_\alpha c^{(q)}_2)] \\
&\times\exp(i\xi[c^{(q)}_1+\ell^{(q)}v-\ell^{(n)}u])d\xi, \qquad (44)
\end{aligned}$$

with $Y_p^{(nq)}(u,v) = \delta_{p1}(c_1^{(q)} + \ell^{(q)}v - \ell^{(n)}\delta_{p1}u) + \delta_{p2}c_2^{(q)} - c_p^{(n)}$ and

$$\begin{aligned}
\mathrm{Shi}(u) &= \int_0^u \frac{\sinh(x)}{x}dx, \\
\mathrm{Ei}(u) &= -⨍_{-u}^{\infty} \frac{\exp(-x)}{x}dx, \\
E_1(u) &= \int_u^{\infty} \frac{\exp(-x)}{x}dx.
\end{aligned} \tag{45}$$

The functions $W_{IK}(\xi, s)$ behave as $O(s^4/\xi^4)$ for very large ξ. Thus, the improper integral over $[0,\infty)$ which appears in the definition of $\Omega_{IK}(u,v,s)$ in (41) is well defined.

The derivations of (39) and (43) make use of the following results:

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_{-1}^{1} \frac{(\epsilon^2 - (v-u)^2)\psi(u)}{(\epsilon^2 + (v-u)^2)^2}du &= -⨍_{-1}^{1} \frac{\psi(u)}{(v-u)^2}du \text{ for } -1 < v < 1, \\
\int_0^{\infty} \frac{\xi}{\xi^2+\eta^2}\cos(a\xi)d\xi &= -\frac{1}{2}\cosh(a\eta)(\mathrm{Ei}(a\eta) - E_1(a\eta)) \\
&\quad + \mathrm{Shi}(a\eta)\sinh(a\eta)\ (a\eta > 0).
\end{aligned} \tag{46}$$

Note that $\mathrm{Ei}(x) - E_1(x)$ tend to $2\ln(x)$ as $x \to 0^+$. This explains the presence of the Cauchy principal integral in (39).

If the cracks are electrically impermeable, the functions $\Delta V_K^{(q)}(u,s)$ ($q = 1, 2, \cdots, N$) in (31) are to be determined by solving the hypersingular integral equations in (39).

5.3 Electrically permeable cracks

From (15) and (34), $\Delta V_4^{(q)}(u,s) = 0$ for $-1 < u < 1$ and $q = 1, 2, \cdots$, N, if the cracks are electrically permeable. According to (13), the unknown functions $\Delta V_1^{(q)}(u,s)$, $\Delta V_2^{(q)}(u,s)$ and $\Delta V_3^{(q)}(u,s)$ are governed by (39) (with $\Delta V_4^{(q)}(u,s)=0$) for $I=1$, 2, 3 (instead of $I=1$, 2, 3, 4).

5.4 Solution of hypersingular integral equations

The hypersingular integral equations in (39) may be solved numerically for $\Delta V_K^{(q)}(u,s)$ using the collocation technique proposed by Kaya and Erdogan [13]. Specifically, we make the approximations

$$\Delta V_K^{(q)}(u,s)\simeq\sqrt{1-u^2}\sum_{j=1}^{J}\omega_K^{(qj)}(s)U^{(j-1)}(u)\text{ for }-1<u<1, \tag{47}$$

where $U^{(j)}(x)=\sin([j+1]\arccos(x))/\sin(\arccos(x))$ is the j^{th} order Chebyshev polynomial of the second kind and $\omega_P^{(nj)}(s)$ are unknown coefficients. Substitution of (47) into (39) yields a system of linear algebraic equations which can be used to determine $\omega_P^{(nj)}(s)$ for any fixed value of s. Some details on how the linear algebraic equations may be set up may be found in Athanasius, Ang and Sridhar [3].

5.5 Generalized crack tip stress intensity factors

At the crack tips $(c_1^{(n)}\pm\ell^{(n)},c_2^{(n)})$, we define the stress and electric displacement intensity factors

$$\begin{aligned}&K_I(c_1^{(n)}\pm\ell^{(n)},c_2^{(n)},t)\\=&\lim_{x_1\to(c_1^{(n)}\pm\ell^{(n)})^{\pm}}\sqrt{2(\pm x_1-(c_1^{(n)}\pm\ell^{(n)})}S_{22}(c_1^{(n)}\pm\ell^{(n)},c_2^{(n)},t),\\&K_{II}(c_1^{(n)}\pm\ell^{(n)},c_2^{(n)},t)\\=&\lim_{x_1\to(c_1^{(n)}\pm\ell^{(n)})^{\pm}}\sqrt{2(\pm x_1-(c_1^{(n)}\pm\ell^{(n)})}S_{12}(c_1^{(n)}\pm\ell^{(n)},c_2^{(n)},t),\end{aligned}$$

$$
\begin{aligned}
& K_{III}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, t) \\
= & \lim_{x_1 \to (c_1^{(n)} \pm \ell^{(n)})^{\pm}} \sqrt{2(\pm x_1 - (c_1^{(n)} \pm \ell^{(n)})} S_{32}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, t), \\
& K_{IV}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, t) \\
= & \lim_{x_1 \to (c_1^{(n)} \pm \ell^{(n)})^{\pm}} \sqrt{2(\pm x_1 - (c_1^{(n)} \pm \ell^{(n)})} S_{42}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, t). \qquad (48)
\end{aligned}
$$

Once the coefficients $\omega_P^{(nj)}(s)$ are known, the Laplace transforms of the generalized stress intensity factors defined in (48) can be easily computed using

$$
\begin{aligned}
\widehat{K}_I(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, s) &= \frac{1}{\sqrt{\ell^{(n)}}} D_{2P}^{(n)} \sum_{j=1}^{J} \omega_P^{(nj)}(s) U^{(j-1)}(\pm 1), \\
\widehat{K}_{II}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, s) &= \frac{1}{\sqrt{\ell^{(n)}}} D_{1P}^{(n)} \sum_{j=1}^{J} \omega_P^{(nj)}(s) U^{(j-1)}(\pm 1), \\
\widehat{K}_{III}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, s) &= \frac{1}{\sqrt{\ell^{(n)}}} D_{3P}^{(n)} \sum_{j=1}^{J} \omega_P^{(nj)}(s) U^{(j-1)}(\pm 1), \\
\widehat{K}_{IV}(c_1^{(n)} \pm \ell^{(n)}, c_2^{(n)}, s) &= \frac{1}{\sqrt{\ell^{(n)}}} D_{4P}^{(n)} \sum_{j=1}^{J} \omega_P^{(nj)}(s) U^{(j-1)}(\pm 1). \qquad (49)
\end{aligned}
$$

We may recover the dynamics stress and electric displacement intensity factors at any time t using the numerical Laplace transform algorithm in Stehfest [26], that is, by the formula

$$
f(t) \simeq \frac{\ln(2)}{t} \sum_{n=1}^{2M} V_n \widehat{f}(\frac{n \ln(2)}{t}), \qquad (50)
$$

where $\widehat{f}(s)$ denotes the Laplace transform of $f(t)$, M is a positive integer and

$$
V_n = (-1)^{n+M} \sum_{m=[(n+1)/2]}^{\min(n,M)} \frac{m^M (2m)!}{(M-m)!m!(m-1)!(n-m)!(2m-n)!}, \qquad (51)
$$

with $[r]$ denoting the integer part of the real number r.

The Stehfest's algorithm requires the problem under consideration to be solved for only real Laplace transform parameter s and it has been widely used by researchers for the numerical inversion of Laplace transforms in solving many problems in engineering (see, for example, Ang [1] and Hemker [11]).

6 Specific problems

In this section, some specific cases of the problem stated in Section 2 are solved. The crack tip stress and electric displacement intensity factors are computed by using (50) to invert (49).

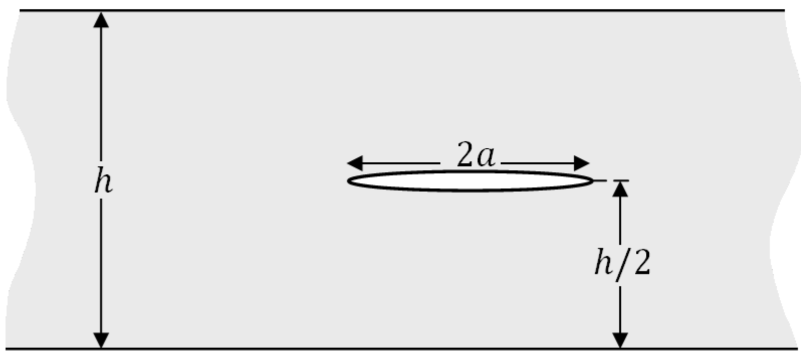

Figure 1. A sketch of Problem 1.

Problem 1. Consider the case of a single electrically impermeable crack of length $2a$ which is centrally located in the piezoelectric strip of width h, as shown in Figure 1. The electrical poling direction is taken to be along the x_2 direction and the crack is acted upon by an internal uniform load such that the generalized stress on the crack is given by $S_{I2} = -H(t)\delta_{I2}\sigma_0$, where σ_0 is a positive constant.

The material constants of the piezoelectric strip are given by

$$
\begin{aligned}
&C_{1111} = C_{3333} = A,\, C_{1133} = C_{3311} = N,\ C_{2222} = C,\\
&C_{1122} = C_{2211} = C_{2233} = C_{3322} = F,\\
&C_{1212} = C_{2112} = C_{2121} = C_{1221} = C_{2323} = C_{3223} = C_{3232} = C_{2332} = L,\\
&C_{1313} = C_{3113} = C_{3131} = C_{1331} = \frac{1}{2}(A - N),\\
&C_{2141} = C_{1241} = C_{3243} = C_{2343} = C_{4121} = C_{4112} = C_{4332} = C_{4323} = e_1,\\
&C_{1142} = C_{3342} = C_{4211} = C_{4233} = e_2,\\
&C_{2242} = C_{4222} = e_3,\ C_{4141} = C_{4343} = -\epsilon_1,\ C_{4242} = -\epsilon_2,
\end{aligned}
$$

where A, N, F, C, L, e_1, e_2, e_3, ϵ_1 and ϵ_2 are independent constants.

For our calculation here, the material constants for piezoelectric material PZT-5H as given by

$$
\begin{aligned}
&A = 12.6 \times 10^{10},\ N = 7.78 \times 10^{10},\ F = 5.3 \times 10^{10},\\
&C = 11.7 \times 10^{10},\, L = 3.53 \times 10^{10},\ e_1 = 17.0,\\
&e_2 = -6.5,\ e_3 = 23.3, \epsilon_1 = 151 \times 10^{-10},\\
&\epsilon_2 = 130 \times 10^{-10}, \rho = 7500,
\end{aligned}
$$

are used. Note that A, N, F, C and L above are in $\mathrm{N/m^2}$, e_1, e_2 and e_3 in $\mathrm{C/m^2}$, ϵ_1 and ϵ_2 in C/(Vm) and ρ in $\mathrm{kg/m^3}$.

The computed stress intensity factor $K_I/(\sigma_0\sqrt{a})$ and electric displacement intensity factor $CK_{IV}/(e_3\sigma_0\sqrt{a})$ at the right crack tip (Figure 1) are plotted against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ (up to $t\sqrt{L/(\rho a^2)} = 15$) in Figures 2 and 3 respectively. The computation here is carried out using $J = 30$ in (47) and $M = 5$ in the Stehfest's formula (50) for inverting Laplace transform.

In Figures 2 and 3, the values of $K_I/(\sigma_0\sqrt{a})$ and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ here are compared with those extracted directly from one of the graphs in Wang and Yu [28]. The two sets of $K_I/(\sigma_0\sqrt{a})$ and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ (the values computed here and those from [28]) exhibit the same general trends and are quite close to each other. At sufficiently large time, the intensity factors computed here settle down to the static values represented by the dashed horizontal lines (Figures 2 and 3). The static values are calculated using the electro-elastostatic analysis in Athanasius, Ang and Sridhar [3]. (Note that the numerical values of the intensity factors are given in [28] for only a narrow range of time, that is, for $t\sqrt{L/(\rho a^2)} < 5$ well before the stress intensity factors become much closer to the corresponding static values. Thus, no data in [28] is available for direct extraction to check if the dynamic stress intensity factors converge to the static values at higher time. Nevertheless, as the stress intensity factors computed here and the ones in [28] are in good agreement for $t\sqrt{L/(\rho a^2)} < 5$, the solution of [28] is expected to be in good agreement with our solution for higher time, as the computation of the transient stress intensity factors is more difficult for small time.)

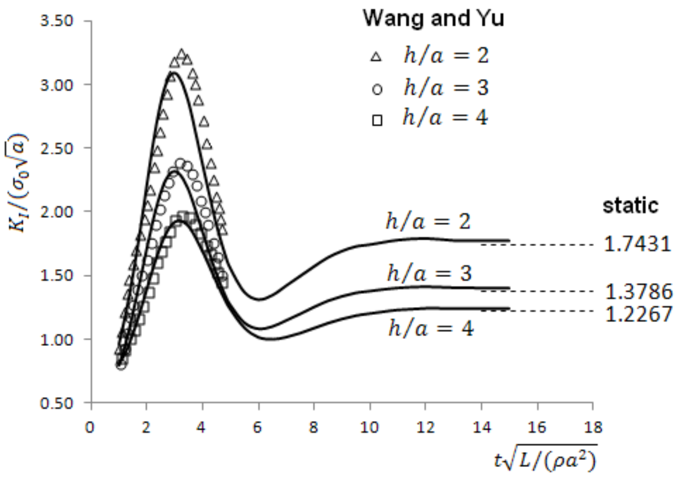

Figure 2. Plots of $K_I/(\sigma_0\sqrt{a})$ against $t\sqrt{L/(\rho a^2)}$.

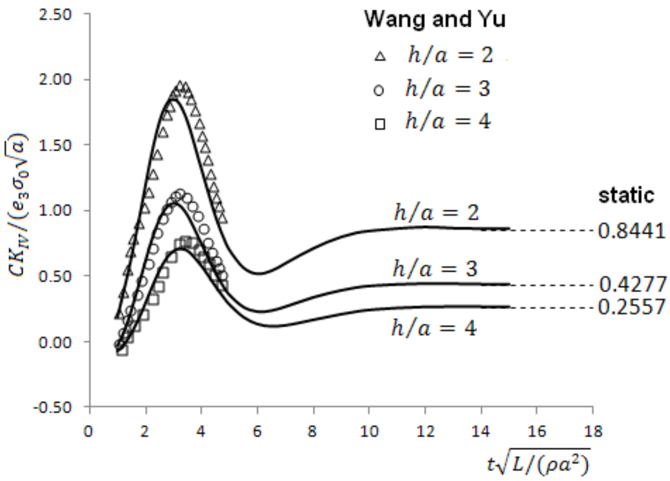

Figure 3. Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a})$ against $t\sqrt{L/(\rho a^2)}$.

Problem 2. Consider a pair of coplanar cracks, each of length $2a$, as shown in Figure 4. The electrical poling direction is taken to be along the x_2 direction and the non-zero components of the generalized stress acting on the crack faces are given by $S_{22} = -H(t)\sigma_0$ and $S_{42} = -H(t)D_0$. The cracks are electrically impermeable. The coefficients C_{IjKp} of the material occupying the strip are as in Problem 1.

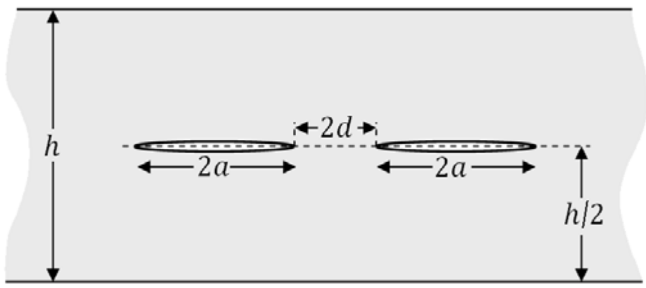

Figure 4. A sketch of Problem 2.

For $d/a = 0.50$ and selected values of h/a, the stress intensity factor $K_I/(\sigma_0\sqrt{a})$ and electric displacement intensity factor $K_{IV}/(D_0\sqrt{a})$ at the inner crack tip $(-d, h/2)$ are plotted against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ in Figures 5 and 6 respectively. It appears that the peak values of the intensity factors are higher and occur at earlier time for smaller h/a. The values of the stress intensity factors for the corresponding static problem, as calculated using the analysis in [3], are also shown using dotted horizontal lines in Figures 5 and 6. It is obvious that the dynamic intensity factors tend to the corresponding static values as time increases.

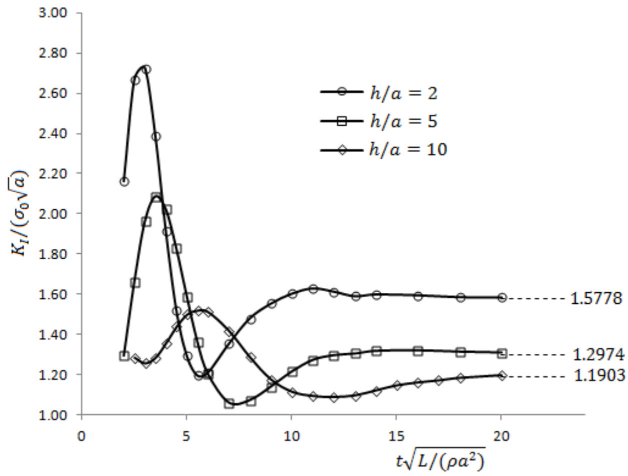

Figure 5. Plots of $K_I/(\sigma_0\sqrt{a})$ at an inner crack tip against $t\sqrt{L/(\rho a^2)}$ for $d/a = 0.50$ and selected values of h/a.

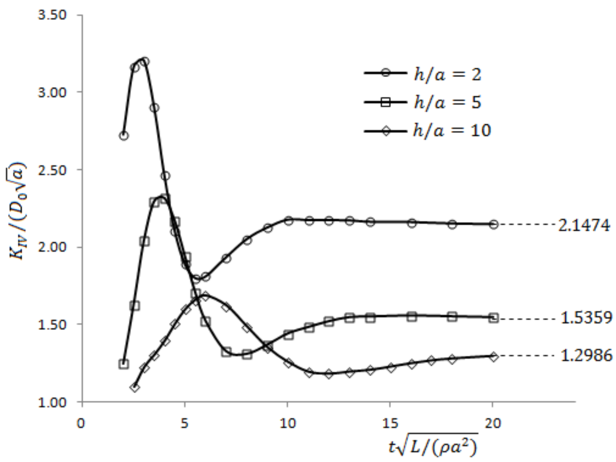

Figure 6. Plots of $K_{IV}/(D_0\sqrt{a})$ at an inner crack tip against $t\sqrt{L/(\rho a^2)}$ for $d/a = 0.50$ and selected values of h/a.

For $h/a = 4.0$ and selected values of d/a, the stress intensity factor $K_I/(\sigma_0\sqrt{a})$ and electric displacement intensity factor $K_{IV}/(D_0\sqrt{a})$ at the inner crack tip $(-d, h/2)$ are plotted against the non-dimensionalized time $t\sqrt{L/(\rho a^2)}$ in Figures 7 and 8 respectively. As may be expected, decreasing the distance between the inner tips of the cracks has the effect of increasing the magnitudes of the generalized stress intensity factors. For a fixed h/a, the time taken for the dynamic intensity factors in Figures 7 and 8 to reach the peak values appear to be roughly the same for the different values of d/a.

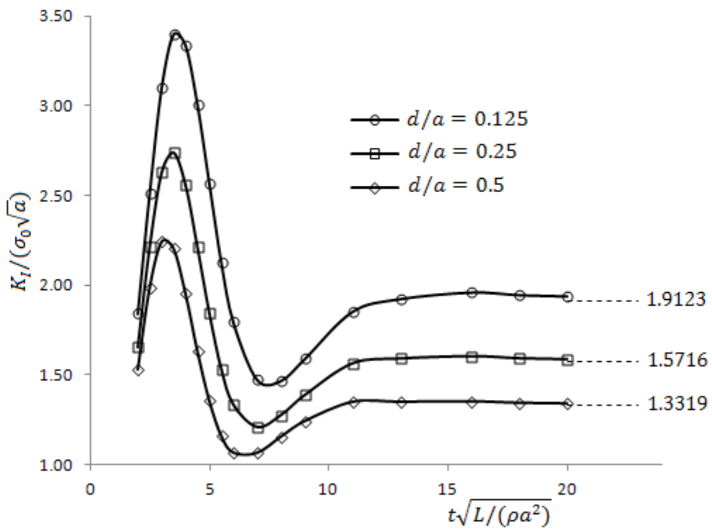

Figure 7. Plots of $K_I/(\sigma_0\sqrt{a})$ at an inner crack tip against $\sqrt{L/(\rho a^2)}$ for $h/a = 4.0$ and selected values of d/a.

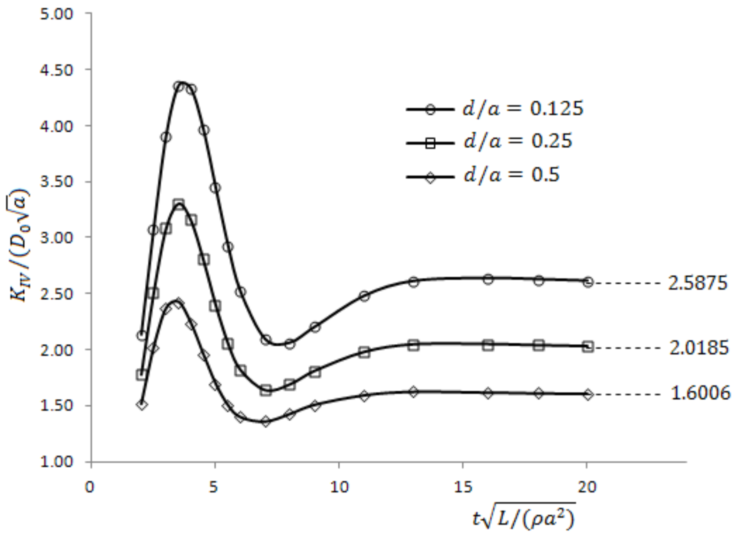

Figure 8. Plots of $K_{IV}/(D_0\sqrt{a})$ at an inner crack tip against $\sqrt{L/(\rho a^2)}$ for $h/a = 4.0$ and selected values of d/a.

Problem 3. Consider a pair of parallel cracks, each of length $2a$, as shown in Figure 9. The electrical poling direction is taken to be along the x_2 direction. The cracks are both taken to be electrically either impermeable or permeable. The non-zero loads acting on the impermeable cracks are given by $S_{22} = -H(t)\sigma_0$ and $S_{42} = -H(t)D_0$. For permeable cracks, the only non-zero load is $S_{22} = -H(t)\sigma_0$. The coefficients C_{IjKp} of the material occupying the strip are as in the last two problems.

For this particular example, plots of the stress intensity factors $K_I/(\sigma_0\sqrt{a})$ and $K_{II}/(\sigma_0\sqrt{a})$ at the crack tips against $t\sqrt{L/(\rho a^2)}$for electrically permeable cracks are almost indistinguishable from the plots for electrically permeable cracks. The plots of $K_I/(\sigma_0\sqrt{a})$ and $K_{II}/(\sigma_0\sqrt{a})$ for $h/a = 4.0$ and selected values of h_1/a are given Figures 10 and 11. When the parallel cracks are farther away from each other, the mutual shielding effect tends to increase $K_I/(\sigma_0\sqrt{a})$. The proximity of a crack to the nearest edge of the strip has the effect of increasing the stress intensity factor $K_I/(\sigma_0\sqrt{a})$. In Figure 10, it is obvious that the stress intensity factor $K_I/(\sigma_0\sqrt{a})$ increases as the cracks move away from each other towards the edges of the strip, that is, as h_1/a decreases. The stress intensity factor $K_{II}/(\sigma_0\sqrt{a})$ is not zero here as the normal stress distribution on the top and bottom faces of each of the cracks is not balanced. For the particular problem here, the distance separating the parallel cracks and the proximity of a crack to the nearest edge of the strip have opposite effect on the stress intensity factor $K_{II}/(\sigma_0\sqrt{a})$. This explains why $K_{II}/(\sigma_0\sqrt{a})$ for $h_1/a = 1.75$ in Figure 11 is larger than for $h_1/a = 1.50$ but smaller than for $h_1/a = 0.75$.

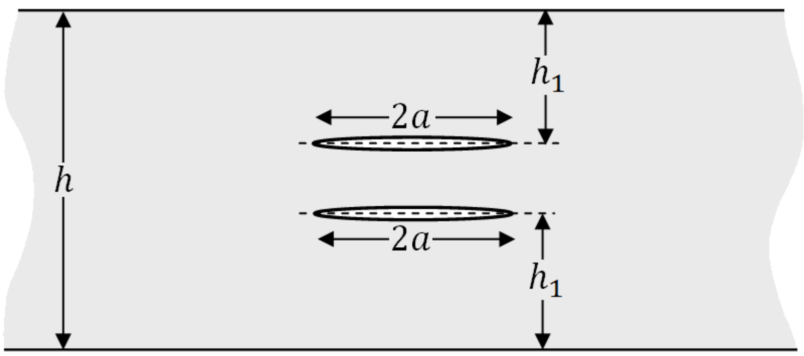

Figure 9. A sketch of Problem 3.

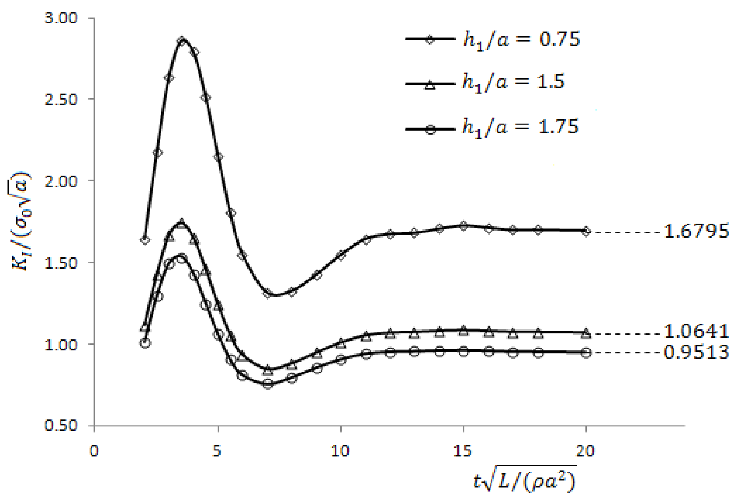

Figure 10. Plots of $K_I/(\sigma_0\sqrt{a})$ against $t\sqrt{L/(\rho a^2)}$ for $h/a = 4.0$ and selected values of h_1/a.

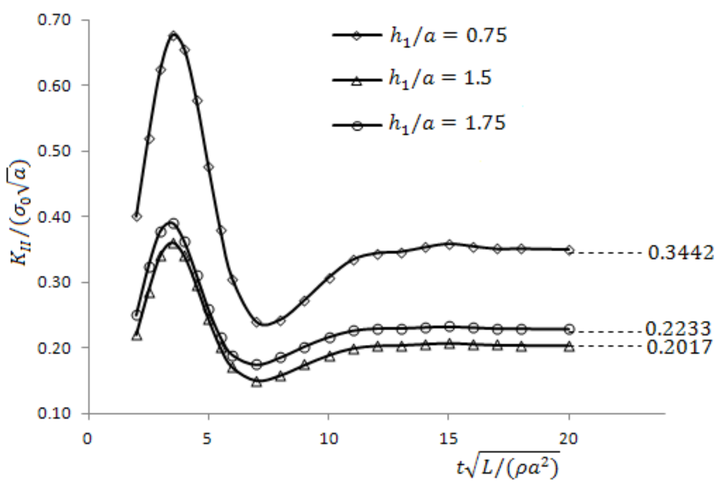

Figure 11. Plots of $K_{II}/(\sigma_0\sqrt{a})$ against $t\sqrt{L/(\rho a^2)}$ for $h/a = 4.0$ and selected values of h_1/a.

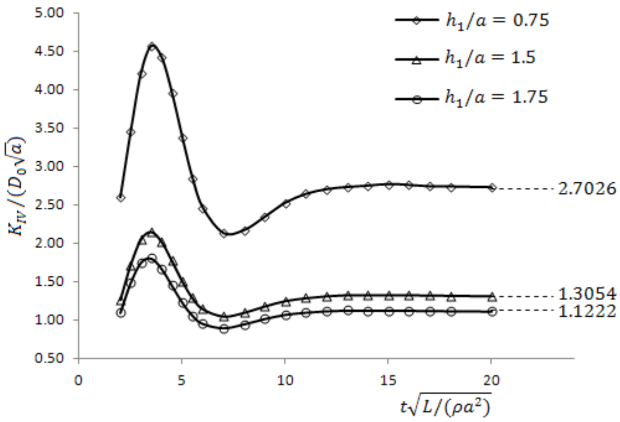

Figure 12. Plots of $K_{IV}/(D_0\sqrt{a})$ against $t\sqrt{L/(\rho a^2)}$ for $h/a = 4.0$ and selected values of h_1/a.

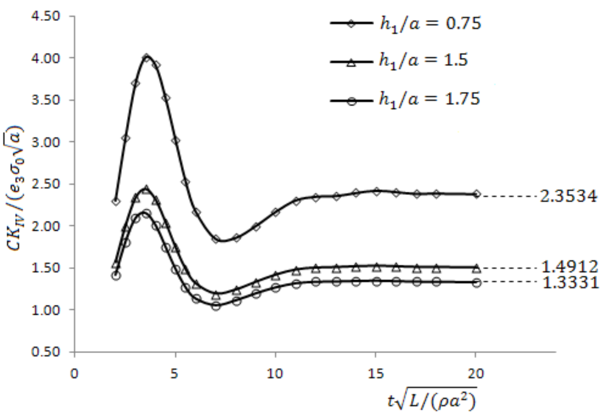

Figure 13. Plots of $CK_{IV}/(e_3\sigma_0\sqrt{a}))$ against $t\sqrt{L/(\rho a^2)}$ for $h/a = 4.0$ and selected values of h_1/a.

For $h/a = 4.0$ and selected values of h_1/a, plots of the non-dimensionalized electric displacement intensity factors $K_{IV}/(D_0\sqrt{a})$ (for electrically impermeable cracks) and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ (for electrically permeable cracks) against time $t\sqrt{L/(\rho a^2)}$ are given in Figures 12 and 13 respectively. It appears that increasing h_1/a has the same effect on $K_{IV}/(D_0\sqrt{a})$ and $CK_{IV}/(e_3\sigma_0\sqrt{a})$ as on $K_I/(\sigma_0\sqrt{a})$ in Figure 10.

7 Summary

A semi-analytic solution is derived in the Laplace transform domain for an electro-elastodynamic problem involving an arbitrary number of arbitrarily located parallel planar cracks in a piezoelectric strip. Although the solution is explicitly expressed in terms of exponential Fourier integral transforms, the solution is regarded as semi-analytic as the integrands in the Fourier integrals contain unknown functions to be determined approximately. The unknown functions are related to the Laplace transforms of the jumps in the displacements and electric potential across opposite crack faces. The task of determining the Laplace transforms of the jumps in the displacements and electric potential across opposite crack faces is reduced to solving numerically a system of hypersingular integral equations. Once the hypersingular integral equations are solved, the crack tip stress and electric displacement intensity factors in the Laplace transform domain can be easily extracted. The intensity factors in the physical domain are then recovered by using a numerical method for inverting Laplace transforms.

To check the solution, the crack tip stress and electric displacement intensity factors are computed for a single crack which is centrally located in the strip and subject to uniform impact loads. The computed stress and electric displacement intensity factors are found to be in reasonably good agreement with those published in the literature. New results for the stress and electric displacement intensity factors are obtained for a pair of coplanar cracks and a pair of parallel cracks in the piezoelectric strip.

Acknowledgements. The first author (WT Ang) acknowledges the sponsorship of Singapore Ministry of Education Tier 1 Research Grant RG9/09.

References

[1] WT Ang, Transient response of a crack in an anisotropic strip, *Acta Mechanica* **70** (1987) 97-109.

[2] L Athanasius and WT Ang, Dynamic interaction of multiple arbitrarily oriented planar cracks in a piezoelectric space: a semi-analytic solution, *European Journal of Mechanics-A/Solids* **30** (2011) 608-618.

[3] L Athanasius, WT Ang and I Sridhar, Electro-elastostatic analysis of multiple cracks in an infinitely long piezoelectric strip: a hypersingular integral approach, *European Journal of Mechanics-A/Solids* **29** (2010) 410-419.

[4] DM Barnett and J Lothe, Dislocations and line charges in anisotropic piezoelectric insulators, *Physica Status Solidi (b)* **67** (1975) 105-111.

[5] JT Chen and HK Hong, Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series, *ASME Applied Mechanics Review* **52** (1999) 17-33.

[6] ZT Chen, Dynamic fracture mechanics study of an electrically impermeable mode III crack in a transversely isotropic piezoelectric material under pure electric load, *International Journal of Fracture* **141** (2006) 395-402.

[7] ZT Chen and BL Karihaloo, Dynamic response of a cracked piezoelectric ceramic under arbitrary electro-mechanical impact, *International Journal of Solids and Structures* **36** (1999) 5125-5133.

[8] ZT Chen and SA Meguid, The transient response of a piezoelectric strip with a vertical crack under electromechanical impact load, *International Journal of Solids and Structures* **37** (2000) 6051-6062.

[9] ZT Chen and MJ Worswick, Antiplane mechanical and inplane electric time-dependent load applied to two coplanar cracks in piezoelectric ceramic material, *Theoretical and Applied Fracture Mechanics* **33** (2000) 173-184.

[10] F García-Sánchez, Ch Zhang, J Sládek and V Sládek, 2-D transient dynamic crack analysis in piezoelectric solids by BEM, *Computational Materials Science* **39** (2007) 179–186.

[11] CJ Hemker, Transient well flow in vertically heterogeneous aquifer, *Journal of Hydrology* **225** (1999) 1-18.

[12] HK Hong and JT Chen, Derivations of integral equations of elasticity, *ASCE Journal of Engineering Mechanics* **114** (1988) 1028-1044.

[13] AC Kaya and F Erdogan, On the solution of integral equations with strongly singular kernels, *Quarterly of Applied Mathematics* **45** (1987) 105-122.

[14] M Kuna, Fracture mechanics of piezoelectric materials-where are we right now? *Engineering Fracture Mechanics* **77** (2010) 309-326.

[15] SM Kwon and KY Lee, Edge cracked piezoelectric ceramic block under electromechanical impact loading, *International Journal of Fracture* **112** (2001): 139-150.

[16] XF Li, Closed-form solution for a piezoelectric strip with two collinear cracks normal to the strip boundaries, *European Journal of Mechanics-A/Solids* **21** (2002) 981-989.

[17] XF Li and TY Fan, Transient analysis of a piezoelectric strip with a permeable crack under anti-plane impact loads, *International Journal of Engineering Science* **40** (2002) 131-143.

[18] XF Li and KY Lee, Dynamic behavior of a piezoelectric ceramic layer with two surface cracks, *International Journal of Solids and Structures* **41** (2004) 3193-3209.

[19] XF Li and KY Lee, Transient response of a semi-infinite piezoelectric layer with a surface permeable crack, *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **57** (2006) 636-651.

[20] XF Li and GJ Tang, Transient response of a piezoelectric ceramic strip with an eccentric crack under electromechanical impacts, *International Journal of Solids and Structures* **40** (2003) 3571-3588.

[21] F Liu and X Zhong, Transient response of two collinear dielectric cracks in a piezoelectric solid under inplane impacts, *Applied Mathematics and Computation* **217** (2010) 3779-3791.

[22] SA Meguid and ZT Chen, Transient response of a finite piezoelectric strip containing coplanar insulating cracks under electromechanical impact, *Mechanics of Materials* **33** (2001) 85-96.

[23] Y Shindo, F Narita and E Ozawa, Impact response of a finite crack in an orthotropic piezoelectric ceramic, *Acta Mechanica* **137** (1999) 99-107.

[24] Y Shindo, K Watanabe and F Narita, Electroelastic analysis of a piezoelectric ceramic strip with a central crack, *International Journal of Engineering Science* **38** (2000) 1-19.

[25] IN Sneddon and M Lowengrub, *Crack Problems in the Classical Theory of Elasticity*, Wiley, New York, 1969.

[26] H Stehfest, Numerical inversion of the Laplace transform, *Communications of ACM* **13** (1970) 47-49 (see also p624).

[27] BL Wang and YW Mai, A piezoelectric material strip with a crack perpendicular to its boundary surfaces, *International Journal of Solids and Structures* **39** (2002) 4501-4524.

[28] X Wang and S Yu, Transient response of a crack in piezoelectric strip subjected to the mechanical and electrical impacts: mode-I problem, *Mechanics of Materials* **33** (2001) 11-20.

[29] CY Wang and Ch Zhang, 3-D and 2-D dynamic Green's functions and time-domain BIEs for piezoelectric solids, *Engineering Analysis with Boundary Elements* **29** (2005) 454-465.