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Tensor Decompositions Applied to Electromagnetics: A Review

Mingyu Wang(1), Cheng Qian(1), and Abdulkadir C. Yucel(1)
(1) School of Electrical and Electronic Engineering, Nanyang Technological University, 639798, Singapore

Abstract

In this paper, tensor decomposition methodologies are revisited. Their applications to the electromagnetic (EM) solvers as well as surrogate modelling for uncertainty quantification are reviewed. The computational and memory savings achieved via such decompositions are discussed.

***Index terms* – EM analysis, EM solvers, surrogate modelling, tensor decompositions.**

1. Introduction

Tensor decomposition methodologies [1-4] have received significant attention from researchers in many branches of computational science and engineering. For EM analyses, these methodologies are primarily used to reduce the computational cost and memory requirements of integral equation (IE) based EM solvers [5-17] and the computational time requirement of uncertainty quantification in EM analysis [18-22]. These methodologies are based on Tucker, tensor train, and canonical polyadic decompositions. Each of these decompositions performs best for certain applications. For example, the Tucker decomposition [4] performs best for compressing the low dimensional arrays storing the samples of the Green's function and its associated integrals on a structured grid [8, 9, 13, 16, 17, 23]. However, its performance degrades when applied to compression of the high-dimensional data because of the curse of dimensionality [15]. High-dimensional arrays arise in EM analysis when the system matrices of the IE solvers are tensorized or when the quantities of interests in uncertainty quantification problems are functions of a large number random variables. For the compression or completion of such high dimensional arrays, the tensor train and canonical polyadic decompositions appear to be highly suitable solutions [15, 24]. Oftentimes, tensor train decomposition is preferred when high accuracy is sought for, while the canonical polyadic decomposition is a perfect candidate for achieving very high efficiency [9].

In what follows, these tensor decomposition techniques and their applications to the EM solvers and surrogate modelling in the EM analysis are reviewed. After introducing each decomposition, methodologies to obtain the low-rank tensors and matrices in decompositions are discussed. Then, the applications of these decompositions to the EM analyses are explained and the computational and memory savings achieved by such decompositions are discussed. In the talk, more details on the decompositions and their potential applications to other EM problems will be discussed. Furthermore, some of our recent results compiling the performance of the Tucker decomposition for compressing the system tensors of EM solvers will be shared.

2. Review of Tensor Decompositions and Their Applications to EM Analysis

2.1 Tucker Decomposition

Let $\mathcal{T}$ represent a complex 3-D array with dimensions n_1, n_2, and n_3, i.e. $\mathcal{T} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. Tucker representation of such array comprises of a core tensor $\mathcal{C} \in \mathbb{C}^{r_1 \times r_2 \times r_3}$ and factor matrices $\bar{\mathbf{U}}^i \in \mathbb{C}^{n_i \times r_i}, i = 1,2,3$, as [2, 4, 13]:

$$\mathcal{T} = \mathcal{C} \times_1 \bar{\mathbf{U}}^1 \times_2 \bar{\mathbf{U}}^2 \times_3 \bar{\mathbf{U}}^3 \quad . \tag{1}$$

Here the symbol $\times_i, i = 1,2,3$, represents the $i-$ mode matrix-tensor multiplication while r_i stands for the multilinear rank pertaining to i^{th} dimension. The tensor $\mathcal{T}$ is considered as highly compressible if $(r_1 r_2 r_3) + \sum_{i=1}^{3} n_i r_i << n_1 n_2 n_3$. The low-rank core tensor $\mathcal{C}$ and factor matrices $\bar{\mathbf{U}}^i, i = 1,2,3$ can be obtained via standard SVD technique [2] for a provided tolerance, tol. However, SVD technique requires excessive computational resources for large-scale tensors. To tackle this issue, a cross approximation based technique [25, 26], which constructs the core tensor and factor matrices by computing some rows, columns, and slices of $\mathcal{T}$, can be used. Furthermore, one can also utilize randomized algorithms to obtain the components of the Tucker decomposition for large tensors [27].

Tucker decomposition has been employed to reduce the memory requirement of various EM solvers. Recently, it was applied to compress the system tensors (circulant/Toeplitz tensors) in the fast Fourier transform (FFT)-accelerated IE solvers [8, 9, 13, 16, 17, 23] during their setup stages. The compressed tensors are then restored to their original format with negligible computational cost during the iterative solution part of the solvers. Doing so substantially reduces the memory requirement of such solvers for many practical applications. For example, the studies in [9, 13] applied the Tucker decompositions to the system tensors arising in the FFT-accelerated volume IE solvers for the EM analysis of magnetic resonance imaging (MRI). The study in [13] applied the Tucker decompositions to the system tensors generated for the piecewise constant discretization of the currents and reported an achieved compression ratio (CR) as 10,000 for $tol = 10^{-8}$. On the other hand, [9] applied the Tucker decomposition to the system tensors arising due to piecewise linear discretization of currents. In this case, Tucker decomposition achieved a CR of 8000 for $tol = 10^{-6}$. In [8], Tucker decomposition was used to compress the off-diagonal blocks of the system matrix. In particular, it compressed the blocks modelling the remote interactions between the MRI coil and the human body and achieved a CR of 1.9 million for $tol = 10^{-3}$. The studies in [16, 17] applied the Tucker decomposition to the compression of the system tensors generated for the piecewise constant and piecewise linear discretizations of currents and charges. [17] and [16] demonstrated that CR of 10,000x can be achieved for $tol = 10^{-4}$ while compressing system tensors used for the electrostatic and magneto-quasi-static analyses, respectively. Apart from using the Tucker decompositions for reducing the memory requirement of solvers, [17] and [16] used these decompositions to accelerate the setup stages of the solvers. To do that, Tucker decomposition was used to compress the large Toeplitz tensors that can be generated with the available computational resources during the installation stage of the solvers. Next, the compressed tensors stored in harddisk were used to obtain circulant tensors during the executions of the setup stages of the simulators. Doing so reduced the setup time of the solvers more than 4,655x and 43x for the electrostatic and magneto-quasi-static analyses, respectively.

All these works referred above make use of Tucker decomposition for compressing system tensors of smoothly varying kernels. Tucker decomposition has also been used to compress the tensors of rapidly varying kernels arising in the high-frequency EM simulators [15, 28]. In [28], the authors used the Tucker decompositions to compress the 3-D arrays holding the FFT'ed translation operator samples on a structured grid. These tensors are used in the translation stage of the fast multipole method (FMM) – FFT accelerated IE solvers and characterize the interactions between FMM groups for each plane-wave direction. The Tucker decomposition was able to reduce the memory requirement of these tensors 80%. Furthermore, such a study was then extended to compress the FFT'ed translation operator tensors stored in a 4-D array [15]. In [15], it was shown that the hierarchical Tucker decompositions [29, 30] applied to the 4-D array yielded 10% more memory reduction compared to the conventional Tucker applied to the 3-D array for $tol = 10^{-6}$. In addition, the Tucker decomposition was used to compress far-fields of the FMM groups [14]. It was shown that the Tucker decomposition achieves more than 87% memory reduction while expediting the aggregation and disaggregation stages of the FMM by a factor of 15.8 and 15.2, respectively.

2.2 Tensor Train (TT) Decomposition

Let $\mathcal{T}$ represent a complex d - dimensional array, i.e., $\mathcal{T} \in \mathbb{C}^{n_1 \times \ldots \times n_d}$. Its TT decomposition reads as [3]:

$$\mathcal{T}(i_1, \ldots, i_d) = \sum_{\alpha_0, \ldots, \alpha_d} \mathcal{G}_1(\alpha_0, i_1, \alpha_1) \ldots \mathcal{G}_d(\alpha_{d-1}, i_d, \alpha_d) . \quad (2)$$

Here $\mathcal{G}_k \in \mathbb{C}^{r_{k-1} \times n_k \times r_k}, k = 1, 2, \ldots, d$ represents TT core tensor and r_k is the TT rank ($r_0 = r_d = 1$). i_k and α_k stand for spatial and auxiliary indices, respectively. SVD [3] or cross approximation [31] can be used to obtain the components of TT decomposition.

TT decomposition has been applied to surrogate modelling for EM analysis. In [10], the TT decomposition was used to obtain near weakly-singular and near singular integral interactions, which are to be computed for the system matrices of the IE solvers. In particular, a low-rank TT decomposition was constructed through a set of pre-computed lookup tables to rapidly interpolate the integral values. The technique was shown to be much faster compared to the conventional singularity subtraction technique. In [22], TT decomposition was employed to perform hierarchical uncertainty quantification with high-dimensional subsystems. For such high dimensionality, the ability and capability of the conventional stochastic methods are limited for extracting a surrogate model for each subsystem. In this work, at the low level, adaptive anchored analysis of variance-based sparse stochastic testing simulator was used to extract the surrogate model of each subsystem efficiently. At the high level, the TT decomposition was employed to construct the basis functions and Gauss quadrature points. For an example of an oscillator circuit with 184 random parameters, the proposed scheme was shown to be 92x faster than the hierarchical MC method while achieving highly accurate results. In [20], spectral quantic TT (SQTT) approximation was used for the uncertainty quantification of EM problems. In particular, TT was used to approximate high dimensional tensors storing the values of quantities of interest on Gauss-Legendre points. The approximated tensors were used to construct the surrogate models of the quantities via generalized polynomial chaos expansion. For an example characterizing EM scattering from a rough surface realized by 25 random parameters, the SQTT required nearly one-third of the simulations required by a high dimensional model representation technique while yielding more accurate results. Also, SQTT has been used to generate the statistical moments preserving reduced order models in [32].

TT decomposition has also been applied to the 2D EM solvers [5, 6, 11, 12]. In [5], TT decomposition was used to compress the Toeplitz matrices in a pre-corrected FFT-accelerated EM solver. In particular, it was applied to the Toeplitz matrices tensorized via hierarchical binary tree representation of the structured grid enclosing the computational domain. The technique was used for the low-frequency inductance extraction problems as well as the high-frequency scattering problems. It also exploited the sparsity of the computational domain to expedite the matrix-vector multiplications. It was shown that the technique allows for further CPU time and memory reduction compared with the conventional pre-corrected - FFT methods. In [6], TT decomposition was used to compress tensorized Toeplitz matrix arising in the 2D FFT-accelerated volume IE solvers. Besides the memory reduction, the proposed technique achieved rapid matrix-vector multiplications. The memory scaling of the technique is reported as $O(N)$ and $O(r^2 N \log N)$ in the low- and high-frequency regimes, respectively, where N is the number of basis functions and r is the highest TT rank. Besides TT-based fast iterative solver, [11] provided a direct solution of TT decomposed MoM matrix equation approach. This methodology exhibited $O(N)$ complexity for the homogenous scatters of arbitrary geometry subjected to incident fields with wavelengths comparable to the scatterer size (in quasi-constant rank regime). In [7, 12], 2-D IE solvers are accelerated by quantized TT (QTT). Compared to the conventional compression schemes, the QTT scheme yielded higher compression. Also, the QTT decomposition was used to construct approximate inverse preconditioners for rapid solution of the linear system of equations in IE solvers [7].

2.3 Canonical Polyadic (CP) Decomposition

CP decomposition represents a $d-$dimensional array $\mathcal{T} \in \mathbb{C}^{n_1 \times n_2 \times \ldots \times n_d}$ with the sum of rank-one tensors $V^k \in \mathbb{C}^{n_k}, k = 1, 2, \ldots, d$ [1, 9, 24]:

$$\mathcal{T} \approx \sum_{l=1}^{r} V_l^1 \circ V_l^2 \circ \ldots \circ V_l^d . \quad (3)$$

Here, r denotes the canonical rank and $\circ$ represents the outer product. The alternating least squares method [24] can be used to obtain the decomposition. For large scale tensors, the low-rank approximation of CP decomposition may not exist and the methodology is ill-posed [33]

CP decomposition has been applied to surrogate modelling. [21] proposed a CP decomposition-based scheme to extract the surrogate model for high-dimensional problems. This work uses a small number of samples to estimate the whole tensor holding the values of quantities of interests. This can be achieved by exploiting the hidden low-rank property of a tensor and the sparsity of a generalized polynomial chaos expansion. The numerical examples in this paper showed that the proposed technique outperforms a sparse grid-based surrogate modelling technique. However, researchers faced two challenges in this study: One is the determination of tensor rank and the other one is the selection of informative simulation samples. [19] provided the solutions to these challenges. First, the researchers proposed l_q / l_2 group-sparsity regularization technique to determine the rank automatically. Second, they proposed a two-stage adaptive sampling method, which performs the exploration based on Voronoi cells/diagrams and exploitation based on the nan-linearity of the quantity of interest.

CP decomposition was also used in conjunction with Tucker decomposition in an EM solver. In [9], the authors applied CP decomposition to the core tensor of Tucker decomposition. The proposed Tucker+CP scheme yielded better compression compared to the Tucker scheme.

3. Conclusions

In this work, a summary of the tensor decomposition methodologies applied to EM problems was provided. Specifically, tensor decompositions applied to the IE solvers for reducing their memory and CPU requirements were revisited. In addition, the tensor decompositions applied to the surrogate modelling for uncertainty quantification in EM analysis were reviewed.

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