CHAOTIC HEDGING WITH ITERATED INTEGRALS AND NEURAL NETWORKS

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ABSTRACT. In this paper, we extend the Wiener-Ito chaos decomposition to the class of diffusion processes, whose drift and diffusion coefficient are of linear growth. By omitting the orthogonality in the chaos expansion, we are able to show that every $p$-integrable functional, for $p \in [1, \infty)$, can be represented as sum of iterated integrals of the underlying process. Using a truncated sum of this expansion and (possibly random) neural networks for the integrands, whose parameters are learned in a machine learning setting, we show that every financial derivative can be approximated arbitrarily well in the $L^p$-sense. Moreover, the hedging strategy of the approximating financial derivative can be computed in closed form.

1. INTRODUCTION

The problem of pricing and hedging financial derivatives is a crucial task for portfolio managers in the financial industry. In complete markets such as the Bachelier and Black-Scholes model (see [Bachelier, 1900] and [Black and Scholes, 1973]), the hedging problem is well understood since in these models there exists only one pricing measure and every financial derivative can be perfectly replicated. However, in incomplete markets, there exist infinitely many pricing measures, which is why the pricing and hedging problem needs to be addressed with an additional criteria such as, for example, “mean-variance hedging” (see e.g. [Duffie and Richardson, 1991], [Schweizer, 1992], and [Pham et al., 1998]), “super-replication” (see e.g. [El Karoui and Quenez, 1995], [Cvitanić et al., 1999], [Cheridito et al., 2005], and [Acciaio et al., 2016]), “quadratic hedging” (see e.g. [Schweizer, 1999] and [Pham, 2000]), or “utility indifference pricing and hedging” (see e.g. [Musiela and Zariphopoulou, 2004] and [Carmona, 2009]).

Our “chaotic hedging” approach solves the hedging problem by minimizing a given loss function and is therefore similar to the idea of “quadratic hedging”. The numerical algorithm relies on the so-called chaos expansion of a financial derivative, which can be seen as Taylor formula for random variables, and has been applied for option pricing (see [Lacoste, 1996] and [Lelong, 2018]) and for solving stochastic differential equations (SDEs) (see [Xiu and Karniadakis, 2002]). Moreover, by proving a universal approximation result for random neural networks, we are able to efficiently learn the financial payoff, which is similar to other successful neural networks applications in finance (see e.g. [Han et al., 2018], [Bühler et al., 2019], [Sirignano and Cont, 2019], [Cuchiero et al., 2020], [Eckstein et al., 2020], [Eckstein et al., 2021], [Neufeld and Sester, 2021], [Neufeld et al., 2022], and [Schmocker, 2022]). Altogether, this paper introduces a two-step approximation of any sufficiently integrable financial derivative, by first using a truncation of its chaos expansion, and then learning (random) neural networks for each order of the expansion. The algorithm can be also applied to high-dimensional options and returns the hedging strategy of the approximation in closed form.

More precisely, we consider a market with finite time horizon $T > 0$ and $d \in \mathbb{N}$ risky assets, whose price processes are modelled by a diffusion process with drift and diffusion coefficient of linear growth. This includes in particular Brownian motion, but also more general affine and polynomial diffusion processes (see [Duffie et al., 2000], [Cuchiero, 2011], and [Filipović and Larsson, 2016]). In this setting, the financial derivative to be replicated is given by a $p$-integrable random variable, for some $p \in [1, \infty)$.

The main idea of this paper relies on the chaos expansion of a financial derivative. In the Brownian motion case, the so-called Wiener-Ito chaos decomposition (see [Ito, 1951, Theorem 4.2]) yields an expression for every square-integrable functional of the Brownian path as infinite sum of iterated integrals of the given Brownian motion, which can be seen as starting point of Malliavin calculus (see [Malliavin, 1978] and [Nualart, 2006]). This property is called the “chaotic representation property (CRP)”.

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later extended to the compensated Poisson process (see [Ito, 1956, Theorem 2]) and Azéma martingales (see [Emery, 1989, Proposition 6]). Moreover, it was shown in [Dellacherie and Meyer, 1975] that the only normal martingales (i.e. martingales with deterministic covariation) having the CRP which are Lévy processes are the Brownian path and the compensated Poisson process. For the CRP of a more general Lévy process, [Nualart and Schoutens, 2000] used the Teugels martingales involving the power jump processes to obtain the chaos expansion. In addition, [Løkka, 2004], [Jamshidian, 2005], [Di Nunno et al., 2008], and [Di Tella and Engelbert, 2016] proved the chaos expansion for particular martingales with independent increments.

However, to the best of our knowledge, a chaos expansion with respect to a diffusion process with stochastic covariation and non-independent increments has not yet been established in the literature. Opposed to the classical Wiener-Ito chaos decomposition, we omit the orthogonality in the chaos expansion and use the direct sum (instead of the orthogonal sum) together with the so-called “time-space harmonic Hermite polynomials” in order to span the chaotic subspaces. Moreover, by extending the Kailath-Segall formula in [Segall and Kailath, 1976], we are able to express every \( p \)-integrable functionals, \( p \in [1, \infty) \), as iterated integrals of the underlying diffusion process, which yields an analogue of the classical Wiener-Ito chaos decomposition in this more general setting with possible drift process.

In the literature, the above mentioned “chaotic representation property (CRP)” implies the so-called “predictable representation property (PRP)”, which can be seen as a generalization of the martingale representation result with respect to Brownian motion (see [Karatzas and Shreve, 1998, Section 3.4]). The latter ensures the existence of a hedging strategy for each square-integrable functional of the Brownian path (see e.g. [Karatzas and Shreve, 1998, Problem 3.4.17]). This has been extended to the compensated Poisson process (see e.g. [Ikeda and Watanabe, 1989, Theorem II.6.7]), Azéma martingales (see [Emery, 1989]), and other martingales with independent increments (see [Nualart and Schoutens, 2000], [Løkka, 2004], [Davis, 2005], [Jamshidian, 2005], and [Di Tella and Engelbert, 2016]). In our setting, we are able to retrieve a similar representation, but with respect to the diffusion process, which relies on the \( L^p \)-closedness of stochastic integrals (see [Grandits and Krawczyk, 1998] and [Delbaen et al., 1997]). This leads us in the martingale case (i.e. without drift) to a martingale representation result similar to the one of Brownian motion (see also [Soner et al., 2011]).

Finally, the machine learning application of this paper consists of learning the integrands of the iterated integrals in the chaos expansion. More precisely, we first approximate a financial derivative by many deterministic integrands by so-called “tensor-valued neural networks”. Inspired by reservoir computing (see e.g. [Grigoryeva and Ortega, 2018] and [Cuchiero et al., 2021]), we extend the universal approximation result to random neural networks with randomly initialized weights and biases (see e.g. [Rahimi and Recht, 2007], [Gonon et al., 2022], and [Kratsios and Bilokopytov, 2020]), leaving us only with the training of the linear readout. In this case, the estimation procedure reduces to a simple linear regression, which can be solved within few minutes, also for high-dimensional financial derivatives.

1.1. Outline. In Section 2, we introduce the setting of this paper and provide the main theoretical results, including the chaos expansion, the corresponding (semi-) martingale representation result, and the theoretical approximation results for financial derivatives. Subsequently, we illustrate in Section 3 how these theoretical results can be applied to learn the hedging strategy of various options in different markets. Finally, all proofs of the theoretical results are given in Section 4.

1.2. Notation. As usual, \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) denote the sets of natural numbers, whereas \( \mathbb{R} \) and \( \mathbb{C} \) represent the real and complex numbers (with complex unit \( i := \sqrt{-1} \)), respectively, where \( s \wedge t := \min(s, t), \) for \( s, t \in \mathbb{R} \). Moreover, \( \mathbb{R}^d \) denotes the Euclidean space equipped with the norm \( \|x\| = \sqrt{x_1^2 + \ldots + x_d^2} \), whereas \( \mathbb{R}^{d \times l} \) represents the vector space of matrices \( A = (a_{ij})_{i=1 \ldots d, j=1 \ldots l} \) equipped with the Frobenius norm \( \|A\|_F = \left( \sum_{i=1}^{d} \sum_{j=1}^{l} |a_{ij}|^2 \right)^{1/2} \). Hereby, \( \mathbb{S}^d_+ \subset \mathbb{R}^{d \times d} \) denotes the cone of symmetric non-negative definite matrices and \( I_d \in \mathbb{S}^d_+ \) represents the identity matrix. In addition, for \( S \subseteq \mathbb{R}, C(S) \) denotes the space of continuous functions \( f : S \to \mathbb{R}, \) and for \( p \in [1, \infty) \) and \( T > 0 \), \( L^p(dt) := L^p([0, T], \mathcal{B}([0, T]), dt) \) is the Banach space of Borel-measurable functions \( f : [0, T] \to \mathbb{R} \) with finite norm \( \|f\|_{L^p(dt)} := \left( \int_0^T |f(t)|^p dt \right)^{1/p} \). Hereby, \( \mathcal{B}([0, T]) \) denotes the Borel \( \sigma \)-algebra of \([0, T], \) and for \( p = 2, \) the inner product \( \langle f, g \rangle_{L^2(dt)} = \int_0^T f(t)g(t) dt \) turns \( L^2(dt) \) into a Hilbert space.
Throughout this paper, we consider a financial market with finite time horizon $T > 0$ consisting of $d \in \mathbb{N}$ risky assets, whose vector of price processes follows a stochastic process $X = (X_t^1, \ldots, X_t^d)^T_{t \in [0,T]}$ with continuous sample paths and values in a subset $E \subseteq \mathbb{R}^d$.

### 2. Setting and Main Results

#### 2.1. Setting

On a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we fix an $l$-dimensional Brownian motion $B = (B_t^1, \ldots, B_t^d)_{t \in [0,T]}$ endowed with the natural $\mathbb{P}$-augmented filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ generated by $B$ such that $\mathcal{F}_T = \mathcal{A}$, and let $b = (b_1, \ldots, b_d)^T : [0,T] \times E \to \mathbb{R}^d$ and $\sigma = (\sigma_{ij})_{i=1,\ldots,d; j=1,\ldots,d} : [0,T] \times E \to \mathbb{R}^{d \times d}$ be continuous. We assume that $X$ is an $E$-valued strong solution (see [Karatzas and Shreve, 1998, Definition 5.2.1]) of the stochastic differential equation (SDE) $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$, with $E \subseteq \mathbb{R}^d$, i.e. $X$ is $\mathbb{F}$-adapted, and for every $i = 1, \ldots, d$ and $t \in [0,T]$, it holds $\mathbb{P}$-a.s. that

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s)ds + \sum_{j=1}^l \int_0^t \sigma_{ij}(s, X_s)dB_s^j,$$

$X_0^i \in E$, (1)

where $\int_0^T (\|b(s, X_s)\| + \|\sigma(s, X_s)\|_F^2) ds < \infty$, $\mathbb{P}$-a.s.

**Definition 2.1.** An $E$-valued strong solution of (1) is called a (time-inhomogeneous) diffusion process $X$ on $E$ with drift coefficient $b : [0,T] \times E \to \mathbb{R}^d$ and diffusion coefficient $a : [0,T] \times E \to \mathbb{S}_+^d$ defined by

$$a(t, x) = \sigma(t, x)\sigma(t, x)^\top,$$

for $(t, x) \in [0,T] \times E$. Moreover, we say that the drift $b : [0,T] \times E \to \mathbb{R}^d$ and diffusion coefficient $a : [0,T] \times E \to \mathbb{S}_+^d$ (shortly "coefficients") are of linear growth if there exists some $C_L > 0$ such that

$$\|b(t, x)\| + \|a(t, x)\|_F \leq C_L (1 + \|x\|),$$

for all $(t, x) \in [0,T] \times E$.

**Remark 2.2.** By [Karatzas and Shreve, 1998, Theorem 5.2.9], (1) admits a strong solution if, e.g., $b : [0,T] \times E \to \mathbb{R}^d$ and $\sigma : [0,T] \times E \to \mathbb{R}^{d \times l}$ satisfy a Lipschitz and linear growth condition, i.e. there exists a constant $C_1 > 0$ such that

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\|_F \leq C_1 \|x - y\|,$$

$$\|b(t, x)\| + \|\sigma(t, x)\|_F \leq C_1 (1 + \|x\|),$$

for all $t \in [0,T]$ and $x, y \in E$.

Moreover, (1) admits by the Yamada conditions in [Yamada and Watanabe, 1971] a strong solution (see also [Ikeda and Watanabe, 1989, Theorem IV.3.2] and [Altay and Schmock, 2016, Theorem 2.1 & Remark 2.7]) for $d, m \geq 2$ if there exists $\gamma > 0$ and $G, \kappa, \eta : [0, \gamma] \to [0, \infty)$ with $G(0) = \kappa(0) = 0$, $G : [0, \gamma] \to [0, \infty)$ non-decreasing, concave, and continuous, $\eta : [0, \gamma] \to [0, \infty)$ non-decreasing, $\eta(u) > 0$ for all $u \in (0, \gamma]$, $\int_0^\gamma \frac{1}{\eta(u)}du = \int_0^\gamma \frac{1}{G(u)}du = \infty$, and $G(u) \geq \kappa(u) + \frac{d-1}{2\gamma} \eta(u)^2$ for all $u \in (0, \gamma]$, such that for every $x \in E$ with $\|x - y\| \leq \gamma$ and $t \in [0,T]$, it holds that

$$\|b(t, x) - b(t, y)\| \leq \kappa(\|x - y\|) \quad \text{and} \quad \|\sigma(t, x) - \sigma(t, y)\|_F \leq \eta(\|x - y\|).$$

**Example 2.3** (Polynomial Diffusions). For $d = m = 1$, let $B = (B_t^1)_{t \in [0,T]}$ be a one-dimensional Brownian motion and let $X$ be an $E$-valued solution of (1) with time-homogeneous drift $b(x) = \beta_0 + \beta_1 x$ and diffusion coefficient $a(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, i.e.

$$dX_t = (\beta_0 + \beta_1 X_t)dt + \sqrt{\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2}dB_t,$$

for $t \in [0,T]$. Hereby, the state space $E \subseteq \mathbb{R}$ depends on $\beta_0, \beta_1, \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$, which we require to satisfy $\alpha_0 + \alpha_1 x + \alpha_2 x^2 \geq 0$, for all $x \in E$. For $\kappa(u) = |\beta_1| u$ and $\eta(u) = \sqrt{(|\alpha_1| + 2|\alpha_2|)u}$, with $\gamma > 0$, it follows by choosing $G(u) := \kappa(u)$ (see [Altay and Schmock, 2016, Remark 2.7]) and inserting
the inequality $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$ for any $x, y \geq 0$, that

$$|b(x) - b(y)| = |(\beta_0 + \beta_1 x) - (\beta_0 + \beta_1 y)| \leq |\beta_1| |x - y| = \kappa(|x - y|),$$

$$|\sqrt{a(x)} - \sqrt{a(y)}| = \sqrt{\alpha_0 + \alpha_1 x + \alpha_2 x^2} - \sqrt{\alpha_0 + \alpha_1 y + \alpha_2 y^2} \leq |\alpha_1| |x - y| + |\alpha_2| (x^2 - y^2) \leq \sqrt{|\alpha_1| |x - y| + 2|\alpha_2| |x - y|} = \eta(|x - y|).$$

Hence, (2) admits by Remark 2.2 a strong solution, which shows that $X$ is a diffusion process, called a polynomial diffusion in the sense of [Filipović and Larsson, 2016, Definition 2.1]. Moreover, in the case of vanishing drift (i.e. $\beta_0 = \beta_1 = 0$), there are in particular three relevant cases:

$$E = \mathbb{R} \quad \text{with } \alpha_0 > 0, \alpha_1 = 0 \text{ and } \alpha_2 \geq 0, \quad \text{e.g. Brownian motion (BM)}, \quad (3a)$$

$$E = [0, \infty) \quad \text{with } \alpha_0 = 0 \text{ and } \alpha_1, \alpha_2 \geq 0, \quad \text{e.g. geometric BM (GBM)}, \quad (3b)$$

$$E = [0, 1] \quad \text{with } \alpha_0 = 0 \text{ and } \alpha_1 = -\alpha_2, \quad \text{e.g. Jacobi process.} \quad (3c)$$

If $\alpha_2 = 0$ or $E$ is compact (e.g. (3c)), the diffusion coefficient $a$ is of linear growth. Moreover, conditional moments $\mathbb{E}[X_k^p | \mathcal{F}_s]$ and higher order correlators $\mathbb{E}[X_{t_1}^{k_1} \cdots X_{t_n}^{k_n} | \mathcal{F}_s]$ can be computed with [Filipović and Larsson, 2016, Theorem 3.1] and [Benth and Lavagnini, 2021, Theorem 4.5], respectively.

In the remaining part of this section, we prove that an $E$-valued time-inhomogeneous diffusion process with drift $b : [0, T] \times E \to \mathbb{R}^d$ and diffusion coefficient $a : [0, T] \times E \to \mathbb{S}_d^+$ of linear growth admits moments of all orders and is exponentially integrable.

**Lemma 2.4.** Let $X$ be a time-inhomogeneous diffusion process, whose coefficients are of linear growth with constant $C_L > 0$. Then, for every $n \in \mathbb{N}$ and $[4, \infty)$, it holds that

$$\sup_{t \in [0, T]} \mathbb{E}[|X_t|^n] \leq \max(\|X_0\|, n)^n e^{4nC_L \sqrt{\Delta T}}. \quad (4)$$

Moreover, for every $0 \leq \varepsilon < \exp\left(-4C_L \sqrt{\Delta T} - 1\right)$ and $t \in [0, T]$, we have that

$$\mathbb{E}[e^{\varepsilon \|X_t\|}] < \infty. \quad (5)$$

Next, we denote by $D_1([0, T]; \mathbb{R}^d)$ the vector space of left-continuous functions $g : [0, T] \to \mathbb{R}^d$ with right limits (LCRL). Moreover, for a given $E$-valued diffusion process with $E \subseteq \mathbb{R}^d$ as well as every $p \in [1, \infty)$ and Borel-measurable function $g : [0, T] \to \mathbb{R}^d$, we define

$$\|g\|_{L^p(X)} := \mathbb{E}\left[\left(\int_0^T |g(t)\top b(t, X_t)| \, dt\right)^p\right]^{\frac{1}{p}} + \mathbb{E}\left[Q(g)^{\frac{p}{2}}\right]^{\frac{1}{p}}, \quad (6)$$

where $Q(g) := \int_0^T g(t)\top a(t, X_t) g(t) \, dt$ is defined below. Hence, by using [Fraňková, 1991, Lemma 1.7] that every LCRL function in $D_1([0, T]; \mathbb{R}^d)$ is bounded, we can show that its $\|\cdot\|_{L^p(X)}$-norm is finite.

**Lemma 2.5.** Let $X$ be a diffusion process, whose coefficients are of linear growth, and let $p \in [1, \infty)$. Then, it holds for every $g \in D_1([0, T]; \mathbb{R}^d)$ that $\|g\|_{L^p(X)} < \infty$.

On $D_1([0, T]; \mathbb{R}^d)$, we define the equivalence relation “$\sim$” by $f \sim g$ if and only if $\|f - g\|_{L^p(X)} = 0$, for any $f, g \in D_1([0, T]; \mathbb{R}^d)$. Moreover, by a slight abuse of notation, we sometimes write $g \in L^p(X)$ as function, which in fact refers to the equivalence class $[g]_\sim := \{f \in D_1([0, T]; \mathbb{R}^d) : f \sim g\}$. Then, we introduce the following quotient space of deterministic LCRL integrands.

**Definition 2.6.** For a diffusion process $X$, whose coefficients are of linear growth, and $p \in [1, \infty)$, we define $L^p(X)$ as the quotient space $L^p(X) := D_1([0, T]; \mathbb{R}^d) / \sim$ equipped with $\|\cdot\|_{L^p(X)}$ defined in (6).

Next, we show that $(L^p(X), \|\cdot\|_{L^p(X)})$ is in fact an (incomplete) normed vector space, on which we define the following operators returning the stochastic integral and its quadratic variation.

**Lemma 2.7.** Let $X$ be a diffusion process, whose coefficients are of linear growth, and let $p \in [1, \infty)$. Then, $(L^p(X), \|\cdot\|_{L^p(X)})$ is a normed vector space.
Definition 2.8. For $p \in [1, \infty)$, we introduce the operators $W : L^p(X) \to L^p(\mathbb{P})$ and $Q : L^p(X) \to L^2(\mathbb{P})$ defined by

$$L^p(X) \ni g \quad \mapsto \quad W(g) := \sum_{i=1}^d \int_0^T g_i(t)dX^i_t \in L^p(\mathbb{P}),$$

$$L^p(X) \ni g \quad \mapsto \quad Q(g) := \int_0^T g(t)^\top a(t, X_t)g(t)dt \in L^2(\mathbb{P}),$$

which return the stochastic integral and its quadratic variation of a function $g \in L^p(X)$.

Remark 2.9. While we show in Lemma 2.10 that the operators $W : L^p(X) \to L^p(\mathbb{P})$ and $Q : L^p(X) \to L^2(\mathbb{P})$ are well-defined and bounded, we first observe that $t \mapsto W(g)_t := \sum_{i=1}^d \int_0^t g_i(s)dX^i_s$ is well-defined as stochastic integral. Indeed, the drift part $t \mapsto \int_0^t g(s)^\top b(s, X_s)ds$ is well-defined as Lebesgue integral. On the other hand, since every $g \in L^p(X)$ is by definition left-continuous, and $t \mapsto \sigma(t, X_t)$ is itself continuous, the integrand $t \mapsto g(t)^\top \sigma(t, X_t)$ is left-continuous and $\mathbb{F}$-adapted, thus $\mathbb{F}$-predictable and locally bounded. Hence, $t \mapsto \int_0^t g(s)^\top \sigma(s, X_s)dB_s$ is well-defined as stochastic integral, which shows that the process $t \mapsto W(g)_t$ is also well-defined.

Moreover, for every $p \in [1, \infty)$ and $g \in L^p(X)$, we introduce the processes $t \mapsto W(g)_t := \sum_{i=1}^d \int_0^t g_i(s)dX^i_s$ and $t \mapsto Q(g)_t := \int_0^t g(s)^\top a(s, X_s)g(s)ds$, which represent the stochastic integral and its quadratic variation at each time $t \in [0, T]$.

Lemma 2.10. Let $X$ be a diffusion process and let $p \in [1, \infty)$. Then, there exists a constant $C_{1,p} > 0$ such that for every $g \in L^p(X)$, it holds that

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |W(g)_t|^p \right]^{\frac{1}{p}} \leq C_{1,p} ||g||_{L^p(X)},$$

(7a)

$$\mathbb{E} \left[ \sup_{t \in [0,T]} Q(g)_t^p \right]^{\frac{1}{p}} \leq ||g||_{L^p(X)},$$

(7b)

with $C_{1,p} := \max(1, C_p) > 0$, where $C_p > 0$ denotes the constant of the classical upper Burkholder-Davis-Gundy inequality with exponent $p \in [1, \infty)$.

In the following, we now use the operators $W : L^p(X) \to L^p(\mathbb{P})$ and $Q : L^p(X) \to L^2(\mathbb{P})$ returning the stochastic integral and its quadratic variation to obtain the chaos expansion.

2.2. Abstract Chaos Expansion. In this section, we introduce the time-space harmonic Hermite polynomials, which are obtained from the classical (probabilist’s) Hermite polynomials $(h_n)_{n \in \mathbb{N}_0}$ defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \left( e^{-\frac{1}{2}x^2} \right),$$

(8)

for $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$ (see [Abramowitz and Stegun, 1970, Section 22] for more details).

Definition 2.11. The time-space harmonic Hermite polynomials $(H_n)_{n \in \mathbb{N}_0}$ are defined by

$$H_n(x, t) = t^{\frac{n}{2}} h_n \left( \frac{x}{\sqrt{t}} \right),$$

for $n \in \mathbb{N}_0$ and $(x, t) \in \mathbb{R} \times [0, \infty)$.

Example 2.12. The first Hermite polynomials and time-space harmonic Hermite polynomials up to order four are given by $h_0(x) = H_0(x, t) = 1$ and

$$h_1(x) = x, \quad h_2(x) = x^2 - 1, \quad h_3(x) = x^3 - 3x, \quad h_4(x) = x^4 - 6x^2 + 3,$$

$$H_1(x, t) = x, \quad H_2(x, t) = x^2 - t, \quad H_3(x, t) = x^3 - 3xt, \quad H_4(x, t) = x^4 - 6x^2t + 3t^2,$$

for $(x, t) \in \mathbb{R} \times [0, \infty)$. 

Remark 2.13. In the usual setting of Malliavin calculus introduced in [Malliavin, 1978] with \( X \) being a Brownian motion, the linear operator \( W : L^2(X) \rightarrow L^2(\mathbb{P}) \) is defined as isonormal process such that \( W(g) \) is a Gaussian random variable, for all \( g \in L^2(X) \) (see [Nualart, 2006, Definition 1.1.1]). Then, [Nualart, 2006, Lemma 1.1.1] shows the orthogonality relation
\[
\mathbb{E} [H_n(W(f), Q(f)) H_m(W(g), Q(g))] = \begin{cases} \frac{1}{n!} (f; g)^n_{L^2(X)} & n = m, \\ 0 & n \neq m. \end{cases}
\]
This relation still holds true for martingales with deterministic quadratic variation, which are by [Dambis, 1965] and [Dubins and Schwarz, 1965] deterministic time changes of Brownian motion. However, as soon as the quadratic variation of \( X \) is stochastic or any additional drift is added, the random variables \( (H_n(W(g), Q(g)))_{n \in \mathbb{N}_0} \) might no longer be orthogonal in \( L^2(\mathbb{P}) \).

Now, we define the chaotic subspaces \( \mathcal{H}_n \) of \( L^p(\mathbb{P}) \), which are the generalization of the orthogonal chaotic subspaces in classical Malliavin calculus used in [Nualart, 2006, Theorem 1.1.1].

Definition 2.14. For every \( n \in \mathbb{N}_0 \), the \( n \)-th chaotic subspace \( \mathcal{H}_n \subset L^p(\mathbb{P}) \) is defined as
\[
\mathcal{H}_0 = \mathbb{R}, \quad \text{and} \quad \mathcal{H}_n = \text{span} \{ H_n(W(g), Q(g)) : g \in L^{np}(X) \}, \quad \text{for } n \geq 1,
\]
where \( (H_n)_{n \in \mathbb{N}_0} \) are the time-space harmonic Hermite polynomials from Definition 2.11.

Next, we show that every chaotic subspace \( \mathcal{H}_n, n \in \mathbb{N}_0, \) is well-defined as vector subspace of \( L^p(\mathbb{P}) \).

Lemma 2.15. For every \( p \in [1, \infty), n \in \mathbb{N}_0, \) and \( g \in L^{np}(X) \), it holds that \( \mathcal{H}_n \subset L^p(\mathbb{P}) \).

Moreover, we denote by “\( \oplus \)” the direct sum, which is the sum over finite tuples defined by
\[
\bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n := \left\{ \sum_{n \in \mathbb{N}} H_n : H_n \in \mathcal{H}_n, N \subset \mathbb{N} \text{ finite} \right\}.
\]
In the classical Wiener-Ito chaos decomposition with \( p = 2 \), the orthogonal sum is used instead of the direct sum because the random variables \( H_n(W(g), Q(g)) \) are orthogonal in \( L^2(\mathbb{P}) \), see also Remark 2.13. However, as soon as \( \langle X \rangle \) becomes stochastic or any drift is added, the chaotic subspaces will no longer be orthogonal to each other in \( L^2(\mathbb{P}) \).

Theorem 2.16 (Chaos Expansion). Let \( X \) be a time-inhomogeneous diffusion process, whose coefficients are of linear growth, and let \( p \in [1, \infty) \). Then, \( \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n \) is dense in \( L^p(\mathbb{P}) \).

The chaos expansion in Theorem 2.16 provides the theoretical foundation for the following machine learning application, where we learn a financial derivative \( G \in L^p(\mathbb{P}) \) and its hedging strategy.

2.3. Chaos Expansion with Iterated Integrals. In this section, we introduce iterated integrals of a diffusion process \( X \), whose drift and diffusion coefficient is of linear growth. By using the time-space harmonic Hermite polynomials, we obtain the chaos expansion with respect to iterated integrals.

For every \( n \in \mathbb{N} \), we define \( (\mathbb{R}^d)^{\otimes n} := \mathbb{R}^d \otimes \cdots \otimes \mathbb{R}^d \), with \( (\mathbb{R}^d)^{\otimes 0} := \mathbb{R} \). Moreover, for every \( p \in [1, \infty) \), let \( L^{np}(X)^{\otimes n} := L^{np}(X)^\otimes \cdots \otimes L^{np}(X) \) be the vector space of linear functionals acting on multilinear forms \( L^{np}(X) \times \cdots \times L^{np}(X) \rightarrow \mathbb{R} \), i.e. we define the linear functional \( g_1 \otimes \cdots \otimes g_n \in L^{np}(X)^{\otimes n} \) by
\[
(g_1 \otimes \cdots \otimes g_n)(A) := A(g_1, \ldots, g_n),
\]
for each multilinear form \( A : L^{np}(X) \times \cdots \times L^{np}(X) \rightarrow \mathbb{R} \). Hence, we can express every tensor \( g \in L^{np}(X)^{\otimes n} \) as \( g = \sum_{j=1}^m \lambda_j g_{j,1} \otimes \cdots \otimes g_{j,n} \), for some \( m \in \mathbb{N} \), \( \lambda_j \in \mathbb{R}, g_{j,1}, \ldots, g_{j,n} \in L^{np}(X), j = 1, \ldots, m \), where the representation might not be unique. From this, we define the linear functional \( g = \sum_{j=1}^m \lambda_j g_{j,1} \otimes \cdots \otimes g_{j,n} \in L^{np}(X)^{\otimes n} \) by
\[
g(A) = \sum_{j=1}^m \lambda_j A(g_{j,1}, \ldots, g_{j,n}),
\]
for each multilinear form \( A : L^{np}(X) \times \cdots \times L^{np}(X) \rightarrow \mathbb{R} \), where the value of the sum in (10) is independent of the representation of \( g \in L^{np}(X)^{\otimes n} \), see also [Ryan, 2002, Chapter 1]. On the vector
Moreover, it holds for every Lemma 2.17.

The operator Lemma 2.21.

upper Burkholder-Davis-Gundy inequality with exponent r with Lemma 2.22.

integrals of a semimartingale, which is similar to [Carlen and Kree, 1991, Theorem 1].

representations of such that J

For \( J \) defined by

\[ J_n(g) := \sum_{j=1}^m \lambda_j J_n(g_{j,1} \otimes \cdots \otimes g_{j,n}), \quad \text{for } g = \sum_{j=1}^m \lambda_j g_{j,1} \otimes \cdots \otimes g_{j,n} \in L^{np}(X)^{\otimes n}. \]

While we prove in Lemma 2.21 and Lemma 2.22 that \( J_n \) does not depend on the representation of \( g \in L^{np}(X)^{\otimes n} \), we first that \( J_n(g) \) is well-defined as iterated stochastic integrals.

Remark 2.20. For \( g_{j,1} = (g_{j,1,1},...,g_{j,1,d})^\top \in L^{np}(X) \), we introduce the stochastic process \( t_2 \mapsto J_1(g_{j,1})_{t_2} \) defined by

\[ [0,T] \ni t_2 \mapsto J_1(g_{j,1})_{t_2} := W(g_{j,1})_{t_2} = \sum_{i_1=1}^d \int_0^{t_2} g_{j,1,i_1}(t_1) dX_{i_1}^{t_1}, \]

such that \( J_1(g_{j,1})_T = J_1(g_{j,1})_0 = W(g_{j,1})_0 \), where \( t_2 \mapsto J_1(g_{j,1})_{t_2} \) is by Remark 2.9 well-defined as stochastic integral. By abbreviating \( g_{j,1:k} := g_{j,1} \otimes \cdots \otimes g_{j,k} \in L^{np}(X)^{\otimes k} \), we iteratively define for every \( k = 2,...,n-1 \) the process \( t_{k+1} \mapsto J_k(g_{j,1:k})_{t_{k+1}} \) by

\[ [0,T] \ni t_{k+1} \mapsto J_k(g_{j,1:k})_{t_{k+1}} := \sum_{i_k=1}^d \int_{t_k}^{t_{k+1}} J_{k-1}(g_{j,1:(k-1)})_{t_k} g_{j,k,i_k}(t_k) dX_{i_k}^{t_k}, \]

such that \( J_k(g_{j,1:k})_T = J_k(g_{j,1:k})_{t_1} \). Firstly, \( t_{k+1} \mapsto J_k(g_{j,1:k})_{t_{k+1}} \) is well-defined as stochastic integrals of \( t_k \mapsto \theta_{t_k}^{i_k} := J_{k-1}(g_{j,1:(k-1)})_{t_k} g_{j,k,i_k}(t_k) \) with respect to \( dX_{i_k}^{t_k} \), for \( i_k = 1,...,d \), because every integrand \( t_k \mapsto \theta_{t_k}^{i_k} \) is left-continuous and \( \mathbb{F} \)-adapted, thus \( \mathbb{F} \)-predictable and locally bounded.

Lemma 2.21. The operator \( J_n : L^{np}(X)^{\otimes n} \to L^0(\mathbb{P}) \) is linear. Moreover, for every \( g \in L^{np}(X)^{\otimes n} \) with representations \( g^{(1)} = \sum_{j=1}^{m_1} \lambda_j^{(1)} g_{j,1}^{(1)} \otimes \cdots \otimes g_{j,n}^{(1)} \in L^{np}(X)^{\otimes n} \) and \( g^{(2)} = \sum_{j=1}^{m_2} \lambda_j^{(2)} g_{j,1}^{(2)} \otimes \cdots \otimes g_{j,n}^{(2)} \in L^{np}(X)^{\otimes n} \), it holds that \( J_n(g^{(1)}) = J_n(g^{(2)}) \), \( \mathbb{P} \)-a.s.

Next, we show that the linear operator \( J_n : L^{np}(X)^{\otimes n} \to L^p(\mathbb{P}) \) is well-defined and bounded, thus continuous. For this purpose, we derive an upper Burkholder-Davis-Gundy type of inequality for iterated integrals of a semimartingale, which is similar to [Carlen and Kree, 1991, Theorem 1].

Lemma 2.22. Let \( X \) be a diffusion process, whose coefficients are of linear growth, let \( p \in [1,\infty) \) and \( n \in \mathbb{N} \). Then, there exists a constant \( C_{n,p} > 0 \) such that for every \( g \in L^{np}(X)^{\otimes n} \), it holds that

\[
E \left[ \sup_{t \in [0,T]} |J_n(g)(t)|^p \right] \leq C_{n,p} \|g\|_{L^{np}(X)^{\otimes n}},
\]

with \( C_{n,p} := \prod_{k=1}^n \max \left( 1, \frac{C_{r,p}}{r} \right) > 0 \), where \( C_r > 0 \), \( r \in [1,\infty) \), denotes the constant of the classical upper Burkholder-Davis-Gundy inequality with exponent \( r \in [1,\infty) \).


Next, we consider for every $p \in [1, \infty)$ and $n \in \mathbb{N}$ the vector subspace of diagonal tensors in $L_{\text{diag}}^p(X)^{\otimes n}$. For this purpose, we define the vector subspace

$$L_{\text{diag}}^p(X)^{\otimes n} := \text{span}\{g_0^{\otimes n} := g_0 \otimes \cdots \otimes g_0 : g_0 \in L_{\text{diag}}^p(X)\} \subseteq L_{\text{diag}}^p(X)^{\otimes n},$$

and show that the iterated integral of such a diagonal tensor can be expressed with the time-space harmonic Hermite polynomials in Definition 2.11. This relies on the Kailath-Segall formula, which was shown for $L^2(\mathbb{P})$-integrable martingales in [Segall and Kailath, 1976], but still holds true for semimartingales.

Lemma 2.23 (Kailath-Segall). Let $X$ be a diffusion process, whose coefficients are of linear growth, let $p \in [1, \infty)$ and $n \in \mathbb{N}$. Then, it follows for every $g_0 \in L_{\text{diag}}^p(X)$ and $t \in [0, T]$ that

$$J_n\left(g_0^{\otimes n}\right)_t = \frac{1}{n!}H_n(W(g_0)_t, Q(g_0)_t), \quad \mathbb{P}\text{-a.s.},$$

where $(H_n)_{n \in \mathbb{N}_0}$ denote the time-space harmonic Hermite polynomials.

Remark 2.24. If $p = 2$ and $X$ is a martingale with deterministic $(X)$, Lemma 2.23 can be generalized to non-diagonal tensors $g := g_1^{\otimes n} \times \cdots \times g_m^{\otimes n} : [0, T]^n \rightarrow (\mathbb{R}^d)^{\otimes n}$, where $\alpha_1 + \cdots + \alpha_m = n$ and $g_1, \ldots, g_m \in L^2(X)$, with $(g_k, g_l)_{L^2(X)} = 1$ if $k = l$, and $(g_k, g_l)_{L^2(X)} = 0$ if $k \neq l$. In this case, it follows from [Ito, 1951, Theorem 3.1] that $J_n(g) = \frac{1}{n!} \prod_{k=1}^m H_{\alpha_k}(W(g_k), Q(g_k))$.

Finally, we prove the following chaos expansion for financial derivatives, which is obtained from the abstract chaos theorem in Theorem 2.16 together with the Kailath-Segall formula in Lemma 2.23.

Theorem 2.25 (Chaos Expansion). Let $X$ be a diffusion process, whose coefficients are of linear growth, and let $G \in L^p(\mathbb{P})$ for some $p \in [1, \infty)$. Then, for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ and a sequence of functions $(g_n)_{n=0,\ldots,N}$, with $g_n \in L_{\text{diag}}^p(X)^{\otimes n}$ for all $n = 0, \ldots, N$, such that

$$\left\| G - \sum_{n=0}^N J_n(g_n) \right\|_{L^p(\mathbb{P})} < \varepsilon.$$  \hspace{1cm} (12)

2.4. (Semi-)Martingale Representation. By using the chaos expansion in Theorem 2.25, we derive the following (semi-) martingale representation theorem. For every $p \in [1, \infty)$, we introduce the space $\Theta_p$ consisting of $d$-dimensional $\mathbb{P}$-predictable stochastic processes $\theta := (\theta_1^T, \ldots, \theta_d^T)_{t \in [0, T]}$ satisfying

$$\|\theta\|_{\Theta_p} := \mathbb{E}\left[\int_0^T \theta_1^T b(t, X_t) \, dt\right]^p + \mathbb{E}\left[\int_0^T \theta_1^T a(t, X_t) \theta_1 \, dt\right]^\frac{p}{2} < \infty.$$  \hspace{1cm} (13)

Then, we observe that $\int_0^T \theta_1^T a(t, X_t) \theta_1 \, dt < \infty$ holds $\mathbb{P}$-a.s. true, which implies that $\int_0^T \theta_1^T \, dX_t$ is well-defined, for all $\theta \in \Theta_p$. Moreover, we define the vector subspace

$$\mathcal{G}_p := \left\{ \int_0^T \theta_1^T \, dX_t : \theta \in \Theta_p \right\} \subseteq L^p(\mathbb{P}).$$

Moreover, we call an equivalent probability measure $Q \sim \mathbb{P}$ an equivalent local martingale measure (ELMM) for $X$ if $X$ is a local martingale under $Q$. Then, by assuming the existence of such an ELMM $Q \sim \mathbb{P}$, we first prove a representation with respect to the semimartingale $X$ under $\mathbb{P}$.

Lemma 2.26 (Grandits and Krawczyk, 1998, Lemma 4.5). Let $X$ be a diffusion process, let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and assume that there exists an ELMM $Q \sim \mathbb{P}$ for $X$ with density $\frac{dQ}{d\mathbb{P}} \in L^q(\mathbb{P})$. Then, $\mathcal{G}_p$ is closed in $L^p(\mathbb{P})$ if and only if there exists some $C > 0$ such that for every $\theta \in \Theta_p$, it holds that

$$\|\theta\|_{\Theta_p} \leq C \left\| \int_0^T \theta_1^T \, dX_t \right\|_{L^p(\mathbb{P})}.$$  \hspace{1cm} (13)

Theorem 2.27 (Semimartingale Representation). Let $X$ be a diffusion process, whose coefficients are of linear growth, and let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, assume that there exists an ELMM $Q \sim \mathbb{P}$ for $X$ with density $\frac{dQ}{d\mathbb{P}} \in L^q(\mathbb{P})$, and suppose that (13) holds true. Then, for every $G \in L^p(\mathbb{P})$ there exists a $d$-dimensional and $\mathbb{P}$-predictable process $\theta \in \Theta_p$ such that

$$G = \mathbb{E}^Q[G] + \int_0^T \theta_1^T \, dX_t, \quad \mathbb{P}\text{-a.s.}$$
9. **Theorem 2.27** yields sufficient conditions for the representation of a random variable $G \in L^p(\mathbb{P})$, for $p \in [1, \infty)$, with respect to the semimartingale diffusion process, which relies on the $L^p$-closedness of $\mathcal{G}_p$, see [Grandits and Krawczyk, 1998] and [Delbaen et al., 1997] for more details.

Moreover, if $X$ is a local martingale diffusion, i.e. a diffusion process with $b(t, x) = 0$, for all $(t, x) \in [0, T] \times E$, we obtain the following martingale representation result.

**Corollary 2.28** (Martingale Representation). Let $X$ be a local martingale diffusion, whose diffusion coefficient $a : [0, T] \times E \to \mathbb{R}^d$ is of linear growth, and let $p \in (1, \infty)$. Then, for every $G \in L^p(\mathbb{P})$ there exists a $d$-dimensional and $\mathbb{P}$-predictable process $\theta \in \Theta_p$ such that

$$G = \mathbb{E}[G] + \int_0^T \theta^\top_t dX_t, \quad \mathbb{P}\text{-a.s.}$$

**Remark 2.29.** The classical martingale representation expresses an $L^2(\mathbb{P})$-integrable random variable in terms of a stochastic integral with respect to Brownian motion, where the integrand $\theta = (\theta_1^t, \ldots, \theta_d^t)_{t \in [0,T]}$ is unique (see [Karatzas and Shreve, 1998, Problem 3.4.17]). This result was extended in [Dudley, 1977, Abstract] to any $\mathcal{F}_T$-measurable random variable without integrability condition, where, however, $\theta$ is not unique anymore. Hence, Corollary 2.28 can be seen as generalization of [Dudley, 1977, Abstract] to diffusion processes with coefficients of linear growth. Moreover, [Soner et al., 2011, Theorem 6.5] derived a similar result when $a(t, X_t)$ is $\mathbb{P}$-a.s. invertible, for all $t \in [0, T]$.

2.5. **Approximation of Financial Derivatives by Neural Networks.** In this section, we approximate a financial derivative $G \in L^p(\mathbb{P})$ with the help of the chaos expansion in Theorem 2.25. More precisely, we first approximate $G \in L^p(\mathbb{P})$ by a truncated sum of iterated integrals, and then replace the integrands of the iterated integrals by tensor-valued neural networks, to obtain a parametric family of random variables. The numerical method returns the hedging strategy of the learned option in closed form.

For a general introduction to neural networks defined on Euclidean spaces and their universal approx-
imation property, we refer to [Cybenko, 1989], [Hornik, 1991], and [Pinkus, 1999]. Moreover, results about approximation rates of neural networks can be found in [Bölskei et al., 2019].

**Definition 2.30.** For $n \in \mathbb{N}$ and $p \in [1, \infty)$, a diagonal tensor $\varphi = \sum_{j=1}^m w_j \varphi^{\otimes n}_j \in L^p_{\text{diag}}(X)^{\otimes n}$ is called an $(\mathbb{R}^d)^{\otimes n}$-valued (resp. tensor-valued) neural network if for every $j = 1, \ldots, m$ it holds that

$$[0, T] \ni t \mapsto \varphi_j(t) = (\varphi_{j,1}(t), \ldots, \varphi_{j,p}(t))^\top := (\rho(a_{j,1}t + b_{j,1}), \ldots, \rho(a_{j,p}t + b_{j,p}))^\top \in \mathbb{R}^d,$$

for some $a_j = (a_{j,1}, \ldots, a_{j,d})^\top \in \mathbb{R}^d$, $b_j = (b_{j,1}, \ldots, b_{j,d})^\top \in \mathbb{R}^d$, and $w_j \in \mathbb{R}$. In this case, the vectors $a_{j,1}, \ldots, a_{j,d} \in \mathbb{R}^d$ and $b_{j,1}, \ldots, b_{j,d} \in \mathbb{R}^d$ are called the weights and biases, respectively, whereas the numbers $w_1, \ldots, w_m \in \mathbb{R}$ are called the linear readouts. Moreover, $\rho : \mathbb{R} \to \mathbb{R}$ is called an activation function.

**Definition 2.31.** For $n \in \mathbb{N}$ and $\rho \in C(\mathbb{R})$, we define $NN_{n,d}^p$ as the set of all $(\mathbb{R}^d)^{\otimes n}$-valued neural networks, whereas for $n = 0$, we set $NN_{0,d}^p := \mathbb{R}$.

**Remark 2.32.** In order that every $(\mathbb{R}^d)^{\otimes n}$-valued neural network $\varphi \in NN_{n,d}^p$ is well-defined in the vector subspace $L^p_{\text{diag}}(X)^{\otimes n}$, we need the activation function $\rho : \mathbb{R} \to \mathbb{R}$ to be continuous. Then, every function $\varphi_j : [0, T] \to \mathbb{R}^d$ in (15) is continuous, thus in $D_t([0, T]; \mathbb{R}^d)$, and in $L^p(X)$ by Lemma 2.5. Hence, $\varphi = \sum_{j=1}^m \varphi^{\otimes n}_j \in L^p_{\text{diag}}(X)^{\otimes n}$, which shows that $NN_{n,d}^p \subset L^p_{\text{diag}}(X)^{\otimes n}$ for all $p \in [1, \infty)$.

For the approximation of a financial derivative $G \in L^p(\mathbb{P})$, we use finitely many iterated integrals and replace the integrands by tensor-valued neural networks.

**Definition 2.33.** Let $\varphi_{0:N} := (\varphi_0, \ldots, \varphi_N) \in \times_{n=0}^N NN_{n,d}^p$ be a tuple of tensor-valued neural networks $\varphi_n \in NN_{n,d}^p$ for all $n = 0, \ldots, N$. Then, we define the financial derivative generated by $\varphi_{0:N}$ as

$$G^{\varphi_{0:N}} := \sum_{n=0}^N J_n(\varphi_n) \in L^p(\mathbb{P}).$$

**Remark 2.34.** Since every $\varphi_n \in NN_{n,d}^p$, $n = 1, \ldots, N$, is by Remark 2.32 contained in $L^p_{\text{diag}}(X)^{\otimes n}$, Lemma 2.22 implies that $J_n(\varphi_n) \in L^p(\mathbb{P})$. Hence, by Minkowski’s inequality, we have $G^{\varphi_{0:N}} \in L^p(\mathbb{P})$.

Now, we use the properties of neural networks to approximate any tensor in $L^p(\mathbb{P})$, for $p \in [1, \infty)$ and $n \in \mathbb{N}$. For this purpose, we first introduce a property related to the activation function.
Definition 2.35. A function \( \rho \in C(\mathbb{R}) \) is called activating if \( \mathcal{NN}_{1,1}^{n} \) is dense in \( C([0, T]) \) with respect to the uniform topology.

Remark 2.36. A function \( \rho \in C(\mathbb{R}) \) is activating if it satisfies one of the following sufficient conditions:

(i) bounded and sigmoidal (see [Cybenko, 1989, Theorem 1 & Lemma 1]),
(ii) discriminatory, i.e. if for every finite signed Radon measure \( \mu : B([0, T]) \to \mathbb{R} \) satisfying
\[
\int_{0}^{T} \rho(at + b) d\mu(t) = 0, \text{ for all } a, b \in \mathbb{R}, \text{ it follows that } \mu = 0 \text{ (see [Cybenko, 1989, Theorem 1])},
\]
(iii) bounded and non-constant (see [Hornik, 1991, Theorem 2]), or
(iv) non-polynomial (see [Pinkus, 1999, Theorem 3.11]).

Now, we use this property of the activation function to lift the universal approximation property of classical neural networks to the normed vector space \( L^{np}(X)^{\otimes n}, \| \cdot \|_{L^{np}(X)^{\otimes n}} \).

Proposition 2.37. Let \( X \) be a diffusion process, whose coefficients are of linear growth, and let \( p \in [1, \infty) \) and \( n \in \mathbb{N} \). If \( \rho \in C(\mathbb{R}) \) is activating, then \( \mathcal{NN}_{n,d}^{p} \) is dense in \( L^{np}(X)^{\otimes n}, \text{ i.e. for every } g \in L^{np}(X)^{\otimes n} \) and \( \varepsilon > 0 \) there exists some \( \varphi \in \mathcal{NN}_{n,d}^{p} \) such that \( \| g - \varphi \|_{L^{np}(X)^{\otimes n}} < \varepsilon \).

This version of the universal approximation property of neural networks presented in Proposition 2.37 allows us to get an approximation result for financial derivatives \( G \in L^{p}(\mathbb{P}) \), where \( p \in [1, \infty) \).

Theorem 2.38 (Universal Approximation). Let \( X \) be a diffusion process, whose coefficients are of linear growth, and let \( G \in L^{p}(\mathbb{P}) \) for some \( p \in [1, \infty) \). If \( \rho \in C(\mathbb{R}) \) is activating, then for every \( \varepsilon > 0 \) there exists some \( N \in \mathbb{N} \) and a tuple \( \varphi_{N} = (\varphi_{0}, \ldots, \varphi_{N}) \in X_{N}^{\otimes n} \mathcal{NN}_{n,d}^{p} \) such that \( G_{\rho_{N}} := \sum_{n=0}^{N} J_{n}(\varphi_{n}) \in L^{p}(\mathbb{P}) \) satisfies \( \| G - G_{\rho_{N}} \|_{L^{p}(\mathbb{P})} < \varepsilon \).

Next, we show that the weights and biases of the tensor-valued neural networks in the approximating derivative can be initialized randomly and only the linear readouts need to be trained to obtain a similar approximation result as in Theorem 2.38. This generalizes random neural networks (see e.g. [Rahimi and Recht, 2007], [Gonon et al., 2022], [Kratsios and Bilokopytov, 2020], and [Schmocker, 2022]) and non-temporal reservoir computing (see e.g. [Grigoryeva and Ortega, 2018]).

For this purpose, we fix an activation function \( \rho \in C(\mathbb{R}) \) and two real-valued random variables \( Y \) and \( Z \) defined on another probability space \( (\Omega', \mathcal{A}', \mathbb{P}') \) with joint density function \( f_{Y,Z} : \mathbb{R}^{2} \to [0, \infty) \) being continuous and satisfying \( f_{Y,Z}(y, z) > 0 \) for all \( (y, z) \in \mathbb{R}^{2} \), as well as for every \( p \in [1, \infty) \) that
\[
\mathbb{E}' \left[ \| \rho(Y \cdot + Z) \|_{L^{p}(d\mu)}^{p} \right] := \int_{\mathbb{R}^{2}} \left( \int_{0}^{T} |\rho(yt + z)|^{p} dt \right)^{\frac{1}{p}} f_{Y,Z}(y, z) dy dz < \infty. \tag{16}
\]

Example 2.39. The condition (16) is satisfied for example if \( \rho \in C(\mathbb{R}) \) is bounded (e.g. sigmoid function \( \rho(x) = \frac{1}{1 + \exp(-x)} \)) or if there exist some \( C, r > 0 \) such that \( |\rho'(x)| \leq C (1 + |x|)^{r} \), for all \( x \in \mathbb{R} \), and \( Y \) and \( Z \) have moments of all orders (e.g. ReLU function \( \rho(x) = \max(x, 0) \) and \( Y, Z \sim N(0, 1) \)).

Now, for every \( n \in \mathbb{N} \), the map \( \Omega' \ni \omega' \mapsto \varphi(\omega') := \sum_{j=1}^{m} w_{j}(\omega') \varphi_{j}(\omega') \in L^{np}(X)^{\otimes n} \) is called a random tensor-valued neural network if for every \( \omega' \in \Omega' \) and \( j = 1, \ldots, m \), it holds that
\[
[0, T] \ni t \mapsto \varphi_{j}(\omega')(t) = (\rho(a_{j,1}(\omega')t + b_{j,1}(\omega'), \ldots, \rho(a_{j,d}(\omega')t + b_{j,d}(\omega')))^{T} \in \mathbb{R}^{d},
\]
where \( (a_{j,i})_{i} \in \mathbb{N} ; i = 1, \ldots, d \sim Y \) and \( (b_{j,i})_{i} \in \mathbb{N} ; i = 1, \ldots, d \sim Z \) are independent and identically distributed (i.i.d.) random variables, and where the map \( \Omega' \ni \omega' \mapsto w_{j}(\omega') := \bar{w}_{j} \mathbb{1} A_{j}'(\omega') \in \mathbb{R} \) consists of some \( \bar{w}_{j} \in \mathbb{R} \) and an event \( A_{j}' \in \sigma \{ (a_{j,0}, b_{j,0} : j_{0} \in \mathbb{N}, i = 1, \ldots, d) \} \), for all \( j = 1, \ldots, m \).

We denote by \( \mathcal{RN}_{n,d}^{p} \) the set of random \( (\mathbb{R}^{d})^{\otimes n} \)-valued neural networks, with \( \mathcal{RN}_{0,d}^{p} := \mathbb{R} \), and show that \( \mathcal{RN}_{n,d}^{p} \) is contained in every Bochner space \( L^{r}(\Omega'; L^{np}(X)^{\otimes n}) \) (see [Hytönen et al., 2016, Definition 1.2.15]), for all \( r \in [1, \infty) \), where \( L^{np}(X)^{\otimes n} \) is the \( \| \cdot \|_{L^{np}(X)^{\otimes n}} \)-completion of \( L^{np}(X)^{\otimes n} \).

Proposition 2.40. For every \( p \in [1, \infty) \), \( n \in \mathbb{N} \), and \( r \in [1, \infty) \), we have \( \mathcal{RN}_{n,d}^{p} \subset L^{r}(\Omega'; L^{np}(X)^{\otimes n}) \).

By the strong law of large numbers for Banach space-valued random variables in [Hytönen et al., 2016, Theorem 3.3.10], we obtain the following universal approximation result for random neural networks.

Proposition 2.41. Let \( X \) be a diffusion process, whose coefficients are of linear growth, and let \( g \in L^{np}(X)^{\otimes n} \) with \( p \in [1, \infty) \) and \( n \in \mathbb{N} \). If \( \rho \in C(\mathbb{R}) \) is activating, then for every \( r \in [1, \infty) \) and \( \varepsilon > 0 \) there exists some \( \varphi \in \mathcal{RN}_{n,d}^{p} \) such that \( \mathbb{E} \left[ \| g - \varphi \|_{L^{np}(X)^{\otimes n}}^{r} \right] < \varepsilon \).
As a consequence of Proposition 2.41, we obtain the following approximation result for financial derivatives \( G \in L^p(\mathbb{P}) \), with \( p \in [1, \infty) \), which is similar to Theorem 2.38 but now with random neural networks instead of classical neural networks.

**Corollary 2.42 (Random Universal Approximation).** Let \( X \) be a diffusion process, whose coefficients are of linear growth, and let \( G \in L^p(\mathbb{P}) \) for some \( p \in [1, \infty) \). If \( \rho \in C(\mathbb{R}) \) is activating, then for every \( r \in [1, \infty) \) and \( \varepsilon > 0 \) there exists some \( N \in \mathbb{N} \) and \( \varphi_{0:N} := (\varphi_0, \ldots, \varphi_N) \in \mathbb{X}_{n=0}^{N} \mathcal{N}_{n,d}^\rho \) such that

\[
(\omega' \mapsto G^{\rho_{0:N}}(\omega') := \sum_{n=0}^{N} J_n (\varphi_n(\omega'))) \in L^r(\Omega' ; L^p(\mathbb{P})) \text{ satisfies } \mathbb{E}^\prime \left[ \left\| G - G^{\rho_{0:N}} \right\|_{L^p(\mathbb{P})}^r \right] < \varepsilon.
\]

Next, we show that the hedging strategy of a financial derivative \( G^{\rho_{0:N}} \in L^p(\mathbb{P}) \) generated by tensor-valued neural networks \( \varphi_{0:N} \in \mathbb{X}_{n=0}^{N} \mathcal{N}_{n,d}^\rho \) can be computed in closed form. In case of random neural networks, the result can still be applied because \( \varphi(\omega') \in \mathcal{N}_{n,d}^\rho \) for \( \varphi \in \mathcal{R} \) and \( \omega' \in \Omega' \).

**Theorem 2.43 (Hedging Strategy).** Let \( X \) be a diffusion process, whose coefficients are of linear growth, and let \( p \in [1, \infty) \), \( N \in \mathbb{N} \), and \( \rho \in C(\mathbb{R}) \). Moreover, let \( \varphi_{0:N} = (\varphi_0, \ldots, \varphi_N) \in \mathbb{X}_{n=0}^{N} L^p(X)^\otimes n \), with \( \varphi_0 \in \mathbb{R} \) and \( \varphi_n := \sum_{j=1}^{m_n} w_{n,j} \varphi_{n,j} \in L^p(X)^\otimes n \) for all \( n = 1, \ldots, N \), and define \( G^{\rho_{0:N}} := \sum_{n=0}^{N} J_n (\varphi_n) \). Then, it holds that

\[
G^{\rho_{0:N}} = \varphi_0 + \sum_{i=1}^{d} \int_0^T \theta^{(N),i}_t \, dX_t^i, \quad \mathbb{P}\text{-a.s.,}
\]

where

\[
\theta^{(N),i}_t := \sum_{n=1}^{N} \frac{1}{n} \left( \sum_{j=1}^{m_n} w_{n,j} H_{n-1} (W(\varphi_{n,j},t), Q(\varphi_{n,j}), t) \varphi_{n,j,i}(t) \right),
\]

for \( i = 1, \ldots, d \) and \( t \in [0,T] \). In particular, (17) and (18) holds true for \( G^{\rho_{0:N}} := \sum_{n=0}^{N} J_n (\varphi_n) \) with \( \varphi_{0:N} = (\varphi_0, \ldots, \varphi_N) \in \mathbb{X}_{n=0}^{N} \mathcal{N}_{n,d}^\rho \) where \( \varphi_0 \in \mathbb{R} \) and \( \varphi_n := \sum_{j=1}^{m_n} w_{n,j} \varphi_{n,j} \in \mathcal{N}_{n,d}^\rho \) for all \( n = 1, \ldots, N \).

3. **Numerical Examples**

In the following, we show how the theoretical results of this paper can be applied to different hedging problems, where we consider various market models and option types. All numerical experiments have been implemented in Python on an average laptop (Lenovo ThinkPad X13 Gen2a with Processor AMD Ryzen 7 PRO 5850U and Radeon Graphics, 1901 Mhz, 8 Cores, 16 Logical Processors). The code can be found under the following link: https://github.com/psc25/ChaoticHedging

3.1. **European Option with Brownian Motion.** In the first example, we consider a one-dimensional Brownian motion \( X = (X_t)_{t \in [0,T]} \) and learn for \( T = 1 \) the European call option

\[
G = \max(X_T - K, 0),
\]

with strike price \( K = -0.5 \). By the Clark-Ocone formula in [Clark, 1979] and [Ocone, 1984] (see also [Di Nunno et al., 2008, Theorem 4.1]), the true hedging strategy \( \theta \) of \( G \) is given by \( \theta_t = \mathbb{E} [D_t G | \mathcal{F}_t] \), for all \( t \in [0,T] \), where \( D_tG \) denotes the Malliavin derivative of \( G \) (see [Di Nunno et al., 2008, Definition 3.1] and [Nualart, 2006, Definition 1.2.1]). Using the chain rule of the Malliavin derivative in [Nualart, 2006, Proposition 1.2.4], it follows for every \( t \in [0,T] \) that \( D_t G = \Phi_{[K,\infty)}(X_t) \) and thus

\[
\theta_t = \mathbb{E} [\Phi_{[K,\infty)}(X_T) | \mathcal{F}_t] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-X_t)^2}{2(T-t)}} \, dx = \Phi \left( \frac{X_t - K}{\sqrt{T-t}} \right),
\]

where \( \Phi : \mathbb{R} \to [0,1] \) denotes the cumulative distribution function of the standard normal distribution. On the other hand, the hedging strategy \( \theta^{(N)}_t \) of the approximating financial derivative \( G^{\rho_{0:N}} \in L^p(\mathbb{P}) \) with \( N \) iterated integrals and tensor-valued neural networks \( \varphi_{0:N} = (\varphi_0, \ldots, \varphi_N) \in \mathbb{X}_{n=0}^{N} \mathcal{N}_{n,d}^\rho \) is computed with (18). The results are plotted in Figure 1, where time is discretized over an equidistant grid of 500 points, and where \( 10^5 \) sample paths of \( X \) are generated, which are split up into 80%/20% for training and testing. Moreover, we use random neural networks of size \( m_n = 50 \) with sigmoid activation function \( \rho(x) = \frac{1}{1 + \exp(-x)} \) so that the learning procedure simplifies to a linear regression.
3.2. Asian Option in Local Volatility Model. In the second example, we consider an Asian option in the CEV model (see [Cox, 1975]), where the price process \( X = (X_t)_{t \in [0,T]} \) with \( T = 1 \) follows

\[
dX_t = \alpha X_t dt + \sigma_0 \sqrt{X_t} dB_t,
\]

for \( t \in [0,T] \), with \( X_0 = 100, \alpha = -0.02, \) and \( \sigma_0 = 0.4 \). Hereby, \( X \) is a diffusion process with drift \( b(t,x) = \alpha x \) and diffusion coefficient \( a(t,x) = \sigma_0^2 x \) of linear growth. Then, we learn the Asian option

\[
G = \max \left( K - \frac{1}{T} \int_0^T X_t dt, 0 \right),
\]

with strike price \( K = 102 \). By using similar Fourier arguments as in [Carr and Madan, 1999], we have

\[
\mathbb{E}^Q [G | \mathcal{F}_t] = (K - I_{0,t}) \mathbb{E}^Q \left[ \mathbb{I} \{ I_{t,T} \leq K \} | \mathcal{F}_t \right] - \mathbb{E}^Q \left[ I_{t,T} \mathbb{I} \{ I_{t,T} \leq K \} | \mathcal{F}_t \right],
\]

\[
= \frac{1}{2} + \frac{1}{2} \int_0^\infty \text{Re} \left( e^{iKu} \frac{\varphi_{I_{t,T}}(0,u)}{ru} \right) du = \frac{1}{2T} X_t + \frac{1}{2} \int_0^\infty \text{Re} \left( e^{-iKu} \frac{\varphi_{I_{t,T}}(0,u)}{ru} \right) du,
\]

where \( I_{r,s} := \frac{1}{T} \int_s^r X_t dt \) and \( \varphi_t : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) denotes the characteristic function of \( (X_T, I_{t,T}) | X_t \) under the unique ELMM \( \mathbb{Q} \sim \mathbb{P} \). Hence, the true hedging strategy \( \theta \) is obtained by taking the derivative in (22) with respect to \( X_t \) and applying the fractional fast Fourier transform (FFT) in [Chourdakis, 2005] to compute the integrals. Hereby, the characteristic function is calculated with the affine transform formula in [Duffie et al., 2000, Section 2.2] and the time-integral extension in [Keller-Ressel, 2009, Section 4.3]. The results are plotted in Figure 2, where we use the same learning setting as in Section 3.1.
K with strike price the estimation of the linear regression (taking less than one second), and the calculation of the approximating hedging strategy.

arguments as in [Carr and Madan, 1999], we conclude that Basket option in Affine Model. In the third example, we learn a Basket option written on \( X \) for \( X \). The running time includes the computation of the regressors (i.e. \( H_n(W(\varphi_{n,j}), Q(\varphi_{n,j}), n = 1, \ldots, N \) and \( j = 1, \ldots, m_n) \), the estimation of the linear regression (taking less than one second), and the calculation of the approximating hedging strategy.

\[
\begin{align*}
\text{Learning performance} & \\
\begin{array}{c}
\text{(a) Learning performance} \\
\text{(b) Payoff distribution on test set}
\end{array} \\
\begin{array}{c}
\text{(c) Running time and number of parameters} \\
\text{(d) } \theta \text{ and } \theta^{(N)} \text{ for two samples of test set}
\end{array}
\end{align*}
\]

\[\text{Figure 2. Learning the Asian option } G \text{ in (21) by } G^{\phi \circ N}, \ N = 0, \ldots, 6. \] The learning performance is displayed in (a) in terms of the mean squared error (MSE) \( \mathbb{E} \left( (G - G^{\phi \circ N})^2 \right) \) and the integrated mean squared error (IMSE) \( \mathbb{E} \left[ \int_0^T \left| \theta_t - \theta_t^{(N)} \right|^2 dt \right] \), where the hedging strategy \( \theta \) of \( G \) is computed with the help of (22) and the approximating hedging strategies \( \theta^{(N)} \) of \( G^{\phi \circ N} \) are calculated with (18). In (c), the running time\(^1\) on an average laptop (see above) and the number of parameters in the linear regression are depicted. The distributions of \( G \) and \( G^{\phi \circ N} \) are compared in (b) on the test set, whereas \( \theta \) and \( \theta^{(N)} \) of two samples in the test set are shown in (d).

3.3. Basket Option in Affine Model. In the third example, we learn a Basket option written on \( d = 10 \) stocks with \( l = 10 \) Brownian motions. More precisely, we consider for \( T = 1 \) the affine SDE

\[dX_t = b(X_t)dt + \sqrt{a(X_t)}dB_t, \quad t \in [0, T], \quad b(x) = \beta_0 + \sum_{k=1}^d \beta_k x_k, \quad a(x) = \alpha_0 + \sum_{k=1}^d \alpha_k x_k\]

for \( X_0 = (10, \ldots, 10)^T \), where \( \beta_k = (\beta_{k,1}, \ldots, \beta_{k,d})^\top \in \mathbb{R}^d \) and \( \alpha_k = r_k^\top r_k \in \mathbb{R}_+^d \), \( k = 0, \ldots, d, \) for some \( r_k = (r_{k,1}, \ldots, r_{k,d})^\top \in \mathbb{R}^d \), \( k = 1, \ldots, d \). The parameters are sampled according to \( \beta_{0,i} \sim \mathcal{N}(0, 3 \cdot 10^{-4}), \beta_{k,i} \sim \mathcal{N}(0, 3 \cdot 10^{-2}), \) and \( r_{0,i} \sim \mathcal{N}(0, 3 \cdot 10^{-2}) \), for all \( k, i = 1, \ldots, d \). In this case, \( X \) is a diffusion process with drift and diffusion coefficient of linear growth. Then, we learn the Basket option

\[G = \max \left( K - w^\top X_T, 0 \right), \quad (23)\]

with strike price \( K = 4 \) and \( w = (1, -0.95, 0.9, -0.85, \ldots, 0.6, -0.55)^\top \in \mathbb{R}^d \). By using similar Fourier arguments as in [Carr and Madan, 1999], we conclude that

\[
\mathbb{E}^Q[G | \mathcal{F}_t] = K - \frac{1}{2} \int_0^T \mathbb{E}^Q \left[ \int_0^T \mathbb{E}^Q \left[ w^\top X_T \mathbb{I}_{\{w^\top X_T \leq K\}} \right] \right] f_t du,
\]

where

\[f_t = \frac{1}{2} \mathbb{E}^Q \left[ w^\top X_t \mathbb{I}_{\{w^\top X_T \leq K\}} - w^\top X_t \mathbb{I}_{\{w^\top X_T > K\}} \right] f_t,
\]

The running time includes the computation of the regressors (i.e. \( H_n(W(\varphi_{n,j}), Q(\varphi_{n,j}), n = 1, \ldots, N \) and \( j = 1, \ldots, m_n) \), the estimation of the linear regression (taking less than one second), and the calculation of the approximating hedging strategy.
where $\varphi_t(u) = \mathbb{E}^Q \left[ \exp \left( iv^\top X_t \right) \mid \mathcal{F}_t \right]$ and $\psi_t(v) = \mathbb{E}^Q \left[ w^\top X_T \exp \left( iv^\top X_T \right) \mid \mathcal{F}_t \right]$, for $v \in \mathbb{R}^d$ and $Q$ an ELMM. Hence, the true hedging strategy $\theta = (\theta_1^t, \ldots, \theta_{10}^t)^\top_{t \in [0,T]}$ is given as the gradient of (24) with respect to $X_t$, where the integrals are computed as in Section 3.2. The results are plotted in Figure 3, where we use the same learning setting as in Section 3.2, except that the sample size is now $5 \cdot 10^5$ and the size of the random neural networks is increased to $m_n = 250$.

**Figure 3.** Learning the Basket option $G$ in (23) by $G^{\varphi_0,N}$, $N = 0, \ldots, 6$. The learning performance is displayed in (a) in terms of the mean squared error (MSE) $\mathbb{E} \left[ (G - G^{\varphi_0,N})^2 \right]$ and the integrated mean squared error (IMSE) $\mathbb{E} \left[ \int_0^T \| \theta_t - \theta_t^{(N)} \|^2 \, dt \right]$, where the hedging strategy $\theta = (\theta_1^t, \ldots, \theta_{10}^t)^\top_{t \in [0,T]}$ of $G$ is computed with the help of (24) and the approximating hedging strategies $\theta^{(N)} = (\theta_1^{(N)}, \ldots, \theta_{10}^{(N)})^\top_{t \in [0,T]}$ of $G^{\varphi_0,N}$ are calculated with (18). In (c), the running time $t^{\text{run}}$ on an average laptop (see above) and the number of parameters in the linear regression are depicted. The distributions of $G$ and $G^{\varphi_0,N}$ are compared in (b) on the test set, whereas $\theta$ and $\theta^{(N)}$ of two samples in the test set are shown in (d1)-(d10).
(d3) $\theta^3$ and $\theta^{(N),3}$ for two samples of test set

(d4) $\theta^4$ and $\theta^{(N),4}$ for two samples of test set

(d5) $\theta^5$ and $\theta^{(N),5}$ for two samples of test set

(d6) $\theta^6$ and $\theta^{(N),6}$ for two samples of test set

(d7) $\theta^7$ and $\theta^{(N),7}$ for two samples of test set

(d8) $\theta^8$ and $\theta^{(N),8}$ for two samples of test set

(d9) $\theta^9$ and $\theta^{(N),9}$ for two samples of test set

(d10) $\theta^{10}$ and $\theta^{(N),10}$ for two samples of test set
In this chapter, we provide all the proofs of the previous results in chronological order.

4.1. Proof of Results in Section 2.1: Stochastic Integral and Quadratic Variation.

Proof of Lemma 2.4. For every $k \in \mathbb{N}$, we define the stopping time $\tau_k := \inf \{ t \geq 0 : \| X_t \| \geq k \} \wedge T$.
Moreover, we fix some $k \in \mathbb{N}$ and $n \in \mathbb{N} \cap [4, \infty)$ in the following. By applying Ito’s formula, it follows for every $t \in [0, T]$, $\mathbb{P}$-a.s., that

$$
\|X_{t \wedge \tau_k}\|^n = \|X_0\|^n + n \int_0^t 1_{\{s \leq \tau_k\}} \|X_s\|^{n-2} X_s^\top b(s, X_s)ds
\]

$$
\ldots + n \int_0^t 1_{\{s \leq \tau_k\}} \|X_s\|^{n-2} X_s^\top \sigma(s, X_s)dB_s
\]

$$
= M_t^{(k)}
$$

$$
\ldots + \frac{n}{2} \int_0^t 1_{\{s \leq \tau_k\}} \|X_s\|^{n-4} \text{tr} \left( \left( (n-2)X_sX_s^\top - \|X_s\|^2 I_d \right) a(s, X_s) \right) ds.
$$

Hereby, we observe by applying the Burkholder-Davis-Gundy inequality (with constant $C_2 > 0$) and using the linear growth condition of $a$ that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} M_t^{(k)} \right]^2 \leq C_2 \mathbb{E} \left[ \int_0^T 1_{\{s \leq \tau_k\}} \|X_s\|^{2n-4} X_s^\top a(s, X_s)X_sds \right]
$$

$$
\leq C_2 \int_0^T \mathbb{E} \left[ 1_{\{s \leq \tau_k\}} \|X_s\|^{2n-2} \|a(s, X_s)\|_F \right] ds
$$

$$
\leq C_2 C_L \int_0^T \mathbb{E} \left[ \|X_s\|^{2n-2} \left( 1 + \|X_s\| \| \right) \right] ds
$$

$$
\leq C_2 C_L T k^{2n-2} (1 + k) < \infty,
$$

which shows that $M_t^{(k)}$ is a martingale with $\mathbb{E} \left[ M_t^{(k)} \right] = 0$, for all $t \in [0, T]$. Hence, by taking the expectation in (25), and using the Cauchy-Schwarz inequality, Fubini’s theorem, the linear growth condition of $a$ and $b$, the inequalities $\text{tr} (A^2 B) \leq \|A\|_F \|B\|_F$, $\|A + B\|_F \leq \|A\|_F + \|B\|_F$ for any $A, B \in \mathbb{R}^{d \times d}$, $\|xx^\top\|_F = \sqrt{\text{tr}(xx^\top xx^\top)} = \|x\|^2$ for any $x \in \mathbb{R}^d$, $\|I_d\|_F = \sqrt{d}$, and Jensen’s inequality, it follows for every $t \in [0, T]$ that

$$
\mathbb{E} \left[ \|X_{t \wedge \tau_k}\|^n \right] \leq \|X_0\|^n + n \mathbb{E} \left[ \int_0^t 1_{\{s \leq \tau_k\}} \|X_s\|^{n-2} X_s^\top b(s, X_s)ds \right]
\]

$$
\ldots + \frac{n}{2} \int_0^t \mathbb{E} \left[ 1_{\{s \leq \tau_k\}} \|X_s\|^{n-4} \text{tr} \left( \left( (n-2)X_sX_s^\top - \|X_s\|^2 I_d \right) a(s, X_s) \right) \right] ds
\]

$$
\leq \|X_0\|^n + n \int_0^t \mathbb{E} \left[ 1_{\{s \leq \tau_k\}} \|X_s\|^{n-1} \|b(s, X_s)\|_F \right] ds
\]

$$
\ldots + \frac{n}{2} \int_0^t \mathbb{E} \left[ 1_{\{s \leq \tau_k\}} \|X_s\|^{n-4} \left( (n-2)X_sX_s^\top - \|X_s\|^2 I_d \right) \right] \|a(s, X_s)\|_F \] ds
\]

$$
\leq \|X_0\|^n + n C_L \sqrt{d} \int_0^t \mathbb{E} \left[ \|X_s\|^{n-1} \left( 1 + \|X_s\| \right) \right] ds
\]

$$
\ldots + \frac{n}{2} C_L \left( n - 2 + \sqrt{d} \right) \int_0^t \mathbb{E} \left[ \|X_s\|^n \left( 1 + \|X_s\| \right) \right] ds
\]

$$
\leq \max \left( \|X_0\|^n, n \right) + \int_0^t \xi_X \left( \mathbb{E} \left[ \|X_s\|^n \right] \right) ds,
$$

where $[0, \infty) \ni y \mapsto \xi_X(y) := n C_L \sqrt{d} (y + \frac{n-1}{n}) + n^2 C_L \sqrt{d} \left( y \frac{n-1}{n} + y \frac{n-2}{n} \right) \in [0, \infty)$ is continuous with $\xi_X(y) > 0$, for all $y \in (0, \infty)$. From this, we define $[u_0, \infty) \ni u \mapsto \Xi_X(u) := \int_{u_0}^u \xi_X(y) dy \in \mathbb{R}$.
where the right hand-side does not depend on which shows that which completes the proof. Moreover, since \( y \geq y \frac{n-1}{n} \geq y \frac{n-2}{n} \) for all \( y \geq u_0 \), as \( u_0 \geq 1 \), and \( y \geq ny \frac{n-1}{n} \) for all \( y \geq u_0 \), as \( u_0 \geq n^n \), we observe for every \( u \in (u_0, \infty) \) that

\[
\Xi_X(u) = \frac{1}{\xi_X(u)} > 0,
\]

which shows that \( \Xi_X^{-1}(v) = \max(\{\|X_0\|, n\}^n \exp(4nC_L\sqrt{v}) \), for all \( v \in (0, \infty) \). Then, by using (26) together with the non-linear Gronwall inequality in [Bihari, 1956, Equation 6-8] and that \( \Xi_X(u_0) = 0 \), it follows for every \( t \in [0, T] \), \( k \in \mathbb{N} \), and \( n \in \mathbb{N} \cap [4, \infty) \) that

\[
\mathbb{E}[\|X_{t\wedge \tau_k}\|^n] \leq \Xi_X^{-1}(\Xi_X(u_0) + t) \leq \Xi_X^{-1}(T) \leq \max(\{\|X_0\|, n\}^n e^{4nC_L\sqrt{T}},
\]

where the right hand-side does not depend on \( k \in \mathbb{N} \). Since \( \|X_{t\wedge \tau_k}\|^n \) converges \( \mathbb{P}\text{-a.s.} \) to \( \|X_t\|^n \), as \( k \to \infty \), Fatou’s lemma implies for every \( t \in [0, T] \) and \( n \in \mathbb{N} \cap [4, \infty) \) that

\[
\mathbb{E}[\|X_t\|^n] \leq \liminf_{k \to \infty} \mathbb{E}[\|X_{t\wedge \tau_k}\|^n] \leq \max(\{\|X_0\|, n\}^n e^{4nC_L\sqrt{T}} < \infty,
\]

which shows the inequality in (4).

In order to show (5), let 0 \( \leq \varepsilon < \exp(-4C_L\sqrt{T} - 1) \). Moreover, we define the sequence \((c_n)_{n \in \mathbb{N}} \) by \( c_n := \max(\{\|X_0\|, n\}^n \exp(4nC_L\sqrt{T}) \), for \( n \in \mathbb{N} \). Then, by using (4), we conclude for every \( t \in [0, T] \) and \( n \in \mathbb{N} \cap [4, \infty) \) that \( \mathbb{E}[\|X_t\|^n] \leq c_n \). In addition, by using Stirling’s inequality \( n! \geq \sqrt{2\pi n} e^{-n} n^n \) for any \( n \in \mathbb{N} \), it follows that

\[
\limsup_{n \to \infty} \sqrt{n} \frac{\varepsilon^n c_n}{n!} \leq \limsup_{n \to \infty} \frac{\varepsilon \max(\{\|X_0\|, n\} e^{4C_L\sqrt{T}}}{(2\pi n)^{\frac{1}{2}} e^{-1} n} = e^{4C_L\sqrt{T} + 1} < 1.
\]

By the root test, we conclude that \( \sum_{n=1}^{\infty} \varepsilon^n c_n \) converges absolutely. Hence, the monotone convergence theorem, Jensen’s inequality, and (4) show for every \( t \in [0, T] \) that

\[
\mathbb{E}[\varepsilon^{\|X_t\|^n}] = \sum_{n=0}^{\infty} \frac{\varepsilon^n \mathbb{E}[\|X_t\|^n]}{n!} \leq 1 + \sum_{n=1}^{3} \frac{\varepsilon^n \mathbb{E}[\|X_t\|^4]}{n!} + \sum_{n=4}^{\infty} \frac{\varepsilon^n c_n}{n!} < \infty,
\]

which completes the proof.

\[ \square \]

**Proof of Lemma 2.5.** Let \( g \in D_t([0, T]; \mathbb{R}^d) \) be LCRL, which is by [Fraňková, 1991, Lemma 1.7] bounded such that \( \|g\|_{\infty} := \sup_{t \in [0, T]} \|g(t)\| < \infty \). Then, Jensen’s inequality for convex and concave functions, Fubini’s theorem, the linear growth condition of \( a \) and \( b \) (with constant \( C_L > 0 \), the
inequality \((x + y)^p \leq 2^{p-1} (x^p + y^p)\) for any \(x, y \geq 0\), and Lemma 2.4 imply that

\[
\|g\|_{L^p(X)} = \mathbb{E} \left[ \left( \int_0^T |g(t)\top b(t, X_t)| \, dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( \int_0^T g(t)\top a(t, X_t) g(t) \, dt \right)^\frac{p}{2} \right]^{\frac{1}{p}} \\
\leq T^{1-\frac{1}{p}} \mathbb{E} \left[ \int_0^T \|g(t)\|^p \|b(t, X_t)\|^p \, dt \right]^{\frac{1}{p}} + T^{\frac{3}{2}-\frac{1}{2p}} \mathbb{E} \left[ \int_0^T \|g(t)\|^{2p} \|a(t, X_t)\|_{\mathcal{F}_t}^p \, dt \right]^{\frac{1}{p}} \\
\leq C_L T^{1-\frac{1}{p}} \left( \int_0^T \|g(t)\|^{p} \mathbb{E} [1 + \|X_t\|^p] \, dt \right)^{\frac{1}{p}} + \sqrt{C_L} T^{\frac{3}{2}-\frac{1}{2p}} \left( \int_0^T \|g(t)\|^{2p} \mathbb{E} [1 + \|X_t\|^p] \, dt \right)^{\frac{1}{p}} \\
\leq C_L T \|g\|_\infty \sup_{t \in [0, T]} \mathbb{E} [1 + \|X_t\|^p]^{\frac{1}{p}} + \sqrt{C_L} T \|g\|_\infty \sup_{t \in [0, T]} \mathbb{E} [1 + \|X_t\|^p]^{\frac{1}{p}} \\
\leq 2C_L T \|g\|_\infty \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \|X_t\|^p \right)^{\frac{1}{p}} + \sqrt{2} C_L T \|g\|_\infty \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \|X_t\|^p \right)^{\frac{1}{p}} < \infty,
\]

which shows that \(g \in L^p(X)\). \(\square\)

**Proof of Lemma 2.7.** For two functions \(f, g \in D_1([0, T]; \mathbb{R}^d)\) and some \(\lambda \in \mathbb{R}\), it follows that \(f + g \in D_1([0, T]; \mathbb{R}^d)\) and \(\lambda \cdot f \in D_1([0, T]; \mathbb{R}^d)\). Hence, we only need to show that \(\| \cdot \|_{L^p(X)}\) is a norm. By definition, \(\| \cdot \|_{L^p(X)}\) is absolutely homogeneous, and since we consider \(L^p(X)\) as quotient space with respect to the kernel of \(\| \cdot \|_{L^p(X)}\), the norm \(\| \cdot \|_{L^p(X)}\) is positive definite. In order to show the triangle inequality, let \(f, g \in L^p(X)\). Then, by applying the Cauchy-Schwarz inequality twice, it holds \(\mathbb{P}\text{-a.s. that}
\]

\[
\int_0^T f(t)\top a(t, X_t) g(t) \, dt = \int_0^T f(t)\top \sigma(t, X_t) \sigma(t, X_t)\top g(t) \, dt \\
\leq \int_0^T \left( f(t)\top \sigma(t, X_t) \sigma(t, X_t)\top f(t) \right)^{\frac{1}{2}} \left( g(t)\top \sigma(t, X_t) \sigma(t, X_t)\top g(t) \right)^{\frac{1}{2}} \, dt \\
\leq \left( \int_0^T f(t)\top a(t, X_t) f(t) \, dt \right)^{\frac{1}{2}} \left( \int_0^T g(t)\top a(t, X_t) g(t) \, dt \right)^{\frac{1}{2}} = Q(f)^{\frac{1}{2}} Q(g)^{\frac{1}{2}},
\]

Hence, by using \(x + 2\sqrt{xy} + y = (\sqrt{x} + \sqrt{y})^2\) for any \(x, y \geq 0\), it follows that

\[
Q(f + g) = Q(f) + 2 \int_0^T f(t)\top a(t, X_t) g(t) \, dt + Q(g) \\
\leq Q(f) + 2Q(f)^{\frac{1}{2}} Q(g)^{\frac{1}{2}} + Q(g) \\
= \left( Q(f)^{\frac{1}{2}} + Q(g)^{\frac{1}{2}} \right)^2.
\]

Finally, by applying Minkowski’s inequality, we conclude that

\[
\|f + g\|_{L^p(X)} = \mathbb{E} \left[ \left( \int_0^T |(f(t) + g(t))\top b(t, X_t)| \, dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( (f + g)^{\frac{1}{2}} \right)^p \right]^{\frac{1}{p}} \\
\leq \mathbb{E} \left[ \left( \int_0^T |f(t)\top b(t, X_t)| \, dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( \int_0^T |g(t)\top b(t, X_t)| \, dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ (Q(f)^{\frac{1}{2}} + Q(g)^{\frac{1}{2}})^2 \right]^{\frac{1}{p}} \\
\leq \mathbb{E} \left[ \left( \int_0^T |f(t)\top b(t, X_t)| \, dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( \int_0^T |g(t)\top b(t, X_t)| \, dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ Q(f)^{\frac{1}{2}} \right]^{\frac{1}{p}} + \mathbb{E} \left[ Q(g)^{\frac{1}{2}} \right]^{\frac{1}{p}} \\
= \|f\|_{L^p(X)} + \|g\|_{L^p(X)},
\]

which completes the proof. \(\square\)
Proof of Lemma 2.10. In order to prove (7b), let \( g \in L^p(X) \). Since \( t \mapsto Q(g)_t \) is non-decreasing, we observe that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} Q(g)_t \right]^{\frac{1}{p}} = \mathbb{E} \left[ Q(g)^{\frac{1}{p}} \right] \leq \|g\|_{L^p(X)},
\]

which shows (7b). Moreover, by using Minkowski’s inequality and the Burkholder-Davis-Gundy inequality (with constant \( C_p > 0 \)), it follows that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |W(g)_t|^p \right]^{\frac{1}{p}} \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t g(s)^\top b(s, X_s) ds \right|^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t g(s)^\top \sigma(s, X_s) dB_s \right|^p \right]^{\frac{1}{p}}
\]

\[
\leq \mathbb{E} \left[ \left( \int_0^T |g(s)^\top b(s, X_s)| ds \right)^p \right]^{\frac{1}{p}} + C_p \mathbb{E} \left[ \left( \int_0^T |g(s)^\top a(s, X_s) g(s)| ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}
\]

\[
\leq \max (1, C_p) \|g\|_{L^p(X)},
\]

which shows the other inequality in (7a).

Lemma 4.1. Let \( X \) be a diffusion process, whose coefficients are of linear growth with constant \( C_L > 0 \), and let \( g \in D_1([0,T]; \mathbb{R}^d) \). Then, it holds for every \( n \in \mathbb{N} \cap [4, \infty) \) that

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ |W(g)_t|^n \right] \leq n^n \|g\|_\infty^n \left( 1 + 8C_{d,L,T,\|X_0\|} T \right)^n, \tag{27a}
\]

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ |Q(g)_t|^n \right] \leq n^n \|g\|_\infty^n \left( 1 + 4C_{d,L,T,\|X_0\|} T \right)^n, \tag{27b}
\]

where \( C_{d,L,T,\|X_0\|} := C_L \max \left( 1, \|X_0\| \exp \left( 4C_L \sqrt{d} T \right) \right) > 0 \) does not depend on \( n \in \mathbb{N} \). Moreover, for every \( g \in D_1([0,T]; \mathbb{R}^d) \) with \( \|g\|_\infty < \left( 1 + 8C_{d,L,T,\|X_0\|} T \right) e^{-1} \), we have for every \( t \in [0,T] \) that

\[
\mathbb{E} \left[ e^{\left|W(g)_t\right|^n} \right] < \infty. \tag{28}
\]

In addition, for every \( g \in D_1([0,T]; \mathbb{R}^d) \) with \( \|g\|_\infty < \left( 1 + 4C_{d,L,T,\|X_0\|} T \right) e^{-1} \), we have for every \( t \in [0,T] \) that

\[
\mathbb{E} \left[ e^{Q(g)_t} \right] < \infty. \tag{29}
\]

Proof. In order to show (27a), we first define the sequence of stopping times \( r^W_k \in \mathbb{N} \) by \( r^W_k = \inf \{ t \geq 0 : \max \{|W(g)_t|, |X_t|\} \geq k \} \wedge T \), for \( k \in \mathbb{N} \). Now, we fix some \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \cap [4, \infty) \). Then, by applying Ito’s formula on \( f(x) = |x|^n \), it follows for every \( t \in [0,T] \), \( \mathbb{P} \)-a.s., that

\[
|W(g)_{t \wedge r^W_k}|^n = n \int_0^t \mathbf{1}_{\{s \leq r^W_k\}} |W(g)_s|^{n-2} W(g)_s g(s)^\top b(s, X_s) ds
\]

\[
+ \cdots + n \int_0^t \mathbf{1}_{\{s \leq r^W_k\}} |W(g)_s|^{n-2} W(g)_s g(s)^\top \sigma(s, X_s) dB_s
\]

\[
= M^{(k)}_t
\]

\[
+ \cdots + \frac{n(n-1)}{2} \int_0^t \mathbf{1}_{\{s \leq r^W_k\}} |W(g)_s|^{n-2} g(s)^\top a(s, X_s) g(s) ds.
\]

Hereby, we observe by applying the Burkholder-Davis-Gundy inequality (with constant \( C_2 > 0 \)) and using the linear growth condition of \( a \) that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |M^{(k)}_t|^2 \right] \leq C_2 \mathbb{E} \left[ \int_0^T \mathbf{1}_{\{s \leq r^W_k\}} |W(g)_s|^{2n-2} g(s)^\top a(s, X_s) g(s) ds \right]
\]

\[
\leq C_2 \int_0^T \mathbb{E} \left[ \mathbf{1}_{\{s \leq r^W_k\}} |W(g)_s|^{2n-2} \|g(s)\|_F^2 \|a(s, X_s)\|_F \right] ds
\]

\[
\leq C_2 C_L \|g\|^2_\infty \int_0^T \mathbb{E} \left[ |W(g)_{s \wedge r^W_k}|^{2n-2} \left( 1 + \|X_{s \wedge r^W_k}\| \right) \right] ds
\]

\[
\leq C_2 C_L \|g\|^2_\infty T k^{2n-2} (1 + k) < \infty,
\]
which shows that $M^{(k)}_t$ is a martingale with $\mathbb{E} \left[ M^{(k)}_t \right] = 0$, for all $t \in [0, T]$. Hence, by taking the expectation in (30), using the linear growth condition of $a$ and $b$, H"older's inequality (with exponents $\frac{n-1}{n} + \frac{1}{n} = 1$ and $\frac{n-2}{n} + \frac{2}{n} = 1$, respectively), the inequalities $(x+y)^n \leq 2^{n-1} (x^n + y^n)$ and $(x+y)^\frac{n}{2} \leq 2^{\frac{n}{2}-1} (x^\frac{n}{2} + y^\frac{n}{2})$ for any $x, y \geq 0$, Lemma 2.4, the inequalities $(x+y)^\frac{n}{2} \leq x^\frac{n}{2} + y^\frac{n}{2}$ and $(x+y)^\frac{n}{2} \leq x^\frac{n}{2} + y^\frac{n}{2}$ for any $x, y \geq 0$, and defining $C_{d_L,T,\|X_0\|} := C_L \max \left( 1, \|X_0\| \exp \left( 4C_L \sqrt{dT} \right) \right) > 0$, it follows for every $t \in [0, T]$ that

\[
\begin{align*}
\mathbb{E} \left[ |W(g)_{t \wedge \tau_k^W}|^n \right] & \leq n \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} W(g)_s g(s)^\top b(s, X_s) ds \right] \\
& \quad + \frac{n(n-1)}{2} \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} g(s)^\top a(s, X_s) g(s) ds \right] \\
& \leq n \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-1} \|b(s, X_s)\| \|g(s)\| ds \right] \\
& \quad + \frac{n(n-1)}{2} \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \|a(s, X_s)\|_F \|g(s)\|^2 ds \right] \\
& \leq n \|g\|_\infty C_L \left( \frac{2n-1}{2} \right) \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \\
& \quad + \left( \frac{2n-1}{2} \right) \frac{n(n-1)}{2} \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \leq 2 \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \\
& \leq 2n \|g\|_\infty C_L \left( 1 + \max (\|X_0\|, n) n e^{4nC_L \sqrt{dT}} \right) \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \\
& \quad + n^2 \|g\|^2_\infty C_L \left( 1 + \max (\|X_0\|, n) n e^{4nC_L \sqrt{dT}} \right) \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \leq 2n \|g\|_\infty C_L \left( 1 + \max (\|X_0\|, n) n e^{4C_L \sqrt{dT}} \right) \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \\
& \quad + n^2 \|g\|^2_\infty C_L \left( 1 + \max (\|X_0\|, n) n e^{4C_L \sqrt{dT}} \right) \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right] ds \\
& \leq \int_0^t \mathbb{E} \left[ \left( \mathbb{I}_{\{s \leq \tau_k^W\}} |W(g)_s|^{n-2} \left( 1 + \|X_s\| \right) \right) \right] ds
\end{align*}
\]
where the right-hand side does not depend on which shows the inequality in (27a).

\[ \text{follows for every } t \in [0, T], k \in \mathbb{N}, \text{ and } n \in \mathbb{N} \cap [4, \infty) \] that

\[ \mathbb{E} \left[ \left| W(g)_{t \wedge \tau_k^W} \right|^n \right] \leq \mathbb{E}^{-1} \left( \mathbb{E} \left[ W(g)_{t \wedge \tau_k^W} \right]^n \right) \leq \mathbb{E}^{-1} \left( 1 + 8C_d, L, T, \|x_0\| T \right)^n, \]

where the right-hand side does not depend on \( k \in \mathbb{N} \). As \( W(g)_{t \wedge \tau_k^W} \) converges \( \mathbb{P} \)-a.s. to \( W(g)_t \), as \( k \to \infty \), Fatou’s lemma implies for every \( t \in [0, T] \) and \( n \in \mathbb{N} \cap [4, \infty) \) that

\[ \mathbb{E} \left[ \left| W(g)_t \right|^n \right] \leq \liminf_{k \to \infty} \mathbb{E} \left[ \left| W(g)_{t \wedge \tau_k^W} \right|^n \right] \leq \mathbb{E}^{-1} \left( 1 + 8C_d, L, T, \|x_0\| T \right)^n, \]

which shows the inequality in (27a).

In order to show (27b), let \( n \in \mathbb{N} \cap [4, \infty) \) and define for every \( k \in \mathbb{N} \) the stopping time \( \tau_k^Q = \inf \{ t \geq 0 : Q(g)_t \geq k \} \) \( \land T \). Then, by applying Itô’s formula, it follows for every \( t \in [0, T] \), \( \mathbb{P} \)-a.s., that \( dQ(g)_t^2 = nQ(g)^{n-1} dQ(g)_t \). Moreover, by using the linear growth condition of \( a \), Hölder’s inequality (with exponents \( \frac{n-1}{n} + \frac{1}{n} = 1 \)), the inequality \( (x + y)^n \leq 2^{n-1} (x^n + y^n) \) for any \( x, y \geq 0 \), Lemma 2.4, and \( (x + y)^{\frac{n}{2}} \leq x^\frac{n}{2} + y^\frac{n}{2} \) for any \( x, y \geq 0 \), it holds for every \( t \in [0, T] \) that

\[ \mathbb{E} \left[ Q(g)_t^n \right] \leq \mathbb{E} \left[ \int_0^t 1_{\{ s \leq \tau_k^Q \}} Q(g)_s^{n-1} \|a(s, X_s)\|_F \|g(s)\|^2 \, ds \right] \]
\[ \leq nC_L \int_0^t \mathbb{E} \left[ 1_{\{ s \leq \tau_k^Q \}} Q(g)_s^{n-1} (1 + \|X_s\|) \right] \|g(s)\|^2 \, ds \]
\[ \leq \left( \frac{n-1}{n} \right) \frac{n}{2} \|g\|_F^2 \int_0^t \mathbb{E} \left[ Q(g)_s^{n-1} \right] \|1 + \|X_s\|^{n-1}\|^n \, ds \]
\[ \leq 2n \|g\|_F^2 C_L \left( 1 + \max \{ \|X_0\|, n \} e^{4nC_L \sqrt{T}} \right) \frac{n}{2} \int_0^t \mathbb{E} \left[ Q(g)_s^{n-1} \right] \, ds \]
\[ \leq 2n \|g\|_F^2 C_L \left( 1 + \max \{ \|X_0\|, n \} e^{4C_d, L, T, \|x_0\| T} \right) \frac{n}{2} \int_0^t \mathbb{E} \left[ Q(g)_s^{n-1} \right] \, ds \]
\[ \leq \frac{n}{2} \|g\|_F^{2n} + \int_0^t \mathbb{E} \left[ Q(g)_s^{n-1} \right] \, ds, \]

where \([0, \infty) \ni y \mapsto \xi_Q(y) := 4n^2 \|g\|_F^2 C_d, L, T, \|x_0\| \|y_0^{n-1}\| \in [0, \infty) \) is continuous with \( \xi_Q(y) > 0 \) for all \( y \in (0, \infty) \). From this, we define \([u_0, \infty) \ni u \mapsto \Xi_Q(u) = \int_{u_0}^u \frac{1}{\xi_Q(y)} \, dy \in [0, \infty) \) satisfying \( \Xi'_{Q}(u) = \frac{1}{\xi_Q^{-1}(u)} > 0 \), for all \( u \in (u_0, \infty) \). By the inverse function theorem, there exists a continuously differentiable inverse \( \Xi^{-1}_Q : (0, \infty) \to (u_0, \infty) \) with \( \left( \Xi^{-1}_Q \right)'(v) = \frac{1}{\xi_Q^{-1}(\Xi^{-1}_Q(v))} > 0 \), for all \( v \in (0, \infty) \), implying that \( \Xi^{-1}_Q : (0, \infty) \to (u_0, \infty) \) is strictly increasing. Moreover, we observe for every \( u \in (u_0, \infty) \) that

\[ \Xi_Q(u) = \int_{u_0}^u \frac{y^{\frac{1}{n}-1}}{4n^2 \|g\|_F^2 C_d, L, T, \|x_0\|} \, dy = \frac{\left( \frac{u^{\frac{1}{n}}}{n} \right)^2 - u_0^{\frac{1}{n}}}{4n^2 \|g\|_F^2 C_d, L, T, \|x_0\|} - \frac{1}{4C_d, L, T, \|x_0\|} > 0, \]

which shows that \( \Xi^{-1}_Q(v) \leq n \|g\|_F^{\frac{n}{2}} \left( 1 + 8C_d, L, T, \|x_0\| v \right) \), for all \( v \in (0, \infty) \). Then, by using (32) together with the non-linear Gronwall inequality in [Bihari, 1956, Equation 6-8] and that \( \Xi_Q(u_0) = 0 \), it
follows for every $t \in [0, T]$, $k \in \mathbb{N}$, and $n \in \mathbb{N} \cap [4, \infty)$ that
\[
\mathbb{E} \left[ Q(g)^n_{t,\tau,\xi} \right] \leq \mathbb{E} Q^1 (\mathbb{E} (u_0) + t) \leq \mathbb{E} Q^1 (T) \leq n^n \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right)^n,
\]
where the right-hand side does not depend on $k \in \mathbb{N}$. Since $Q(g)^n_{t,\tau,\xi}$ converges $\mathbb{P}$-a.s. to $Q(g)_t^n$, as $k \to \infty$, Fatou’s lemma implies for every $t \in [0, T]$ and $n \in \mathbb{N} \cap [4, \infty)$ that
\[
\mathbb{E} \left[ Q(g)_t^n \right] \leq \liminf_{k \to \infty} \mathbb{E} \left[ Q(g)^n_{t,\tau,\xi} \right] \leq n^n \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right)^n,
\]
which proves the inequality in (27b).

In order to show (28), let $g \in D_t^1([0, T] ; \mathbb{R}^d)$ with $\| g \|_{\infty} < \left( \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right) e \right)^{-1}$. By defining $c_n := n^n \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right)^n$, for $n \in \mathbb{N}$, it follows by (27a) that $\mathbb{E} \left[ |W(g)_t|^n \right] \leq c_n$, for all $t \in [0, T]$ and $n \in \mathbb{N} \cap [4, \infty)$. Then, by Stirling’s inequality $n! \geq \sqrt{2\pi n} e^{-n} n^n$ for any $n \in \mathbb{N}$, we have
\[
\limsup_{n \to \infty} \sqrt[n]{c_n} \leq \limsup_{n \to \infty} \frac{n^n \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right)}{(2\pi n)^{n+1/2} e^{-n}} = \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right) e < 1.
\]

By the root test, we conclude that $\sum_{n=1}^{\infty} \frac{c_n}{n!}$ converges absolutely. Hence, the monotone convergence theorem, Jensen’s inequality, and (27a) show for every $t \in [0, T]$ that
\[
\mathbb{E} \left[ e^{\mid W(g)_t \mid} \right] = \sum_{n=0}^{\infty} \mathbb{E} \left[ |W(g)_t|^n \right] \leq 1 + 3 \sum_{n=1}^{\infty} \mathbb{E} \left[ |W(g)_t|^n \right] + \sum_{n=1}^{\infty} c_n n! < \infty.
\]

On the other hand, for (29), let $g \in D_t^1([0, T] ; \mathbb{R}^d)$ with $\| g \|_{\infty} < \left( \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right) e \right)^{-1}$. By defining $d_n := n^n \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right)^n$, for $n \in \mathbb{N}$, it follows by (27b) that $\mathbb{E} \left[ Q(g)_t^n \right] \leq d_n$, for all $t \in [0, T]$ and $n \in \mathbb{N} \cap [4, \infty)$. Then, Stirling’s inequality $n! \geq \sqrt{2\pi n} e^{-n} n^n$ for any $n \in \mathbb{N}$, implies
\[
\limsup_{n \to \infty} \sqrt[n]{d_n} \leq \limsup_{n \to \infty} \frac{n^n \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right)}{(2\pi n)^{n+1/2} e^{-n}} = \| g \|^2_{\infty} \left( 1 + 4C_{d, L, T, \| X_0 \|} T \right) e < 1.
\]

By the root test, we conclude that $\sum_{n=1}^{\infty} \frac{d_n}{n!}$ converges absolutely. Hence, the monotone convergence theorem, Jensen’s inequality, and (27b) show for every $t \in [0, T]$ that
\[
\mathbb{E} \left[ e^{\mid Q(g)_t \mid} \right] = \sum_{n=0}^{\infty} \mathbb{E} \left[ Q(g)_t^n \right] \leq 1 + 3 \sum_{n=1}^{\infty} \mathbb{E} \left[ Q(g)_t^n \right] + \sum_{n=1}^{\infty} d_n n! < \infty,
\]
which completes the proof. \hfill \Box

### 4.2. Proof of Results in Section 2.2.

#### 4.2.1. Auxiliary Results: Time-Space Harmonic Hermite Polynomials.

In the following, we show some properties of the time-space harmonic Hermite polynomials introduced in Definition 2.11.

**Lemma 4.2.** The time-space harmonic Hermite polynomials $(H_n)_{n \in \mathbb{N}_0}$ satisfy for every $(x, t) \in \mathbb{R} \times [0, \infty)$ and $n \in \mathbb{N}$ that
\[
\frac{\partial H_n}{\partial x} (x, t) = n H_{n-1} (x, t), \tag{33a}
\]
\[
\frac{\partial H_n}{\partial t} (x, t) = \frac{n(n-1)}{2} H_{n-2} (x, t), \tag{33b}
\]
\[
H_n (x, t) = x H_{n-1} (x, t) - (n-1)t H_{n-2} (x, t), \tag{33c}
\]
(resp. $n \in \mathbb{N} \cap [2, \infty)$). Moreover, it holds for every $(x, t) \in \mathbb{R} \times [0, \infty)$ that
\[
\sum_{n=0}^{\infty} \frac{1}{n!} H_n (x, t) = e^{x - \frac{t}{2}}, \tag{34a}
\]
\[
\sum_{n=0}^{\infty} \frac{1}{n!} |H_n (x, t)| \leq e^{x + \frac{t}{2}}. \tag{34b}
\]

**Remark 4.3.** From (33a) and (33b), we conclude that $\frac{\partial^2 H_n}{\partial x^2} (x, t) + \frac{\partial H_n}{\partial t} (x, t) = 0$, which explains the term “time-space harmonic” of the polynomials $(H_n)_{n \in \mathbb{N}_0}$. 


Proof of Lemma 4.2. In order to show (33a), we use the chain rule and [Abramowitz and Stegun, 1970, Equation 22.8.8], i.e. \( h_n(x) = n h_{n-1}(x) \) for any \( x \in \mathbb{R} \), to conclude for every \((x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
\frac{\partial H_n}{\partial x}(x, t) = \frac{\partial}{\partial x} \left( t^{n/2} h_n \left( \frac{x}{\sqrt{t}} \right) \right) = t^{n/2} h'_n \left( \frac{x}{\sqrt{t}} \right) \frac{1}{\sqrt{t}} = t^{n/2} n h_{n-1} \left( \frac{x}{\sqrt{t}} \right) = n H_{n-1}(x, t).
\]
For (33b), we apply the product rule, use again \( h'_n(x) = nh_{n-1}(x) \) for any \( x \in \mathbb{R} \), and the recurrence relation in [Abramowitz and Stegun, 1970, Equation 22.7.14], i.e. \( h_n(x) = x h_{n-1}(x) - (n-1) h_{n-2}(x) \) for any \( x \in \mathbb{R} \), to conclude for every \((x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
\frac{\partial H_n}{\partial t}(x, t) = \frac{\partial}{\partial t} \left( t^{n/2} h_n \left( \frac{x}{\sqrt{t}} \right) \right) = \frac{n}{2} t^{n/2-1} h_n \left( \frac{x}{\sqrt{t}} \right) + t^{n/2} h'_n \left( \frac{x}{\sqrt{t}} \right) \left( -\frac{x}{2t^{3/2}} \right) = -\frac{n(n-1)}{2} t^{n/2-2} h_{n-2} \left( \frac{x}{\sqrt{t}} \right) = -\frac{n(n-1)}{2} H_{n-2}(x, t).
\]
Moreover, \( h_n(x) = x h_{n-1}(x) - (n-1) h_{n-2}(x) \) implies for every \((x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
H_n(x, t) = t^{n/2} h_n \left( \frac{x}{\sqrt{t}} \right) = t^{n/2} x h_{n-1} \left( \frac{x}{\sqrt{t}} \right) - (n-1) t^{n/2} h_{n-2} \left( \frac{x}{\sqrt{t}} \right) = x H_{n-1}(x, t) - (n-1) t H_{n-2}(x, t),
\]
which proves (33c). For (34a), we define the function \( f(x) := e^{-\frac{1}{2}x^2}, \) for which (8) implies \( \frac{d^n f}{dx^n}(x) = (-1)^n e^{-\frac{1}{2}x^2} h_n(x). \) Then, Taylor’s theorem around \( y_0 := x \) shows for every \((x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
f(x - t) = f(y) \bigg|_{y = x - t} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}(y_0)(y - y_0)^n \bigg|_{y = x - t} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \frac{d^n f}{dx^n}(x)
\]
\[
= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} (-1)^n e^{-\frac{1}{2}x^2} h_n(x) = e^{-\frac{1}{2}x^2} \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x),
\]
which yields \( \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = f(x - t) e^{\frac{1}{2}x^2} = e^{xt - \frac{1}{2}t^2}. \) Hence, for every \((x, t) \in \mathbb{R} \times [0, \infty) \), we have
\[
\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, t) = \sum_{n=0}^{\infty} \frac{(\sqrt{t})^n}{n!} h_n \left( \frac{x}{\sqrt{t}} \right) = e^{\frac{x}{\sqrt{t}} \sqrt{t} - \frac{1}{2}(\sqrt{t})^2} = e^{x - \frac{1}{2}t}.
\]
On the other hand, for (34b), we use the explicit expression of \( h_n(x) \) in [Abramowitz and Stegun, 1970, Equation 22.3.11] given by \( h_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{2^{k}k!(n-2k)!} x^{n-2k} \), for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N}_0 \), where \( \lfloor z \rfloor := \max \{ n \in \mathbb{Z} : n \leq z \} \), for \( z \in \mathbb{R} \). Then, by using the definition of the time-space harmonic Hermite polynomials in Definition 2.11, it follows for every \((x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
H_n(x, t) = t^{n/2} h_n \left( \frac{x}{\sqrt{t}} \right) = t^{n/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{2^{k}k!(n-2k)!} \left( \frac{x}{\sqrt{t}} \right)^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^{k}k!(n-2k)!} x^{n-2k} (-t)^k.
\]
Moreover, we observe that \( k \leq \lfloor n/2 \rfloor \) if and only if 2k \leq n, for all \((n, k) \in \mathbb{N}_0^2 \). Hence, by using (35) and reordering the sums, we conclude for every \( N \in \mathbb{N} \) and \((x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
\sum_{n=0}^{N} \frac{1}{n!} |H_n(x, t)| \leq \sum_{n=0}^{N} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{|x|^{n-2k} k!}{2^{k}k!(n-2k)!} = \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{k!}{2^{k}k!(N-2k)!} \sum_{n=2k}^{N} |x|^{n-2k} \leq \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(t/2)^k}{k!} \sum_{n=0}^{\infty} \frac{|x|^n}{n!} = e^{\frac{1}{2}t} e^{|x|} = e^{|x| + \frac{1}{2}t}.
\]
Finally, by taking the limit \( N \to \infty \), it follows that
\[
\lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} |H_n(x, t)| \leq e^{|x| + \frac{1}{2}t},
\]
which completes the proof. \( \square \)
The following lemma is a modification of [Jamshidian, 2005, Proposition 5.1] to this setting of $L^p(\mathbb{P})$, with $1 < p < \infty$, where we show that polynomial increments of $X$ are dense in $L^p(\mathbb{P})$, which was similarly applied in [Nualart and Schoutens, 2000, Proposition 2] for Lévy processes.

**Lemma 4.4.** Let $X$ be a diffusion process, whose coefficients are of linear growth, and let $p \in [1, \infty)$. Then,

$$\mathcal{P} := \text{span}\left\{ (\Delta X^i_{t_1})^{k_1} \cdots (\Delta X^i_{t_m})^{k_m} : m \in \mathbb{N}_0, (i_1, \ldots, i_m) \in \{1, \ldots, d\}^m, (k_1, \ldots, k_m) \in \mathbb{N}^m_0, 0 \leq t_0 \leq \ldots \leq \sum k_m \leq T \right\}$$

is dense in $L^p(\mathbb{P}) := L^p(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathcal{A} = \mathcal{F}_T$, where $\Delta X^i_{t_k} := X^i_{t_k} - X^i_{t_{k-1}}$.

**Proof.** Let $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ with convention $\frac{1}{\infty} := 0$. Now, we assume by contradiction that $\mathcal{P}$ is not dense in $L^p(\mathbb{P})$. Then, there exists by Hahn-Banach a non-zero continuous linear functional $l \in L^p(\mathbb{P})^*$, which vanishes on $\mathcal{P}$. By Riesz representation, $l : L^p(\mathbb{P}) \to \mathbb{R}$ can be represented with some $Z \in L^p(\mathbb{P})$ such that $l(Y) = \mathbb{E}[ZY]$, for all $Y \in L^p(\mathbb{P})$, which implies that $l(Y) = \mathbb{E}[ZY] = 0$, for all $Y \in \mathcal{P}$. Thus, it suffices to show that $Z = 0$, which contradicts that $l : L^p(\mathbb{P}) \to \mathbb{R}$ is non-zero.

Since the $\mathbb{P}$-augmentation of $\sigma \{ |X^i - X^i_s| : i = 1, \ldots, d, 0 \leq s < t \leq T \}$ is equal to $\mathcal{F}_T$, we have

$$L^p(\mathbb{P}) = \{ f : \Xi \in \mathbb{E}[|f(\xi)|^p] < \infty \},$$

where $\Xi := \{ \xi = (\Delta X^i_{t_1}, \ldots, \Delta X^i_{t_m})^\top : (i_1, \ldots, i_m) \in \{1, \ldots, d\}^m, 0 \leq t_0 \leq \ldots \leq \sum k_m \leq T \}$. Then, it follows by [Brézis, 2011, Theorem 4.12] that $\{ f(\xi) : \Xi \ni \xi, f \in C_c(\mathbb{R}^m) \}$ is dense in $L^p(\mathbb{P})$, where $C_c(\mathbb{R}^m)$ denotes the space of continuous functions with compact support. By mollification, every $f \in C_c(\mathbb{R}^m)$ can be uniformly approximated by smooth functions, which implies that $\{ f(\xi) : \Xi \ni \xi, f \in C_c^\infty(\mathbb{R}^m; \mathbb{C}) \}$ is dense in $L^p(\mathbb{P})$, see [Brézis, 2011, Corollary 4.23]), where $C_c^\infty(\mathbb{R}^m; \mathbb{C})$ denotes the space of smooth functions with compact support (see [Rudin, 1991, Example 11.46]). Thus, it suffices to show that $\mathbb{E}[Zf(\xi)] = 0$ for all $\Xi \ni \xi, f \in C_c^\infty(\mathbb{R}^m; \mathbb{C})$.

Now, for every fixed $m \in \mathbb{N}$ and $\xi \in \Xi$, we define the distribution $C_c^\infty(\mathbb{R}^m; \mathbb{C}) \ni f \mapsto \hat{U}(f) := \mathbb{E}[Zf(\xi)] \in \mathbb{C}$ and let $C_c^\infty(\mathbb{R}^m; \mathbb{C}) \ni f \mapsto \hat{U}(f) := \hat{U}(f) \in \mathbb{C}$ be its Fourier transform in the sense of distributions (see [Rudin, 1991, Definition 7.14]), where $\mathbb{R}^m \ni \lambda \mapsto \hat{f}(\lambda) := \int_{\mathbb{R}^m} e^{-ix \lambda} f(x) dx \in \mathbb{C}$ denotes the Fourier transform of $f$. Then, by Fubini’s theorem, follows for every $f \in C_c^\infty(\mathbb{R}^m; \mathbb{C})$

$$\hat{U}(f) = U(f) = \mathbb{E}[\hat{Z}(f)] = \mathbb{E}\left[ \int_{\mathbb{R}^m} e^{-i\xi \lambda} f(x) dx \right] = \int_{\mathbb{R}^m} \hat{u}(x) f(x) dx, \quad (36)$$

where we define the function $\mathbb{R}^m \ni x \mapsto \hat{u}(x) := \mathbb{E}[Ze^{-ix \xi}] \in \mathbb{C}$. Next, we show that $\hat{u} : \mathbb{R}^m \to \mathbb{C}$ is constant. For this purpose, we apply Hölder’s inequality, the Cauchy-Schwarz inequality, and the generalized Hölder inequality (with exponents $\sum_{k=1}^{m+1} \frac{1}{p_k} = 1$) to conclude for every $z \in \mathbb{C}^m$ that

$$\mathbb{E}\left[ \left| Ze^{-iz \xi} \right|^p \right] \leq \| Z \|_{L^p(\mathbb{P})} \mathbb{E}\left[ e^{c|z|^2} \right] \leq \| Z \|_{L^p(\mathbb{P})} \mathbb{E}\left[ e^{c|z|^2} \right] \leq \| Z \|_{L^p(\mathbb{P})} \prod_{k=0}^{m+1} \mathbb{E}\left[ e^{c|z|^2} \right] \leq \| Z \|_{L^p(\mathbb{P})} \prod_{k=0}^{m+1} \mathbb{E}\left[ e^{c|z|^2} \right],$$

where $\text{Im}(z) := (\text{Im}(z_1), \ldots, \text{Im}(z_m))^\top$, for $z \in \mathbb{C}^m$. Then, by using Lemma 2.4, there exists some $\varepsilon > 0$ such that $\max_{k=0,\ldots,m} \mathbb{E}\left[ e^{c|z|^2} \right] < \infty$, which implies that $|\mathbb{E}[Ze^{-iz \xi}]| < \infty$, for all $z \in V := \{ z \in \mathbb{C}^m : |\text{Im}(z)| < \frac{\varepsilon}{2(2m+1)^p} \}$.

From this, we see that the function $z \mapsto \hat{v}(z) := \mathbb{E}[Ze^{-iz \xi}]$ is holomorphic on $V$. Hence, the restriction $\hat{v}|_{\mathbb{R}^m} = \hat{u} : \mathbb{R}^m \to \mathbb{C}$ is real analytic and satisfies

$$\frac{\partial^n \hat{u}}{\partial \lambda_{j_1} \cdots \partial \lambda_{j_n}}(0) = (-i)^n \mathbb{E}[Z\xi_{j_1} \cdots \xi_{j_n}] = 0,$$

for all $n \in \mathbb{N}$ and $(j_1, \ldots, j_n) \in \{1, \ldots, d\}^n$. Since $\hat{u} : \mathbb{R}^m \to \mathbb{C}$ is analytic, $\hat{u} : \mathbb{R}^m \to \mathbb{C}$ is hence constant equal to $\hat{u}(0)$, which implies together with (36) that $\hat{U}(f) = \int_{\mathbb{R}^m} \hat{u}(x) f(x) dx = \hat{u}(0) \int_{\mathbb{R}^m} f(x) dx$, for all $f \in C_c^\infty(\mathbb{R}^m; \mathbb{C})$. Then, $U : C_c^\infty(\mathbb{R}^m; \mathbb{C}) \to \mathbb{C}$ is by [Rudin, 1991, Example II.7.16] a scalar multiple of the Dirac delta, i.e. $U(f) = \hat{u}(0) \mathbb{E}[Z] f(0)$, for all $f \in C_c^\infty(\mathbb{R}^m; \mathbb{C})$. However, by choosing $Y = 1 \in \mathcal{P}$, we have $\mathbb{E}[Z] = \mathbb{E}[ZY] = l(Y) = 0$, which implies $U(f) = \hat{u}(0) \mathbb{E}[Z] f(0) = 0$, for all $f \in C_c^\infty(\mathbb{R}^m; \mathbb{C})$. Finally, since $m \in \mathbb{N}$ and $\xi \in \Xi_m$ were chosen arbitrarily, it follows that $\mathbb{E}[Zf(\xi)] = 0$ for all $m \in \mathbb{N}$, $\xi \in \Xi_m$, and $f \in C_c^\infty(\mathbb{R}^m; \mathbb{C})$, which implies that $Z = 0$ and completes the proof. \qed
Proof of Lemma 2.15. Let \( p \in [1, \infty) \). For \( n = 0 \), this follows immediately. For \( n \in \mathbb{N} \), we use the explicit expression of the time-space harmonic Hermite polynomials \( (H_n)_{n \in \mathbb{N}_0} \) derived in (35) to conclude for every \( (x, t) \in \mathbb{R} \times [0, \infty) \) that
\[
|H_n(x, t)| \leq \sum_{k=0}^{[n/2]} \frac{n!}{2^k k!} \lambda^m \sum_{m=0}^{n} \frac{n!}{m! (n-m)!} \| \lambda^m \|_{L^p(X)} \leq \frac{n!}{2^n (n-2k)!} \sum_{k=0}^{[n/2]} \frac{n!}{2^k k!} \lambda^m \sum_{m=0}^{n} \frac{n!}{m! (n-m)!} \| \lambda^m \|_{L^p(X)}
\]
where \( \lambda := \max \{ n \in \mathbb{N} : n \leq x \} \), for \( x \in \mathbb{R} \). By using this, Minkowski’s inequality, and Lemma 2.10, it follows for every \( g \in L^{np}(X) \) that
\[
\mathbb{E}[|H_n(W(g), Q(g))|^p]^{1/p} \leq n! \mathbb{E}\left[\left(|W(g)| + \sqrt{Q(g)}\right)^{np}\right]^{1/p} \leq n! \left( \mathbb{E}\left[\sum_{t \in [0,T]} |W(g)_t|^{np}\right]\right)^{1/p} + \mathbb{E}\left[\sup_{t \in [0,T]} Q(g)^{np}\right]^{1/p} \leq n! C_1 np \|g\|_{L^{np}(X)} + n! \|g\|_{L^{np}(X)} < \infty.
\]
Hence, by definition of \( \mathcal{H}_n \) in (9), we conclude that \( \mathcal{H}_n \subset L^p(\mathbb{P}) \).

4.2.2. Proof of Theorem 2.16.

Proof of Theorem 2.16. Fix some \( p \in [1, \infty) \) and let \( q \in (1, \infty) \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) with convention \( \frac{1}{\infty} := 0 \). Now, we assume by contradiction that \( \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n \) is not dense in \( L^p(\mathbb{P}) \). Then, there exists by Hahn-Banach a non-zero continuous linear functional \( l : L^p(\mathbb{P}) \to \mathbb{R} \) can be represented with some \( Z \in L^q(\mathbb{P}) \) such that \( l(Z) = \mathbb{E}[ZY] \), for all \( Y \in L^p(\mathbb{P}) \), and thus \( l(H) = \mathbb{E}[ZH] = 0 \), for all \( H \in \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n \). Therefore, it suffices to show that \( Z = 0 \), which contradicts the assumption that \( l : L^p(\mathbb{P}) \to \mathbb{R} \) is non-zero.

Now, we fix some \( m \in \mathbb{N} \), \( i_1, \ldots, i_m \in \{1, \ldots, d\} \), \( (k_1, \ldots, k_m) \in \mathbb{N}^m \), and \( 0 \leq t_0 \leq \ldots \leq t_m \leq T \), and define the tuple
\[
j = (j_1, \ldots, j_k) := (1, \ldots, 1, \ldots, m, \ldots, m),
\]
where \( |j| := k_1 + \ldots + k_m \). Moreover, we choose some \( h > 0 \) with \( h < \left( 2p \left( 1 + 8C_{d,L,T,\|X_0\|} T \right) e \right)^{-1} \) and \( h^2 < \left( p \left( 1 + 4C_{d,L,T,\|X_0\|} T \right) e \right)^{-1} \), where the constant \( C_{d,L,T,\|X_0\|} > 0 \) was introduced in Lemma 4.1. In addition, we define the subsets \( Q := \times_{t=1}^{k} \left( \frac{k}{k+t}, \frac{k}{k+t} \right) \) and \( \overline{Q} := \times_{t=1}^{k} \left[ \frac{k}{k+t}, \frac{k}{k+t} \right] \) be of \( \mathbb{R}^{|j|} \). Then, we consider for every fixed \( \lambda \in \mathcal{Q} \) the simple function \( g_\lambda : [0, T] \to \mathbb{R}^d \) defined by
\[
g_\lambda(t) = \sum_{i=1}^{k} \lambda_i \mathbf{1}_{(t_{i-1}, t_i)}(t)e_{i,j_i},
\]
for \( t \in [0, T] \), where \( e_i \) denotes the \( i \)-th unit vector of \( \mathbb{R}^d \). Since \( g_\lambda \in D_t([0, T]; \mathbb{R}^d) \), Lemma 2.5 implies that \( g_\lambda \in L^{np}(X) \), for all \( n \in \mathbb{N} \). Moreover, we conclude that \( W(g_\lambda) = \sum_{t=1}^{k} \lambda_i \Delta^2 X^{i,j_i}_t \) and \( Q(g_\lambda) = \sum_{t=1}^{k} \lambda_i \Delta^2 \Delta(X^{i,j_i})_{t_{i-1}} \), with \( \Delta X^{i,j_i}_t = X^{i,j_i}_t - X^{i,j_i}_{t_{i-1}} \) and \( \Delta(X^{i,j_i})_{t_{i-1}} = (X^{i,j_i})_{t_{i}} - (X^{i,j_i})_{t_{i-1}} \).

In addition, by definition of \( \mathcal{H}_n \) in (9), it follows that \( H_n(W(g_\lambda), Q(g_\lambda)) \in \mathcal{H}_n \), which implies that \( \mathbb{E}[ZH_n(W(g_\lambda), Q(g_\lambda))] = 0, \) for all \( n \in \mathbb{N} \). Hence, by defining \( p\mathbb{E}(x, t) := \sum_{n=0}^{N} \frac{1}{n!} H_n(x, t) \), we conclude for every \( N \in \mathbb{N} \) that
\[
\mathbb{E}[Zp_N(W(g_\lambda), Q(g_\lambda))] = \sum_{n=0}^{N} \frac{1}{n!} \mathbb{E}[ZH_n(W(g_\lambda), Q(g_\lambda))] = 0.
\]
Hereby, the polynomial \((x, t) \mapsto p_N(x, t)\) converges by Lemma 4.2 pointwise to the function \((x, t) \mapsto \exp(x - \frac{1}{2}t)\), as \(N \to \infty\), and for every \(N \in \mathbb{N}_0\) and \((x, t) \in \mathbb{R} \times [0, \infty)\), it holds that

\[
|p_N(x, t)| \leq \sum_{n=0}^{N} \frac{1}{n!} |H_n(x, t)| \leq \sum_{n=0}^{\infty} \frac{1}{n!} |H_n(x, t)| \leq e^{\frac{|x|}{2} + \frac{1}{2}t}.
\]  

(38)

Moreover, since \(\|g_{\lambda}\|_\infty \leq \max_{l=1, \ldots, |k|} k_{l,l} |\lambda_l| \leq h\), we have \(\|(2p) \cdot g_{\lambda}\|_\infty \leq \left( (1 + 8C_{d,L,T} \cdot x_0 T) e^{-1} \right)^{-1}\) and \(\|\sqrt{p} \cdot g_{\lambda}\|_\infty^2 < \left( (1 + 4C_{d,L,T} \cdot x_0 T) e^{-1} \right)^{-1}\). Hence, we can apply Lemma 4.1 to conclude that

\[
\mathbb{E} \left[ e^{2p|W(g_{\lambda})|} \right] = \mathbb{E} \left[ e^{W((2p) \cdot g_{\lambda}) r} \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ e^{pQ(g_{\lambda})} \right] = \mathbb{E} \left[ e^{Q(\sqrt{p} g_{\lambda}) r} \right] < \infty.
\]  

(39)

Then, by using the generalized Hölder inequality (with exponents \(\frac{1}{q} + \frac{1}{r} + \frac{1}{p} = 1\)), it follows that

\[
\mathbb{E} \left[ |Z| e^{W(g_{\lambda}) + \frac{1}{2}Q(g_{\lambda})} \right] \leq \|Z\|_{L^q(p)} \mathbb{E} \left[ e^{2p|W(g_{\lambda})|} \right] \mathbb{E} \left[ e^{pQ(g_{\lambda})} \right] < \infty.
\]

Moreover, it holds by (38) that \(\|Z p_N(W(g_{\lambda}), Q(g_{\lambda}))\| \leq \|Z\| e^{W(g_{\lambda}) + \frac{1}{2}Q(g_{\lambda})} \cdot p\)-a.s., for all \(N \in \mathbb{N}_0\), where the latter is integrable. Thus, we can apply dominated convergence on (37) to conclude that

\[
\mathbb{E} \left[ Z e^{W(g_{\lambda}) - \frac{1}{2}Q(g_{\lambda})} \right] = \lim_{N \to \infty} \mathbb{E} \left[ Z p_N(W(g_{\lambda}), Q(g_{\lambda})) \right] = 0.
\]  

(40)

From this, we define the function \(\overline{U} \ni \lambda \mapsto u(\lambda) := \mathbb{E} \left[ Z e^{W(g_{\lambda}) - \frac{1}{2}Q(g_{\lambda})} \right] =: \mathbb{E} \left[ U(\lambda) \right] \in \mathbb{R}\), where

\[
U(\lambda) := Z e^{W(g_{\lambda}) - \frac{1}{2}Q(g_{\lambda})} = Z \prod_{l=1}^{[k]} \exp \left( \lambda_l \Delta X_{t_l}^{i_l} - \frac{1}{2} \lambda_l^2 \Delta(X_{t_l})_{t_l} \right)
\]

is a random function. Then, it follows from (40) that \(u(\lambda) = 0\), for all \(\lambda \in \overline{Q}\).

Now, we show by induction on \(l = 1, \ldots, |k|\) that the \(|k|\)-th partial derivative of \(u\) with respect to each but different direction \(\lambda_l\), for \(l = 1, \ldots, |k|\), evaluated at \(\lambda = 0 \in \mathbb{R}^{|k|}\), is equal to the expected \(|k|\)-th partial derivative of \(U\) with respect to each but different direction, evaluated at \(\lambda = 0 \in \mathbb{R}^{|k|}\), which means that we can interchange expectation and partial derivatives such that

\[
\frac{\partial^{[k]} u}{\partial \lambda_1 \cdots \partial \lambda_{|k|}}(0) = \mathbb{E} \left[ \frac{\partial^{[k]} U}{\partial \lambda_1 \cdots \partial \lambda_{|k|}}(0) \right] = \mathbb{E} \left[ Z \Delta X_{t_1}^{i_1} \cdots \Delta X_{t_l}^{i_l} \cdots \Delta X_{t_{m}}^{i_m} \right]_{\text{k}_l\text{-times}} = \mathbb{E} \left[ Z \left( \Delta X_{t_1}^{i_1} \right)^{k_1} \cdots \left( \Delta X_{t_m}^{i_m} \right)^{k_m} \right]
\]  

(41)

In order to prove (41), we choose some \(n \in \mathbb{N}\) with \(n > 2|k|/\rho\), and define \(r \in \mathbb{R}\) by \(\frac{1}{r} := \frac{1}{q} + \frac{1}{p} + \frac{|k|}{\rho}\), which implies that \(\frac{1}{r} < \frac{1}{q} + \frac{1}{p} = 1\), and thus \(r > 1\). Then, we conclude by the generalized Hölder inequality (with exponents \(\frac{1}{q} + \frac{1}{2p} + \sum_{k=1}^{[|k|]} \frac{1}{n} = 1\)), the inequalities \(e^{x-t} \leq e^{|x|}\) and \(|x-t|^n \leq 2^n (|x|^n + t^n)\) for any \((x, t) \in \mathbb{R} \times [0, \infty)\), that \(\|g_{\lambda}\|_\infty \leq h\), together with (39), and Lemma 4.1 that

\[
\sup_{\lambda \in \overline{Q}} \mathbb{E} \left[ \left| \frac{\partial^{[k]} U}{\partial \lambda_1 \cdots \partial \lambda_{|k|}}(\lambda) \right| \right] = \sup_{\lambda \in \overline{Q}} \mathbb{E} \left[ Z^r e^{r W(g_{\lambda}) - \frac{1}{2} Q(g_{\lambda})} \right] \prod_{l=1}^{[k]} \left| \Delta X_{t_l}^{i_l} - \lambda_l \Delta X_{t_l}^{i_l} \right|_{\text{n-times}}
\]

\[
\leq \|Z\|_{L^q(p)} \sup_{\lambda \in \overline{Q}} \mathbb{E} \left[ e^{2p|W(g_{\lambda})| - pQ(g_{\lambda})} \right] \prod_{l=1}^{[k]} \mathbb{E} \left[ \left| \Delta X_{t_l}^{i_l} - \lambda_l \Delta X_{t_l}^{i_l} \right|_{\text{n-times}}^n \right]^{\frac{r}{n}}
\]

\[
\leq \|Z\|_{L^q(p)} \mathbb{E} \left[ e^{W((2p) \cdot g_{\lambda})} \right]^{\frac{r}{p}} \prod_{l=1}^{[k]} \left( \mathbb{E} \left[ |W(g_{\lambda})|^n \right] + \left( \frac{h}{k_l} \right)^n \mathbb{E} \left[ Q(g_{\lambda})^n \right] \right)^{\frac{r}{n}} < \infty,
\]

where \(g_l := 1_{(t_{l-1}, t_l]} e_{i_l} \in D_t([0, T], \mathbb{R}^d)\). Hence, the family of random variables

\[
\left\{ \frac{\partial^{[k]} U}{\partial \lambda_1 \cdots \partial \lambda_{|k|}}(\lambda) : \lambda \in \overline{Q} \right\}
\]  

(42)
is by de la Vallée-Poussin’s theorem uniformly integrable. Since \( Q \ni \lambda \mapsto \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \in \mathbb{R} \) is continuous on \( Q \), Vitali’s convergence theorem shows for every \( \lambda \in Q \) and every sequence \( (\lambda^{(n)})_{n \in \mathbb{N}} \subseteq \mathbb{R}^{[k]} \) with
\[
\limsup_{n \to \infty} \| \lambda^{(n)} - \lambda \| = 0 \quad \text{that} \quad \limsup_{n \to \infty} \mathbb{E} \left[ \left| \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda^{(n)}) - \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right| \right] = 0, \quad \text{i.e. for every} \quad \lambda \in Q \quad \text{and} \quad \varepsilon > 0 \quad \text{there exists} \quad \delta > 0 \quad \text{with} \quad v \in [-\delta, \delta] \mathbb{E} \left[ \left| \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + v) - \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right| \right] \leq \varepsilon.
\]
This implies that
\[
\limsup_{n \to \infty} \sup_{v \in [-|h_n|, |h_n|]} \mathbb{E} \left[ \left| \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + v) - \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right| \right] = 0, \quad \text{for all} \quad \lambda \in Q \quad \text{and sequences} \quad (h_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \quad \text{with} \quad \limsup_{n \to \infty} |h_n| = 0.
\]
Now, we prove the induction initialization that \( Q \ni \lambda \mapsto \frac{\partial}{\partial \lambda_1} \mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (v) \right] \bigg|_{v = \lambda} \in \mathbb{R} \) is continuous on \( Q \) and
\[
\mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (\lambda) \right] = \frac{\partial}{\partial \lambda_1} \mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (v) \right] \bigg|_{v = \lambda}, \quad \text{for all} \quad \lambda \in Q.
\]
To this end, we apply the fundamental theorem of calculus and use Fubini’s theorem to conclude that
\[
\mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (\lambda + v) \bigg|_{v = 0} ^{\tilde{h} \varepsilon_1} \right] = \mathbb{E} \left[ \int_0 ^{\tilde{h}} \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (\lambda + s \varepsilon_1) ds \right] = \int_0 ^{\tilde{h}} \mathbb{E} \left[ \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + s \varepsilon_1) ds \right],
\]
for all \( \tilde{h} \in \mathbb{R} \) and \( \lambda \in Q \) such that \( \lambda + \tilde{h} \varepsilon_1 \in Q \). This and (43) imply for every \( \lambda \in Q \) and every sequence \( (h_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \setminus \{0\} \) with \( \limsup_{n \to \infty} |h_n| = 0 \) that
\[
\limsup_{n \to \infty} \left| \frac{1}{h_n} \mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (\lambda + v) \bigg|_{v = 0} ^{h_n \varepsilon_1} \right] - \mathbb{E} \left[ \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right] \right| = \limsup_{n \to \infty} \left| \frac{1}{h_n} \int_{\min(h_n,0)} ^{\max(h_n,0)} \mathbb{E} \left[ \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + v) - \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right] ds \right| = \limsup_{n \to \infty} \sup_{v \in [-|h_n|, |h_n|]} \mathbb{E} \left[ \left| \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + v) - \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right| \right] = 0.
\]
Since \( \lambda \mapsto \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \) is continuous on \( Q \) and (42) is uniformly integrable, Vitali’s convergence theorem implies for every \( \lambda \in Q \) and every sequence \( (\lambda^{(n)})_{n \in \mathbb{N}} \subseteq \mathbb{R}^{[k]} \) with \( \limsup_{n \to \infty} \| \lambda^{(n)} - \lambda \| = 0 \) that
\[
\limsup_{n \to \infty} \mathbb{E} \left[ \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda^{(n)}) - \mathbb{E} \left[ \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right] \right] \leq \limsup_{n \to \infty} \mathbb{E} \left[ \left| \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda^{(n)}) - \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right| \right] = 0.
\]
This and (44) show the induction initialization that \( \mathbb{E} \left[ \frac{\partial^{[k]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right] = \frac{\partial}{\partial \lambda_1} \mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (v) \right] \bigg|_{v = \lambda}, \quad \text{for all} \quad \lambda \in Q, \quad \text{and} \quad \lambda \mapsto \frac{\partial}{\partial \lambda_1} \mathbb{E} \left[ \frac{\partial^{[k-1]} \mathbf{U}}{\partial \lambda_2 \cdots \partial \lambda_{[k]}} (v) \right] \bigg|_{v = \lambda} \in \mathbb{R} \) is continuous on \( Q \).
Next, we prove the induction step, i.e. if for some \( l = 1, \ldots, |k| - 1 \), the function \( Q \ni \lambda \mapsto \frac{\partial}{\partial \lambda_1} \mathbb{E} \left[ \frac{\partial^{[l]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_l} (v) \right] \bigg|_{v = \lambda} \in \mathbb{R} \) is continuous on \( Q \) and for every \( \lambda \in Q \), it holds that
\[
\mathbb{E} \left[ \frac{\partial^{[l]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_l} (\lambda) \right] = \frac{\partial}{\partial \lambda_1} \mathbb{E} \left[ \frac{\partial^{[l]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_l} (v) \right] \bigg|_{v = \lambda},
\]
then \( Q \ni \lambda \mapsto \frac{\partial^{[l+1]} \mathbf{U}}{\partial \lambda_1 \cdots \partial \lambda_{l+1}} \mathbb{E} \left[ \frac{\partial^{[l]} \mathbf{U}}{\partial \lambda_{l+1} \cdots \partial \lambda_{[k]}} (v) \right] \bigg|_{v = \lambda} \in \mathbb{R} \) is also continuous on \( Q \) and (46) still holds true with \( l + 1 \) replacing \( l \). To see this, we use the induction hypothesis, apply the fundamental theorem
of calculus, and Fubini’s theorem such that
\[
\frac{\partial^j}{\partial \lambda_1 \cdots \partial \lambda_l} \mathbb{E} \left[ \frac{\partial^{[k]-l-1}U}{\partial \lambda_{l+2} \cdots \partial \lambda_{[k]}} (\lambda + v) \right]_{v=0}^{\hbar_{e+1}} = \mathbb{E} \left[ \frac{\partial^{[k]-1}U}{\partial \lambda_1 \cdots \partial \lambda_{l+2} \cdots \partial \lambda_{[k]}} (\lambda + v) \right]_{v=0}^{\hbar_{e+1}}
\]
\[
= \mathbb{E} \left[ \int_0^{\hbar} \frac{\partial^{[k]}U}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + se_{e+1}) ds \right] = \int_0^{\hbar} \mathbb{E} \left[ \frac{\partial^{[k]}U}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + se_{e+1}) \right] ds,
\]
for all \( \hbar \in \mathbb{R} \) and \( \lambda \in Q \) such that \( \lambda + \hbar e_{e+1} \in \overline{Q} \). Using this and the same arguments as in (44) and (45), we conclude that
\[
\limsup_{n \to \infty} \mathbb{E} \left[ \frac{1}{h_n} \frac{\partial^j}{\partial \lambda_1 \cdots \partial \lambda_l} \mathbb{E} \left[ \frac{\partial^{[k]-l-1}U}{\partial \lambda_{l+2} \cdots \partial \lambda_{[k]}} (\lambda + v) \right]_{v=0}^{h_n e_{e+1}} - \mathbb{E} \left[ \frac{\partial^{[k]}U}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right] \right]
\]
\[
\leq \limsup_{n \to \infty} \frac{1}{h_n} \int_{\min(h_n,0)}^{\max(h_n,0)} \mathbb{E} \left[ \frac{\partial^{[k]}U}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda + se_{e+1}) - \frac{\partial^{[k]}U}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (\lambda) \right] ds = 0,
\]
for all \( \lambda \in Q \) and sequences \( (h_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \{0\} \) with \( \limsup_{n \to \infty} |h_n| = 0 \). This together with (45) establishes the induction step such that (46) is true with \( l + 1 \) instead of \( l \).

Finally, by taking \( l = [k] \) in (46) and using that \( u : \overline{Q} \to \mathbb{R} \) satisfies \( u(\lambda) = 0 \), for all \( \lambda \in \overline{Q} \), we have
\[
0 = \frac{\partial^{[k]}u}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (0) = \mathbb{E} \left[ \frac{\partial^{[k]}U}{\partial \lambda_1 \cdots \partial \lambda_{[k]}} (0) \right] = \mathbb{E} \left[ Z \left( \Delta X_{1i}^1 \cdots \Delta X_{tm}^m \right)^k \right].
\]
Moreover, it holds that \( 0 = u(0) = \mathbb{E}[Z] \). Hence, by using that the linear span of polynomial increments of the form \( \left( \Delta X_{1i}^{1k} \right)^{k_1} \cdots \left( \Delta X_{tm}^{mk} \right)^{k_m} \), for \( m \in \mathbb{N}_0 \), \( i_1, \ldots, i_m \in \{1, \ldots, d\} \), \( k_1, \ldots, k_m \in \mathbb{N}^m \), and \( 0 \leq t_0 \leq \cdots \leq t_m \leq T \), is by Lemma 4.4 dense in \( L^p(\mathbb{P}) \), it follows that \( Z = 0 \). This contradicts however the assumption that \( \mathbb{I} : L^p(\mathbb{P}) \to \mathbb{R} \) is non-zero, which shows that \( \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n \) is dense in \( L^p(\mathbb{P}) \). \( \square \)

4.3. Proof of Results in Section 2.3.

4.3.1. Auxiliary Results: Iterated Integrals.

**Proof of Lemma 2.17.** We follow the proof of [Ryan, 2002, Proposition 2.1] and denote by \( \overline{L}^{np}(X)^\otimes \) the dual space of the \( \| \cdot \|_{L^{np}(X)} \)-completion of \( L^{np}(X) \). Since \( L^{np}(X)^\otimes \) is a vector space, it suffices to show that \( \| \cdot \|_{L^{np}(X)^\otimes} \) is a norm. First, we prove that \( \| \cdot \|_{L^{np}(X)^\otimes} \) is absolutely homogeneous, i.e. \( \| \alpha g \|_{L^{np}(X)^\otimes} = |\alpha| \| g \|_{L^{np}(X)^\otimes} \). For \( \alpha = 0 \), this is obvious, whereas for \( \alpha \neq 0 \) and \( g \in L^{np}(X)^\otimes \) with representation \( g = \sum_{j=1}^m \lambda_j g_{j,1} \otimes \cdots \otimes g_{j,n} \), it follows that
\[
\| \alpha g \|_{L^{np}(X)^\otimes} \leq \sum_{j=1}^m |\lambda_j| \| \alpha g_{j,1} \|_{L^{np}(X)} \prod_{k=2}^n \| g_{j,k} \|_{L^{np}(X)} = |\alpha| \sum_{j=1}^m |\lambda_j| \prod_{k=1}^n \| g_{j,k} \|_{L^{np}(X)}.
\]
Since this holds true for every representation, we have \( \| \alpha g \|_{L^{np}(X)^\otimes} \leq |\alpha| \| g \|_{L^{np}(X)^\otimes} \). Similarly,
\[
\| g \|_{L^{np}(X)^\otimes} = \| \alpha^{-1} \alpha g \|_{L^{np}(X)^\otimes} \leq \frac{1}{|\alpha|} \| \alpha g \|_{L^{np}(X)^\otimes},
\]
which implies that \( \| g \|_{L^{np}(X)^\otimes} \leq \| \alpha g \|_{L^{np}(X)^\otimes} \). This together with the reverse inequality shown above proves that \( \| \alpha g \|_{L^{np}(X)^\otimes} = |\alpha| \| g \|_{L^{np}(X)^\otimes} \).

Secondly, we show that \( \| \cdot \|_{L^{np}(X)^\otimes} \) is positive definite, i.e. \( \| g \|_{L^{np}(X)^\otimes} \geq 0 \), for all \( g \in L^{np}(X)^\otimes \), and \( \| g \|_{L^{np}(X)^\otimes} = 0 \) if and only if \( g = 0 \in L^{np}(X)^\otimes \). Since \( \| \cdot \|_{L^{np}(X)} \) is by Lemma 4.2 positive
Then, the multilinear form \( l = \sum_{j=1}^{m} \lambda_j g_{j,1} \otimes \cdots \otimes g_{j,n} \) is chosen arbitrarily and \( \{g_{j,k}\}_{j=1}^{\tilde{m}} \subset L^{np}(X) \) the multilinear form \( L^{np}(X) \times \cdots \times L^{np}(X) \ni (f_1, \ldots, f_n) \mapsto A(f_1, \ldots, f_n) := \prod_{k=1}^{n} l_k(f_k) \in \mathbb{R} \), which implies that

\[
|g(A)| \leq \sum_{j=1}^{m} |\lambda_j| \prod_{k=1}^{n} |l_k(g_{j,k})| \leq \prod_{k=1}^{n} \|l_k\|_{L^{np}(X)^{\otimes n}} \sum_{j=1}^{m} |\lambda_j| \prod_{k=1}^{n} \|g_{j,k}\|_{L^{np}(X)} \leq \varepsilon \prod_{k=1}^{n} \|l_k\|_{L^{np}(X)^{\otimes n}}.
\]

Since the value of \( g(A) \) does not depend on the representation and \( \varepsilon > 0 \) was chosen arbitrarily, we conclude that \( g(A) = 0 \). Next, we consider for every \( k = 1, \ldots, n \) the finite dimensional vector subspace span \( \{g_{j,k}\}_{j=1}^{\tilde{m}} \subset L^{np}(X) \) with basis \( \{g_{j,k}\}_{j=1}^{\tilde{m}} \subset L^{np}(X) \), for some \( \tilde{m} \in \mathbb{N} \). Hence, \( g \in L^{np}(X)^{\otimes n} \) can be represented as \( g = \sum_{j=1}^{\tilde{m}} \sum_{k=1}^{\tilde{m}} \tilde{\lambda}_{j,\ldots,j,n} \tilde{g}_{j,1} \otimes \cdots \otimes \tilde{g}_{j,n,n} \), for some \( \tilde{\lambda}_{j,\ldots,j,n} \in \mathbb{R} \), which implies that

\[
0 = g(A) = \sum_{j=1}^{\tilde{m}} \sum_{j=1}^{\tilde{m}} \tilde{\lambda}_{j_1,\ldots,j_n} \prod_{k=1}^{n} l_k(\tilde{g}_{j,k}) = l_1 \left( \sum_{j=1}^{\tilde{m}} \sum_{j=1}^{\tilde{m}} \tilde{\lambda}_{j_1,\ldots,j_n,1} \tilde{g}_{j,1} \prod_{k=2}^{n} l_k(\tilde{g}_{j,k}) \right).
\]

Now, we iteratively apply for every \( k = 2, \ldots, n - 1 \) that \( l_k \in L^{np}(X)^{*} \) was chosen arbitrarily and \( \{g_{j,k}\}_{j=1}^{\tilde{m}} \subset L^{np}(X) \) are linearly independent to conclude that \( \sum_{j=1}^{\tilde{m}} \tilde{\lambda}_{j_1,\ldots,j_n,1} \prod_{k=2}^{n} l_k(\tilde{g}_{j,k}) = 0 \). By using that \( \{\tilde{g}_{j,1}\}_{j=1}^{\tilde{m}} \subset L^{np}(X) \) are linearly independent, it follows for every \( j_1 \in I_1 \) that

\[
0 = \sum_{j_2=1}^{\tilde{m}} \sum_{j_1=1}^{\tilde{m}} \tilde{\lambda}_{j_1,\ldots,j_n,1} \prod_{k=2}^{n} l_k(\tilde{g}_{j,k}) = l_2 \left( \sum_{j_2=1}^{\tilde{m}} \sum_{j_1=1}^{\tilde{m}} \tilde{\lambda}_{j_1,\ldots,j_n,2} \tilde{g}_{j,2} \prod_{k=3}^{n} l_k(\tilde{g}_{j,k}) \right).
\]

Then, for every \( \varepsilon > 0 \), we have \( \|f + g\|_{L^{np}(X)^{\otimes n}} \leq \|f\|_{L^{np}(X)^{\otimes n}} + \varepsilon \).
which shows that \(|g(t)| \leq \|g\|_{L^{np}(X)^\otimes n}\), for all \(g \in L^{np}(X)^\otimes n\). Hence, \(l : L^{np}(X)^\otimes n \to \mathbb{R}\) is a bounded linear functional on \((L^{np}(X)^\otimes n, \|\cdot\|_{L^{np}(X)^\otimes n})\) of norm at most 1 such that \(\prod_{k=1}^{n} \|g_k\|_{L^{np}(X)} = l(g_1 \otimes \cdots \otimes g_n) \leq \|g_1 \otimes \cdots \otimes g_n\|_{L^{np}(X)^\otimes n}\). This shows together with the reverse inequality from above that \(\|g_1 \otimes \cdots \otimes g_n\|_{L^{np}(X)^\otimes n} = \prod_{k=1}^{n} \|g_k\|_{L^{np}(X)}\). \(\square\)

**Proof of Lemma 2.21.** In order to show that \(J_n : L^{np}(X)^\otimes n \to L^0(\mathbb{P})\) is linear, we define the multilinear map \(L^{np}(X) \times \cdots \times L^{np}(X) \ni (g_1, ..., g_n) \mapsto A(g_1, ..., g_n) := J_n(g_1 \otimes \cdots \otimes g_n) \to L^0(\mathbb{P})\). Then, there exists by the universal property of the tensor product (see [Ryan, 2002, Proposition 1.4]) a linear map \(\tilde{A} : L^{np}(X)^\otimes n \to L^0(\mathbb{P})\) such that \(\tilde{A}(g_1 \otimes \cdots \otimes g_n) = A(g_1, ..., g_n)\), for all \(g_1, ..., g_n \in L^{np}(X)\). However, since \(\tilde{A}(g_1 \otimes \cdots \otimes g_n) = J_n(g_1 \otimes \cdots \otimes g_n)\), for all \(g_1, ..., g_n \in L^{np}(X)\), we have \(\tilde{A} = J_n\) on \(L^{np}(X)^\otimes n\), which shows that \(J_n\) is also linear.

For the second part, by linearity of \(J_n : L^{np}(X)^\otimes n \to L^0(\mathbb{P})\), it suffices to show that \(J_n(0) = 0\), \(\mathbb{P}\)-a.s., for each representation \(g = \sum_{k=1}^{m} \lambda_j g_{k_1} \otimes \cdots \otimes g_{k_n}\) of \(0 \in L^{np}(X)^\otimes n\). For some fixed \(C \in A\), we define the multilinear form \(L^{np}(X) \times \cdots \times L^{np}(X) \ni (f_1, ..., f_n) \mapsto \tilde{A}(f_1, ..., f_n) := \mathbb{E} [\mathbb{I}_C J_n(f_1 \otimes \cdots \otimes f_n)] \in \mathbb{R}\). Then, it follows for every representation \(g := \sum_{j=1}^{m} \lambda_j g_{j_1} \otimes \cdots \otimes g_{j_n} \in L^{np}(X)^\otimes n\) of \(0 \in L^{np}(X)^\otimes n\) that

\[
\mathbb{E} [\mathbb{I}_C J_n(g)] = \sum_{j=1}^{m} \lambda_j \mathbb{E} [\mathbb{I}_C J_n(g_{j_1} \otimes \cdots \otimes g_{j_n})] = \sum_{k=1}^{m} \lambda_j A(g_{j_1}, ..., g_{j_n}) = g(A).
\]

Since the value of \(g(A)\) is independent of the representation of \(0 \in L^{np}(X)^\otimes n\), it follows by choosing \(g = 0 \in L^{np}(X)^\otimes n\) that \(\mathbb{E} [\mathbb{I}_C J_n(g)] = g(A) = 0\). However, because \(C \in A\) was chosen arbitrarily, we obtain \(J_n(0) = 0\), \(\mathbb{P}\)-a.s., which completes the proof. \(\square\)

**Proof of Lemma 2.22.** For \(n = 1\), (11) follows from Lemma 2.10 and since \(J_1(g_t) = W(g_t)\), for all \(t \in [0, T]\). For \(n \in \mathbb{N} \cap [2, \infty)\), let \(g_1, ..., g_n \in L^{np}(X)\) and define \(g_{1:k} := g_1 \otimes \cdots \otimes g_k \in L^{np}(X)^\otimes k\).

For fixed \(k = 2, ..., n\), Hölder’s inequality (with exponents \(\frac{k-1}{k} + \frac{1}{k} = 1\)) implies that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^T J_{k-1}(g_{1:(k-1)}) t g_k(t)^\top b(t, X_t) dt \right|^{\frac{n}{np}} \right]^{\frac{k}{n}} \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left| J_{k-1}(g_{1:(k-1)}) t \right|^{\frac{n}{np}} \right]^{\frac{k}{n}} \mathbb{E} \left[ \left| \int_0^T g_k(t)^\top b(t, X_t) dt \right|^{np} \right]^{\frac{1}{np}} \quad (47)
\]

Similarly, by the Burkholder-Davis-Gundy inequality (with exponent \(\frac{np}{n}\) and constant \(C_{np} > 0\)) and Hölder’s inequality (with exponents \(\frac{k-1}{k} + \frac{1}{k} = 1\)), it follows that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^T J_{k-1}(g_{1:(k-1)}) t g_k(t)^\top \sigma(t, X_t) dB_t \right|^{\frac{np}{np}} \right]^{\frac{k}{n}} \leq C_{np} \mathbb{E} \left[ \left| \int_0^T J_{k-1}(g_{1:(k-1)}) t^2 g_k(t)^\top a(t, X_t) g_k(t) dt \right|^{\frac{np}{np}} \right]^{\frac{k}{n}} \quad (48)
\]
Hence, by using Minkowski’s inequality as well as (47) and (48), it follows for every \( k = 2, \ldots, n \) that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_k(g_{1,k}) t^{\frac{np}{2}}| \right]^\frac{k}{np} \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^T J_{k-1}(g_{1,(k-1)}) t g_k(t) ^\top b(t, X_t) dt \right|^{\frac{np}{2}} \right]^\frac{k}{np}
\]

\[\ldots + \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^T J_{k-1}(g_{1,(k-1)}) t g_k(t) ^\top \sigma(t, X_t) dB_t \right|^{\frac{np}{2}} \right]^\frac{k}{np} \]

\[
\leq \max \left( 1, C_{np} \right) \mathbb{E} \left[ \sup_{t \in [0,T]} |J_{k-1}(g_{1,(k-1)}) t^{\frac{np}{2}}| \right]^\frac{k-1}{np} \|g_k\|_{L^{np}(X)}.
\]

By applying this argument iteratively backwards for \( k = n, n-1, \ldots, 3, 2 \), and using that \( J_1(g_1)_t = W(g_1)_t \), for all \( t \in [0, T] \), it follows with the help of Lemma 2.10 that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_n(g)_t| \right]^\frac{1}{p} \leq \max \left( 1, C_p \right) \mathbb{E} \left[ \sup_{t \in [0,T]} |J_{n-1}(g_{1,(n-1)}) t^{\frac{np}{2}}| \right]^\frac{n-1}{np} \|g_n\|_{L^{np}(X)}
\]

\[
\leq \cdots \leq \left( \prod_{k=2}^{n} \max \left( 1, C_{np} \right) \|g_k\|_{L^{np}(X)} \right) \mathbb{E} \left[ \sup_{t \in [0,T]} |J_1(g_1)_t|^{np} \right]^{\frac{1}{np}}
\]

\[
\leq \left( \prod_{k=2}^{n} \max \left( 1, C_{np} \right) \|g_k\|_{L^{np}(X)} \right) C_{1, np} \|g_1\|_{L^{np}(X)} \leq C_{n, np} \prod_{k=1}^{n} \|g_k\|_{L^{np}(X)}.
\]

For the general case of \( g \in L^{np}(X)^{\otimes n} \), we fix \( \varepsilon > 0 \) and assume that \( \sum_{j=1}^{m} \lambda_j g_{j,1} \otimes \cdots \otimes g_{j,n} \) is a representation of \( g \in L^{np}(X)^{\otimes n} \) such that \( \sum_{j=1}^{m} |\lambda_j| \prod_{k=1}^{n} \|g_{j,k}\|_{L^{np}(X)} \leq \|g\|_{L^{np}(X)^{\otimes n}} + \varepsilon/C_{n, p} \). Then, it follows by using Minkowski’s inequality and (50) that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_n(g)_t| \right]^\frac{1}{p} \leq \sum_{j=1}^{m} |\lambda_j| \mathbb{E} \left[ \sup_{t \in [0,T]} |J_n(g_{j,1} \otimes \cdots \otimes g_{j,n})_t| \right]^\frac{1}{p}
\]

\[
\leq C_{n, p} \prod_{k=1}^{m} |\lambda_j| \prod_{k=1}^{n} \|g_{j,k}\|_{L^{np}(X)} \leq C_{n, p} \|g\|_{L^{np}(X)^{\otimes n}} + \varepsilon.
\]

Since \( \varepsilon > 0 \) was chosen arbitrarily, we obtain the conclusion in (11). \( \square \)

**Proof of Lemma 2.23.** We follow the proof of [Segall and Kailath, 1976, Theorem 1]. For \( g_0 \in L^{np}(X) \), we define for every \( n \in \mathbb{N} \) the process \( t \mapsto P^{(n)}_t := J_n(g^{(n)}_0)_t \), with \( P^{(0)}_t := 1 \) and \( P^{(-1)}_t := 0 \). First, we show the Kailath-Segall identity, i.e. it holds for every \( t \in [0, T] \) and \( n \in \mathbb{N} \), \( \mathbb{P} \)-a.s., that

\[
nP^{(n)}_t = W(g_0)_t P^{(n-1)}_t - \left( W(g_0)_t, P^{(n-2)}_t \right) _{t = Q(g_0)_t},\]

which is by definition true for \( n = 1 \). In order to show the induction step, we assume that (51) holds true for \( n - 1 \) with \( n \in \mathbb{N} \cap [2, \infty) \). Then, by applying Ito’s formula on the function \( f(x, y) = xy \) twice, it follows for every \( t \in [0, T] \), \( \mathbb{P} \)-a.s., that

\[
P^{(n)}_t = \int_0^t P^{(n-1)}_s dW(g_0)_s
\]

\[
= W(g_0)_t P^{(n-1)}_t - \int_0^t W(g_0)_s P^{(n-2)}_s dW(g_0)_s - \int_0^t P^{(n-2)}_s dQ(g_0)_s
\]

\[
= W(g_0)_t P^{(n-1)}_t - \int_0^t W(g_0)_s P^{(n-2)}_s dW(g_0)_s - Q(g_0)_t P^{(n-2)}_t
\]

\[
\ldots + \int_0^t Q(g_0)_s P^{(n-3)}_s dW(g_0)_s.
\]
Moreover, by using the induction hypothesis $W(g_0)_t P^{(n-2)}_s - Q(g_0)_t P^{(n-3)}_s = (n - 1) P^{(n-1)}_s$, and the fact that $\int_0^t P^{(n-1)}_s dW(g_0)_t = P^{(n)}_t$, we conclude for every $t \in [0, T]$, $\mathbb{P}$-a.s., that
\[
\begin{align*}
P^{(n)}_t &= W(g_0)_t P^{(n-1)}_t - \int_0^t \left( W(g_0)_s P^{(n-2)}_s - Q(g_0)_s P^{(n-3)}_s \right) dW(g_0)_t - Q(g_0)_t P^{(n-2)}_t \\
&= W(g_0)_t P^{(n-1)}_t - (n - 1) \int_0^t P^{(n-1)}_s dW(g_0)_t - Q(g_0)_t P^{(n-2)}_t \\
&= W(g_0)_t P^{(n-1)}_t - (n - 1) P^{(n)}_t - Q(g_0)_t P^{(n-2)}_t.
\end{align*}
\]
Hence, by adding $(n - 1) P^{(n)}_t$ on both sides, the Kailath-Segall identity in (51) follows for each $n \in \mathbb{N}$. Moreover, we recall that the rescaled time-space harmonic Hermite polynomials $\left( \frac{1}{n!} H_n(x, t) \right)_{n \in \mathbb{N}}$ satisfy by Lemma 4.2 the recurrence relation
\[
n \left( \frac{1}{n!} H_n(x, t) \right) = x \left( \frac{1}{(n - 1)!} H_{n-1}(x, t) \right) - t \left( \frac{1}{(n - 2)!} H_{n-2}(x, t) \right),
\]
for all $(x, t) \in \mathbb{R} \times [0, \infty)$, with $H_1(x, t) = x$ and $H_0(x, t) = 1$. Therefore, by choosing $x := W(g_0)_t$ and $t := Q(g_0)_t$, we see that $\left( \frac{1}{n!} H_n(W(g_0)_t, Q(g_0)_t) \right)_{n \in \mathbb{N}}$ satisfies the same recurrence as $\left( P^{(n)}_t \right)_{n \in \mathbb{N}}$ in (51), which implies the desired result.

4.3.2. Proof of Theorem 2.25.

**Proof of Theorem 2.25.** As $\bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n$ is by Theorem 2.16 dense in $L^p(\mathbb{P})$, there exists some $S \in \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n$ of the form $S = \sum_{n=0}^N H_n$, with $N \in \mathbb{N}$ and $H_n \in \mathcal{H}_n$, for all $n = 0, \ldots, N$, such that
\[
||G - S||_{L^p(\mathbb{P})} = \left\| G - \sum_{n=0}^N H_n \right\|_{L^p(\mathbb{P})} < \varepsilon. \tag{52}
\]
By Definition 2.14, every $H_n \in \mathcal{H}_n$ is a linear combination of random variables of the form $H_n(W(g), Q(g))$ with $g \in L^{np}(X)$, which implies $H_n = \frac{1}{m} \sum_{j=1}^m \lambda_{n,j} H_n(W(g_{n,j}), Q(g_{n,j}))$, for some $m \in \mathbb{N}_0$, $\lambda_{n,j} \in \mathbb{R}$ and $g_{n,j} \in L^{np}(X)$. From this, we define for every $n = 0, \ldots, N$ the function $g_n = \sum_{j=1}^m \lambda_{n,j} g_{n,j} \in L^{np}(X)^{\otimes n}$. Then, Lemma 2.23 and linearity of $J_n$ show
\[
H_n = \frac{1}{m} \sum_{j=1}^m \lambda_{n,j} H_n(W(g_{n,j}), Q(g_{n,j})) = \sum_{j=1}^m \lambda_{n,j} J_n \left( g_{n,j}^{\otimes n} \right) = J_n(g_n),
\]
$\mathbb{P}$-a.s., for all $n = 0, \ldots, N$. This proves together with (52) the conclusion in (12). \qed

4.4. Proof of Results in Section 2.4.

4.4.1. Auxiliary Results: Iterated Integral is a martingale under ELMM.

**Lemma 4.5.** Let $X$ be a diffusion process, let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, and assume that there exists an ELMM $\mathbb{Q} \sim \mathbb{P}$ for $X$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^p(\mathbb{P})$. Then, for every $n \in \mathbb{N}$ and $g \in L^{np}(X)^{\otimes n}$, the process $t \mapsto J_n(g)_t$ is a martingale under $\mathbb{Q}$.

**Proof.** First, we observe for some $g \in L^{np}(X)^{\otimes n}$ that $t \mapsto J_n(g)_t$ is a local martingale under $\mathbb{Q}$. Moreover, by applying Hölder’s inequality and Lemma 2.22, we conclude that
\[
\mathbb{E}^\mathbb{Q} \left[ \sup_{t \in [0, T]} |J_n(g)_t| \right] = \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \sup_{t \in [0, T]} |J_n(g)_t| \right] \leq \left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|_{L^q(\mathbb{P})} \mathbb{E} \left[ \sup_{t \in [0, T]} |J_n(g)_t|^p \right]^{\frac{1}{p}} \leq C_{n,p} \left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|_{L^q(\mathbb{P})} \|g\|_{L^{np}(X)^{\otimes n}} < \infty,
\]
which shows that $t \mapsto J_n(g)_t$ is a true martingale under $\mathbb{Q}$. \qed
4.4.2. Proof of Theorem 2.27 and Corollary 2.28.

Proof of Theorem 2.27. For every \( M \in \mathbb{N} \), we apply Theorem 2.25 with \( \varepsilon = 1/M \) to obtain some \( N(M) \in \mathbb{N} \), \( g_0^{(M)} \in \mathbb{R} \) and \( g_n^{(M)} \in L_{\text{diag}}^{np}(X)^{\otimes n} \), for \( n = 1, \ldots, N(M) \), such that

\[
\left\| G - \sum_{n=0}^{N(M)} J_n \left( g_n^{(M)} \right) \right\|_{L^p(\mathbb{P})} < \frac{1}{M}. \tag{53}
\]

Hereby, we can write \( g_n^{(M)} = \sum_{j=1}^{m_n^{(M)}} \lambda_{n,j}^{(M)} \left( g_{n,j}^{(M)} \right)^{\otimes n} \in L_{\text{diag}}^{np}(X)^{\otimes n} \), for some \( m_n^{(M)} \in \mathbb{N} \), \( \lambda_{n,j}^{(M)} \in \mathbb{R} \), and \( g_{n,j}^{(M)} = \left( g_{n,j}^{(M)} \right)^{\top} \in L^{np}(X) \). Now, for every \( M \in \mathbb{N} \), \( n = 1, \ldots, N(M) \), and \( j = 1, \ldots, m_n^{(M)} \), we define \( \theta_{n,j}^{(M)} = \left( \theta_{n,j,t}^{(M)} \right)^{\top} \) by \( \theta_{n,j,t}^{(M)} := \lambda_{n,j}^{(M)} J_{n-1} \left( \left( g_{n,j}^{(M)} \right)^{\otimes(n-1)} \right) g_{n,j}^{(M)}(t) \), for \( i = 1, \ldots, d \) and \( t \in [0, T] \), which is \( \mathbb{F} \)-adapted and left-continuous, thus \( \mathbb{F} \)-predictable and locally bounded. Then, by Minkowski’s inequality, Hölder’s inequality (with exponents \( \frac{p}{n-1} \) and \( \frac{p}{n} = 1 \)), iteratively applying (49) as in the proof of Lemma 2.22, and using Lemma 2.10, it follows that

\[
\| g_{n,j}^{(M)} \|_{\Theta_p} = \mathbb{E} \left[ \left\| \int_0^T \left( \theta_{n,j,t}^{(M)} \right)^{\top} a(t, X_t) g_{n,j}^{(M)}(t) dt \right\|^p \right]^{\frac{1}{p}} \leq \lambda_{n,j}^{(M)} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| J_{n-1} \left( \left( g_{n,j}^{(M)} \right)^{\otimes(n-1)} \right) \right| \mathbb{E} \left[ \left| \int_0^T g_{n,j}^{(M)}(t)^{\top} a(t, X_t) g_{n,j}^{(M)}(t) dt \right|^p \right]^{\frac{1}{p}} \right] \leq \cdots \leq \lambda_{n,j}^{(M)} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| J_{n-1} \left( \left( g_{n,j}^{(M)} \right)^{\otimes(n-1)} \right) \right| \mathbb{E} \left[ \left| \int_0^T g_{n,j}^{(M)}(t)^{\top} a(t, X_t) g_{n,j}^{(M)}(t) dt \right|^p \right]^{\frac{1}{p}} \right] \leq C_{n,p} \lambda_{n,j}^{(M)} \mathbb{E} \left[ \left| \left( g_{n,j}^{(M)} \right)^{\otimes n} \right| \right]_{L^{np}(X)} < \infty,
\]

which shows that \( \theta_{n,j}^{(M)} \in \Theta_p \), and thus \( J_n \left( g_{n,j}^{(M)} \right) = \sum_{i=1}^d J_0 \theta_{n,j,i}^{(M)} dX_i \in \mathcal{G}_p \), for all \( M \in \mathbb{N} \), \( n = 1, \ldots, N(M) \), and \( j = 1, \ldots, m_n^{(M)} \). Hence, by using that \( \mathcal{G}_p \) is a vector space, we conclude that \( \sum_{n=1}^{N(M)} J_n \left( g_{n,j}^{(M)} \right) = \sum_{n=1}^{N(M)} \sum_{j=1}^{m_n^{(M)}} \lambda_{n,j}^{(M)} J_n \left( g_{n,j}^{(M)} \right) \in \mathcal{G}_p \), for all \( M \in \mathbb{N} \).

Now, let \( \mathbb{Q} \) be the ELMM with \( \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^p(\mathbb{P}) \). Then, every process \( t \rightarrow J_n \left( g_{n,j}^{(M)} \right) \) is by Lemma 4.4.5 a martingale under \( \mathbb{Q} \), which implies that \( \mathbb{E}^{\mathbb{Q}} \left[ J_n \left( g_{n,j}^{(M)} \right) \right] = 0 \). Hence, by Minkowski’s inequality, Hölder’s inequality, and (53), it follows for every \( M \in \mathbb{N} \) that

\[
\left\| G - \mathbb{E}^{\mathbb{Q}}[G] - \sum_{n=1}^{N(M)} J_n \left( g_{n,j}^{(M)} \right) \right\|_{L^p(\mathbb{P})} \leq \left\| G - \sum_{n=0}^{N(M)} J_n \left( g_n^{(M)} \right) \right\|_{L^p(\mathbb{P})} + \left\| \mathbb{E}^{\mathbb{Q}}[G] - g_0^{(M)} \right\|_{L^p(\mathbb{P})} < \frac{1}{M} + \left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|_{L^p(\mathbb{P})} \left\| G - \sum_{n=0}^{N(M)} J_n \left( g_n^{(M)} \right) \right\|_{L^p(\mathbb{P})} < \left( 1 + \left\| \frac{d\mathbb{Q}}{d\mathbb{P}} \right\|_{L^p(\mathbb{P})} \right) \frac{1}{M}.
\]
Then, there exists some \( \theta \) which shows that the condition in (13) is satisfied. Hence, the assumptions of Theorem 2.27 are fulfilled, which contains functions of \( L \) and Jensen’s inequality, the inequality \( \sum_{i=1}^{d} x_i \leq \frac{1}{2} \sum_{i=1}^{d} x_i^2 \) for any \( x_i, y_i \geq 0 \) and Doob’s inequality (with constant \( C_{D,p} > 0 \)), it follows for every \( \theta \in \Theta \) that

\[
\|\theta\|_{\Theta} \leq \frac{1}{c_p} E \left[ \sup_{t \in [0, T]} \left| \int_{0}^{T} \theta_s dX_s \right|^p \right] \leq \frac{C_{D,p}}{c_p} E \left[ \left( \int_{0}^{T} \theta_t dX_t \right)^p \right],
\]

which shows that the condition in (13) is satisfied. Hence, the assumptions of Theorem 2.27 are fulfilled, which immediately yields the representation in (14).

4.5. Proof of Results in Section 2.5. For the results of this section, we define for every \( p \in [1, \infty) \) the vector subspace of \( L^p(X) \) given by

\[
L^p_f(X) := \left\{ g := (g_1, \ldots, g_d)^\top \in L^p(X) : \|g_i\|_{L^{\max(p,2)}(dt)} < \infty, \text{ for all } i = 1, \ldots, d \right\} \subseteq L^p(X)
\]

which contains functions of \( L^p(X) \), whose components belong to \( L^{\max(p,2)}(dt) \).

4.5.1. Auxiliary Results: Universal Approximation.

Lemma 4.6. Let \( X \) be a diffusion process, whose coefficients are of linear growth, and let \( p \in [1, \infty) \). Then, there exists some \( C_{L^p} > 0 \) such that for every \( t \mapsto g(t) = (g_1(t), \ldots, g_d(t))^\top \in L^p_f(X) \), we have

\[
\|g\|_{L^p(X)} \leq C_{L^p} \sum_{i=1}^{d} \|g_i\|_{L^{\max(p,2)}(dt)}.
\]

Proof. Fix some \( g \in L^p_f(X) \) for some \( p \in [1, \infty) \). Then, in the case of \( p \in [1, 2] \), it follows by Jensen’s inequality, the inequality \( \sum_{i=1}^{d} x_i \leq \sum_{i=1}^{d} x_i^2 \) for any \( x_1, \ldots, x_d \geq 0 \), and the inequality \( \sqrt{x} + \sqrt{y} \leq \sqrt{2} \sqrt{x+y} \) for any \( x, y \geq 0 \) that

\[
\|g\|_{L^p(X)} \leq E \left[ \left( \int_{0}^{T} |g(t)^\top b(t, X_t)| dt \right)^\frac{1}{p} \right] + E \left[ \left( \int_{0}^{T} |g(t)^\top a(t, X_t)g(t)| dt \right)^\frac{2}{p} \right] \leq \sqrt{T} E \left[ \left( \int_{0}^{T} \|g(t)^\top b(t, X_t)\| dt \right)^\frac{2}{p} \right] + E \left[ \left( \int_{0}^{T} \|g(t)^\top a(t, X_t)\|_{F} dt \right)^\frac{2}{p} \right]
\]

\[
\leq \sqrt{T} E \left[ \sum_{i=1}^{d} \int_{0}^{T} |g_i(t)|^2 \|b(t, X_t)\|^2 dt \right] + E \left[ \sum_{i=1}^{d} \int_{0}^{T} |g_i(t)|^2 \|a(t, X_t)\|_{F} dt \right] \leq \sqrt{2} \max \left( \sqrt{T}, 1 \right) \left( \sum_{i=1}^{d} E \left[ \int_{0}^{T} |g_i(t)|^2 \left( \|b(t, X_t)\|^2 + \|a(t, X_t)\|_{F} \right) dt \right] \right)^\frac{1}{2}
\]

\[
\leq 2\sqrt{d} \max \left( T, 1 \right) \sum_{i=1}^{d} \left( \int_{0}^{T} |g_i(t)|^2 E \left[ \|b(t, X_t)\|^2 + \|a(t, X_t)\|_{F} \right] dt \right)^\frac{1}{2}.
\]

On the other hand, for the case of \( p \in (2, \infty) \), we conclude by Jensen’s inequality, the inequalities \( x^\frac{1}{p} + y^\frac{1}{p} \leq 2^{-\frac{1}{p}}(x + y)^\frac{1}{p} \) for any \( x, y \geq 0 \), \( \sum_{i=1}^{d} x_i \leq d^{\frac{1}{p}-1} \sum_{i=1}^{d} x_i^\frac{2}{p} \) and \( \left( \sum_{i=1}^{d} x_i \right)^\frac{2}{p} \leq
Theorem 7.3.2.1(5) that the number of discontinuity points of \(\sum \phi\) follows that

\[
\sum_{i=1}^{d} x_i^\frac{1}{p} \text{ for any } x_1, \ldots, x_d \geq 0 \text{ that }
\]

\[
\|g\|_{L^p(X)} = \mathbb{E} \left[ \left( \int_0^T |g(t)^\top b(t, X_t)| dt \right)^p \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( \int_0^T |g(t) a(t, X_t) g(t)| dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}
\]

\[
\leq \sqrt{T} \mathbb{E} \left[ \left( \sum_{i=1}^{d} \int_0^T |g_i(t)|^2 \|b(t, X_t)\|^2 dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} + \mathbb{E} \left[ \left( \sum_{i=1}^{d} \int_0^T |g_i(t)|^2 \|a(t, X_t)\|_{F} dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}
\]

\[
\leq 2^{1-\frac{1}{p}} \max \left( \sqrt{T}, 1 \right) \mathbb{E} \left[ \left( \sum_{i=1}^{d} \int_0^T |g_i(t)|^2 \|b(t, X_t)\|^2 dt \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \left( \sum_{i=1}^{d} \int_0^T |g_i(t)|^2 \|a(t, X_t)\|_{F} dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}
\]

\[
\leq 2d^{\frac{1}{2}-\frac{1}{p}} \max \left( \sqrt{T}, 1 \right) \mathbb{E} \left[ \sum_{i=1}^{d} \left( \int_0^T |g_i(t)|^{p} \|b(t, X_t)\|^p + \|a(t, X_t)\|_{F}^{p} \|t, X_t\| \right) dt \right]^{\frac{1}{p}}
\]

\[
\leq 2 \sqrt{d} \max \left( \sqrt{T}, 1 \right) \sum_{i=1}^{d} \left( \int_0^T |g_i(t)|^{p} \mathbb{E} \left[ \|b(t, X_t)\|^p + \|a(t, X_t)\|_{F}^{p} \right] dt \right)^{\frac{1}{p}}.
\]

(55)

Moreover, by the linear growth condition of \(a\) and \(b\) (with constant \(C_L > 0\)), the inequalities \(x^{\max(p,2)} \leq 1 + x^{p+1}\) and \(x^{\max(p/2,1)} \leq 1 + x^{p+1}\) for any \(x \geq 0\), the inequality \((x + y)^p \leq 2^{p-1} (x^p + y^p)\) for any \(x, y \geq 0\), and Lemma 2.4, it follows for every \(t \in [0, T]\) that

\[
\mathbb{E} \left[ \|b(t, X_t)\|^{\max(p,2)} + \|a(t, X_t)\|_{F}^{\max(\frac{p}{2},1)} \right] \leq 2 + \mathbb{E} \left[ \|b(t, X_t)\|^{p+1} \right] + \mathbb{E} \left[ \|a(t, X_t)\|_{F}^{p+1} \right]
\]

\[
\leq 2 + C_{L+1}^{p+1} \mathbb{E} \left[ (1 + \|X_t\|)^{p+1} \right] + C_{L+1}^{p+1} \mathbb{E} \left[ (1 + \|X_t\|)^{p+1} \right]
\]

\[
\leq 2 + 2C_{L+1}^{p+1} \left( 1 + \sup_{t \in [0, T]} \mathbb{E} \left[ \|X_t\|^{p+1} \right] \right) =: C_{p, X} < \infty.
\]

(56)

Hence, by defining \(C_{L,p} := 2 \sqrt{d} \max(T, 1) C_{p, X}^{\max(p,2)} > 0\) and inserting (56) into (54) resp. (55), it follows that

\[
\|g\|_{L^p(X)} \leq 2 \sqrt{d} \max(T, 1) C_{p, X}^{\max(p,2)} \sum_{i=1}^{d} \left( \int_0^T |g_i(t)|^{\max(p,2)} \mathbb{E} \left[ \|b(t, X_t)\|^{\max(p,2)} + \|a(t, X_t)\|_{F}^{\max(\frac{p}{2},1)} \right] dt \right)^{\frac{1}{\max(p,2)}}
\]

\[
\leq 2 \sqrt{d} \max(T, 1) C_{p, X}^{\max(p,2)} \sum_{i=1}^{d} \left( \int_0^T |g_i(t)|^{\max(p,2)} dt \right)^{\frac{1}{\max(p,2)}}
\]

\[
= C_{L,p} \sum_{i=1}^{d} \|g_i\|_{L^{\max(p,2)}(dt)}.
\]

which completes the proof. \(\square\)

**Proposition 4.7.** Let \(X\) be a diffusion process, whose coefficients are of linear growth, and let \(p \in [1, \infty)\). If \(\rho \in C(\mathbb{R})\) is activating, then \(\mathcal{N} N_{1,d}^p\) is dense in \(L^p(X)\), i.e. for every \(g \in L^p(X)\) and \(\varepsilon > 0\) there exists some \(\varphi \in \mathcal{N} N_{1,d}^p\) such that \(\|g - \varphi\|_{L^p(X)} < \varepsilon\).

**Proof.** Fix some \(\varepsilon > 0\) and let \(t \mapsto g(t) = (g_1(t),...,g_d(t))^\top \in L^p(X)\) for some \(p \in [1, \infty)\). Since every component \(g_i : [0, T] \rightarrow \mathbb{R}, i = 1, \ldots, d\), is left-continuous with right limits, it follows by [Aumann, 1954, Theorem 7.3.2.1(5)] that the number of discontinuity points of \(g_i : [0, T] \rightarrow \mathbb{R}\) is at most countable. Let
\((t^n_i)_{n\in N_{0,i}}\) be the set of discontinuity points of \(g_i: [0,T] \to \mathbb{R}\) (with \(N_{0,i} = \mathbb{N}\) if there are countably many, with \(N_{0,i} = \{1, \ldots, M\}\) if there are finitely many, and with \(N_{0,i} = \emptyset\) if there are none). By left-continuity of \(g_i: [0,T] \to \mathbb{R}\), there exists for every \(n \in N_{0,i}\) some \(\delta^n_i > 0\) such that
\[
\forall t \in [t^n_i - \delta^n_i, t^n_i]: \quad |g_i(t) - g_i(t^n_i) - g_i(t^n_i +)| < |g_i(t^n_i) - g_i(t^n_i +)|, \tag{57}
\]
where \(g_i(t^n_i +) := \lim_{h \to 0+} g_i(t^n_i + h)\). Now, let \(N_i \subseteq N_{0,i}\) be such that \([t^n_i - \delta^n_i, t^n_i] \cap [t^m_i - \delta^m_i, t^m_i] = \emptyset\), for all \(n, m \in N_i\) with \(n \neq m\). Moreover, for every \(n \in N_i\), we define
\[
d^n_i := \min\left(\delta^n_i, \frac{1}{2^n} \left(\frac{\varepsilon}{6C_{L^p} d} |g_i(t^n_i) - g_i(t^n_i +)|\right)^{\max(p,2)}\right),
\]
where \(C_{L^p} > 0\) was defined in Lemma 4.6. Furthermore, we define the function \(f_i: [0,T] \to \mathbb{R}\) by
\[
f_i(t) = \begin{cases} \frac{t^n_i - t}{d^n_i} g_i(t^n_i - d^n_i) + \frac{t - (t^n_i - d^n_i)}{d^n_i} g_i(t^n_i +) & t \in [t^n_i - d^n_i, t^n_i], n \in N_i, \\ 0 & \text{otherwise} \end{cases}
\]
for \(t \in [0,T]\). Then, we observe that \(f_i: [0,T] \to \mathbb{R}\) is continuous. In addition, by using the inequality 
\((x + y + z)^{\max(p,2)} \leq 3^{\max(p,2)} (x^{\max(p,2)} + y^{\max(p,2)} + z^{\max(p,2)})\) for all \(x, y, z \geq 0\), and (57), it follows for every \(n \in N_i\) and \(t \in [t^n_i - d^n_i, t^n_i] \subseteq [t^n_i - \delta^n_i, t^n_i]\) that
\[
|g_i(t) - f_i(t)|^{\max(p,2)} \leq \left| g_i(t) - g_i(t^n_i) + \frac{t^n_i - t}{d^n_i} (g_i(t^n_i) - g_i(t^n_i - d^n_i)) + \right. \\
\left. \ldots + \frac{t - (t^n_i - d^n_i)}{d^n_i} (g_i(t^n_i) - g_i(t^n_i +)) \right|^{\max(p,2)}
\]
\[
\leq 3^{\max(p,2)-1} \left( |g_i(t) - g_i(t^n_i)|^{\max(p,2)} + \frac{|t^n_i - t|}{d^n_i} \right)^{\max(p,2)} |g_i(t^n_i) - g_i(t^n_i - d^n_i)|^{\max(p,2)}
\]
\[
\ldots + \left( \frac{|t - (t^n_i - d^n_i)|}{d^n_i} \right)^{\max(p,2)} |g_i(t^n_i) - g_i(t^n_i +)|^{\max(p,2)}
\]
\[
< 3^{\max(p,2)} |g_i(t^n_i) - g_i(t^n_i +)|^{\max(p,2)}.
\]
Using this and that \(N_i \subseteq N_{0,i} \subseteq \mathbb{N}\), it follows for every \(i = 1, \ldots, d\) that
\[
\|g - f_i\|_{L^{\max(p,2)}(\Omega)} \leq \sum_{n \in N_i} \int_{t^n_i - d^n_i}^{t^n_i} |g_i(t) - f_i(t)|^{\max(p,2)} dt
\]
\[
\leq \sum_{n \in N_i} d^n_i \sup_{t \in [t^n_i - d^n_i, t^n_i]} |g_i(t) - f_i(t)|^{\max(p,2)}
\]
\[
\leq \sum_{n \in N_{0,i}} \frac{1}{2^n} \left( \frac{\varepsilon}{6C_{L^p} d} |g_i(t^n_i) - g_i(t^n_i +)|\right)^{\max(p,2)} |g_i(t^n_i) - g_i(t^n_i +)|^{\max(p,2)}
\]
\[
< \left( \frac{\varepsilon}{2C_{L^p} d} \right)^{\max(p,2)} \sum_{n \in \mathbb{N}} \frac{1}{2^n} = \left( \frac{\varepsilon}{2C_{L^p} d} \right)^{\max(p,2)}.
\]
Hence, by defining \([0,T] \ni t \mapsto f(t) := (f_1(t), \ldots, f_d(t))^T \in \mathbb{R}^d\) and using Lemma 4.6, we have
\[
\|g - f\|_{L^{p}(X)} \leq C_{L^p} \sum_{i=1}^{d} \|g - f_i\|_{L^{\max(p,2)}(\Omega)} < \frac{\varepsilon}{2} = \frac{\varepsilon}{2}, \quad \text{(58)}
\]
On the other hand, since every \( f_i : [0, T] \rightarrow \mathbb{R} \) is continuous and \( \rho \in C(\mathbb{R}) \) is activating, there exists for every \( i = 1, \ldots, d \) some neural network \( \varphi_i \in \mathcal{N}_{1,d}^p \) such that
\[
\sup_{t \in [0,T]} |f_i(t) - \varphi_i(t)| < \frac{\varepsilon}{2CL_\rho dT_{\max(p,\varphi)}}.
\] (59)

By defining \( \varphi := (\varphi_1, \ldots, \varphi_d)^T \in \mathcal{N}_{1,d}^p \), it follows for every \( i = 1, \ldots, d \) that
\[
\|f_i - \varphi_i\|_{L_{\max(p,2)}} \leq \frac{1}{T_{\max(p,2)}} \sup_{t \in [0,T]} |f_i(t) - \varphi_i(t)| < \frac{\varepsilon}{2CL_\rho dT_{\max(p,\varphi)}} < \infty,
\]
which implies that \( f - \varphi \in L^p_f(X) \). Hence, we can apply Lemma 4.6 and use (58) to conclude that
\[
\|g - \varphi\|_{L^p(X)} \leq \|g - f\|_{L^p(X)} + \|f - \varphi\|_{L^p(X)} < \frac{\varepsilon}{2} + C_{L^p} \sum_{i=1}^d \|f_i - \varphi_i\|_{L_{\max(p,2)}(dt)} < \frac{\varepsilon}{2} + C_{L^p} dT_{\max(p,\varphi)} \frac{\varepsilon}{2CL_\rho dT_{\max(p,\varphi)}} = \varepsilon.
\]

Since \( \varepsilon > 0 \) and \( g \in L^p(X) \) were chosen arbitrarily, it follows that \( \mathcal{N}_{1,d}^p \) is dense in \( L^p(X) \).

**Proof of Proposition 2.37.** Fix some \( \varepsilon > 0 \) and let \( g \in L^{np}_{\text{diag}}(X)^{\otimes n} \) with representation \( \sum_{j=1}^m \lambda_j g_j^{\otimes n} \), for \( 0 \neq \lambda_j \in \mathbb{R} \) and \( g_j \in L^{np}(X) \), such that
\[
\left\| g - \sum_{j=1}^m \lambda_j g_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}} < \frac{\varepsilon}{2},
\] (60)
which exists as \( g - \sum_{j=1}^m \lambda_j g_j^{\otimes n} \) is a representation of \( 0 \in L^{np}(X)^{\otimes n} \). Then, for every \( j = 1, \ldots, m \), we apply Proposition 4.7 to obtain some \( \varphi_j \in \mathcal{N}_{1,d}^p \) such that
\[
\|g_j - \varphi_j\|_{L^{np}(X)^{\otimes n}} < \min \left( 1, \frac{\varepsilon}{2mn|\lambda_j| (1 + \|g_j\|_{L^{np}(X)})^{n-1}} \right),
\]
which implies that \( \|\varphi_j\|_{L^{np}(X)^{\otimes n}} \leq 1 + \|g_j\|_{L^{np}(X)} \). Using this, the telescoping sum \( g_j^{\otimes n} - \varphi_j^{\otimes n} = \sum_{l=1}^n g_j^{\otimes (n-l)} \otimes (g_j - \varphi_j) \otimes \varphi_j^{\otimes (l-1)} \), the triangle inequality, and Lemma 2.17, it follows that
\[
\left\| g_j^{\otimes n} - \varphi_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}} \leq \sum_{l=1}^n \left\| g_j^{\otimes (n-l)} \otimes (g_j - \varphi_j) \otimes \varphi_j^{\otimes (l-1)} \right\|_{L^{np}(X)^{\otimes n}}
\]
\[
\leq \sum_{l=1}^n \|g_j\|_{L^{np}(X)} \|g_j - \varphi_j\|_{L^{np}(X)} \|\varphi_j\|_{L^{np}(X)}^{l-1} 
\]
\[
\leq n \left( 1 + \|g_j\|_{L^{np}(X)} \right)^{n-1} \|g_j - \varphi_j\|_{L^{np}(X)} < \frac{\varepsilon}{2m|\lambda_j|}.
\] (61)

Finally, by defining \( \varphi := \sum_{j=1}^m \lambda_j \varphi_j^{\otimes n} \in \mathcal{N}_{n,d}^p \), it follows by combining (60) and (61) with the triangle inequality that
\[
\|g - \varphi\|_{L^{np}(X)^{\otimes n}} \leq \left\| g - \sum_{j=1}^m \lambda_j g_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}} + \sum_{j=1}^m |\lambda_j| \left\| g_j^{\otimes n} - \varphi_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}}
\]
\[
< \frac{\varepsilon}{2} + \sum_{j=1}^m |\lambda_j| \frac{\varepsilon}{2m|\lambda_j|} = \varepsilon.
\]

Since \( \varepsilon > 0 \) and \( g \in L^{np}_{\text{diag}}(X)^{\otimes n} \) were chosen arbitrarily, \( \mathcal{N}_{n,d}^p \) is dense in \( L^{np}(X)^{\otimes n} \).
4.5.2. Proof of Theorem 2.38.

Proof of Theorem 2.38. Fix some \( \varepsilon > 0 \) and let \( G \in L^p(\mathbb{P}) \) for some \( p \in [1, \infty) \). By Theorem 2.25, there exists \( N \in \mathbb{N} \) and a sequence \((g_n)_{n=0,...,N} \), with \( g_n \in L^{np}_{\text{diag}}(X)^{\otimes n} \) for all \( n = 0, ..., N \), such that

\[
\left\| G - \sum_{n=0}^{N} J_n(g_n) \right\|_{L^p(\mathbb{P})} < \frac{\varepsilon}{2},
\]

(62)

Moreover, for every \( n = 1, ..., N \), we obtain by applying Proposition 2.37 some \( \varphi_n \in \mathcal{N}_{n,d} \) such that \( \|g_n - \varphi_n\|_{L^{np}(X)^{\otimes n}} < \frac{\varepsilon}{2C_{n,p}} \), where \( C_{n,p} > 0 \) was introduced in Lemma 2.22. From this, we define \( G^{\varphi_0:N} := \sum_{n=0}^{N} J_n(\varphi_n) \in L^p(\mathbb{P}) \) with \( \varphi_0 := g_0 \in \mathbb{R} \). Hence, by using (62), Minkowski’s inequality, and the Burkholder-Davis-Gundy type of inequality in Lemma 2.22, it follows that

\[
\left\| G - G^{\varphi_0:N} \right\|_{L^p(\mathbb{P})} \leq \left\| G - \sum_{n=0}^{N} J_n(g_n) \right\|_{L^p(\mathbb{P})} + \left\| \sum_{n=0}^{N} J_n(g_n - \varphi_n) \right\|_{L^p(\mathbb{P})} < \frac{\varepsilon}{2} + \sum_{n=1}^{N} C_{n,p} \|g_n - \varphi_n\|_{L^{np}(X)^{\otimes n}} < \frac{\varepsilon}{2} + \sum_{n=1}^{N} C_{n,p} \frac{\varepsilon}{2C_{n,p}}N = \varepsilon,
\]

which completes the proof. \( \square \)

4.5.3. Auxiliary Results: Random Universal Approximation.

Proof of Proposition 2.40. First, we observe that \((L^{np}(X)^{\otimes n}, \| \cdot \|_{L^{np}(X)^{\otimes n}})\) is a Banach space, which we claim to be separable. Indeed, by the Stone-Weierstrass theorem, the countable set \( \mathbb{Q}[t] := \left\{ 0, T \right\} \ni t \mapsto \sum_{n=0}^{N} q_n t^n \in \mathbb{R} : N \in \mathbb{N}, q_n \in \mathbb{Q} \) is dense in \( C([0, T]) \) with respect to the uniform topology. Hence, by following the arguments in the proof of Proposition 4.7, \( \mathbb{Q}[t] \) can also be used to approximate the components of any LCRL function in \( L^{np}(X) \) with respect to \( \| \cdot \|_{L^{np}(X)} \). Then, by the same arguments as in Proposition 2.37, we conclude that the countable set

\[
\left\{ p_1 \otimes \cdots \otimes p_n \in L^{np}(X)^{\otimes n} : p_k = (p_{k,1,1}, \ldots, p_{k,d,d})^T \in L^{np}(X), p_{k,i} \in \mathbb{Q}[t], \right. \\
\left. \text{for all } k = 1, \ldots, n \text{ and } i = 1, \ldots, d \right\}
\]

is dense in \( L^{np}(X)^{\otimes n} \).

Now, we observe that random tensor-valued neural networks are linear combinations of maps of the form \( \Omega' \ni \omega' \mapsto \mathbb{1}_{A'}(\omega') R(\omega')^{\otimes n} \in L^{np}_{\text{diag}}(X)^{\otimes n} \subseteq L^{np}(X)^{\otimes n} \), with some \( A' \in A' \) and

\[
\Omega' \ni \omega' \mapsto R(\omega') := (R_1(\omega'), ..., R_d(\omega'))^T := (\rho(a_{1,i}(\omega'),+b_{1,i}(\omega')))_{i=1,...,d} \in L^{np}(X),
\]

where \( \rho(a_{1,i}(\omega'),+b_{1,i}(\omega')) \) denotes the function \( [0,T] \ni t \mapsto \rho(a_{1,i}(\omega')t+b_{1,i}(\omega')) \in \mathbb{R} \). Hence, it suffices to show that \( (\omega' \mapsto \mathbb{1}_{A'}(\omega') R(\omega')^{\otimes n} \in L^{np}_{\text{diag}}(X)^{\otimes n}) \), for some fixed \( A' \in A' \) and for all \( r \in [1, \infty) \), to obtain the conclusion. By the definition of the Bochner space \( L^r(\Omega'; L^{np}(X)^{\otimes n}) \) in [Hytönen et al., 2016, Definition 1.2.15], this means that we need to prove that the map \( \Omega' \ni \omega' \mapsto \mathbb{1}_{A'}(\omega') R(\omega')^{\otimes n} \in L^{np}(X)^{\otimes n} \) is \( L^{p'} \)-strongly measurable (see [Hytönen et al., 2016, Definition 1.1.14]) and satisfies\( E^\mathbb{F} \left[ \left\| \mathbb{1}_{A'}(\cdot)^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}} \right] < \infty \), for all \( r \in [1, \infty) \).

In order to prove that the map \( \Omega' \ni \omega' \mapsto \mathbb{1}_{A'}(\omega') R(\omega')^{\otimes n} \in L^{np}(X)^{\otimes n} \) is \( L^{p'} \)-strongly measurable, we define the sequence of \( L^{np}(X) \)-valued random variables \( (R^{(M)})_{M \in \mathbb{N}} \) by

\[
\Omega' \ni \omega' \mapsto R^{(M)}(\omega') := \left( R_1^{(M)}(\omega'), ..., R_d^{(M)}(\omega') \right)^T \in L^{np}(X),
\]

with \( \Omega' \ni \omega' \mapsto R_i^{(M)}(\omega') := \sum_{k=-M^2}^{M^2} \sum_{l=-M^2}^{M^2} \mathbb{1}_{\left\{ \frac{k}{M} \leq a_{1,i}(\omega') < \frac{k+1}{M} \right\}} \mathbb{1}_{\left\{ \frac{l}{M} \leq b_{1,i}(\omega') < \frac{l+1}{M} \right\}} \cdot \rho \left( \frac{k}{M} \cdot + \frac{l}{M} \right) \in L^{np}(dt),
\]
for \( i = 1, \ldots, d \) and \( M \in \mathbb{N} \). Hence, by multilinearity of the tensor product, the map \( \Omega' \ni \omega' \mapsto \mathbb{I}_{A'}(\omega') R(M)(\omega')^{\otimes n} \in L^{\omega'}(X)^{\otimes n} \) is a \( \omega' \)-simple function (see [Hytönen et al., 2016, Definition 1.1.13]), for all \( M \in \mathbb{N} \). Moreover, for \( |z| := \max \{ k \in \mathbb{Z} : z \geq k \} \), we observe for every \( \omega' \in \Omega' \), \( i = 1, \ldots, d \), \( t \in [0, T] \) and \( M \in \mathbb{N} \cap [M_{\omega', i}, \infty) \) with \( M_{\omega', i} := \max \{ |a_{i,1}(\omega')|, |b_{i,1}(\omega')| \} \) that
\[
R_{i}^{(M)}(\omega')(t) = \rho \left( \frac{M_{a_{i,1}(\omega')}}{M} t + \frac{M_{b_{i,1}(\omega')}}{M} \right).
\]
Since \( \rho \in C(\mathbb{R}) \) is continuous, \( \left( R_{i}^{(M)}(\omega') \right)_{M \in \mathbb{N}} \) converges for every \( \omega' \in \Omega' \) and \( i = 1, \ldots, d \) pointwise to \( R_{i}(\omega') \) on \([0, T]\). We claim that this convergence is even uniform on \([0, T]\). Indeed, we observe for every \( \omega' \in \Omega' \) and \( i = 1, \ldots, d \) that
\[
\sup_{t \in [0, T]} \left| a_{i,1}(\omega') t + b_{i,1}(\omega') \right| - \left( \frac{M_{a_{i,1}(\omega')}}{M} t + \frac{M_{b_{i,1}(\omega')}}{M} \right) \leq \frac{1}{M} \left( |M_{a_{i,1}(\omega')}| + |M_{b_{i,1}(\omega')}| \right) \max(T, 1) \leq 2 \max(T, 1) \leq 0.
\]
Hence, \( \left( \frac{M_{a_{i,1}(\omega')}}{M} + \frac{M_{b_{i,1}(\omega')}}{M} \right)_{M \in \mathbb{N}} \) converges for every \( \omega' \in \Omega' \) and \( i = 1, \ldots, d \) uniformly to \( a_{i,1}(\omega') + b_{i,1}(\omega') \) on \([0, T]\), as \( M \to \infty \). Moreover, we observe for every \( \omega' \in \Omega' \) and \( i = 1, \ldots, d \) that
\[
\left( \frac{M_{a_{i,1}(\omega')}}{M} t + \frac{M_{b_{i,1}(\omega')}}{M} \right) \leq \left( \frac{M_{a_{i,1}(\omega')}}{M} t + \frac{M_{b_{i,1}(\omega')}}{M} \right) \leq \frac{1}{M} \left( |M_{a_{i,1}(\omega')}| + |M_{b_{i,1}(\omega')}| \right) \max(T, 1) \leq 2 \max(T, 1) \leq 0.
\]
for all \( t \in [0, T] \) and \( M \in \mathbb{N} \). Since \( \rho \in C(\mathbb{R}) \) is continuous, thus uniformly continuous on \([a_{i,1}(\omega'), b_{i,1}(\omega')]\), there exists for every fixed \( \varepsilon > 0 \) some \( \delta_{i}' \) such that for every \( u, v \in [a_{i,1}(\omega'), b_{i,1}(\omega')] \) with \( |u - v| < \delta_{i}' \), it holds that \( |\rho(u) - \rho(v)| < \frac{\varepsilon}{T_{\max(n, p, d)} C_{L^{p}}} \). Thus, by choosing \( M_{0, \omega', i} := M_{0, \omega', i}(\delta_{i}') \in \mathbb{N} \) with \( M_{0, \omega', i} \geq \max \left( \frac{2 \max(T, 1)}{\delta_{i}'}, M_{0, i} \right) \), it holds for every \( M \geq M_{0, \omega', i} \) that
\[
\sup_{t \in [0, T]} \left( \rho(a_{i,1}(\omega') \cdot + b_{i,1}(\omega')) - \rho \left( \frac{M_{a_{i,1}(\omega')}}{M} \cdot + \frac{M_{b_{i,1}(\omega')}}{M} \right) \right) < \frac{\varepsilon}{T_{\max(n, p, d)} C_{L^{p}}}
\]
From this, we conclude for every \( \omega' \in \Omega' \), \( i = 1, \ldots, d \), and \( M \geq M_{0, \omega', i} \) that
\[
\left\| \mathbb{I}_{A'}(\omega') R_{i}(\omega') - \mathbb{I}_{A'}(\omega') R_{i}^{(M)}(\omega') \right\|_{L_{\max(n, p, 2)}^{0}(dt)} \leq \left\| \mathbb{I}_{A'}(\omega') \left( \rho(a_{i,1}(\omega') \cdot + b_{i,1}(\omega')) - \rho \left( \frac{M_{a_{i,1}(\omega')}}{M} \cdot + \frac{M_{b_{i,1}(\omega')}}{M} \right) \right) \right\|_{L_{\max(n, p, 2)}^{0}(dt)} \leq \frac{1}{T_{\max(n, p, 2)}} \sup_{t \in [0, T]} \left( \rho(a_{i,1}(\omega') \cdot + b_{i,1}(\omega')) - \rho \left( \frac{M_{a_{i,1}(\omega')}}{M} \cdot + \frac{M_{b_{i,1}(\omega')}}{M} \right) \right) \right\|_{L_{\max(n, p, 2)}^{0}(dt)} < \frac{\varepsilon}{C_{L^{p}}}
\]
which implies that \( \mathbb{I}_{A'}(\omega') R(\omega') - \mathbb{I}_{A'}(\omega') R^{(M)}(\omega') \in L_{T}^{0}(X) \). Hence, for every \( \omega' \in \Omega' \), we can apply Lemma 4.6 and use (63) to conclude that for every \( M \geq M_{0, \omega', i} := \max_{i=1, \ldots, d} M_{0, i} \), we have
\[
\left\| \mathbb{I}_{A'}(\omega') R(\omega') - \mathbb{I}_{A'}(\omega') R^{(M)}(\omega') \right\|_{L_{\omega'}^{p}(X)} \leq C_{L^{p}} \sum_{i=1}^{d} \left\| \mathbb{I}_{A'}(\omega') R_{i}(\omega') - \mathbb{I}_{A'}(\omega') R_{i}^{(M)}(\omega') \right\|_{L_{\max(n, p, 2)}^{0}(dt)} < C_{L^{p}} \sum_{i=1}^{d} \frac{\varepsilon}{C_{L^{p}}} = \varepsilon.
\]
Since \( \varepsilon > 0 \) was chosen arbitrarily, it follows that \( \Omega' \ni \omega' \mapsto \mathbb{I}_{A'}(\omega') R^{(M)}(\omega') \in L_{\omega'}^{p}(X) \) converges pointwise to \( \Omega' \ni \omega' \mapsto \mathbb{I}_{A'}(\omega') R(\omega') \in L_{\omega'}^{p}(X) \) on \( L_{\omega'}^{p}(X) \), as \( M \to \infty \). Hence, for every \( \omega' \in \Omega' \).
and every $\varepsilon > 0$ there exists some $M_{1,\omega'} := M_{1,\omega'}(\varepsilon) \in \mathbb{N}$ such that for every $M \geq M_{1,\omega'}$, it holds that

$$\left\| \mathbbm{1}_{A'}(\omega') R^{(M)}(\omega') - \mathbbm{1}_{A'}(\omega') R^{(\omega')} \right\|_{L^{np}(X)} \leq n \left( 1 + \left\| \mathbbm{1}_{A'}(\omega') R^{(\omega')} \right\|_{L^{np}(X)} \right)^{n-1},$$

which implies that $\left\| \mathbbm{1}_{A'}(\omega') R^{(M)}(\omega') \right\|_{L^{np}(X)} \leq 1 + \left\| \mathbbm{1}_{A'}(\omega') R^{(\omega')} \right\|_{L^{np}(X)}$, for all $M \geq M_{1,\omega'}$. Then, by using the telescoping sum $R(\omega')^{\otimes n} - R^{(M)}(\omega')^{\otimes n} = \sum_{i=1}^{n} R(\omega')^{\otimes (n-i)} \otimes (R(\omega') - R^{(M)}(\omega')) \otimes R^{(M)}(\omega')^{\otimes (i-1)}$, the triangle inequality, and Lemma 2.17, it follows for every $M \geq M_{1,\omega'}$ that

$$\left\| \mathbbm{1}_{A'}(\omega') R^{(\omega')} - \mathbbm{1}_{A'}(\omega') R^{(M)}(\omega')^{\otimes n} \right\|_{L^{np}(X)\otimes n} \leq \sum_{i=1}^{n} \left\| \mathbbm{1}_{A'}(\omega') R(\omega')^{\otimes (i-1)} \otimes (R(\omega') - R^{(M)}(\omega')) \otimes R^{(M)}(\omega')^{\otimes (i-1)} \right\|_{L^{np}(X)}$$

$$\leq \sum_{i=1}^{n} \left\| \mathbbm{1}_{A'}(\omega') R^{(\omega')} \right\|_{L^{np}(X)} \left\| \mathbbm{1}_{A'}(\omega') R^{(M)}(\omega')^{\otimes (i-1)} \right\|_{L^{np}(X)} \leq n \left( 1 + \left\| \mathbbm{1}_{A'}(\omega') R^{(\omega')} \right\|_{L^{np}(X)} \right)^{n-1} \left\| \mathbbm{1}_{A'}(\omega') R^{(\omega')} - \mathbbm{1}_{A'}(\omega') R^{(M)}(\omega')^{\otimes n} \right\|_{L^{np}(X)} < \varepsilon. \quad (64)$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\Omega' \ni \omega' \mapsto \mathbbm{1}_{A'}(\omega') R^{(M)}(\omega')^{\otimes n} \in L^{np}(X)\otimes n$ converges pointwise to $\Omega' \ni \omega' \mapsto \mathbbm{1}_{A'}(\omega') R(\omega')^{\otimes n} \in L^{np}(X)\otimes n$, as $M \to \infty$, which shows that $\Omega' \ni \omega' \mapsto \mathbbm{1}_{A'}(\omega') R^{(\omega')} \in L^{np}(X)\otimes n$ is $\mathbb{P}$-strongly measurable.

Finally, by using that $\rho \in C(\mathbb{R})$ is continuous, we observe for every $\omega' \in \Omega'$ and $i = 1, \ldots, d$ that

$$\left\| \rho(a_{1,i}(\omega') \cdot + b_{1,i}(\omega')) \right\|_{L^{\max(\rho,2)}(dt)} \leq T^{\max(\rho,2)} \sup_{t \in [0,T]} \left\| \rho(a_{1,i}(\omega') t + b_{1,i}(\omega')) \right\| < \infty,$$

which implies that $(\rho(a_{1,i}(\omega') \cdot + b_{1,i}(\omega')))_{i=1,\ldots,d}^{\top} \in L^{p}(X)$ for all $\omega' \in \Omega'$. Hence, by using Lemma 2.17, Lemma 4.6, the inequality $\left( \sum_{i=1}^{d} x_{i}^{nr} \right)_{i=1,\ldots,d}^{\top} \leq d^{nr-1} \sum_{i=1}^{d} x_{i}^{nr}$ for any $x_{1}, \ldots, x_{n} \geq 0$, that $(a_{1,i})_{i=1,\ldots,d}^{\top} \sim Y$ and $(b_{1,i})_{i=1,\ldots,d}^{\top} \sim Z$ are identically distributed, and (16), it follows for every $r \in [1, \infty]$ that

$$\mathbb{E}' \left[ \left\| \mathbbm{1}_{A'} R^{(\omega')} \right\|_{L^{np}(X)}^{\otimes n} \right] = \mathbb{E}' \left[ \left\| \mathbbm{1}_{A'} \right\|_{L^{np}(X)}^{\otimes n} \right] \leq \mathbb{E}' \left[ \left( \left\| \rho(a_{1,i} \cdot + b_{1,i}) \right\|_{L^{\max(\rho,2)}}^{\otimes n(n)} \right)_{i=1,\ldots,d}^{\top} \right]^{nr} \leq C_{L^{np}}^{nr} \mathbb{E}' \left[ \left( \sum_{i=1}^{d} \left\| \rho(a_{1,i} \cdot + b_{1,i}) \right\|_{L^{\max(\rho,2)}}^{\otimes n(n)} \right)_{i=1,\ldots,d}^{\top} \right]^{nr} \leq C_{L^{np}}^{nr} d^{nr-1} \sum_{i=1}^{d} \mathbb{E}' \left[ \left\| \rho(a_{1,i} \cdot + b_{1,i}) \right\|_{L^{\max(\rho,2)}}^{\otimes n(n)} \right]^{nr} \leq C_{L^{np}}^{nr} d^{nr-1} \mathbb{E}' \left[ \left\| \rho(Y + Z) \right\|_{L^{\max(\rho,2)}}^{\otimes n(n)} \right]^{nr} < \infty.$$

This shows that $(\omega' \mapsto \mathbbm{1}_{A'}(\omega') R^{(\omega')} \in L^{p}(\Omega'; L^{np}(X)\otimes n))$ for all $r \in [1, \infty)$.

**Proposition 4.8.** Let $X$ be a diffusion process, whose coefficients are of linear growth, and let $g \in L^{p}(X)$ with $p \in [1, \infty]$. If $\rho \in C(\mathbb{R})$ is activating, then for every $r \in [1, \infty)$ and $\varepsilon > 0$ there exists some $\varphi \in \mathcal{R}_{N_{1,d}}^{p}$ such that $\mathbb{E}' \left[ \left\| g - \varphi \right\|_{L^{p}(X)}^{1/r} \right] < \varepsilon.$

**Proof.** Fix some $r \in [1, \infty)$ and $\varepsilon > 0$, and let $g \in L^{p}(X)$ for some $p \in [1, \infty)$. Since $\rho \in C(\mathbb{R})$ is activating, there exists by Proposition 4.7 a classical neural network $\tilde{\varphi} \in \mathcal{N} \mathcal{N}_{1,d}$ of the form $[0, T] \ni t \mapsto \tilde{\varphi}(t) = \sum_{n=1}^{N} \tilde{w}_{n} \left( \rho \left( \tilde{a}_{n,i} t + \tilde{b}_{n,i} \right) \right)_{i=1,\ldots,d}^{\top} \in \mathbb{R}^{d}$, with $N \in \mathbb{N}$ and $\tilde{a}_{n} = \left( \tilde{a}_{n,i} \right)_{i=1,\ldots,d}^{\top} \in \mathbb{R}^{d}$, $\tilde{b}_{n} = \left( \tilde{b}_{n,i} \right)_{i=1,\ldots,d} \in \mathbb{R}^{d}$, such that

$$\left\| g - \tilde{\varphi} \right\|_{L^{p}(X)} < \frac{\varepsilon}{3}. \quad (65)$$
Now, we define for every \( n = 1, \ldots, N \) and \( m, j \in \mathbb{N} \) a random neural network \( \varphi_{n,m,j} \in \mathcal{R}N_{1,d}^n \) by
\[
\Omega' \ni \omega' \mapsto \varphi_{n,m,j}(\omega') := (\varphi_{n,m,j,1}(\omega'), \ldots, \varphi_{n,m,j,d}(\omega'))^\top \in L^p(X)
\]
with \( \Omega' \ni \omega' \mapsto w_{n,m,j,i}(\omega') := w_{n,m,j,i}(\omega') \rho(a_{j,i}(\omega') \cdot b_{j,i}(\omega')) \in L^p(dt) \),
where the random variable
\[
\Omega' \ni \omega' \mapsto w_{n,m,j,i}(\omega') := \frac{m^2 \tilde{w}_n}{4f_{Y,Z}} \mathbb{I}_{\{ \max(|a_{j,i}(\omega') - a_{n,i}|, |b_{j,i}(\omega') - b_{n,i}|) < \frac{1}{m} \}} \in \mathbb{R}
\]
is \( \sigma(\{a_{j,i}, b_{j,i} : j_0 \in \mathbb{N}\}) \)-measurable and bounded. Hereby, we observe for every \( n = 1, \ldots, N \) and \( m \in \mathbb{N} \) that the sequence of \( L^p(X) \)-valued random variables \( \varphi_{n,m,j} \) defined in (66) is i.i.d., and
belongs by Proposition 2.40 to \( L^1(\Omega'; L^p(X)) \), where we use \( L^p(X) \otimes L^q(Y) \cong L^p(X \times Y) \) (see also Remark 2.18).

Now, we show that for every \( n = 1, \ldots, N \) the expectations \( (t \mapsto \mathbb{E}'[|\varphi_{n,m,1}(\omega(t))|]) \in L^p(X) \), \( m \in \mathbb{N} \), of the random neural networks in (66) approximate with the \( \| \cdot \|_{L^p(X)} \)-norm, as \( m \to \infty \), for this purpose, we define for every \( m \in \mathbb{N}, n = 1, \ldots, N \), \( i = 1, \ldots, d \) the subsets \( A_{n,m,i} = [a_{n,i} - \frac{1}{m}, a_{n,i} + \frac{1}{m}] \times [b_{n,i} - \frac{1}{m}, b_{n,i} + \frac{1}{m}] \subset \mathbb{R}^2 \) satisfying \( A_{n,m+1,i} \subset A_{n,m,i} \), for all \( n \in \mathbb{N} \) and \( \bigcap_{m \in \mathbb{N}} A_{n,m,i} = \{ \bar{a}_{n,i}, \bar{b}_{n,i} \} \). Moreover, we fix some \( \varepsilon > 0 \) and let \( n = 1, \ldots, N \) and \( i = 1, \ldots, d \). Since \( \rho \in C(\mathbb{R}) \) is continuous, thus uniformly continuous on the compact interval \( K_{n,i} := \{ yt + z : t \in [0, T], (y, z) \in A_{n,1,i} \} \), there exists some \( \delta_{\rho,n,i} := \delta_{\rho,n,i}(\varepsilon) > 0 \) such that for every \( u, v \in K_{n,i} \) with \( |u - v| < \delta_{\rho,n,i} \), it holds that
\[
|\rho(u) - \rho(v)| < \varepsilon \frac{6C_{L_p}d\max|\omega_n|}{\sup_{(y,z) \in A_{n,m,i}} |f_{Y,Z}(y,z)|} \mathbb{I}_{|\tilde{w}_n|} |\bar{a}_{n,i}, \bar{b}_{n,i}|.
\]
Similarly, as \( f_{Y,Z} : \mathbb{R}^2 \to (0, \infty) \) is continuous, thus uniformly continuous on the compact set \( A_{n,m,i} \), there exists some \( \delta_{f,n,i} := \delta_{f,n,i}(\varepsilon) > 0 \) such that for every \( u, v \in A_{n,1,i} \) with \( |u - v| < \delta_{f,n,i} \), we have
\[
|f_{Y,Z}(u) - f_{Y,Z}(v)| < \varepsilon \frac{6C_{L_p}d\max|\omega_n|}{\sup_{t \in [0, T]} |f_{Y,Z}(\bar{a}_{n,i}, \bar{b}_{n,i})|} \mathbb{I}_{|\tilde{w}_n|} |\bar{a}_{n,i}, \bar{b}_{n,i}|.
\]
Now, for every \( n = 1, \ldots, N \), we choose some \( m_n \in \mathbb{N} \) with \( m_n = \max\{\sqrt{\max|\omega_n|}, T+1\} \), where \( \delta_n := \min_{i=1,...,d} \min(\delta_{\rho,n,i}, \delta_{f,n,i}) > 0 \). Then, it follows from (67) and (68) that
Hence, by using that \( |A_{n,m,i}| := \int_{A_{n,m,i}} dydz = \frac{4}{m^2} \) for any \( m \in \mathbb{N}, n = 1, \ldots, N, \) and \( i = 1, \ldots, d, \) we conclude for every \( n = 1, \ldots, N \) and \( i = 1, \ldots, d \) that

\[
\begin{align*}
\|\tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) &- \mathbb{E}'[\varphi_{n,m,1,i}]\|_{L^{\max(p,2)}(dt)} \\
\leq & \left\| \tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) - \int_{\mathbb{R}^2} \frac{m^2 \tilde{w}_n \mathbb{I}_{A_{n,m,i}}(y,z)}{4f_{Y,Z}(y,z)} \rho(y+z) f_{Y,Z}(y,z) dydz \right\|_{L^{\max(p,2)}(dt)} \\
\leq & \left\| \int_{A_{n,m,i}} \frac{m^2 \tilde{w}_n}{4} \left( \rho(\tilde{a}_{n,i} \cdot \tilde{b}_{n,i}) - \frac{f_{Y,Z}(y,z)}{f_{Y,Z}(\tilde{a}_{n,i}, \tilde{b}_{n,i})} \rho(y+z) \right) dydz \right\|_{L^{\max(p,2)}(dt)} \\
\leq & T^{\frac{1}{p}} \sup_{t \in [0,T]} \left( t \mapsto \frac{1}{m} \right) \left( t \mapsto \tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) - \mathbb{E}'[\varphi_{n,m,1,i}(t)] \right) \end{align*}
\]

which shows that \( t \mapsto \tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) \mathbb{E}'[\varphi_{n,m,1,i}(\cdot)(t)] \in L^p_t(X). \) Thus, we can apply Lemma 4.6 and use (69) to conclude that for every \( n = 1, \ldots, N, \) it holds that

\[
\begin{align*}
\|\tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) &- \mathbb{E}'[\varphi_{n,m,1}]\|_{L^p(X)} \\
\leq & C_{L^p} \sum_{i=1}^d \left\| \tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) - \mathbb{E}'[\varphi_{n,m,1}] \right\|_{L^{\max(p,2)}(dt)} \\
\leq & C_{L^p} \frac{\varepsilon}{3C_{L^p} dN} = \frac{\varepsilon}{3N}.
\end{align*}
\]

This shows that for every \( n = 1, \ldots, N \) the expectations \( t \mapsto \mathbb{E}'[\varphi_{n,m,1,i}(\cdot)(t)] \) are indeed close to \( t \mapsto \tilde{w}_n \rho \left( \tilde{a}_{n,i} \cdot \tilde{b}_{n,i} \right) \) in \( L^p(X) \) in the \( \| \cdot \|_{L^p(X)} \)-norm.

Now, we approximate the constant random variable \( \{ \omega' \mapsto \mathbb{E}'[\varphi_{n,m,1,j}(\omega')] \}_{j \in \mathbb{N}} \in L^1(\Omega'; L^p(X)) \) by the average of the i.i.d. sequence \( \{ \omega' \mapsto \varphi_{n,m,1,j}(\omega') \}_{j \in \mathbb{N}} \subseteq L^1(\Omega'; L^p(X)) \) defined in (66). More precisely, by applying for every \( n = 1, \ldots, N \) the strong law of large numbers for Banach space-valued random variables in [Hytonen et al., 2016, Theorem 3.3.10], we conclude that

\[
\frac{1}{J} \sum_{j=1}^J \varphi_{n,m,1,j} \overset{J \to \infty}{\longrightarrow} \mathbb{E}'[\varphi_{n,m,1}] \text{ in } L^1(\Omega'; L^p(X)). \tag{71}
\]

In order to generalize the convergence to \( L^r(\Omega'; L^p(X)) \), we define for every \( n = 1, \ldots, N \) the sequence of real-valued non-negative random variables \( \{ Z_{n,j} \}_{j \in \mathbb{N}} \) by \( Z_{n,j} := \mathbb{E}'[\varphi_{n,m,1,j}] - \frac{1}{J} \sum_{j=1}^J \varphi_{n,m,1,j} \|_{L^p(X)}, \) \( J \in \mathbb{N}. \) Then, for every \( n = 1, \ldots, N \) and every fixed \( \eta > 0, \) it follows by applying Chebyshev’s inequality with the non-negative non-decreasing function \( 0, \infty) \ni s \mapsto s^\frac{1}{p} \in (0, \infty) \) and by using (71) that

\[
\mathbb{P}'[\|Z_{n,j} - 0\| \geq \eta] \leq \frac{\mathbb{E}'[\|Z_{n,j}^+\|^\frac{p}{p-1}]}{\eta^\frac{p}{p-1}} = \frac{1}{\eta^\frac{p}{p-1}} \mathbb{E}'\left[ \left\| \mathbb{E}'[\varphi_{n,m,1,j}] - \frac{1}{J} \sum_{j=1}^J \varphi_{n,m,1,j} \right\|^r_{L^p(X)} \right] \overset{J \to \infty}{\longrightarrow} 0, \tag{72}
\]

which shows that \( Z_{n,j} \) converges for every \( n = 1, \ldots, N \) in probability to 0, as \( J \to \infty. \) Moreover, we now fix some \( q > r. \) Then, by using the triangle inequality of \( \| \cdot \|_{L^p(X)} \), Minkowski’s inequality, [Hytonen et al., 2016, Proposition 1.2.2], that \( \{ \varphi_{n,m,1,j} \}_{j \in \mathbb{N}} \) defined in (66) is identically distributed, that \( \varphi_{n,m,1} \in \mathcal{R}N_{1,d}^q \) with \( \mathcal{R}N_{1,d}^q \subset L^1(\Omega'; L^p(X)) \) and \( \mathcal{R}N_{1,d}^r \subset L^r(\Omega'; L^p(X)) \) by Proposition 2.40, it
follows for every \( n = 1, \ldots, N \) and \( J \in \mathbb{N} \) that

\[
\mathbb{E}' \left[ Z_{n,j}^2 \right]^{\frac{1}{2}} = \mathbb{E}' \left[ \left\| \mathbb{E}' [\varphi_{n,m,n,1}] - \frac{1}{J} \sum_{j=1}^{J} \varphi_{n,m,n,j} \right\|_{L^p(X)}^q \right]^{\frac{1}{q}} \\
\leq \left\| \mathbb{E}' [\varphi_{n,m,n,1}] \right\|_{L^p(X)} + \mathbb{E}' \left[ \left( \frac{1}{J} \sum_{j=1}^{J} \left\| \varphi_{n,m,n,j} \right\|_{L^p(X)} \right)^q \right]^{\frac{1}{q}} \\
\leq \mathbb{E}' \left[ \left\| \varphi_{n,m,n,1} \right\|_{L^p(X)} \right] + \frac{1}{J} \sum_{j=1}^{J} \mathbb{E}' \left[ \left\| \varphi_{n,m,n,j} \right\|_{L^p(X)}^q \right]^{\frac{1}{q}} \\
\leq \mathbb{E}' \left[ \left\| \varphi_{n,m,n,1} \right\|_{L^p(X)} \right] + \mathbb{E}' \left[ \left\| \varphi_{n,m,n,1} \right\|_{L^p(X)}^q \right]^{\frac{1}{q}} < \infty.
\]

Since the right-hand side is finite and does not depend on \( J \in \mathbb{N} \), we conclude for every \( n = 1, \ldots, N \) that \( \sup_{J \in \mathbb{N}} \mathbb{E}' \left[ Z_{n,j}^2 \right] < \infty \). Hence, for every \( n = 1, \ldots, N \), the family of random variables \( (Z_{n,j})_{J \in \mathbb{N}} \) is by de la Vallée-Poussin’s theorem uniformly integrable. Then, by using (72), i.e. \( Z_{n,j} \to 0 \) in probability as \( J \to \infty \), together with Vitali’s convergence theorem, it follows for every \( n = 1, \ldots, N \) that indeed

\[
\mathbb{E} \left[ \left\| \mathbb{E}' [\varphi_{n,m,n,1}] - \frac{1}{J} \sum_{j=1}^{J} \varphi_{n,m,n,j} \right\|_{L^p(X)}^r \right] = \lim_{J \to \infty} \mathbb{E}' [Z_{n,j}] = 0.
\]

From this, we conclude that for every \( n = 1, \ldots, N \) there exists some \( J_n \in \mathbb{N} \) such that

\[
\mathbb{E}' \left[ \left\| \mathbb{E}' [\varphi_{n,m,n,1}] - \frac{1}{J_n} \sum_{j=1}^{J_n} \varphi_{n,m,n,j} \right\|_{L^p(X)}^r \right]^{\frac{1}{r}} < \frac{\varepsilon}{3N}.
\]  

Finally, by defining \( \varphi := \sum_{n=1}^{N} \frac{1}{J_n} \sum_{j=1}^{J_n} \varphi_{n,m,n,j} \in \mathcal{R} \mathcal{N}_{\epsilon,d}^{p,r} \), it follows by combining (65), (70), and (73) with the triangle inequality and Minkowski’s inequality that

\[
\mathbb{E}' \left[ \left\| g - \varphi \right\|_{L^p(X)} \right]^{\frac{1}{r}} \leq \mathbb{E}' \left[ \left( \left\| g - \tilde{g} \right\|_{L^p(X)} + \sum_{n=1}^{N} \left\| \tilde{w}_n \rho \left( \tilde{a}_n \cdot + \tilde{b}_n \right) - \mathbb{E}' [\varphi_{n,m,n,1}] \right\|_{L^p(X)} \right. \right. \\
\left. \left. \ldots + \sum_{n=1}^{N} \mathbb{E}' \left[ \left\| \varphi_{n,m,n,1} \right\|_{L^p(X)}^q \right]^{\frac{1}{q}} \right)^{\frac{1}{r}} \right]^{\frac{1}{r}} \\
\leq \left\| g - \tilde{g} \right\|_{L^p(X)} + \sum_{n=1}^{N} \left\| \tilde{w}_n \rho \left( \tilde{a}_n \cdot + \tilde{b}_n \right) - \mathbb{E}' [\varphi_{n,m,n,1}] \right\|_{L^p(X)} \\
\ldots + \sum_{n=1}^{N} \mathbb{E}' \left[ \left\| \mathbb{E}' [\varphi_{n,m,n,1}] - \frac{1}{J_n} \sum_{j=1}^{J_n} \varphi_{n,m,n,j} \right\|_{L^p(X)}^r \right]^{\frac{1}{r}} \\
< \frac{\varepsilon}{3} + N \frac{\varepsilon}{3N} + N \frac{\varepsilon}{3N} = \varepsilon,
\]

which completes the proof.
Proof of Proposition 2.41. Fix some $\varepsilon > 0$ and let $g \in L^{np}_{\text{diag}}(X)^{\otimes n}$ with representation $\sum_{j=1}^{m} \lambda_j g_j^{\otimes n}$ for some $m \in \mathbb{N}$, $0 \neq \lambda_j \in \mathbb{R}$, and $g_j \in L^{np}(X)$, such that

$$ \left\| g - \sum_{j=1}^{m} \lambda_j g_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}} < \frac{\varepsilon}{2}, \quad (74) $$

which exists as $g - \sum_{j=1}^{m} \lambda_j g_j^{\otimes n}$ is a representation of $0 \in L^{np}(X)^{\otimes n}$. Then, for every $j = 1, \ldots, m$, we apply Proposition 4.8 to obtain some $\varphi_j \in \mathcal{R}N_{p,d}^{\otimes n}$ such that

$$ E^{r} \left[ \| g_j - \varphi_j \|_{L^{np}(X)^{\otimes n}}^{r} \right]^{\frac{1}{r}} < \min \left( 1, \frac{\varepsilon}{2mn|\lambda_j| (\|g_j\|_{L^{np}(X)} + 1)^{n-1}} \right). \quad (75) $$

Thus, by applying the triangle inequality and Minkowski’s inequality, we conclude that

$$ E^{r} \left[ \| \varphi_j \|_{L^{np}(X)^{\otimes n}}^{r} \right]^{\frac{1}{r}} \leq E^{r} \left[ (\| g_j \|_{L^{np}(X)} + \| g_j - \varphi_j \|_{L^{np}(X)})^{n} \right]^{\frac{1}{n}} \quad (76) $$

Hence, by using the telescoping sum $g_j^{\otimes n} - \varphi_j^{\otimes n} = \sum_{l=1}^{n} (g_j^{\otimes (n-l)} \otimes (g_j - \varphi_j^{\otimes (l-1)}))$, Minkowski’s inequality, Lemma 2.17, Hölder’s inequality, Jensen’s inequality, (76), and (75), it follows for every $j = 1, \ldots, m$ that

$$ E^{r} \left[ \| g_j^{\otimes n} - \varphi_j^{\otimes n} \|_{L^{np}(X)^{\otimes n}}^{r} \right]^{\frac{1}{r}} \leq \frac{2mn|\lambda_j|}{\varepsilon} \left( \| g_j \|_{L^{np}(X)} + 1 \right)^{n-1} \quad (77) $$

Finally, by defining $\varphi := \sum_{j=1}^{m} \lambda_j \varphi_j^{\otimes n} \in \mathcal{R}N_{n,d}^{\otimes n}$, it follows by combining (74) and (77) with the triangle inequality and Minkowski’s inequality that

$$ E^{r} \left[ \| g - \varphi \|_{L^{np}(X)^{\otimes n}}^{r} \right]^{\frac{1}{r}} \leq \left\| g - \sum_{j=1}^{m} \lambda_j g_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}}^{r} + \sum_{j=1}^{m} |\lambda_j| \left\| g_j^{\otimes n} - \varphi_j^{\otimes n} \right\|_{L^{np}(X)^{\otimes n}}^{r}^{\frac{1}{r}} \quad (78) $$

which completes the proof. $\square$
4.5.4. Proof of Corollary 2.42 and Theorem 2.43.

Proof of Corollary 2.42. Fix some \( \varepsilon > 0 \) and \( r \in [1, \infty) \), and let \( G \in L^p(\mathbb{P}) \) for some \( p \in [1, \infty) \).

By Theorem 2.25, there exists \( N \in \mathbb{N} \) and a sequence \( (g_n)_{n=0,\ldots,N} \), with \( g_n \in L_{\text{diag}}^{np}(X)^{\otimes n} \) for all \( n = 0, \ldots, N \), such that

\[
\left\| G - \sum_{n=0}^{N} J_n(g_n) \right\|_{L^p(\mathbb{P})} < \frac{\varepsilon}{2}.
\]

(78)

Then, for every \( n = 1, \ldots, N \), we apply Proposition 2.41 to obtain some \( \varphi_n \in \mathcal{N}^{np}_{n;d} \) with

\[
\mathbb{E}^{\prime} \left[ \left\| g_n - \varphi_n \right\|_{L^{np}(X)^{\otimes n}} \right] \leq \frac{\varepsilon}{2C_{n,p}N},
\]

(79)

where \( C_{n,p} > 0 \) was introduced in Lemma 2.22. From this, we define the map \( \Omega' \ni \omega' \mapsto G^{\varphi_0;N}(\omega') := \sum_{n=0}^{N} J_n(\varphi_n(\omega')) \in L^p(\mathbb{P}) \) with \( \varphi_0 := g_0 \in \mathbb{R} \) and show that \( \omega' \mapsto G^{\varphi_0;N}(\omega') \in L^r(\Omega'; L^p(\mathbb{P})) \).

Indeed, since \( \Omega' \ni \omega' \mapsto \varphi_n(\omega') \in L^{np}(X)^{\otimes n} \) is by Proposition 2.40 strongly measurable and since \( J_n : L^{np}(X)^{\otimes n} \rightarrow L^p(\mathbb{P}) \) is by Lemma 2.21 and Lemma 2.22 linear and bounded, this continuous and therefore measurable, it follows by [Hyttönen et al., 2016, Corollary 1.1.11] that \( \omega' \mapsto G^{\varphi_0;N}(\omega') := \sum_{n=0}^{N} J_n(\varphi_n(\omega')) \in L^p(\mathbb{P}) \) is also strongly measurable. Moreover, by applying Minkowski’s inequality and using Lemma 2.22, we conclude that

\[
\mathbb{E}^{\prime} \left[ \left\| G^{\varphi_0;N} \right\|_{L^p(\mathbb{P})}^{r} \right] \leq \sum_{n=1}^{N} \mathbb{E}^{\prime} \left[ \left\| J_n(\varphi_n(\cdot)) \right\|_{L^p(\mathbb{P})}^{r} \right] \leq \sum_{n=1}^{N} C_{n,p} \varepsilon \frac{\varepsilon}{2C_{n,p}N} < \infty,
\]

which shows that indeed \( (\omega' \mapsto G^{\varphi_0;N}(\omega')) \in L^r(\Omega'; L^p(\mathbb{P})) \). Finally, by combining (78) and (79) with Minkowski’s inequality, and using Lemma 2.22, it follows that

\[
\mathbb{E}^{\prime} \left[ \left\| G - G^{\varphi_0;N} \right\|_{L^p(\mathbb{P})}^{r} \right] \leq \mathbb{E}^{\prime} \left[ \left( \left\| G - \sum_{n=0}^{N} J_n(g_n) \right\|_{L^p(\mathbb{P})} + \sum_{n=0}^{N} \left\| J_n(g_n - \varphi_n) \right\|_{L^p(\mathbb{P})} \right)^{\frac{r}{2}} \right]
\]

\[
\leq \left\| G - \sum_{n=0}^{N} J_n(g_n) \right\|_{L^p(\mathbb{P})}^{r} + \sum_{n=0}^{N} \mathbb{E}^{\prime} \left[ \left\| J_n(g_n - \varphi_n) \right\|_{L^p(\mathbb{P})}^{r} \right]^{\frac{r}{2}}
\]

\[
< \frac{\varepsilon}{2} + \sum_{n=1}^{N} C_{n,p} \varepsilon \frac{\varepsilon}{2C_{n,p}N} = \varepsilon,
\]

which completes the proof.

Proof of Theorem 2.43. First, we derive for every \( n = 1, \ldots, N \) and \( g_0 \in L^{np}(X) \) the hedging strategy of \( J_n(g_0^\otimes n) \). For this purpose, we use Lemma 2.23. Ito’s formula applied on \( (x, t) \mapsto \frac{1}{n!} H_n(x, t) \), Lemma 4.2 (i.e. the identities (33a), (33b), and (33c)), that \( dW(g_0)_t = g_0(t)^T dX_t \) \( \mathbb{P} \)-a.s., and that \( \langle W(g_0) \rangle_t = Q(g_0)_t \) \( \mathbb{P} \)-a.s., to conclude \( \mathbb{P} \)-a.s. that

\[
J_n(g_0^\otimes n)_T = \frac{1}{n!} \frac{H_n(W(g_0)_T, Q(g_0)_T)}{H_n(W(g_0)_0, Q(g_0)_0) + n \int_0^T \frac{1}{n!} H_{n-1}(W(g_0)_t, Q(g_0)_t) dW(g_0)_t}
\]

\[
\frac{1}{2} \int_0^T \frac{1}{n!} H_{n-2}(W(g_0)_t, Q(g_0)_t) dQ(g_0)_t
\]

\[
+ \frac{n(n-1)}{2} \int_0^T \frac{1}{n!} H_{n-2}(W(g_0)_t, Q(g_0)_t) d(W(g_0))_t.
\]

(80)
Now, in order to show (17), we conclude by using $J_0(\varphi_0) = \varphi_0$ and (80) that, $\mathbb{P}$-a.s.,

\[
G^{\varphi_0,N} = J_0(\varphi_0) + \sum_{n=1}^{N} J_n(\varphi_n) = \varphi_0 + \sum_{n=1}^{N} \sum_{j=1}^{m_n} w_{n,j} J_n\left(\varphi_{n,j}\right)_T
\]

\[
= \varphi_0 + \sum_{n=1}^{N} \frac{1}{(n-1)!} \sum_{j=1}^{m_n} \frac{d}{dt} \int_{0}^{T} H_{n-1}(W(\varphi_{n,j}), t, Q(\varphi_{n,j}), t) \varphi_{n,j,i}(t) dX^i_t,
\]

which completes the proof.

\[
\square
\]

REFERENCES


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