Properties of the Conditional Expectation

- Let $X_0, X_1, X_2, \dots$ be random variables
- For each $n \in \mathbb{N}_0$ let $\mathcal{F}_n := \sigma(X_0, X_1, \dots, X_n)$ be the information generated by $(X_k)_{k\in\mathbb{N}}$ up to time n.
- Let Y, Z be any integrable random variables (meaning $E[|Y|] < \infty$ and $E[|Z|] < \infty$).
- Recall that a random variable U is called $\mathcal{F}_n$-measurable, if it only depends on $X_0, X_1, \dots, X_n$, which means that U can be written as $U = f(X_0, X_1, \dots, X_n)$ for some function f.
- Consider the conditional expectation $E[Y\,|\mathcal{F}_n] := E[Y\,|X_0, X_1, \dots, X_n]$, $\;n \in \mathbb{N}_0$.

Then the conditional expectation satisfies the following properties:

1) $E[Y\,|\mathcal{F}_n]$ is a $\mathcal{F}_n$-measurable random variable

2) Tower property: $E\big[E[Y \mid \mathcal{F}_n]\big] = E[Y]$

as well as: for every $k \in \mathbb{N}_0$, we have $E\big[E[Y \mid \mathcal{F}_{n+k}] \,\big|\, \mathcal{F}_n\big] = E[Y\,|\mathcal{F}_n]$.

3) If Y is $\mathcal{F}_n$-measurable, then $E[Y \mid \mathcal{F}_n] = Y$.

4) If Y is $\mathcal{F}_n$-measurable, then $E[YZ \mid \mathcal{F}_n] = Y E[Z \mid \mathcal{F}_n]$.

5) If Y is independent of $(X_0, \dots, X_n)$, then $E[Y \mid \mathcal{F}_n] = E[Y]$.

In addition, the conditional expectation satisfies the following properties
like the classical expectation:

6) Linearity: For any $a, b \in \mathbb{R}$ we have $E[aY + bZ \mid \mathcal{F}_n] = aE[Y \mid \mathcal{F}_n] + bE[Z \mid \mathcal{F}_n]$

7) Monotonicity: If $Y \leq Z$, then $E[Y \mid \mathcal{F}_n] \leq E[Z \mid \mathcal{F}_n]$.