Properties of the Conditional Expectation

- Let $X_0, X_1, X_2, \ldots$ be random variables.
- For each $n \in \mathbb{N}_0$ let $\mathcal{F}_n := \sigma(X_0, X_1, \ldots, X_n)$ be the information generated by $(X_k)_{k \in \mathbb{N}}$ up to time $n$.
- Let $Y, Z$ be any integrable random variables (meaning $E[|Y|] < \infty$ and $E[|Z|] < \infty$).
- Recall that a random variable $U$ is called $\mathcal{F}_n$-measurable, if it only depends on $X_0, X_1, \ldots, X_n$, which means that $U$ can be written as $U = f(X_0, X_1, \ldots, X_n)$ for some function $f$.
- Consider the conditional expectation $E[Y | \mathcal{F}_n] := E[Y | X_0, X_1, \ldots, X_n]$, $n \in \mathbb{N}_0$.

Then the conditional expectation satisfies the following properties:

1) $E[Y | \mathcal{F}_n]$ is a $\mathcal{F}_n$-measurable random variable

2) Tower property: $E[E[Y | \mathcal{F}_n]] = E[Y]$

   as well as: for every $k \in \mathbb{N}_0$, we have $E[E[Y | \mathcal{F}_{n+k}] | \mathcal{F}_n] = E[Y | \mathcal{F}_n]$.

3) If $Y$ is $\mathcal{F}_n$-measurable, then $E[Y | \mathcal{F}_n] = Y$.

4) If $Y$ is $\mathcal{F}_n$-measurable, then $E[YZ | \mathcal{F}_n] = YE[Z | \mathcal{F}_n]$.

5) If $Y$ is independent of $(X_0, \ldots, X_n)$, then $E[Y | \mathcal{F}_n] = E[Y]$.

In addition, the conditional expectation satisfies the following properties like the classical expectation:

6) Linearity: For any $a, b \in \mathbb{R}$ we have $E[aY + bZ | \mathcal{F}_n] = aE[Y | \mathcal{F}_n] + bE[Z | \mathcal{F}_n]$

7) Monotonicity: If $Y \leq Z$, then $E[Y | \mathcal{F}_n] \leq E[Z | \mathcal{F}_n]$. 