# DEEP RELU NEURAL NETWORKS OVERCOME THE CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we consider PIDEs with gradient-independent Lipschitz continuous nonlinearities and prove that deep neural networks with ReLU activation function can approximate solutions of such semilinear PIDEs without curse of dimensionality in the sense that the required number of parameters in the deep neural networks increases at most polynomially in both the dimension d of the corresponding PIDE and the reciprocal of the prescribed accuracy  $\epsilon$ .

#### 1. INTRODUCTION

Nonlinear partial integro-differential equations (PIDEs) have many important applications in finance, physics, biology, economics, and engineering; we refer to [9, 10, 11, 32] and the references therein, as well as [5] for an excellent survey on applications of nonlocal partial differential equations (PDEs) in various fields. The development of numerical methods and their complexity analysis for PIDEs are only at their infancy and are still emerging. [17, 18] proposed both machine learning based and Monte Carlo based methods for solving *linear* PIDEs and showed that their algorithms do not suffer from the curse of dimensionality. In [2] the deep Galerkin algorithm of [33] has been extended to PIDEs. In [34] physics informed neural networks for solving nonlinear PIDEs have been constructed. Deep neural network (DNN) based algorithms for solving nonlinear PIDEs have been presented in [2, 6, 13, 14]. Recently, [5] introduced for non-local nonlinear PDEs with Neumann boundary conditions both a deep splitting algorithm as well as a multilevel Picard (MLP) algorithm to numerically solve the non-local PDEs under consideration. Furthermore, [16] introduced a DNN based algorithm for backward stochastic differential equations with jumps.

The impressive computational performance of deep learning methods has raised questions concerning their theoretical foundations. However, compared to the field of applications where these methods are successfully applied extensively, there exists only one theoretical result proving that DNNs do overcome the curse of dimensionality when approximating *linear* PIDEs, see [18]. The main contribution of our paper is a proof that DNNs with ReLU activation function do overcome the curse of dimensionality when approximating *semilinear* PIDEs whose nonlinear part, linear part, jump term, and initial condition are Lipschitz continuous. Especially, Theorem 1.2 below proves that the number of parameters in the approximating DNN increases at most polynomially in both the reciprocal of the described approximation accuracy  $\epsilon \in (0, 1)$  and the dimension  $d \in \mathbb{N}$  of the corresponding PIDE.

1.1. A mathematical framework for DNNs. First of all in Setting 1.1 below we introduce a mathematical framework for DNNs with ReLU activation function.

Setting 1.1 (A mathematical framework for DNNs). Let  $\mathbf{A}_d \colon \mathbb{R}^d \to \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that

$$\mathbf{A}_{d}(x) = \left(\max\{x_{1}, 0\}, \max\{x_{2}, 0\}, \dots, \max\{x_{d}, 0\}\right),\tag{1}$$

let  $\mathbf{D} = \bigcup_{H \in \mathbb{N}} \mathbb{N}^{H+2}$ , let

$$\mathbf{N} = \bigcup_{H \in \mathbb{N}} \bigcup_{(k_0, k_1, \dots, k_{H+1}) \in \mathbb{N}^{H+2}} \left[ \prod_{n=1}^{H+1} \left( \mathbb{R}^{k_n \times k_{n-1}} \times \mathbb{R}^{k_n} \right) \right],$$
(2)

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and let  $\mathcal{D}: \mathbf{N} \to \mathbf{D}$ ,  $\mathcal{P}: \mathbf{N} \to \mathbb{N}$ ,  $\mathcal{R}: \mathbf{N} \to (\cup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l))$  satisfy that for all  $H \in \mathbb{N}$ ,  $k_0, k_1, \ldots, k_H, k_{H+1} \in \mathbb{N}$ ,  $\Phi = ((W_1, B_1), \ldots, (W_{H+1}, B_{H+1})) \in \prod_{n=1}^{H+1} (\mathbb{R}^{k_n \times k_{n-1}} \times \mathbb{R}^{k_n})$ ,  $x_0 \in \mathbb{R}^{k_0}, \ldots, x_H \in \mathbb{R}^{k_H}$  with the property that  $\forall n \in \mathbb{N} \cap [1, H]: x_n = \mathbf{A}_{k_n}(W_n x_{n-1} + B_n)$  we have that

$$\mathcal{P}(\Phi) = \sum_{n=1}^{H+1} k_n (k_{n-1} + 1), \quad \mathcal{D}(\Phi) = (k_0, k_1, \dots, k_H, k_{H+1}), \tag{3}$$

 $\mathcal{R}(\Phi) \in C(\mathbb{R}^{k_0}, \mathbb{R}^{k_{H+1}}), and$ 

$$(\mathcal{R}(\Phi))(x_0) = W_{H+1}x_H + B_{H+1}.$$
(4)

Let us comment on the mathematical objects in Setting 1.1. For all  $d \in \mathbb{N}$ ,  $\mathbf{A}_d \colon \mathbb{R}^d \to \mathbb{R}^d$  refers to the componentwise rectified linear unit (ReLU) activation function. By N we denote the set of all parameters characterizing artificial feed-forward DNNs, by  $\mathcal{R}$  we denote the operator that maps each parameters characterizing a DNN to its corresponding function, by  $\mathcal{P}$  we denote the function that counts the number of parameters of the corresponding DNN, and by  $\mathcal{D}$  we denote the function that maps the parameters characterizing a DNN to the vector of its layer dimensions.

## 1.2. Main result.

**Theorem 1.2.** Consider the notations in Subsection 1.4, assume Setting 1.1, let  $T, c \in (0, \infty)$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $v \in \mathbb{R}^d$  let  $\beta_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^{d \times d})$ ,  $\Phi_{\beta_{\varepsilon}^d}, \Phi_{\sigma_{\varepsilon}^d, v} \in \mathbb{N}$  satisfy that  $\beta_{\varepsilon}^d = \mathcal{R}(\Phi_{\beta_{\varepsilon}^d})$ ,  $\sigma_{\varepsilon}^d(\cdot)v = \mathcal{R}(\Phi_{\sigma_{\varepsilon}^d,v})$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $\gamma_{\varepsilon}^d \colon \mathbb{R}^{2d} \to \mathbb{R}^d$ ,  $F_{\varepsilon}^d \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ ,  $G^d \colon \mathbb{R}^d \to \mathbb{R}^d$  be measurable and satisfy for all  $y, z \in \mathbb{R}^d$  that  $\gamma_{\varepsilon}^d(y, z) = F_{\varepsilon}^d(y)G^d(z)$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $v \in \mathbb{R}^d$  let  $\Phi_{F_{\varepsilon}^d,v} \in \mathbb{N}$  satisfy  $F_{\varepsilon}^d(\cdot)v = \mathcal{R}(\Phi_{F_{\varepsilon}^d,v})$ , assume for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1]$ ,  $v \in \mathbb{R}^d$  that  $\mathcal{D}(\Phi_{\sigma_{\varepsilon}^d,v}) = \mathcal{D}(\Phi_{\sigma_{\varepsilon}^d,0})$  and  $\mathcal{D}(\Phi_{F_{\varepsilon}^d,v}) = \mathcal{D}(\Phi_{F_{\varepsilon}^d,v})$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $g_{\varepsilon}^d \in C(\mathbb{R}^d,\mathbb{R})$ ,  $\Phi_{g_{\varepsilon}^d} \in \mathbb{N}$  satisfy that  $\mathcal{R}(\Phi_{g_{\varepsilon}^d}) = g_{\varepsilon}^d$ , for every  $d \in \mathbb{N}$  let  $\beta^d \in C(\mathbb{R}^d,\mathbb{R}^d)$ ,  $\sigma^d \in C(\mathbb{R}^d,\mathbb{R}^{d \times d})$ ,  $g^d \in C(\mathbb{R}^d,\mathbb{R})$ , let  $f \in C(\mathbb{R},\mathbb{R})$ , for every  $d \in \mathbb{N}$  let  $\gamma^d \in \mathbb{N}$  be a Lévy measure, assume that for all  $d \in \mathbb{N}$  there exists  $C_d \in (0,\infty)$  such that for all  $x, y, z \in \mathbb{R}^d$ ,  $t \in [0, T]$  we have that

$$\left\|\gamma^{d}(x,z)\right\| \le C_{d}\left(1 \wedge \|z\|^{2}\right), \quad \left\|\gamma^{d}(x,z) - \gamma^{d}(y,z)\right\|^{2} \le C_{d}\|x-y\|^{2}\left(1 \wedge \|z\|^{2}\right), \tag{5}$$

assume that for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x, z \in \mathbb{R}^d$  the Jacobian matrix  $(D_x \gamma^d)(x, z)$  exists, assume that for all  $d \in \mathbb{N}$  there exists  $\lambda_d \in (0, \infty)$  such that for all  $t \in [0, T]$ ,  $x, z \in \mathbb{R}^d$ ,  $\delta \in [0, 1]$  we have that

$$\lambda_d \le \left| \det(I_d + \delta(D_x \gamma^d)(x, z)) \right|,\tag{6}$$

where  $I_d$  denotes the  $d \times d$  identity matrix, and assume for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}$ ,  $\varepsilon \in (0, 1)$  that

$$\left\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(y)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma_{\varepsilon}^{d}(y)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(y,z)\right)\right\|^{2} \nu^{d}(dz) \le c\|x - y\|^{2}, \tag{7}$$

$$f(w_1) - f(w_2)|^2 \le c|w_1 - w_2|^2, \quad \left|g_{\varepsilon}^d(x) - g_{\varepsilon}^d(y)\right|^2 \le cd^c ||x - y||^2,$$
(8)

$$\left\|\beta_{\varepsilon}^{d}(0)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(0,z)\right\|^{2} \nu^{d}(dz) + |f(0)|^{2} + |g_{\varepsilon}^{d}(0)|^{2} \le cd^{c},\tag{9}$$

$$\begin{aligned} \left\|\beta_{\varepsilon}^{d}(x) - \beta^{d}(x)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma^{d}(x)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma^{d}(x,z)\right\|^{2} \nu^{d}(dz) + \left|g_{\varepsilon}^{d}(x) - g^{d}(x)\right|^{2} \\ \leq \varepsilon c d^{c} (d^{c} + \|x\|^{2}), \end{aligned}$$

$$\tag{10}$$

and

$$\mathcal{P}(\Phi_{\beta^d_{\varepsilon}}) + \mathcal{P}(\Phi_{\sigma^d_{\varepsilon},0}) + \mathcal{P}(\Phi_{F^d_{\varepsilon},0}) + \mathcal{P}(\Phi_{g^d_{\varepsilon}}) \le d^c \varepsilon^{-c}.$$
(11)

Then

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES (i) for every  $d \in \mathbb{N}$  there exists a unique viscosity solution  $u^{1} u^{d} : [0,T] \times \mathbb{R}^{d} \to \mathbb{R}$  to the PIDE

$$\left(\frac{\partial}{\partial t}u^{d}\right)(t,x) + \left\langle\beta^{d}(x), (\nabla_{x}u^{d})(t,x)\right\rangle \\
+ \frac{1}{2}\operatorname{trace}\left(\sigma^{d}(t,x)(\sigma^{d}(t,x))^{\top}\operatorname{Hess}_{x}u^{d}(t,x)\right) + f(u^{d}(t,x)) \\
+ \int_{\mathbb{R}^{d}}\left(u^{d}(x+\gamma^{d}(x,z)) - u^{d}(t,x) - \left\langle(\nabla_{x}u^{d})(t,x),\gamma^{d}(x,z)\right\rangle\right)\nu^{d}(dz) = 0 \quad (12) \\
\forall t \in [0,T), x \in \mathbb{R}^{d}, \\
u^{d}(T,x) = g^{d}(x) \quad \forall x \in \mathbb{R}^{d}$$

satisfying that  $\sup_{s \in [0,T], y \in \mathbb{R}^d} \frac{|u^d(s,y)|}{1+||y||} < \infty$  and (ii) there exist  $\eta \in (0,\infty)$  and  $(\Psi_{d,\epsilon})_{d \in \mathbb{N}, \epsilon \in (0,1)} \subseteq \mathbb{N}$  such that for all  $d \in \mathbb{N}, \epsilon \in (0,1)$  we have that  $\mathcal{R}(\Psi_{d,\epsilon}) \in C(\mathbb{R}^d,\mathbb{R}), \ \mathcal{P}(\Psi_{d,\epsilon}) \leq \eta d^{\eta} \epsilon^{-\eta}, \ and$ 

$$\left(\int_{[0,1]^d} \left| (\mathcal{R}(\Psi_{d,\epsilon}))(x) - u^d(t,x) \right|^2 dx \right)^{\frac{1}{2}} \le \epsilon.$$
(13)

Let us make some comments on the mathematical objects in Theorem 1.2. The functions  $\mu_d \colon \mathbb{R}^d \to \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , and  $\sigma_d \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ ,  $d \in \mathbb{N}$ , describe the linear part of the family of PIDEs indexed by  $d \in \mathbb{N}$  in (12). The functions  $g_d \colon \mathbb{R}^d \to \mathbb{R}$ ,  $d \in \mathbb{N}$ , describe the initial condition, while the function  $f \colon \mathbb{R} \to \mathbb{R}$  describes the nonlinearity of the PIDEs in (12). Condition (5) and (6) are needed for the existence and uniqueness of the solution to the PIDE (12) (see the proof of Theorem 5.1). Conditions (7)–(8) are globally Lipschitz condition. Condition (9) states that the initial values of the input functions grow polynomially in the dimension  $d \in \mathbb{N}$ . Condition (10) ensures that the input functions  $\beta^d, \sigma^d, \gamma^d, g^d$  can be approximated by the functions  $\beta^d_{\varepsilon}, \sigma^d_{\varepsilon}, \gamma^d_{\varepsilon}, g^d_{\varepsilon}$ . The bound  $d^c \varepsilon^{-c}$ , which is a polynomial of d and  $\varepsilon^{-1}$ , in condition (11) ensures that the functions  $\beta^d_{\varepsilon}, \sigma^d_{\varepsilon}, \gamma^d_{\varepsilon}, g^d_{\varepsilon}$  can be represented by DNNs without curse of dimensionality. Under these assumptions Theorem 1.2 states that, roughly speaking, if DNNs can approximate the initial condition, the linear part, the nonlinearity, and the jump part of the PIDEs in (12) without the curse of dimensionality, then they can also approximate its solution without the curse of dimensionality. We refer to [1, 7, 8, 19, 24, 30] for similar results obtained for PDEs without any non-local/ jump term.

1.3. Outline of the proof and organization of the paper. Theorem 1.2 follows directly from Theorem 5.1, see the proof of Theorem 1.2 which is provided right after the proof of Theorem 5.1. Although the result presented in Theorem 1.2 is purely deterministic, we use probabilistic arguments to prove its statement. More precisely, we employ the theory of full history recursive MLP approximations, which are numerical approximation methods for which it is known that they overcome the curse of dimensionality. We refer to [31] for the convergence analysis of MLP algorithms for semilinear PIDEs and to [3, 4, 12, 15, 21, 22, 23, 25, 26, 27, 28] for corresponding results proving that MLP algorithms overcome the curse of dimensionality for PDEs without any non-local/ jump term.

The main strategy of the proof, roughly speaking, is to demonstrate that these MLP approximations can be represented by DNNs, if the coefficients determining the linear part, the jump term, the initial condition, and the nonlinearity are represented by DNNs (cf. Lemma 4.12). Such ideas have been successfully applied to prove that DNNs overcome the curse of dimensionality in the numerical approximations of *semilinear* heat equations (see [24]) as well as *semilinear* Kolmogorov PDEs (see [8]). We also refer to [19, 30] for results proving that DNNs overcome the curse of dimensionality when approximating *linear* PDEs.

In order to introduce the outline of the proof we first need an MLP setting. For every  $K \in \mathbb{N}$  let  $|\cdot|_K \colon \mathbb{R} \to \mathbb{R}$ R satisfy for all  $t \in \mathbb{R}$  that  $\lfloor t \rfloor_K = \max(\{0, \frac{T}{K}, \frac{2T}{K}, \dots, T\} \cap ((-\infty, t) \cup \{0\}))$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ be a probability space satisfying the usual conditions, let  $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$ , let  $\mathfrak{t}^{\theta} \colon \Omega \to [0, 1], \theta \in \Theta$ , be identically independently distributed random variables which satisfy for all  $t \in (0, 1)$  that  $\mathbb{P}(\mathfrak{t}^0 \leq t) = t$ , for every  $\theta \in \Theta$ ,  $t \in [0,T]$  let  $\mathfrak{T}_t^{\theta} \colon \Omega \to \mathbb{R}$  satisfy for all  $\theta \in \Theta$  that  $\mathfrak{T}_t^{\theta} = t + (T-t)\mathfrak{t}^{\theta}$ , for every  $d \in \mathbb{N}$  let  $W^{d,\theta} \colon \Omega \times [0,T] \to \mathbb{R}^d$ ,  $\theta \in \Theta$ , be identically independently distributed standard  $(\mathbb{F}_t)_{t \in [0,T]}$ . Brownian motions, for every  $d \in \mathbb{N}$  let  $N^{d,\theta}$ ,  $\theta \in \Theta$ , be independent  $(\mathbb{F}_t)_{t \in [0,T]}$ -Poisson random measures on  $[0,\infty) \times (\mathbb{R}^d \setminus \{0\})$  with intensity  $\nu^d$ , for every  $d \in \mathbb{N}, \theta \in \Theta$  let

$$\tilde{N}^{d,\theta}(dt,dz) = N^{d,\theta}(dt,dz) - dt\,\nu^d(dz),\tag{14}$$

<sup>&</sup>lt;sup>1</sup>For the definition of a viscosity solution see, e.g., [31, Definition 2.7].

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and assume for all  $d \in \mathbb{N}$  that  $\mathcal{F}_0$ ,  $(\mathfrak{t}^{\theta})_{\theta \in \Theta}$ ,  $(N^{d,\theta})_{\theta \in \Theta}$  and  $(W^{d,\theta})_{\theta \in \Theta}$  are independent. First, the viscosity solution to (12) can be represented by the following stochastic fixed point equation (SFPE) (cf. [31, Proposition 5.16]):

$$u^{d}(t,x) = \mathbb{E}\left[g^{d}(X_{T}^{d,0,t,x})\right] + \int_{t}^{T} \mathbb{E}\left[f(u^{d}(X_{s}^{d,0,t,x}))\right] ds$$

$$(15)$$

where

$$X_{s}^{d,\theta,t,x} = x + \int_{t}^{s} \beta^{d}(X_{u-}^{d,\theta,t,x}) du + \int_{t}^{s} \sigma^{d}(X_{u-}^{d,\theta,t,x}) dW_{u}^{d,\theta} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma^{d}(X_{u-}^{d,\theta,t,x},z) \tilde{N}^{d,\theta}(du,dz).$$

$$(16)$$

We approximate the input functions  $\beta^d$ ,  $\sigma^d$ ,  $\gamma^d$ ,  $g^d$ , f by  $\beta^d_{\varepsilon}$ ,  $\sigma^d_{\varepsilon}$ ,  $\gamma^d_{\varepsilon}$ ,  $g^d_{\varepsilon}$ ,  $f_{\varepsilon}$  where the functions with index  $\varepsilon$  are represented by DNNs. We then get the following SFPE:

$$u^{d,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,0,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,\varepsilon}(X^{d,0,\varepsilon,t,x}_s))\right] ds$$
(17)

where

$$\begin{aligned} X_{s}^{d,\theta,\varepsilon,t,x} &= x + \int_{t}^{s} \beta_{\varepsilon}^{d}(X_{u-}^{d,\theta,\varepsilon,t,x}) du \\ &+ \int_{t}^{s} \sigma_{\varepsilon}^{d}(X_{u-}^{d,\theta,\varepsilon,t,x}) dW_{u}^{d,\theta} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d}(X_{u-}^{d,\theta,\varepsilon,t,x},z) \tilde{N}^{d,\theta}(du,dz). \end{aligned}$$
(18)

Next, we approximate the stochastic differential equation (SDE) in (18) by the Euler-Maruyama discretization:

$$X_{s}^{d,\theta,K,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d} (X_{\max\{t,\lfloor u-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}) du + \int_{t}^{s} \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor u-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}) dW_{u}^{d,\theta} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor u-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}, z) \tilde{N}^{d,\theta} (du, dz).$$

$$(19)$$

The latter is associated to the following SFPE:

$$u^{d,K,\varepsilon}(t,x) = \mathbb{E}\Big[g^d_{\varepsilon}(X^{d,0,K,\varepsilon,t,x}_T)\Big] + \int_t^T \mathbb{E}\Big[f_{\varepsilon}(u^{d,K,\varepsilon}(X^{d,0,K,\varepsilon,t,x}_s))\Big]\,ds.$$
(20)

The DNNs that approximate  $u^d$ ,  $d \in \mathbb{N}$ , will be constructed from the following MLP approximation:

$$U_{n,m}^{d,\theta,K,\varepsilon}(t,x) = \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^{n}} \sum_{i=1}^{m^{n}} g_{\varepsilon}^{d} (X_{T}^{d,(\theta,0,-i),K,\varepsilon,t,x}) + \sum_{\ell=0}^{n-1} \frac{T-t}{m^{n-\ell}} \sum_{i=1}^{m^{n}} (f \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} - f \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon}) (\mathfrak{T}_{t}^{(\theta,\ell,i)}, X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}}^{d,(\theta,0,i),K,\varepsilon,t,x})$$
(21)

(cf. Lemma 4.12). We then decompose the error  $U_{n,m}^{d,\theta,K,\varepsilon}(t,x) - u^d(t,x)$  as follows:

$$U_{n,m}^{d,\theta,K,\varepsilon}(t,x) - u^{d}(t,x) = \underbrace{U_{n,m}^{d,\theta,K,\varepsilon}(t,x) - u^{d,K,\varepsilon}(t,x)}_{=:E_{1}} + \underbrace{u^{d,K,\varepsilon}(t,x) - u^{d,\varepsilon}(t,x)}_{=:E_{2}} + \underbrace{u^{d,\varepsilon}(t,x) - u^{d}(t,x)}_{=:E_{3}}$$
(22)

where  $E_1$  is estimated in Lemma 3.3,  $E_2$  is estimated in Lemma 3.2, and  $E_3$  is estimated in Lemma 2.1. In the proof of our main result, Theorem 5.1, we combine these results and construct a DNN realization on a probability space that suffices the prescribed approximation accuracy  $\epsilon \in (0, 1)$  between  $U_{n,m}^{d,\theta,K,\varepsilon}(t,x)$  and  $u^d(t,x)$ . Note that Lemmas 3.3, 3.2, and 2.1 are stability and perturbation results that can be read without knowledge on DNNs.

The remaining part of the paper is organized as follows. In Section 2 we establish a stability result on SFPEs that demonstrates the error  $E_3$ . In Section 3 we recall basic facts on Euler-Maruyama and MLP approximations and establish the error  $E_1$  of the MLP approximation as well as the discretization error  $E_2$ . Section 4 introduces a mathematical framework for DNNs and demonstrates the connection between DNNs and MLP approximations, see Lemma 4.12. Finally, Section 5 combines the results of the previous sections to prove the main results, Theorem 5.1 and Theorem 1.2.

1.4. Notations. Let  $\|\cdot\|, \|\cdot\| \colon (\bigcup_{d\in\mathbb{N}}\mathbb{R}^d) \to [0,\infty)$ , dim:  $(\bigcup_{d\in\mathbb{N}}\mathbb{R}^d) \to \mathbb{N}$  satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$  that  $\|x\| = \sqrt{\sum_{i=1}^d (x_i)^2}$ ,  $\|x\| = \max_{i\in[1,d]\cap\mathbb{N}} |x_i|$ , and dim(x) = d and let  $\|\cdot\|_{\mathrm{F}} \colon \bigcup_{d\in\mathbb{N}} \mathbb{R}^{d\times d} \to [0,\infty)$  satisfy for all  $d \in \mathbb{N}$ ,  $x = (x_{ij})_{i,j\in[1,d]\cap\mathbb{Z}} \in \mathbb{R}^{d\times d}$  that  $\|x\|_{\mathrm{F}}^2 = \sum_{i,j=1}^d |x_{ij}|^2$ .

# 2. Approximation of the coefficients

In Lemma 2.1 below we approximate the solution to the SFPE (30) through solution to the SFPE (31), whose linear part, initial condition, and nonlinearity can be exactly represented through suitable DNNs.

**Lemma 2.1** (A stability result). Consider the notations given in Subsection 1.4, let  $T \in (0, \infty)$ ,  $c \in [1, \infty)$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ , let  $\beta_{\varepsilon}^{d}$ ,  $\beta^{d} \in C(\mathbb{R}^{d}, \mathbb{R}^{d})$ ,  $\sigma_{\varepsilon}^{d}$ ,  $\sigma^{d} \in C(\mathbb{R}^{d}, \mathbb{R}^{d \times d})$ ,  $g_{\varepsilon}^{d}$ ,  $g^{d} \in C(\mathbb{R}^{d}, \mathbb{R})$ ,  $f_{\varepsilon}$ ,  $f \in C(\mathbb{R}, \mathbb{R})$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  let  $\gamma^{d}, \gamma_{\varepsilon}^{d} \colon \mathbb{R}^{2d} \to \mathbb{R}^{d}$  be measurable, let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_{t})_{t \in [0,T]})$  be a filtered probability space satisfying the usual conditions, for every  $d \in \mathbb{N}$  let  $W^{d} \colon \Omega \times [0,T] \to \mathbb{R}^{d}$ , be standard  $(\mathbb{F}_{t})_{t \in [0,T]}$ -Brownian motion, for every  $d \in \mathbb{N}$  let  $N^{d}$  be an  $(\mathbb{F}_{t})_{t \in [0,T]}$ -Poisson random measure on  $[0,\infty) \times (\mathbb{R}^{d} \setminus \{0\})$  with intensity  $\nu^{d}$ , for every  $d \in \mathbb{N}$  let  $\tilde{N}^{d}(dt, dz) = N^{d}(dt, dz) - dt \nu^{d}(dz)$ , assume for all  $d \in \mathbb{N}$  that  $\mathcal{F}_{0}$ ,  $N^{d}$  and  $W^{d}$  are independent, assume for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^{d}$ ,  $w_{1}, w_{2} \in \mathbb{R}$ ,  $\varepsilon \in (0, 1)$  that

$$\left\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(y)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma_{\varepsilon}^{d}(y)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(y,z)\right)\right\|^{2} \nu^{d}(dz) \le c\|x - y\|^{2}, \quad (23)$$

$$f_{\varepsilon}(w_1) - f_{\varepsilon}(w_2)|^2 \le c|w_1 - w_2|^2, \quad T \left| g_{\varepsilon}^d(x) - g_{\varepsilon}^d(y) \right|^2 \le cd^c ||x - y||^2, \tag{24}$$

$$\left\|\beta_{\varepsilon}^{d}(0)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(0,z)\right\|^{2} \nu^{d}(dz) + T^{3}|f_{\varepsilon}(0)|^{2} + T|g_{\varepsilon}^{d}(0)|^{2} \le cd^{c},\tag{25}$$

and

$$\begin{aligned} \left\| \beta_{\varepsilon}^{d}(x) - \beta^{d}(x) \right\|^{2} + \left\| \sigma_{\varepsilon}^{d}(x) - \sigma^{d}(x) \right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\| \gamma_{\varepsilon}^{d}(x,z) - \gamma^{d}(x,z) \right\|^{2} \nu^{d}(dz) \\ + \left| g_{\varepsilon}^{d}(x) - g^{d}(x) \right|^{2} + |f_{\varepsilon}(w_{1}) - f(w_{1})|^{2} \\ \leq \varepsilon c d^{c} (d^{c} + \|x\|^{2}) + \varepsilon |w_{1}|^{4}, \end{aligned}$$

$$(26)$$

for every  $d \in \mathbb{N}$ ,  $t \in [0,T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0,1)$  let  $(X_s^{d,t,x})_{s \in [t,T]}$ ,  $(X_s^{d,\varepsilon,t,x})_{s \in [t,T]}$  be  $(\mathbb{F}_t)_{t \in [0,T]}$ -adapted càdlàg processes which satisfy for all  $s \in [t,T]$  that  $\mathbb{P}$ -a.s.

$$X_{s}^{d,t,x} = x + \int_{t}^{s} \beta^{d}(X_{r-}^{d,t,x})dr + \int_{t}^{s} \sigma^{d}(X_{r-}^{d,t,x})dW_{r}^{d} + \int_{t}^{s} \int_{\mathbb{R}^{d}\setminus\{0\}} \gamma^{d}(X_{r-}^{d,t,x},z)\tilde{N}^{d}(dr,dz)$$
(27)

and

$$X_{s}^{d,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) dr + \int_{t}^{s} \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) dW_{r}^{d} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z) \tilde{N}^{d}(dr,dz), \quad (28)$$

and for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $u^d, u^{d,\varepsilon} \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$  be measurable and satisfy for all  $t \in [0,T]$ ,  $x \in \mathbb{R}^d$  that  $\mathbb{E}\left[\left|g^d(X_T^{d,t,x})\right|\right] + \int_t^T \mathbb{E}\left[\left|f(u^d(s,X_s^{d,t,x}))\right|\right] ds < \infty$ ,  $\mathbb{E}\left[\left|g^d_\varepsilon(X_T^{d,\varepsilon,t,x})\right|\right] + \int_t^T \mathbb{E}\left[\left|f_\varepsilon(u^{d,\varepsilon}(s,X_s^{d,\varepsilon,t,x}))\right|\right] ds < \infty$ ,

$$\sup_{s \in [0,T]} \sup_{y \in \mathbb{R}^d} \frac{|u^d(s,y)| + |u^{d,\varepsilon}(s,y)|}{1 + \|y\|} < \infty,$$
(29)

$$u^{d}(t,x) = \mathbb{E}\left[g^{d}(X_{T}^{d,t,x})\right] + \int_{t}^{T} \mathbb{E}\left[f(u^{d}(s,X_{s}^{d,t,x}))\right] ds,$$
(30)

and

$$u^{d,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,\varepsilon}(s,X^{d,\varepsilon,t,x}_s))\right] ds.$$
(31)

Then

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(i) for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\max\left\{\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right], \mathbb{E}\left[d^{c} + \left\|X_{s}^{d,t,x}\right\|^{2}\right]\right\} \le (d^{c} + \|x\|^{2})e^{7c(s-t)},\tag{32}$$

(ii) for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^d$  we have that

$$\left| u^{d,\varepsilon}(t,x) - u^{d,\varepsilon}(t,y) \right| \le 2(cd^{c}T^{-1})^{\frac{1}{2}} \|x - y\| e^{5cT + 2cT^{2}},$$
(33)

and

(iii) for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left| u^{d,\varepsilon}(t,x) - u^{d}(t,x) \right| \le 2cd^{c}\varepsilon^{\frac{1}{2}}(d^{c} + \|x\|^{2})e^{24cT + 5cT^{2}}.$$
(34)

*Proof of Lemma 2.1.* Throughout this proof let  $\langle \cdot, \cdot \rangle \colon \bigcup_{d \in \mathbb{R}} \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$  satisfy for every  $d \in \mathbb{N}$ ,  $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$  that  $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ . First, the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d \colon ||x + y||^2 \le 2||x||^2 + 2||y||^2$ , (25), and (23) show for all  $d \in \mathbb{N}, x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\|\beta_{\varepsilon}^{d}(x)\|^{2} \leq 2\|\beta_{\varepsilon}^{d}(0)\|^{2} + 2\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(0)\|^{2} \leq 2cd^{c} + 2c\|x\|^{2} = 2c(d^{c} + \|x\|^{2}).$$
(35)

Similarly, we have for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\left\|\sigma_{\varepsilon}^{d}(x)\right\|_{\mathrm{F}}^{2} \leq 2c(d^{c} + \|x\|^{2}).$$
(36)

Next, the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d$ :  $||x+y||^2 \le 2||x||^2 + 2||y||^2$ , (25), and (23) show for all  $d \in \mathbb{N}, x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\int_{\mathbb{R}^{d}\setminus\{0\}} \left\| \gamma_{\varepsilon}^{d}(x,z) \right\|^{2} \nu^{d}(dz) \leq \int_{\mathbb{R}^{d}\setminus\{0\}} 2 \left\| \gamma_{\varepsilon}^{d}(0,z) \right\|^{2} + 2 \left\| \gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(0,z) \right\|^{2} \nu^{d}(dz) \\
\leq 2cd^{c} + 2c \|x^{2}\| = 2c(d^{c} + \|x\|^{2}).$$
(37)

Next, Itô's formula (see, e.g., [20, Theorem 3.1]) and (28) show for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that  $\mathbb{P}$ -a.s. we have that

$$\begin{split} \left\|X_{s}^{d,\varepsilon,t,x}\right\|^{2} &= \left\|x\right\|^{2} + \int_{t}^{s} \left(2\left\langle X_{r}^{d,\varepsilon,t,x},\beta_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x})\right\rangle + \left\|\sigma_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x})\right\|_{\mathrm{F}}^{2}\right) dr \\ &+ 2\int_{t}^{s} \sum_{i,j=1}^{d} \left(X_{r-}^{d,\varepsilon,t,x}\right)_{i} \left(\sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right)_{ij} d(W_{j}^{d})_{r} \\ &+ 2\int_{t}^{s} \int_{\mathbb{R}^{d}\setminus\{0\}} \left\langle X_{r-}^{d,\varepsilon,t,x},\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z)\right\rangle \tilde{N}^{d}(dz,dr) \\ &+ \int_{t}^{s} \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z)\right\|^{2} N^{d}(dz,dr). \end{split}$$
(38)

Next, for every  $d,n\in\mathbb{N},\,x\in\mathbb{R}^d,\,\varepsilon\in(0,1)$  let  $\tau_n^{d,\varepsilon,x}\colon\Omega\to\mathbb{R}$  satisfy that

$$\begin{aligned} \tau_n^{d,\varepsilon,x} &= \inf\left\{s \in [t,T] \colon \int_t^s \left(2\left\langle X_r^{d,\varepsilon,t,x}, \beta_\varepsilon^d(X_r^{d,\varepsilon,t,x})\right\rangle + \left\|\sigma_\varepsilon^d(X_r^{d,\varepsilon,t,x})\right\|_F^2\right) dr \\ &+ \int_t^s \sum_{i,j=1}^d \left|\left(X_r^{d,\varepsilon,t,x}\right)_i \left(\sigma_\varepsilon^d(X_r^{d,\varepsilon,t,x})\right)_{ij}\right|^2 dr \\ &+ \int_t^s \int_{\mathbb{R}^d \setminus \{0\}} \sum_{i=1}^d \left|\left(X_r^{d,\varepsilon,t,x}\right)_i \left(\gamma_\varepsilon^d(X_r^{d,\varepsilon,t,x},z)\right)_i\right|^2 \nu^d(dz) dr \\ &+ \int_t^s \int_{\mathbb{R}^d \setminus \{0\}} \left\|\gamma_\varepsilon^d(X_r^{d,\varepsilon,t,x},z)\right\|_F^2 \nu^d(dz) dr \ge n \right\} \wedge T \end{aligned}$$
(39)

$$\mathbb{E}\left[d^{c} + \left\|X_{s\wedge\tau_{n}^{d,\varepsilon,x}}^{d,\varepsilon,x}\right\|^{2}\right] \\
= d^{c} + \left\|x\right\|^{2} + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \left(2\left\langle X_{r}^{d,\varepsilon,t,x},\beta_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x})\right\rangle + \left\|\sigma_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x})\right\|_{\mathrm{F}}^{2}\right)dr\right] \\
+ \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x},z)\right\|^{2}\nu^{d}(dz)dr\right] \\
\leq d^{c} + \left\|x\right\|^{2} + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \left\|X_{r}^{d,\varepsilon,t,x}\right\|^{2}dr\right] + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \left\|\beta_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x})\right\|^{2}dr\right] \\
+ \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \left\|\sigma_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x})\right\|_{\mathrm{F}}^{2}dr\right] + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(X_{r}^{d,\varepsilon,t,x},z)\right\|^{2}\nu^{d}(dz)dr\right]$$
(40)

and

$$\mathbb{E}\left[d^{c} + \left\|X_{s\wedge\tau_{n}^{d,\varepsilon,x}}^{d,\varepsilon,x}\right\|^{2}\right] \\
\leq d^{c} + \|x\|^{2} + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} \left(d^{c} + \left\|X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right)dr\right] + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} 2c\left(d^{c} + \left\|X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right)dr\right] \\
+ \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} 2c\left(d^{c} + \left\|X_{r}^{d,\varepsilon,t,x}\right\|_{F}^{2}\right)dr\right] + \mathbb{E}\left[\int_{t}^{s\wedge\tau_{n}^{d,\varepsilon,x}} 2c\left(d^{c} + \left\|X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right)dr\right] \\
\leq d^{c} + \|x\|^{2} + 7c\int_{t}^{s} \mathbb{E}\left[d^{c} + \left\|X_{r\wedge\tau_{n}^{d,\varepsilon,x}}^{d,\varepsilon,t,x}\right\|^{2}dr\right].$$
(41)

This, Fatou's lemma, and Grönwall's inequality show for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[d^{c} + \left\|X_{s \wedge \tau_{n}^{d,\varepsilon,x}}^{d,\varepsilon,t,x}\right\|^{2}\right] \leq (d^{c} + \|x\|^{2})e^{7c(s-t)}.$$
(42)

Next, using (26) and letting  $\varepsilon$  tend to zero in (23) and (25) we obtain that

$$\left\|\beta^{d}(x) - \beta^{d}(y)\right\|^{2} + \left\|\sigma^{d}(x) - \sigma^{d}(y)\right\|^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma^{d}(x,z) - \gamma^{d}(y,z)\right\|^{2} \nu^{d}(dz) \, dr \le c \|x - y\|^{2} \quad (43)$$

and

$$\left\|\beta^{d}(0)\right\|^{2} + \left\|\sigma^{d}(0)\right\|^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma^{d}(0,z)\right\|^{2} \nu^{d}(dz) \le cd^{c}.$$
(44)

Using a similar argument as that for (42) we then obtain for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,t,x}\right\|^{2}\right] \le (d^{c} + \|x\|^{2})e^{7c(s-t)}.$$
(45)

This and (42) show (i).

Next, (26) and (45) show for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\int_{t}^{s} \left\|\beta_{\varepsilon}^{d}(X_{r-}^{d,t,x}) - \beta^{d}(X_{r-}^{d,t,x})\right\|^{2} dr\right] \\
\leq \int_{t}^{s} \varepsilon cd^{c} \left(d^{c} + \mathbb{E}\left[\left\|X_{r}^{d,t,x}\right\|^{2}\right]\right) dr \qquad (46) \\
\leq \int_{t}^{s} \varepsilon cd^{c} (d^{c} + \|x\|^{2})e^{7c(r-t)} dr \leq \varepsilon cd^{c} (d^{c} + \|x\|^{2})(s-t)e^{7cT}$$

and similarly

$$\int_{t}^{s} \mathbb{E}\left[\left\|\sigma_{\varepsilon}^{d}(X_{r-}^{d,t,x}) - \sigma^{d}(X_{r-}^{d,t,x})\right\|_{\mathrm{F}}^{2}\right] dr \leq \varepsilon c d^{c} (d^{c} + \|x\|^{2})(s-t)e^{7cT}.$$
(47)

Next, (26) and (45) show for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \mathbb{E} \left[ \left\| \gamma_{\varepsilon}^{d}(X_{r-}^{d,t,x},z) - \gamma^{d}(X_{r-}^{d,t,x},z) \right\|^{2} \right] \nu^{d}(dz) dr$$

$$\leq \int_{t}^{s} \varepsilon cd^{c} \left( d^{c} + \mathbb{E} \left[ \left\| X_{r}^{d,t,x} \right\|^{2} \right] \right) dr$$

$$\leq \int_{t}^{s} \varepsilon cd^{c}(d^{c} + \|x\|^{2})e^{7c(r-t)} dr \leq \varepsilon cd^{c}(d^{c} + \|x\|^{2})(s-t)e^{7cT}.$$
(48)

Next, Hölder's inequality, the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d$ :  $||x + y||^2 \leq 2||x||^2 + 2||y||^2$ , (23), and (46) show for all  $d \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\begin{split} & \mathbb{E}\left[\left\|\int_{t}^{s}(\beta_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \beta^{d}(X_{r_{-}}^{d,t,x}))\,dr\right\|^{2}\right] \\ &\leq \mathbb{E}\left[\left(\int_{t}^{s}\left\|\beta_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \beta^{d}(X_{r_{-}}^{d,t,x})\right\|\,dr\right)^{2}\right] \\ &\leq \mathbb{E}\left[\left(\int_{t}^{s}dr\right)\left(\int_{t}^{s}\left\|\beta_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \beta^{d}(X_{r_{-}}^{d,t,x})\right\|^{2}\,dr\right)\right] \\ &\leq T\mathbb{E}\left[\int_{t}^{s}\left\|\beta_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \beta^{d}(X_{r_{-}}^{d,t,x})\right\|^{2}\,dr\right] \\ &\leq 2T\mathbb{E}\left[\int_{t}^{s}\left\|\beta_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \beta_{\varepsilon}^{d}(X_{r_{-}}^{d,t,x})\right\|^{2}\,dr\right] + 2T\mathbb{E}\left[\int_{t}^{s}\left\|\beta_{\varepsilon}^{d}(X_{r_{-}}^{d,t,x}) - \beta^{d}(X_{r_{-}}^{d,t,x})\right\|^{2}\,dr\right] \\ &\leq 2Tc\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x} - X_{r}^{d,t,x}\right\|^{2}\right]\,dr + 2T\cdot\varepsilon cd^{c}(d^{c} + \|x\|^{2})(s-t)e^{7cT} \\ &= 2cT\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x} - X_{r}^{d,t,x}\right\|^{2}\right]\,dr + 2\varepsilon cd^{c}(d^{c} + \|x\|^{2})T(s-t)e^{7cT}. \end{split}$$

Furthermore, Itô's isometry, the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^{d \times d}$ :  $||x + y||_{\mathrm{F}}^2 \leq 2||x||_{\mathrm{F}}^2 + 2||y||_{\mathrm{F}}^2$ , (23), and (47) show for all  $d \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{t}^{s} \left(\sigma_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \sigma^{d}(X_{r_{-}}^{d,t,x})\right) dW_{r}^{d}\right\|^{2}\right] \\
= \mathbb{E}\left[\int_{t}^{s} \left\|\sigma_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \sigma^{d}(X_{r_{-}}^{d,t,x})\right\|_{F}^{2} dr\right] \\
\leq 2\int_{t}^{s} \mathbb{E}\left[\left\|\sigma_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x}) - \sigma_{\varepsilon}^{d}(X_{r_{-}}^{d,t,x})\right\|_{F}^{2}\right] dr + 2\int_{t}^{s} \mathbb{E}\left[\left\|\sigma_{\varepsilon}^{d}(X_{r_{-}}^{d,t,x}) - \sigma^{d}(X_{r_{-}}^{d,t,x})\right\|_{F}^{2}\right] dr \\
\leq 2c\int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x} - X_{r}^{d,t,x}\right\|_{F}^{2}\right] dr + 2\varepsilon cd^{c}(d^{c} + \|x\|^{2})(s-t)e^{7cT}.$$
(50)

Next, Itô's isometry (see, e.g., [9, Proposition 8.8]), the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d \colon ||x + y||^2 \leq 2||x||^2 + 2||y||^2$ , (23), and (48) show for all  $d \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\left(\gamma_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x},z)-\gamma^{d}(X_{r_{-}}^{d,t,x},z)\right)\tilde{N}^{d}(dr,dz)\right\|^{2}\right] \\
=\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\mathbb{E}\left[\left\|\gamma_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x},z)-\gamma^{d}(X_{r_{-}}^{d,t,x},z))\right\|^{2}\right]\nu^{d}(dz)\,dr \\
\leq 2\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\mathbb{E}\left[\left\|\gamma_{\varepsilon}^{d}(X_{r_{-}}^{d,\varepsilon,t,x},z)-\gamma_{\varepsilon}^{d}(X_{r_{-}}^{d,t,x},z))\right\|^{2}\right]\nu^{d}(dz)\,dr \\
+2\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\mathbb{E}\left[\left\|\gamma_{\varepsilon}^{d}(X_{r_{-}}^{d,t,x},z)-\gamma^{d}(X_{r_{-}}^{d,t,x},z)\right\|^{2}\right]\nu^{d}(dz)\,dr \\
\leq 2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x}-X_{r}^{d,t,x}\right\|^{2}\right]dr + 2\varepsilon cd^{c}(d^{c}+\|x\|^{2})(s-t)e^{7cT}.$$
(51)

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES Next, (27) and (28) show for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that  $\mathbb{P}$ -a.s.

$$\begin{aligned} X_{s}^{d,\varepsilon,t,x} &- X_{s}^{d,t,x} \\ &= \int_{t}^{s} (\beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) - \beta^{d}(X_{r-}^{d,t,x})) \, dr + \int_{t}^{s} (\sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) - \sigma^{d}(X_{r-}^{d,t,x})) \, dW_{r}^{d} \\ &+ \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} (\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z) - \gamma^{d}(X_{r-}^{d,t,x},z)) \, \tilde{N}^{d}(dr,dz). \end{aligned}$$

$$(52)$$

This, the fact that  $\forall d \in \mathbb{N}, x, y, z \in \mathbb{R}^d$ :  $||x + y + z||^2 \leq 3||x||^2 + 3||y||^2 + 3||z||^2$ , (49), (50), and (51) show for all  $d \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\begin{split} & \mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x}-X_{s}^{d,t,x}\right\|^{2}\right] \\ &\leq 3\mathbb{E}\left[\left\|\int_{t}^{s}(\beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})-\beta^{d}(X_{r-}^{d,t,x}))dr\right\|^{2}\right] + 3\mathbb{E}\left[\left\|\int_{t}^{s}(\sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})-\sigma^{d}(X_{r-}^{d,t,x}))dW_{r}^{d}\right\|^{2}\right] \\ &\quad + 3\mathbb{E}\left[\left\|\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}(\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z)-\gamma^{d}(X_{r-}^{d,t,x},z))\tilde{N}^{d}(dr,dz)\right\|^{2}\right] \\ &\leq 3\left[2cT\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x}-X_{r}^{d,t,x}\right\|^{2}\right]dr + 2\varepsilon cd^{c}(d^{c}+\|x\|^{2})T(s-t)e^{7cT}\right] \\ &\quad + 3\left[2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x}-X_{r}^{d,t,x}\right\|^{2}\right]dr + 2\varepsilon cd^{c}(d^{c}+\|x\|^{2})(s-t)e^{7cT}\right] \\ &\quad + 3\left[2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x}-X_{r}^{d,t,x}\right\|^{2}\right]dr + 2\varepsilon cd^{c}(d^{c}+\|x\|^{2})(s-t)e^{7cT}\right] \\ &\quad = (12c+6cT)\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x}-X_{r}^{d,t,x}\right\|^{2}\right]dr + 3\left(2T+4\right)\varepsilon cd^{c}(d^{c}+\|x\|^{2})(s-t)e^{7cT}. \end{split}$$

This, Grönwall's inequality, (42), (45), and the fact that  $3(2T+4) \leq 12(T+1) \leq 12e^{cT}$  show for all  $d \in \mathbb{N}$ ,  $t \in [0,T]$ ,  $s \in [t,T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0,1)$  that

$$\mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x}-X_{s}^{d,t,x}\right\|^{2}\right] \leq 3\left(2T+4\right)\varepsilon cd^{c}(d^{c}+\|x\|^{2})(s-t)e^{7cT}e^{(12c+6cT)T}$$

$$\leq 12e^{cT}\varepsilon cd^{c}(d^{c}+\|x\|^{2})e^{7cT}e^{(12c+6cT)T}(s-t)$$

$$= 12\varepsilon cd^{c}(d^{c}+\|x\|^{2})e^{20cT+6cT^{2}}(s-t).$$
(54)

Next, (28), the fact that  $\forall d \in \mathbb{N}, x_1, x_2, x_3, x_4 \in \mathbb{R}^d$ :  $\|\sum_{i=1}^4 x_i\|^2 \leq 4 \sum_{i=1}^4 \|x_i\|^2$ , Jensen's inequality, Itô's isometry, and (23) show for all  $d \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x}-X_{s}^{d,\varepsilon,t,y}\right\|^{2}\right] \leq 4\|x-y\|^{2} + 4\mathbb{E}\left[\left\|\int_{t}^{s}(\beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) - \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,y}))dr\right\|^{2}\right] + 4\mathbb{E}\left[\left\|\int_{t}^{s}(\sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) - \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,y}))dW_{r}^{d}\right\|^{2}\right] + 4\mathbb{E}\left[\left\|\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}(\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z) - \gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,y},z))\tilde{N}^{d}(dr,dz)\right\|^{2}\right]$$
(55)

and

$$\mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x}-X_{s}^{d,\varepsilon,t,y}\right\|^{2}\right] \\
\leq 4\|x-y\|^{2} + 4T\mathbb{E}\left[\int_{t}^{s}\left\|\beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) - \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,y})\right\|^{2} dr\right] \\
+ 4\mathbb{E}\left[\int_{t}^{s}\left\|\sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) - \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,y})\right\|_{\mathrm{F}}^{2} dr\right] \\
+ 4\mathbb{E}\left[\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\left\|\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z) - \gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,y},z)\right\|^{2} \nu^{d}(dz) dr\right]$$

$$\leq 4\|x-y\|^{2} + 4T\mathbb{E}\left[\int_{t}^{s}c\left\|X_{r-}^{d,\varepsilon,t,x} - X_{r-}^{d,\varepsilon,t,y}\right\|^{2} dr\right] + 4\mathbb{E}\left[\int_{t}^{s}c\left\|X_{r-}^{d,\varepsilon,t,x} - X_{r-}^{d,\varepsilon,t,y}\right\|^{2} dr\right] \\
+ 4\mathbb{E}\left[\int_{t}^{s}c\left\|X_{r-}^{d,\varepsilon,t,x} - X_{r-}^{d,\varepsilon,t,y}\right\|^{2} dr\right] \\
= 4\|x-y\|^{2} + (4Tc+8c)\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,y}\right\|^{2}\right] dr.$$
(56)

This, Grönwall's lemma, and (42) show for all  $d \in \mathbb{N}$ ,  $t \in [0,T]$ ,  $s \in [t,T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0,1)$  that

$$\mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x} - X_{s}^{d,\varepsilon,t,y}\right\|^{2}\right] \leq 4\|x - y\|^{2}e^{(4Tc + 8c)(s-t)} \leq 4\|x - y\|^{2}e^{(4Tc + 8c)T} = 4\|x - y\|^{2}e^{8cT + 4cT^{2}}.$$
(57)

Next, (24), Jensen's inequality, and (54) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x}) - g_{\varepsilon}^{d}(X_{T}^{d,t,x})\right|\right] \leq \mathbb{E}\left[(cd^{c})^{\frac{1}{2}}T^{-\frac{1}{2}} \left\|X_{T}^{d,\varepsilon,t,x} - X_{T}^{d,t,x}\right\|\right] \\
\leq (cd^{c})^{\frac{1}{2}}T^{-\frac{1}{2}} \left(\mathbb{E}\left[\left\|X_{T}^{d,\varepsilon,t,x} - X_{T}^{d,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}} \\
\leq (cd^{c})^{\frac{1}{2}}T^{-\frac{1}{2}} \left(12\varepsilon d^{c}(d^{c} + \|x\|^{2})e^{20cT + 6cT^{2}}T\right)^{\frac{1}{2}} \\
\leq 4(\varepsilon cd^{2c})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{10cT + 3cT^{2}}.$$
(58)

Next, (26), Jensen's inequality, and (45) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,t,x}) - g^{d}(X_{T}^{d,t,x})\right|\right] \leq \mathbb{E}\left[\left|\varepsilon cd^{c}\left(d^{c} + \left\|X_{T}^{d,t,x}\right\|^{2}\right)\right|^{\frac{1}{2}}\right] \leq (\varepsilon cd^{c})^{\frac{1}{2}}\left(\mathbb{E}\left[d^{c} + \left\|X_{T}^{d,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}} \\ \leq (\varepsilon cd^{c})^{\frac{1}{2}}\left((d^{c} + \|x\|^{2})e^{7cT}\right)^{\frac{1}{2}} \\ = (\varepsilon cd^{c})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT}.$$
(59)

This, the triangle inequality, and (58) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x}) - g^{d}(X_{T}^{d,t,x})\right|\right] \leq \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x}) - g_{\varepsilon}^{d}(X_{T}^{d,t,x})\right|\right] + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,t,x}) - g^{d}(X_{T}^{d,t,x})\right|\right] \\ \leq 4(\varepsilon cd^{2c})^{\frac{1}{2}}(d^{c} + ||x||^{2})^{\frac{1}{2}}e^{10cT + 3cT^{2}} + (\varepsilon cd^{c})^{\frac{1}{2}}(d^{c} + ||x||^{2})^{\frac{1}{2}}e^{3.5cT} \quad (60) \\ \leq 5(\varepsilon cd^{2c})^{\frac{1}{2}}(d^{c} + ||x||^{2})^{\frac{1}{2}}e^{10cT + 3cT^{2}}.$$

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This, Jensen's inequality, and (45) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,\tilde{x}}) - g^{d}(X_{T}^{d,t,\tilde{x}})\right|\right]\right|_{\tilde{x}=X_{t}^{d,s,x}}\right] \leq 5(\varepsilon cd^{2c})^{\frac{1}{2}} \mathbb{E}\left[\left(d^{c} + \left\|X_{t}^{d,s,x}\right\|^{2}\right)^{\frac{1}{2}}\right]e^{10cT+3cT^{2}}$$

$$\leq 5(\varepsilon cd^{2c})^{\frac{1}{2}}\left(\mathbb{E}\left[d^{c} + \left\|X_{t}^{d,s,x}\right\|^{2}\right]\right)^{\frac{1}{2}}e^{10cT+3cT^{2}}$$

$$\leq 5(\varepsilon cd^{2c})^{\frac{1}{2}}\left((d^{c} + \|x\|^{2})e^{7cT}\right)^{\frac{1}{2}}e^{10cT+3cT^{2}}$$

$$\leq 5(\varepsilon cd^{2c})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{14cT+3cT^{2}}.$$
(61)

Next, the triangle inequality, (25), and (24) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $x \in \mathbb{R}^d$ ,  $w \in \mathbb{R}$  that

$$\begin{aligned} \left| g_{\varepsilon}^{d}(x) \right| &\leq \left| g_{\varepsilon}^{d}(0) \right| + \left| g_{\varepsilon}^{d}(x) - g_{\varepsilon}^{d}(0) \right| \leq (cd^{c}T^{-1})^{\frac{1}{2}} + (cd^{c}T^{-1})^{\frac{1}{2}} \|x\| \\ &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} (d^{c} + \|x\|^{2})^{\frac{1}{2}} \end{aligned}$$
(62)

and

$$|f_{\varepsilon}(w)| \le |f_{\varepsilon}(0)| + |f_{\varepsilon}(w) - f_{\varepsilon}(0)| \le (cd^{c}T^{-3})^{\frac{1}{2}} + c^{\frac{1}{2}}|w|.$$

$$(63)$$

This and (26) show for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that

$$|g^{d}(x)| = \lim_{\varepsilon \to 0} \left| g^{d}_{\varepsilon}(x) \right| \le 2(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + ||x||^{2})^{\frac{1}{2}}.$$
(64)

This and (45) show for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \left\| g^{d}(X_{T}^{d,t,x}) \right\|_{L^{2}(\mathbb{P})} &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \left\| \left( d^{c} + \left\| X_{T}^{d,t,x} \right\|^{2} \right)^{\frac{1}{2}} \right\|_{L^{2}(\mathbb{P})} \\ &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \left( \mathbb{E} \left[ d^{c} + \left\| X_{T}^{d,t,x} \right\|^{2} \right] \right)^{\frac{1}{2}} \\ &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \left( (d^{c} + \|x\|^{2})e^{7cT} \right)^{\frac{1}{2}} \\ &= 2(cd^{c}T^{-1})^{\frac{1}{2}} \left( d^{c} + \|x\|^{2} \right)^{\frac{1}{2}} e^{3.5cT}. \end{aligned}$$

$$(65)$$

This, (31), the triangle inequality, the disintegration theorem, the flow property, and (63) show for all  $d \in \mathbb{N}$ ,  $s \in [0, T], t \in [s, T], x \in \mathbb{R}^d$  that

$$\begin{aligned} \left\| u^{d}(t, X_{t}^{d,s,x}) \right\|_{L^{2}(\mathbb{P})} &= \left\| u^{d}(t,\tilde{x}) \right\|_{\tilde{x}=X_{t}^{d,s,x}} \left\|_{L^{2}(\mathbb{P})} \\ &\leq \left\| \mathbb{E} \left[ \left| g^{d}(X_{T}^{d,t,\tilde{x}}) \right| \right] \right\|_{\tilde{x}=X_{t}^{d,s,x}} \left\|_{L^{2}(\mathbb{P})} + \int_{t}^{T} \left\| \mathbb{E} \left[ \left| f(u^{d}(X_{r}^{d,t,\tilde{x}})) \right| \right] \right\|_{\tilde{x}=X_{t}^{d,s,x}} \left\|_{L^{2}(\mathbb{P})} dr \\ &\leq \left\| \left\| g^{d}(X_{T}^{d,t,\tilde{x}}) \right\|_{L^{2}(\mathbb{P})} \right\|_{\tilde{x}=X_{t}^{d,s,x}} \left\|_{L^{2}(\mathbb{P})} + \int_{t}^{T} \left\| \left\| f(u^{d}(X_{r}^{d,t,\tilde{x}})) \right\|_{L^{2}(\mathbb{P})} \right\|_{\tilde{x}=X_{t}^{d,s,x}} \left\|_{L^{2}(\mathbb{P})} dr \\ &= \left\| g^{d}(X_{T}^{d,s,x}) \right\|_{L^{2}(\mathbb{P})} + \int_{t}^{T} \left\| f(u^{d}(X_{r}^{d,s,x})) \right\|_{L^{2}(\mathbb{P})} dr \\ &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \left( d^{c} + \|x\|^{2} \right)^{\frac{1}{2}} e^{3.5cT} + \int_{t}^{T} \left( (cd^{c}T^{-3})^{\frac{1}{2}} + c^{\frac{1}{2}} \left\| u^{d}(X_{r}^{d,s,x}) \right\|_{L^{2}(\mathbb{P})} \right) dr \\ &\leq 3(cd^{c}T^{-1})^{\frac{1}{2}} \left( d^{c} + \|x\|^{2} \right)^{\frac{1}{2}} e^{3.5cT} + \int_{t}^{T} c^{\frac{1}{2}} \left\| u^{d}(X_{r}^{d,s,x}) \right\|_{L^{2}(\mathbb{P})} dr. \end{aligned}$$

This, Grönwall's lemma, (29), (45), and the fact that  $c \in [1, \infty)$  show for all  $d \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| u^{d}(t, X_{t}^{d,s,x}) \right\|_{L^{2}(\mathbb{P})} \leq 3(cd^{c}T^{-1})^{\frac{1}{2}} \left( d^{c} + \|x\|^{2} \right)^{\frac{1}{2}} e^{3.5cT} e^{c^{1/2}T} \leq 3(cd^{c}T^{-1})^{\frac{1}{2}} \left( d^{c} + \|x\|^{2} \right)^{\frac{1}{2}} e^{4.5cT}.$$
 (67)

$$|u^{d}(t,x)| \le 3(cd^{c}T^{-1})^{\frac{1}{2}} \left(d^{c} + ||x||^{2}\right)^{\frac{1}{2}} e^{4.5cT}.$$
(68)

Next, (31) and the triangle inequality show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^d$  that

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$$\begin{aligned} \left| u^{d,\varepsilon}(t,x) - u^{d,\varepsilon}(t,y) \right| &\leq \mathbb{E} \left[ \left| g^{d}_{\varepsilon}(X^{d,\varepsilon,t,x}_{T}) - g^{d}_{\varepsilon}(X^{d,\varepsilon,t,y}_{T}) \right| \right] \\ &+ \int_{t}^{T} \mathbb{E} \left[ \left| f_{\varepsilon}(u^{d,\varepsilon}(X^{d,\varepsilon,t,x}_{r})) - f_{\varepsilon}(u^{d,\varepsilon}(X^{d,\varepsilon,t,y}_{r})) \right| \right] dr. \end{aligned}$$
(69)

This, the triangle inequality, the disintegration theorem, (24), Jensen's inequality, the fact that  $c \in [1, \infty)$ , and (57) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{split} & \mathbb{E}\Big[\Big|u^{d,\varepsilon}(t,X_{t}^{d,\varepsilon,s,x}) - u^{d,\varepsilon}(t,X_{t}^{d,\varepsilon,s,y})\Big|\Big] \\ &= \mathbb{E}\Big[\Big|u^{d,\varepsilon}(t,\tilde{x}) - u^{d,\varepsilon}(t,\tilde{y})\Big|\Big|_{(\tilde{x},\tilde{y})=(X_{t}^{d,\varepsilon,s,x},X_{t}^{d,\varepsilon,s,y})}\Big] \\ &\leq \mathbb{E}\Big[\mathbb{E}\Big[\Big|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,\tilde{x}}) - g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,\tilde{y}})\Big|\Big]\Big|_{(\tilde{x},\tilde{y})=(X_{t}^{d,\varepsilon,s,x},X_{t}^{d,\varepsilon,s,y})}\Big] \\ &+ \int_{t}^{T} \mathbb{E}\Big[\mathbb{E}\Big[\Big|f_{\varepsilon}(u^{d,\varepsilon}(X_{r}^{d,\varepsilon,t,\tilde{x}})) - f_{\varepsilon}(u^{d,\varepsilon}(X_{r}^{d,\varepsilon,t,\tilde{y}}))\Big|\Big]\Big|_{(\tilde{x},\tilde{y})=(X_{t}^{d,\varepsilon,s,x},X_{t}^{d,\varepsilon,s,y})}\Big] dr \\ &= \mathbb{E}\Big[\Big|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,s,x}) - g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,s,y})\Big|\Big] + \int_{t}^{T} \mathbb{E}\Big[\Big|f_{\varepsilon}(u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,x})) - f_{\varepsilon}(u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,y}))\Big|\Big] dr \\ &\leq \mathbb{E}\Big[(cd^{c}T^{-1})^{\frac{1}{2}}\,\Big\|X_{T}^{d,\varepsilon,s,x} - X_{T}^{d,\varepsilon,s,y}\Big\|\Big] + \int_{t}^{T} c^{\frac{1}{2}}\mathbb{E}\Big[\Big|u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,x}) - u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,y})\Big|\Big] dr \\ &\leq (cd^{c}T^{-1})^{\frac{1}{2}}\,\Big(\mathbb{E}\Big[\Big\|X_{T}^{d,\varepsilon,s,x} - X_{T}^{d,\varepsilon,s,y}\Big\|^{2}\Big]\Big)^{\frac{1}{2}} + \int_{t}^{T} c^{\frac{1}{2}}\mathbb{E}\Big[\Big|u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,x}) - u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,y})\Big|\Big] dr \\ &\leq (cd^{c}T^{-1})^{\frac{1}{2}}\,\Big(4\|x-y\|^{2}e^{8cT+4cT^{2}}\Big)^{\frac{1}{2}} + \int_{t}^{T} c\mathbb{E}\Big[\Big|u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,x}) - u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,y})\Big|\Big] dr \\ &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}}\,\Big(x-y\|e^{4cT+2cT^{2}} + \int_{t}^{T} c\mathbb{E}\Big[\Big|u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,x}) - u^{d,\varepsilon}(X_{r}^{d,\varepsilon,s,y})\Big|\Big] dr. \end{split}$$

This, Grönwall's inequality, (29), and (42) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\left|u^{d,\varepsilon}(t, X_t^{d,\varepsilon,s,x}) - u^{d,\varepsilon}(t, X_t^{d,\varepsilon,s,y})\right|\right] \le 2(cd^cT^{-1})^{\frac{1}{2}} \|x - y\| e^{4cT + 2cT^2} \cdot e^{cT} = 2(cd^cT^{-1})^{\frac{1}{2}} \|x - y\| e^{5cT + 2cT^2}$$
(71)

and hence

$$\left| u^{d,\varepsilon}(t,x) - u^{d,\varepsilon}(t,y) \right| \le 2(cd^{c}T^{-1})^{\frac{1}{2}} \|x - y\| e^{5cT + 2cT^{2}}.$$
(72)

This shows (ii).

Next, (24), (72), Jensen's inequality, and (54) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ that

$$\mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(s, X_{s}^{d,\varepsilon,t,x})) - f_{\varepsilon}(u^{d,\varepsilon}(s, X_{s}^{d,t,x}))\right|\right] \\
\leq c^{\frac{1}{2}}\mathbb{E}\left[\left|u^{d,\varepsilon}(s, X_{s}^{d,\varepsilon,t,x}) - u^{d,\varepsilon}(s, X_{s}^{d,t,x})\right|\right] \\
\leq c^{\frac{1}{2}}2(cd^{c}T^{-1})^{\frac{1}{2}}\mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x} - X_{s}^{d,t,x}\right\|\right]e^{5cT+2cT^{2}} \\
\leq 2c^{\frac{1}{2}}(cd^{c}T^{-1})^{\frac{1}{2}}\left(\mathbb{E}\left[\left\|X_{s}^{d,\varepsilon,t,x} - X_{s}^{d,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}}e^{5cT+2cT^{2}} \\
\leq 2c^{\frac{1}{2}}(cd^{c}T^{-1})^{\frac{1}{2}}\left(12\varepsilon cd^{c}(d^{c} + \|x\|^{2})e^{20cT+6cT^{2}}T\right)^{\frac{1}{2}}e^{5cT+2cT^{2}} \\
\leq 8c^{\frac{3}{2}}d^{c}\varepsilon^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{15cT+5cT^{2}}.$$
(73)

$$\begin{split} &\int_{t}^{T} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(s,X_{s}^{d,\varepsilon,t,x})) - f(u^{d}(s,X_{s}^{d,t,x}))\right|\right] ds \\ &\leq T \sup_{s\in[t,T]} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(s,X_{s}^{d,\varepsilon,t,x})) - f_{\varepsilon}(u^{d,\varepsilon}(s,X_{s}^{d,t,x}))\right|\right] \\ &+ \int_{t}^{T} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(s,X_{s}^{d,t,x})) - f_{\varepsilon}(u^{d}(s,X_{s}^{d,t,x}))\right|\right] ds \\ &+ T \sup_{s\in[t,T]} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d}(s,X_{s}^{d,t,x})) - f(u^{d}(s,X_{s}^{d,t,x}))\right|\right] \\ &\leq T \cdot 8c^{\frac{3}{2}}d^{\varepsilon}\varepsilon^{\frac{1}{2}}(d^{\varepsilon} + ||x||^{2})^{\frac{1}{2}}e^{15cT + 5cT^{2}} + \int_{t}^{T}c^{\frac{1}{2}}\mathbb{E}\left[\left|u^{d,\varepsilon}(s,X_{s}^{d,t,x}) - u^{d}(s,X_{s}^{d,t,x})\right|\right] ds \\ &+ T \sup_{s\in[t,T]}\left[\varepsilon^{\frac{1}{2}}\mathbb{E}\left[\left|u^{d}(s,X_{s}^{d,t,x})\right|^{2}\right]\right] \end{split}$$
(74)

and

$$\begin{split} &\int_{t}^{T} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(s,X_{s}^{d,\varepsilon,t,x})) - f(u^{d}(s,X_{s}^{d,t,x}))\right|\right] ds \\ &\leq 8cd^{c}\varepsilon^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{16cT + 5cT^{2}} + \int_{t}^{T}c^{\frac{1}{2}}\mathbb{E}\left[\left|u^{d,\varepsilon}(s,X_{s}^{d,t,x}) - u^{d}(s,X_{s}^{d,t,x})\right|\right] ds \\ &\quad + T\varepsilon^{\frac{1}{2}}\left(3(cd^{c}T^{-1})^{\frac{1}{2}}\left(d^{c} + \|x\|^{2}\right)^{\frac{1}{2}}e^{4.5cT}\right)^{2} \\ &= 8cd^{c}\varepsilon^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{16cT + 5cT^{2}} + \int_{t}^{T}c^{\frac{1}{2}}\mathbb{E}\left[\left|u^{d,\varepsilon}(s,X_{s}^{d,t,x}) - u^{d}(s,X_{s}^{d,t,x})\right|\right] ds \\ &\quad + 9cd^{c}\varepsilon^{\frac{1}{2}}e^{9cT}(d^{c} + \|x\|^{2}) \\ &\leq 17cd^{c}\varepsilon^{\frac{1}{2}}(d^{c} + \|x\|^{2})e^{16cT + 5cT^{2}} + \int_{t}^{T}c\mathbb{E}\left[\left|u^{d,\varepsilon}(s,X_{s}^{d,t,x}) - u^{d}(s,X_{s}^{d,t,x})\right|\right] ds. \end{split}$$

This, (45), the disintegration theorem, and the flow property show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $s \in [0,T]$ ,  $t \in [s,T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{split} &\int_{t}^{T} \mathbb{E} \left[ \mathbb{E} \left[ \left| f_{\varepsilon}(u^{d,\varepsilon}(r,X_{r}^{d,\varepsilon,t,\tilde{x}})) - f(u^{d}(r,X_{r}^{d,t,\tilde{x}})) \right| \right] \right|_{\tilde{x}=X_{t}^{d,s,x}} \right] dr \\ &\leq 17cd^{c} \varepsilon^{\frac{1}{2}} \mathbb{E} \left[ d^{c} + \left\| X_{t}^{d,s,x} \right\|^{2} \right] e^{16cT + 5cT^{2}} + \int_{t}^{T} c \mathbb{E} \left[ \mathbb{E} \left[ \left| u^{d,\varepsilon}(r,X_{r}^{d,t,\tilde{x}}) - u^{d}(r,X_{r}^{d,t,\tilde{x}}) \right| \right] \right|_{\tilde{x}=X_{t}^{d,s,x}} \right] dr \\ &\leq 17cd^{c} \varepsilon^{\frac{1}{2}} \left( (d^{c} + \|x\|^{2})e^{7cT} \right) e^{16cT + 5cT^{2}} + c \int_{t}^{T} \mathbb{E} \left[ \left| u^{d,\varepsilon}(r,X_{r}^{d,s,x}) - u^{d}(r,X_{r}^{d,s,x}) \right| \right] dr. \end{split}$$

$$(76)$$

$$\mathbb{E}\left[\left|u^{d,\varepsilon}(t,X_{t}^{d,s,x}) - u^{d}(t,X_{t}^{d,s,x})\right|\right] = \mathbb{E}\left[\left|u^{d,\varepsilon}(t,\tilde{x}) - u^{d}(t,\tilde{x})\right|\right|_{\tilde{x}=X_{t}^{d,s,x}}\right] \\
\leq \mathbb{E}\left[\mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,\tilde{x}}) - g^{d}(X_{T}^{d,t,\tilde{x}})\right|\right]\right|_{\tilde{x}=X_{t}^{d,s,x}}\right] \\
+ \int_{t}^{T} \mathbb{E}\left[\mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(r,X_{r}^{d,\varepsilon,t,\tilde{x}})) - f(u^{d}(r,X_{r}^{d,t,\tilde{x}}))\right|\right]\right|_{\tilde{x}=X_{t}^{d,s,x}}\right] dr \\
\leq 5(\varepsilon cd^{2c})^{\frac{1}{2}}(d^{c} + ||x||^{2})^{\frac{1}{2}}e^{14cT+3cT^{2}} \\
+ 17cd^{c}\varepsilon^{\frac{1}{2}}\left(d^{c} + ||x||^{2})e^{7cT}\right)e^{16cT+5cT^{2}} + c\int_{t}^{T} \mathbb{E}\left[\left|u^{d,\varepsilon}(r,X_{r}^{d,s,x}) - u^{d}(r,X_{r}^{d,s,x})\right|\right] dr$$
(77)

$$\leq 22cd^{c}\varepsilon^{\frac{1}{2}}(d^{c} + \|x\|^{2})e^{23cT + 5cT^{2}} + c\int_{t}^{T}\mathbb{E}\left[\left|u^{d,\varepsilon}(r, X_{r}^{d,s,x}) - u^{d}(r, X_{r}^{d,s,x})\right|\right]dr.$$

This, Grönwall's lemma, (29), and (45) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\left|u^{d,\varepsilon}(t, X_t^{d,s,x}) - u^d(t, X_t^{d,s,x})\right|\right] \le 22cd^c \varepsilon^{\frac{1}{2}} (d^c + \|x\|^2) e^{23cT + 5cT^2} \cdot e^{cT} = 2cd^c \varepsilon^{\frac{1}{2}} (d^c + \|x\|^2) e^{24cT + 5cT^2}$$
(78)

and hence  $|u^{d,\varepsilon}(t,x) - u^{d}(t,x)| \leq 2cd^{c}\varepsilon^{\frac{1}{2}}(d^{c} + ||x||^{2})e^{24cT + 5cT^{2}}$ . This shows (iii). The proof of Lemma 2.1 is thus completed.

# 3. EULER-MARUYAMA AND MLP APPROXIMATIONS REVISITED

In Lemma 3.2 below we approximate the solution to the SFPE (86), associated to (83), through solution to the SFPE (85), associated to the Euler-Maruyama approximation (82).

**Setting 3.1.** Consider the notations given in Subsection 1.4, let  $T \in (0, \infty)$ ,  $c \in [1, \infty)$ , for every  $K \in \mathbb{N}$ let  $\lfloor \cdot \rfloor_K \colon \mathbb{R} \to \mathbb{R}$  satisfy for all  $t \in \mathbb{R}$  that  $\lfloor t \rfloor_K = \max(\{0, \frac{T}{K}, \frac{2T}{K}, \dots, T\} \cap ((-\infty, t) \cup \{0\}))$ , for every  $d \in \mathbb{N}, \varepsilon \in (0, 1), let \beta_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^d), \sigma_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^{d \times d}), f_{\varepsilon} \in C(\mathbb{R}, \mathbb{R}), g_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}), for every <math>d \in \mathbb{N}, \varepsilon \in (0, 1)$  let  $\gamma_{\varepsilon}^d \colon \mathbb{R}^{2d} \to \mathbb{R}^d$ , be measurable, for every  $d \in \mathbb{N}$  let  $\nu^d \colon \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$  be a Lévy measure, and assume for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}$ ,  $\varepsilon \in (0, 1)$  that

$$\left\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(y)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma_{\varepsilon}^{d}(y)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(y,z)\right)\right\|^{2} \nu^{d}(dz) \le c\|x - y\|^{2}, \quad (79)$$

$$|f_{\varepsilon}(w_1) - f_{\varepsilon}(w_2)|^2 \le c|w_1 - w_2|^2, \quad \left|g_{\varepsilon}^d(x) - g_{\varepsilon}^d(y)\right|^2 \le cd^c T^{-1}||x - y||^2, \tag{80}$$

and

that

$$\left\|\beta_{\varepsilon}^{d}(0)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(0,z)\right\|^{2} \nu^{d}(dz) + T^{3}|f_{\varepsilon}(0)|^{2} + T|g_{\varepsilon}^{d}(0)|^{2} \le cd^{c}.$$
(81)

**Lemma 3.2** (Discretization error). Assume Setting 3.1, for every  $d, K \in \mathbb{N}, \theta \in \Theta, x \in \mathbb{R}^d, \varepsilon \in (0, 1)$ ,  $t \in [0,T)$  let  $(X_s^{d,K,\varepsilon,t,x})_{s \in [t,T]}$ ,  $(X_s^{d,\varepsilon,t,x})_{s \in [t,T]}$  be adapted càdlàg processes which satisfy for all  $s \in [t,T]$ that  $\mathbb{P}$ -a.s.

$$X_{s}^{d,K,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) dr + \int_{t}^{s} \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) dW_{r}^{d} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}, z) \tilde{N}^{d}(dr,dz),$$

$$(82)$$

and

$$X_{s}^{d,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) dr + \int_{t}^{s} \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) dW_{r}^{d} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z) \tilde{N}^{d}(dr,dz), \quad (83)$$

and for every  $d, K \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ , let  $u^{d,K,\varepsilon}, u^{d,\varepsilon} \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$  be measurable functions satisfying for all  $t \in [0,T]$ ,  $x \in \mathbb{R}^d$  that

$$\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}\frac{|u^{d,K,\varepsilon}(s,y)|+|u^{d,\varepsilon}(s,y)|}{1+\|y\|}<\infty,$$
(84)

$$u^{d,K,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,K,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,K,\varepsilon}(r,X^{d,K,\varepsilon,t,x}_r))\right]dr,\tag{85}$$

and

$$u^{d,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,\varepsilon}(r,X^{d,\varepsilon,t,x}_r))\right] dr.$$
(86)

Then

(i) for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  we have that

$$\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,K,\varepsilon,t,x}\right\|^{2}\right] \leq (d^{c} + \|x\|^{2})e^{7c(s-t)},\tag{87}$$

and

(ii) for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$ , we have that

$$\left| u^{d,K,\varepsilon}(t,x) - u^{d,\varepsilon}(t,x) \right| \le 12c^{\frac{3}{2}}d^{\frac{c}{2}}(T+2)e^{21cT+5cT^{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}\frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}}.$$
(88)

*Proof of Lemma 3.2.* First, for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^d$  we have that

$$\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right] \leq (d^{c} + \|x\|^{2})e^{7c(s-t)}$$

$$\tag{89}$$

and

$$\left| u^{d,\varepsilon}(t,x) - u^{d,\varepsilon}(t,y) \right| \le 2(cd^{c}T^{-1})^{\frac{1}{2}} \|x - y\| e^{5cT + 2cT^{2}}$$
(90)

(cf. Lemma 2.1). Next, the triangle inequality, the fact that  $\forall a_1, a_2 \in \mathbb{R}$ :  $(a_1 + a_2)^2 \leq 2|a_1|^2 + 2|a_2|^2$ , (79), and (81) show for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\|\beta_{\varepsilon}^{d}(x)\|^{2} \leq 2\|\beta_{\varepsilon}^{d}(0)\|^{2} + 2\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(0)\|^{2} \leq 2cd^{c} + 2c\|x\|^{2} = 2c(d^{c} + \|x\|^{2}).$$
(91)

Similarly, we have for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\left\|\sigma_{\varepsilon}^{d}(x)\right\|_{\mathrm{F}}^{2} \leq 2c(d^{c} + \|x\|^{2}).$$

$$\tag{92}$$

Next, the triangle inequality, the fact that  $\forall a_1, a_2 \in \mathbb{R}$ :  $(a_1 + a_2)^2 \leq 2|a_1|^2 + 2|a_2|^2$ , (79), and (81) show for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\int_{\mathbb{R}^{d}\setminus\{0\}} \left\| \gamma_{\varepsilon}^{d}(x,z) \right\|^{2} \nu^{d}(dz) \leq \int_{\mathbb{R}^{d}\setminus\{0\}} 2 \left\| \gamma_{\varepsilon}^{d}(0,z) \right\|^{2} + 2 \left\| \gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(0,z) \right\|^{2} \nu^{d}(dz) \\
\leq 2cd^{c} + 2c \|x\|^{2} = 2c(d^{c} + \|x\|^{2}).$$
(93)

Next, Itô's formula (see, e.g., [20, Theorem 3.1]) and (82) show for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that  $\mathbb{P}$ -a.s.

$$\begin{split} \left\| X_{s}^{d,K,\varepsilon,t,x} \right\|^{2} &= \left\| x \right\|^{2} + \int_{t}^{s} \left( 2 \left\langle X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}, \beta_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) \right\rangle + \left\| \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) \right\|_{F}^{2} \right) dr \\ &+ 2 \int_{t}^{s} \sum_{i,j=1}^{d} \left( X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x} \right)_{i} \left( \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) \right)_{ij} d(W_{j}^{d})_{r} \\ &+ 2 \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \left\langle X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}, \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z) \right\rangle \tilde{N}^{d} (dz,dr) \\ &+ \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \left\| \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z) \right\|^{2} N^{d} (dz,dr). \end{split}$$

$$(94)$$

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES Next, for every  $d, n, K \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  let  $\tau_n^{d, K, \varepsilon, x} \colon \Omega \to \mathbb{R}$  satisfy that

$$\tau_{n}^{d,K,\varepsilon,x} = \inf\left\{s \in [t,T] \colon \int_{t}^{s} \left(2\left\langle X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}, \beta_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\rangle + \left\|\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\|_{\mathrm{F}}^{2}\right)dr + \int_{t}^{s} \sum_{i,j=1}^{d} \left|\left(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right)_{i}\left(\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right)_{ij}\right|^{2}dr + \int_{t}^{s} \int_{\mathbb{R}^{d}\setminus\{0\}} \sum_{i=1}^{d} \left|\left(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right)_{i}\left(\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)\right)_{i}\right|^{2}\nu^{d}(dz)dr + \int_{t}^{s} \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)\right\|_{\mathrm{F}}^{2}\nu^{d}(dz)dr \ge n\right\} \wedge T$$

$$(95)$$

(with the convention that  $\inf \emptyset = \infty$ ). Then (94), the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d \colon 2\langle x, y \rangle \leq ||x||^2 + ||y||^2$ , (91), (92), (93) show for all  $d, n, K \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\max\left\{ \mathbb{E}\left[d^{c} + \left\|X_{\max\{t, \lfloor s \land \tau_{n}^{d,K,\varepsilon,t,x}}\right\|^{2}\right], \mathbb{E}\left[d^{c} + \left\|X_{\max\{t, s \land \tau_{n}^{d,K,\varepsilon,x}\}}\right\|^{2}\right]\right\} \\ \leq d^{c} + \|x\|^{2} + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \left(2\left\langle X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}, \beta_{\varepsilon}^{d}(X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\rangle + \left\|\sigma_{\varepsilon}^{d}(X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\|^{2}\right)dr\right] \\ + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma_{\varepsilon}^{d}(X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}, z)\right\|^{2}\nu^{d}(dz)dr\right] \\ \leq d^{c} + \|x\|^{2} + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \left\|X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}dr\right] + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,t,x}} \left\|\beta_{\varepsilon}^{d}(X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\|^{2}dr\right] \\ + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \left\|\sigma_{\varepsilon}^{d}(X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}, z)\right\|^{2}\nu^{d}(dz)dr\right] \\ + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \left\|\sigma_{\varepsilon}^{d}(X_{\max\{t, \lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}, z)\right\|^{2}\nu^{d}(dz)dr\right].$$
(96)

This, (35)–(37), and the fact that  $c \ge 1$  show for all  $d, K, n \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\max\left\{ \mathbb{E}\left[d^{c} + \left\|X_{\max\{t,\lfloor s \land \tau_{n}^{d,K,\varepsilon,x} \rfloor_{K}\}}^{d,K,\varepsilon,x}\right\|^{2}\right], \mathbb{E}\left[d^{c} + \left\|X_{s \land \tau_{n}^{d,K,\varepsilon,x}}^{d,K,\varepsilon,x}\right\|^{2}\right]\right\} \\ \leq d^{c} + \|x\|^{2} + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \left(d^{c} + \left\|X_{\max\{t,\lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right)dr\right] \\ + \mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} 6c\left(d^{c} + \left\|X_{\max\{t,\lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right)dr\right] \\ \leq d^{c} + \|x\|^{2} + 7c\mathbb{E}\left[\int_{t}^{s \land \tau_{n}^{d,K,\varepsilon,x}} \left(d^{c} + \left\|X_{\max\{t,\lfloor r \land \tau_{n}^{d,K,\varepsilon,x}\rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right)dr\right] \\ \leq d^{c} + \|x\|^{2} + 7c\int_{t}^{s}\mathbb{E}\left[d^{c} + \left\|X_{\max\{t,\lfloor r \land \tau_{n}^{d,K,\varepsilon,x}\rfloor_{K}\}}^{d,K,\varepsilon,x}\right\|^{2}dr\right]$$
(97)

This, Fatou's lemma, and Grönwall's inequality show for all  $d, K \in \mathbb{N}$ ,  $t \in [0,T]$ ,  $s \in [t,T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0,1)$  that

$$\mathbb{E}\left[d^{c} + \left\|X_{\max\{t,\lfloor s \rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[d^{c} + \left\|X_{\max\{t,\lfloor s \land \tau_{n}^{d,K,\varepsilon,x}\rfloor_{K}\}}^{d,K,\varepsilon,x}\right\|^{2}\right] \leq (d^{c} + \|x\|^{2})e^{7c(s-t)}.$$
(98)

$$\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,K,\varepsilon,t,x}\right\|^{2}\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[d^{c} + \left\|X_{s \wedge \tau_{n}^{d,K,\varepsilon,t,x}}\right\|^{2}\right] \\
\leq d^{c} + \|x\|^{2} + 7c \mathbb{E}\left[\int_{t}^{s} \left(d^{c} + \left\|X_{\max\{t,\lfloor r \rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right) dr\right] \\
\leq d^{c} + \|x\|^{2} + 7c \int_{t}^{s} (d^{c} + \|x\|^{2}) e^{7c(r-t)} dr \\
= (d^{c} + \|x\|^{2})(1 + 7c \int_{t}^{s} e^{7c(r-t)} dr) \\
= (d^{c} + \|x\|^{2}) e^{7c(s-t)}.$$
(99)

This shows (i).

Next, Hölder's inequality, (91), and (99) show for all  $d, K \in \mathbb{N}, t \in [0, T], s, s' \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{s}^{s'}\beta_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\,dr\right\|^{2}\right] \leq \mathbb{E}\left[\left(\int_{s}^{s'}\left\|\beta_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\|\,dr\right)^{2}\right]\right]$$

$$\leq \mathbb{E}\left[|s'-s|\int_{s}^{s'}\left\|\beta_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\|^{2}\,dr\right]$$

$$\leq T|s'-s|\sup_{r\in[s,s']}\mathbb{E}\left[2c\left(d^{c}+\left\|X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right)\right]$$

$$\leq T|s'-s|\cdot 2c(d^{c}+\|x\|^{2})e^{7cT}.$$
(100)

Next, Itô's isometry, (92), and (99) show for all  $d, K \in \mathbb{N}, t \in [0, T], s, s' \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{s}^{s'}\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\,dW_{r}^{d}\right\|^{2}\right] = \mathbb{E}\left[\int_{s}^{s'}\left\|\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\right\|_{F}^{2}\,dr\right]$$

$$\leq |s'-s|\sup_{r\in[s,s']}\mathbb{E}\left[2c\left(d^{c}+\left\|X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right)\right]$$

$$\leq |s'-s|\cdot 2c(d^{c}+\|x\|^{2})e^{7cT}.$$
(101)

Next, Itô's isometry, (93), and (99) show for all  $d, K \in \mathbb{N}, t \in [0, T], s, s' \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{s}^{s'}\int_{\mathbb{R}^{d}\setminus\{0\}}\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)\,\tilde{N}^{d}(dr,dz)\right\|^{2}\right] \\
=\mathbb{E}\left[\int_{s}^{s'}\int_{\mathbb{R}^{d}\setminus\{0\}}\left\|\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)\right\|^{2}\nu^{d}(dz)\,dr\right] \\
\leq |s'-s|\sup_{r\in[s,s']}\mathbb{E}\left[2c\left(d^{c}+\left\|X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}\right\|^{2}\right)\right] \\
\leq |s'-s|\cdot 2c(d^{c}+\|x\|^{2})e^{7cT}.$$
(102)

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This, (82), the fact that  $\forall d \in \mathbb{N}, x, y, z \in \mathbb{R}^d$ :  $||x + y + z||^2 \leq 3||x||^2 + 3||y||^2 + 3||z||^2$ , (100), and (101) show for all  $d, K \in \mathbb{N}, t \in [0, T], s, s' \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|X_{s'}^{d,K,\varepsilon,t,x} - X_{s}^{d,K,\varepsilon,t,x}\right\|^{2}\right] \\
\leq 3\mathbb{E}\left[\left\|\int_{s}^{s'}\beta_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\,dr\right\|^{2}\right] + 3\mathbb{E}\left[\left\|\int_{s}^{s'}\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})\,dW_{r}^{d}\right\|^{2}\right] \\
+ 3\mathbb{E}\left[\left\|\int_{s}^{s'}\int_{\mathbb{R}^{d}\setminus\{0\}}\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)\,\tilde{N}^{d}(dr,dz)\right\|^{2}\right] \\
\leq 3 \cdot T|s'-s| \cdot 2c(d^{c}+\|x\|^{2})e^{7cT} + 3 \cdot |s'-s| \cdot 2c(d^{c}+\|x\|^{2})e^{7cT} \\
+ 3 \cdot |s'-s| \cdot 2c(d^{c}+\|x\|^{2})e^{7cT} \\
= 6c(T+2)e^{7cT}(d^{c}+\|x\|^{2})|s'-s|.$$
(103)

This, Hölder's inequality, (79), the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d \colon ||x+y||^2 \le 2||x||^2 + 2||y||^2$ , and (103) show for all  $d, K \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{t}^{s} \left(\beta_{\varepsilon}^{d}(X_{\max\{t, [r] \mid K\}}^{d,K,\varepsilon,t,x}) - \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right) dr\right\|^{2}\right] \\
\leq \mathbb{E}\left[\left(\int_{t}^{s} \left\|\beta_{\varepsilon}^{d}(X_{\max\{t, [r] \mid K\}}^{d,K,\varepsilon,t,x}) - \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right\| dr\right)^{2}\right] \\
\leq T\mathbb{E}\left[\int_{t}^{s} \left\|\beta_{\varepsilon}^{d}(X_{\max\{t, [r] \mid K\}}^{d,K,\varepsilon,t,x}) - \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right\|^{2} dr\right] \\
\leq T\mathbb{E}\left[\int_{t}^{s} c\left\|X_{\max\{t, [r] \mid K\}}^{d,K,\varepsilon,t,x} - X_{r-}^{d,\varepsilon,t,x}\right\|^{2}\right] + 2cT\int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr \\
\leq 2cT(s-t) \cdot 6c(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K} + 2cT\int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr \\
\leq 12c^{2}T^{2}(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K} + 2cT\int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr$$

Next, Itô's isometry, (79), the fact that  $\forall d \in \mathbb{N}, x, y \in \mathbb{R}^d$ :  $||x + y||^2 \leq 2||x||^2 + 2||y||^2$ , and (103) show for all  $d, K \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|\int_{t}^{s} \left(\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) - \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right) dW_{r}^{d}\right\|^{2}\right] \\
= \mathbb{E}\left[\int_{t}^{s} \left\|\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}) - \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right\|^{2} dr\right] \\
\leq \mathbb{E}\left[\int_{t}^{s} c\left\|X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x} - X_{r-}^{d,\varepsilon,t,x}\right\|^{2} dr\right] \\
\leq 2c \int_{t}^{s} \mathbb{E}\left[\left\|X_{\max\{t,\lfloor r\rfloor_{K}\}}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr + 2c \int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr \\
\leq 2c(s-t) \cdot 6c(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K} + 2c \int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr \\
= 12c^{2}T(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K} + 2c \int_{t}^{s} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right] dr.$$
(105)

$$\mathbb{E}\left[\left\|\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\left(\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)-\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z)\right)\tilde{N}^{d}(dr,dz)\right\|^{2}\right] \\
=\mathbb{E}\left[\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\left\|\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)-\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z)\right\|^{2}\nu^{d}(dz)dr\right] \\
\leq\mathbb{E}\left[\int_{t}^{s}c\left\|X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x}-X_{r-}^{d,\varepsilon,t,x}\right\|^{2}dr\right] \\
\leq 2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{\max\{t,\lfloor r\rfloor_{K}\}}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}dr\right] dr + 2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}dr\right] dr \\
\leq 2c(s-t)\cdot 6c(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}+2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}dr\right] dr \\
\leq 12c^{2}T(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}+2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}dr.$$
(106)

This, the fact that  $\forall d \in \mathbb{N}, x, y, z \in \mathbb{R}^d$ :  $||x + y + z||^2 \leq 3||x||^2 + 3||y||^2 + 3||z||^2$ , (104), (105) show for all  $d, K \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|X_{s}^{d,K,\varepsilon,t,x}-X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right] \leq 3\mathbb{E}\left[\left\|\int_{t}^{s}\left(\beta_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\varepsilon,t,x})-\beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right)dr\right\|^{2}\right] + 3\mathbb{E}\left[\left\|\int_{t}^{s}\left(\sigma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x})-\sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x})\right)dW_{r}^{d}\right\|^{2}\right] + 3\mathbb{E}\left[\left\|\int_{t}^{s}\int_{\mathbb{R}^{d}\setminus\{0\}}\left(\gamma_{\varepsilon}^{d}(X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,K,\varepsilon,t,x},z)-\gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z)\right)\tilde{N}^{d}(dr,dz)\right\|^{2}\right]$$

$$(107)$$

and

$$\begin{split} & \mathbb{E}\left[\left\|X_{s}^{d,K,\varepsilon,t,x}-X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right] \\ &\leq 3\left[12c^{2}T^{2}(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}+2cT\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right]dr\right] \\ &\quad + 3\left[12c^{2}T(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}+2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right]dr\right] \\ &\quad + 3\left[12c^{2}T(T+2)e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}+2c\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right]dr\right] \\ &\quad = 36c^{2}T(T+2)^{2}e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}+6c(T+2)\int_{t}^{s}\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x}-X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right]dr. \end{split}$$

This, (89), (99), and Grönwall's inequality show for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $s \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left\|X_{s}^{d,K,\varepsilon,t,x}-X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right] \leq 36c^{2}T(T+2)^{2}e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}e^{6c(T+2)T}.$$
(109)

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES This, (80), and Jensen's inequality show for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\begin{split} \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,K,\varepsilon,t,x}) - g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] &\leq (cd^{c})^{\frac{1}{2}}T^{-\frac{1}{2}}\mathbb{E}\left[\left\|X_{T}^{d,K,\varepsilon,t,x} - X_{T}^{d,\varepsilon,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}} \\ &\leq (cd^{c})^{\frac{1}{2}}T^{-\frac{1}{2}}\left(\mathbb{E}\left[\left\|X_{T}^{d,K,\varepsilon,t,x} - X_{T}^{d,\varepsilon,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}} \\ &\leq (cd^{c})^{\frac{1}{2}}T^{-\frac{1}{2}}\left(36c^{2}T(T+2)^{2}e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}e^{6c(T+2)T}\right)^{\frac{1}{2}} (110) \\ &\leq 6c^{\frac{3}{2}}d^{\frac{c}{2}}(T+2)e^{10cT+3cT^{2}}(d^{c}+\|x\|^{2})^{\frac{1}{2}}\frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}}. \end{split}$$

Next, (90), Jensen's inequality, and (109) show for all  $d, K \in \mathbb{N}, t \in [0, T], r \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left|u^{d,\varepsilon}(r, X_{r}^{d,K,\varepsilon,t,x}) - u^{d,\varepsilon}(r, X_{r}^{d,\varepsilon,t,x})\right|\right] \\
\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|\right] e^{5cT+2cT^{2}} \\
\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \left(\mathbb{E}\left[\left\|X_{r}^{d,K,\varepsilon,t,x} - X_{r}^{d,\varepsilon,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}} e^{5cT+2cT^{2}} \\
\leq 2(cd^{c}T^{-1})^{\frac{1}{2}} \left(36c^{2}T(T+2)^{2}e^{7cT}(d^{c}+\|x\|^{2})\frac{T}{K}e^{6c(T+2)T}\right)^{\frac{1}{2}} e^{5cT+2cT^{2}} \\
\leq 12c^{\frac{3}{2}}d^{\frac{c}{2}}(T+2)e^{15cT+5cT^{2}}(d^{c}+\|x\|^{2})^{\frac{1}{2}}\frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}}.$$
(111)

Next, Jensen's inequality and (99) show for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $r \in [t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[\left|u^{d,K,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x}) - u^{d,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x})\right|\right] \\
\leq e^{3.5c(T-r)} \mathbb{E}\left[\left(d^{c} + \left\|X_{r}^{d,K,\varepsilon,t,x}\right\|^{2}\right)^{\frac{1}{2}}\right] \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + ||y||^{2})^{\frac{1}{2}}} \\
\leq e^{3.5c(T-r)} \left(\mathbb{E}\left[d^{c} + \left\|X_{r}^{d,K,\varepsilon,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}} \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + ||y||^{2})^{\frac{1}{2}}} \\
\leq e^{3.5c(T-r)}(d^{c} + ||x||^{2})^{\frac{1}{2}}e^{3.5c(r-t)} \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + ||y||^{2})^{\frac{1}{2}}} \\
= (d^{c} + ||x||^{2})^{\frac{1}{2}}e^{3.5c(T-t)} \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + ||y||^{2})^{\frac{1}{2}}}.$$
(112)

This, the triangle inequality, (80), the fact that  $c \ge 1$ , (110), (111), the fact that  $1 + cT \le e^{cT}$  show for all  $d, K \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\begin{aligned} \left| u^{d,K,\varepsilon}(t,x) - u^{d,\varepsilon}(t,x) \right| \\ &\leq \mathbb{E} \left[ \left| g_{\varepsilon}^{d}(X_{T}^{d,K,\varepsilon,t,x}) - g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x}) \right| \right] + \int_{t}^{T} \mathbb{E} \left[ \left| f_{\varepsilon}(u^{d,K,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x})) - f_{\varepsilon}(u^{d,\varepsilon}(r,X_{r}^{d,\varepsilon,t,x})) \right| \right] dr \\ &\leq \mathbb{E} \left[ \left| g_{\varepsilon}^{d}(X_{T}^{d,K,\varepsilon,t,x}) - g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x}) \right| \right] + \int_{t}^{T} c \mathbb{E} \left[ \left| u^{d,K,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x}) - u^{d,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x}) \right| \right] \\ &+ \int_{t}^{T} c \mathbb{E} \left[ \left| u^{d,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x}) - u^{d,\varepsilon}(r,X_{r}^{d,\varepsilon,t,x}) \right| \right] dr \end{aligned}$$
(113)

and

$$\begin{aligned} \left| u^{d,K,\varepsilon}(t,x) - u^{d,\varepsilon}(t,x) \right| &\leq 6c^{\frac{3}{2}} d^{\frac{c}{2}}(T+2)e^{10cT+3cT^{2}} (d^{c} + \|x\|^{2})^{\frac{1}{2}} \frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}} \\ &+ \int_{t}^{T} c(d^{c} + \|x\|^{2})^{\frac{1}{2}} e^{3.5c(T-t)} \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y)|}{e^{3.5c(T-r)} (d^{c} + \|y\|^{2})^{\frac{1}{2}}} dr \\ &+ Tc \cdot 12c^{\frac{3}{2}} d^{\frac{c}{2}}(T+2)e^{15cT+5cT^{2}} (d^{c} + \|x\|^{2})^{\frac{1}{2}} \frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}} \end{aligned}$$
(114)  
$$&\leq 12c^{\frac{3}{2}} d^{\frac{c}{2}}(T+2)e^{16cT+5cT^{2}} (d^{c} + \|x\|^{2})^{\frac{1}{2}} \frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}} \\ &+ \int_{t}^{T} c(d^{c} + \|x\|^{2})^{\frac{1}{2}} e^{3.5c(T-t)} \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y)|}{e^{3.5c(T-r)} (d^{c} + \|y\|^{2})^{\frac{1}{2}}} dr. \end{aligned}$$

Dividing by  $(d^c + ||x||^2)^{\frac{1}{2}} e^{3.5c(T-t)}$  shows for all  $d, K \in \mathbb{N}, t \in [0, T], \varepsilon \in (0, 1)$  that

$$\sup_{x \in \mathbb{R}^{d}} \frac{\left| u^{d,K,\varepsilon}(t,x) - u^{d,\varepsilon}(t,x) \right|}{e^{3.5c(T-t)} (d^{c} + \|y\|^{2})^{\frac{1}{2}}} \\
\leq 12c^{\frac{3}{2}} d^{\frac{c}{2}} (T+2)e^{16cT+5cT^{2}} \frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}} + \int_{t}^{T} c \sup_{y \in \mathbb{R}^{d}} \frac{\left| u^{d,K,\varepsilon}(r,y) - u^{d,\varepsilon}(r,y) \right|}{e^{3.5c(T-r)} (d^{c} + \|y\|^{2})^{\frac{1}{2}}} dr.$$
(115)

This, (84), (89), (99), and Grönwall's inequality show for all  $d, K \in \mathbb{N}, t \in [0, T], \varepsilon \in (0, 1)$  that

$$\sup_{y \in \mathbb{R}^{d}} \frac{\left| u^{d,K,\varepsilon}(t,y) - u^{d,\varepsilon}(t,y) \right|}{e^{3.5c(T-t)} (d^{c} + \|y\|^{2})^{\frac{1}{2}}} \leq 12c^{\frac{3}{2}} d^{\frac{c}{2}} (T+2)e^{16cT+5cT^{2}} \frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}} \cdot e^{cT}$$

$$= 12c^{\frac{3}{2}} d^{\frac{c}{2}} (T+2)e^{17cT+5cT^{2}} \frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}}.$$
(116)

Hence, for all  $d, K \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  we have that

$$\left| u^{d,K,\varepsilon}(t,x) - u^{d,\varepsilon}(t,x) \right| \le 12c^{\frac{3}{2}}d^{\frac{c}{2}}(T+2)e^{21cT+5cT^{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}\frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}}.$$
(117)

This shows (ii). The proof of Lemma 3.2 is thus completed.

In Lemma 3.3 below we approximate the solution to SFPE (120), associated to (118), by the MLP approximation (121).

**Lemma 3.3.** Assume Setting 3.1, let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$  be a probability space satisfying the usual conditions, let  $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$ , for every  $d \in \mathbb{N}$  let  $W^{d,\theta} \colon \Omega \times [0,T] \to \mathbb{R}^d$ ,  $\theta \in \Theta$ , be identically independently distributed standard  $(\mathbb{F}_t)_{t \in [0,T]}$ -Brownian motions, for every  $d \in \mathbb{N}$  let  $N^{d,\theta}$ ,  $\theta \in \Theta$ , be independent  $(\mathbb{F}_t)_{t \in [0,T]}$ -Poisson random measures on  $[0,\infty) \times (\mathbb{R}^d \setminus \{0\})$  with intensity  $\nu^d$ , for every  $d \in \mathbb{N}$ ,  $\theta \in \Theta$  let  $\tilde{N}^{d,\theta}(dt,dz) = N^{d,\theta}(dt,dz) - dt \nu^d(dz)$ , assume for all  $d \in \mathbb{N}$  that  $\mathcal{F}_0$ ,  $(N^{d,\theta})_{\theta \in \Theta}$  and  $(W^{d,\theta})_{\theta \in \Theta}$ , are independent, for every  $d, K \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0,1)$ ,  $t \in [0,T)$  let  $(X_s^{d,\theta,K,\varepsilon,t,x})_{s \in [t,T]}$  satisfy that  $X_t^{d,\theta,K,\varepsilon,t,x} = x$  and

$$X_{s}^{d,\theta,K,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}) dr + \int_{t}^{s} \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}) dW_{r}^{d,\theta} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}, z) \tilde{N}^{d,\theta}(dr,dz),$$
(118)

for every  $d, K \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ , let  $u^{d,K,\varepsilon} \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$  be measurable functions satisfying for all  $t \in [0,T]$ ,  $x \in \mathbb{R}^d$  that  $\mathbb{E}\left[\left|g_{\varepsilon}^d(X_{t,T}^{d,0,K,\varepsilon,x})\right|\right] + \int_t^T \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,K,\varepsilon}(r,X_{t,r}^{d,0,K,\varepsilon,x}))\right|\right] < \infty$ ,

$$\sup_{s \in [0,T]} \sup_{y \in \mathbb{R}^d} \frac{\left| u^{d,K,\varepsilon}(s,y) \right|}{1 + \|y\|} < \infty$$
(119)

and

$$u^{d,K,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,0,K,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,K,\varepsilon}(r,X^{d,0,K,\varepsilon,t,x}_r))\right]dr,$$
(120)

let  $\mathfrak{t}^{\theta}: \Omega \to [0, 1]$ ,  $\theta \in \Theta$ , be i.i.d random variables which satisfy for all  $t \in (0, 1)$  that  $\mathbb{P}(\mathfrak{t}^{0} \leq t) = t$ , for every  $\theta \in \Theta$ ,  $t \in [0, T]$  let  $\mathfrak{T}^{\theta}_{t}: \Omega \to \mathbb{R}$  satisfy for all  $\theta \in \Theta$  that  $\mathfrak{T}^{\theta}_{t} = t + (T-t)\mathfrak{t}^{\theta}$ , assume for all  $d \in \mathbb{N}$  that  $(\mathfrak{t}^{\theta})_{\theta \in \Theta}$ ,  $(N^{d,\theta})_{\theta \in \Theta}$  and  $(W^{d,\theta})_{\theta \in \Theta}$ , are independent, for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  let  $U^{d,\theta,K,\varepsilon}_{n,m}: [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ ,  $\theta \in \Theta$ ,  $n, m \in \mathbb{Z}$ , satisfy for all  $\theta \in \Theta$ ,  $n \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ ,  $t \in [0,T]$ ,  $x \in \mathbb{R}^d$  that

$$U_{n,m}^{d,\theta,K,\varepsilon}(t,x) = \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^{n}} \sum_{i=1}^{m^{n}} g_{\varepsilon}^{d} \left( X_{T}^{d,(\theta,0,-i),K,\varepsilon,t,x} \right)$$

$$+ \sum_{\ell=0}^{n-1} \frac{(T-t)}{m^{n-\ell}} \sum_{i=1}^{m^{n-\ell}} \left( f_{\varepsilon} \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} - \mathbb{1}_{\mathbb{N}}(\ell) f_{\varepsilon} \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}, X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}}^{d,(\theta,\ell,i),K,\varepsilon,t,x} \right).$$

$$(121)$$

Then for all  $d, K, n, m \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that  $U_{n,m}^{d,\theta,K,\varepsilon}$  is measurable and  $\mathbb{E}\left[\left(\left|U_{n,m}^{d,\theta,K,\varepsilon}(t,x) - u^{d,K,\varepsilon}(t,x)\right|^2\right]\right)^{\frac{1}{2}} \leq 6e^{\frac{m}{2}}m^{-\frac{n}{2}}e^{12cTn}(cd^cT^{-1})^{\frac{1}{2}}\left(d^c + \|x\|^2\right)^{\frac{1}{2}}.$ 

*Proof of Lemma 3.3.* For measurability see [23, Lemma 3.2]. Next, (25) and the triangle inequality show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $x \in \mathbb{R}^d$ ,  $w \in \mathbb{R}$  that

$$\left|g_{\varepsilon}^{d}(x)\right| \leq \left|g_{\varepsilon}^{d}(0)\right| + (cd^{c}T^{-1})^{\frac{1}{2}} \|x\| \leq (cd^{c}T^{-1})^{\frac{1}{2}} + (cd^{c}T^{-1})^{\frac{1}{2}} \|x\| \leq 2(cd^{c}T^{-1})^{\frac{1}{2}} (d^{c} + \|x\|^{2})^{\frac{1}{2}}$$
(122)

and

$$|f_{\varepsilon}(w)| \le |f_{\varepsilon}(0)| + c^{\frac{1}{2}}|w| \le (cd^{c}T^{-3})^{\frac{1}{2}} + c^{\frac{1}{2}}|w|.$$
(123)

First, for all  $d, K \in \mathbb{N}, t \in [0, T], s \in [t, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  we have that

$$\mathbb{E}\left[d^{c} + \left\|X_{t,s}^{d,0,K,\varepsilon,x}\right\|^{2}\right] \le (d^{c} + \|x\|^{2})e^{7c(s-t)}$$
(124)

(cf. Lemma 3.2). This, (122), and Jensen's inequality show for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T]$   $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\mathbb{E}\left[g_{\varepsilon}^{d}(X_{T}^{d,0,K,\varepsilon,t,x})\right] \leq 2(cd^{c}T^{-1})^{\frac{1}{2}}\mathbb{E}\left[\left(d^{c} + \left\|X_{T}^{d,0,K,\varepsilon,t,x}\right\|^{2}\right)^{\frac{1}{2}}\right]$$

$$\leq 2(cd^{c}T^{-1})^{\frac{1}{2}}\left(\mathbb{E}\left[d^{c} + \left\|X_{T}^{d,0,K,\varepsilon,t,x}\right\|^{2}\right]\right)^{\frac{1}{2}}$$

$$\leq 2(cd^{c}T^{-1})^{\frac{1}{2}}\left((d^{c} + \|x\|^{2})e^{7cT}\right)^{\frac{1}{2}}$$

$$= 2(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT}.$$
(125)

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES 23 This, (120), (123), the fact that  $c \ge 1$ , Jensen's inequality, (124) show for all  $d, K \in \mathbb{N}, t \in [0, T] x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$  that

$$\begin{split} \left| u^{d,K,\varepsilon}(t,x) \right| \\ &\leq \mathbb{E} \left[ \left| g_{\varepsilon}^{d}(X_{T}^{d,0,K,\varepsilon,t,x}) \right| \right] + \int_{t}^{T} \mathbb{E} \left[ \left| f_{\varepsilon}(u^{d,K,\varepsilon}(r,X_{r}^{d,0,K,\varepsilon,t,x})) \right| \right] dr \\ &\leq 2(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT} + \int_{t}^{T} \left( (cd^{c}T^{-3})^{\frac{1}{2}} + c^{\frac{1}{2}}\mathbb{E} \left[ \left| u^{d,K,\varepsilon}(r,X_{r}^{d,0,K,\varepsilon,t,x}) \right| \right] \right) dr \\ &\leq 3(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT} \\ &+ \int_{t}^{T} c \left[ \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + \|y\|^{2})^{\frac{1}{2}}} \right] e^{3.5c(T-r)}\mathbb{E} \left[ \left( d^{c} + \left\| X_{r}^{d,0,K,\varepsilon,t,x} \right\|^{2} \right)^{\frac{1}{2}} \right] dr \\ &\leq 3(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT} \\ &+ \int_{t}^{T} c \left[ \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + \|y\|^{2})^{\frac{1}{2}}} \right] e^{3.5c(T-r)} \left( \mathbb{E} \left[ d^{c} + \left\| X_{r}^{d,0,K,\varepsilon,t,x} \right\|^{2} \right] \right)^{\frac{1}{2}} dr \\ &\leq 3(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT} \\ &+ \int_{t}^{T} c \left[ \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + \|y\|^{2})^{\frac{1}{2}}} \right] e^{3.5c(T-r)} \left( (d^{c} + \|x\|^{2})e^{7c(r-t)} \right)^{\frac{1}{2}} dr \\ &= 3(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT} \\ &+ \int_{t}^{T} c \left[ \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + \|y\|^{2})^{\frac{1}{2}}} \right] e^{3.5c(T-r)} \left( (d^{c} + \|x\|^{2})e^{7c(r-t)} \right)^{\frac{1}{2}} dr \\ &= 3(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}e^{3.5cT} \\ &+ \int_{t}^{T} c \left[ \sup_{y \in \mathbb{R}^{d}} \frac{|u^{d,K,\varepsilon}(r,y)|}{e^{3.5c(T-r)}(d^{c} + \|y\|^{2})^{\frac{1}{2}}} \right] e^{3.5c(T-t)} (d^{c} + \|x\|^{2})^{\frac{1}{2}} dr. \end{aligned}$$

Dividing by  $e^{3.5c(T-t)}(d^c + \|x\|^2)^{\frac{1}{2}}$  we then obtain that for all  $d, K \in \mathbb{N}, t \in [0,T], \varepsilon \in (0,1)$  we have that

$$\sup_{y \in \mathbb{R}^d} \frac{\left| u^{d,K,\varepsilon}(t,y) \right|}{e^{3.5c(T-r)} (d^c + \|y\|^2)^{\frac{1}{2}}} \le 3(cd^cT^{-1})^{\frac{1}{2}} e^{3.5cT} + \int_t^T c \sup_{y \in \mathbb{R}^d} \frac{\left| u^{d,K,\varepsilon}(r,y) \right|}{e^{3.5c(T-r)} (d^c + \|y\|^2)^{\frac{1}{2}}} \, dr \tag{127}$$

This, (119), and Grönwall's inequality show for all  $d, K \in \mathbb{N}, t \in [0, T] \in (0, 1)$  that

$$\sup_{y \in \mathbb{R}^d} \frac{\left| u^{d,K,\varepsilon}(t,y) \right|}{e^{3.5c(T-r)} (d^c + \|y\|^2)^{\frac{1}{2}}} \le 3(cd^cT^{-1})^{\frac{1}{2}} e^{3.5cT} \cdot e^{cT}.$$
(128)

This shows for all  $d, K \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\left| u^{d,K,\varepsilon}(t,x) \right| \le 3(cd^{c}T^{-1})^{\frac{1}{2}}e^{8cT}(d^{c} + \|x\|^{2})^{\frac{1}{2}}.$$
(129)

Next, (81) and (122) show for all  $d \in \mathbb{N}, x \in \mathbb{R}^d, \varepsilon \in (0, 1)$  that

$$\frac{T|f_{\varepsilon}(0)| + |g_{\varepsilon}^{d}(x)|}{(d^{c} + \|x\|^{2})^{\frac{1}{2}}} \le \frac{T \cdot T^{-\frac{3}{2}}c^{\frac{1}{2}}d^{\frac{c}{2}} + 2(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + \|x\|^{2})^{\frac{1}{2}}}{(d^{c} + \|x\|^{2})^{\frac{1}{2}}} \le 3T^{-\frac{1}{2}}(cd^{c})^{\frac{1}{2}}$$
(130)

This, [23, Corollary 3.12] (applied for all  $d, K \in \mathbb{N}, \varepsilon \in (0,1)$  with  $f \leftarrow f_{\varepsilon}, g \leftarrow g_{\varepsilon}^{d}, \varphi \leftarrow (\mathbb{R}^{d} \ni x \mapsto (d^{c} + \|x\|^{2})^{\frac{1}{2}}) \in (0,\infty), (Y_{\cdot,\cdot}^{\theta}(\cdot))_{\theta \in \Theta} \leftarrow (X_{\cdot,\cdot}^{d,\theta,K,\varepsilon,\cdot})_{\theta \in \Theta}, (U_{n,m}^{\theta})_{\theta \in \Theta, n,m \in \mathbb{Z}} \leftarrow (U_{n,m}^{d,\theta,K,\varepsilon})_{\theta \in \Theta, n,m \in \mathbb{Z}}$  in the

$$\sup_{t\in[0,T],x\in\mathbb{R}^{d}} \frac{\mathbb{E}\left[\left(\left|U_{n,m}^{d,\theta,K,\varepsilon}(t,x)-u^{d,K,\varepsilon}(t,x)\right|^{2}\right]\right)^{\frac{1}{2}}}{(d^{c}+\|x\|^{2})^{\frac{1}{2}}}\right]$$

$$\leq 2e^{\frac{m}{2}}m^{-\frac{n}{2}}(1+2Tc)^{N-1}e^{3.5cT}\left(\sup_{x\in\mathbb{R}^{d}}\frac{T|f_{\varepsilon}(0)|+|g_{\varepsilon}^{d}(x)|}{(d^{c}+\|x\|^{2})^{\frac{1}{2}}}+Tc\sup_{t\in[0,T],x\in\mathbb{R}^{d}}\frac{|u^{d,K,\varepsilon}(t,x)|}{(d^{c}+\|x\|^{2})^{\frac{1}{2}}}\right)$$

$$\leq 2e^{\frac{m}{2}}m^{-\frac{n}{2}}(e^{2cT})^{N-1}e^{3.5cT}\left(3T^{-\frac{1}{2}}(cd^{c})^{\frac{1}{2}}+Tc\cdot3(cd^{c}T^{-1})^{\frac{1}{2}}e^{8cT}\right)$$

$$\leq 2e^{\frac{m}{2}}m^{-\frac{n}{2}}(e^{2cT})^{N-1}e^{3.5cT}(1+Tc)\cdot3(cd^{c}T^{-1})^{\frac{1}{2}}e^{8cT}$$

$$\leq 6e^{\frac{m}{2}}m^{-\frac{n}{2}}(e^{2cT})^{N-1}e^{12cT}(cd^{c}T^{-1})^{\frac{1}{2}}.$$
(131)

This shows for all  $\theta \in \Theta$ ,  $d, K, n, m \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\mathbb{E}\left[\left(\left|U_{n,m}^{d,\theta,K,\varepsilon}(t,x) - u^{d,K,\varepsilon}(t,x)\right|^{2}\right]\right)^{\frac{1}{2}} \le 6e^{\frac{m}{2}}m^{-\frac{n}{2}}e^{12cTn}(cd^{c}T^{-1})^{\frac{1}{2}}\left(d^{c} + \|x\|^{2}\right)^{\frac{1}{2}}.$$
 (132)

This completes the proof of Lemma 3.3.

### 4. DNNs

4.1. **Properties of operations associated to DNNs.** In Setting 4.1 below we introduce operations which are important for constructing the random DNN that represents the MLP approximations in the proof of Lemma 4.12.

**Setting 4.1.** Assume Setting 1.1, let  $\mathfrak{n}_n^d \in \mathbf{D}$ ,  $n \in [3, \infty) \cap \mathbb{Z}$ ,  $d \in \mathbb{N}$ , satisfy for all  $n \in [3, \infty) \cap \mathbb{N}$ ,  $d \in \mathbb{N}$  that

$$\mathfrak{n}_n^d = (d, \underbrace{2d, \dots, 2d}_{(n-2) \text{ times}}, d) \in \mathbb{N}^n, \tag{133}$$

let  $\mathfrak{n}_n \in \mathbf{D}$ ,  $n \in [3,\infty)$ , satisfy for all  $n \in [3,\infty)$  that  $\mathfrak{n}_n = \mathfrak{n}_n^1$ , let  $\boxplus: \mathbf{D} \times \mathbf{D} \to \mathbf{D}$  satisfy for all  $H \in \mathbb{N}$ ,  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_H, \alpha_{H+1}) \in \mathbb{N}^{H+2}$ ,  $\beta = (\beta_0, \beta_1, \beta_2, \dots, \beta_H, \beta_{H+1}) \in \mathbb{N}^{H+2}$ that  $\alpha \boxplus \beta = (\alpha_0, \alpha_1 + \beta_1, \dots, \alpha_H + \beta_H, \beta_{H+1}) \in \mathbb{N}^{H+2}$ , and let  $\odot: \mathbf{D} \times \mathbf{D} \to \mathbf{D}$  satisfy for all  $H_1, H_2 \in \mathbb{N}$ ,  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{H_1}, \alpha_{H_1+1}) \in \mathbb{N}^{H_1+2}$ ,  $\beta = (\beta_0, \beta_1, \dots, \beta_{H_2}, \beta_{H_2+1}) \in \mathbb{N}^{H_2+2}$  that  $\alpha \odot \beta = (\beta_0, \beta_1, \dots, \beta_{H_2}, \beta_{H_2+1} + \alpha_0, \alpha_1, \alpha_2, \dots, \alpha_{H_1+1}) \in \mathbb{N}^{H_1+H_2+3}$ .

To prove our main result in this section presented in Lemma 4.12 we employ several results presented in Lemmas 4.2–4.10, which are basic facts on DNNs. The proof of Lemmas 4.2–4.9 can be found in [8, 24] and therefore omitted.

**Lemma 4.2** ( $\odot$  is associative–[24, Lemma 3.3]). *Assume Setting 4.1 and let*  $\alpha, \beta, \gamma \in \mathbf{D}$ . *Then we have that*  $(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$ .

**Lemma 4.3** ( $\boxplus$  and associativity–[24, Lemma 3.4]). Assume Setting 4.1, let  $H, k, l \in \mathbb{N}$ , and let  $\alpha, \beta, \gamma \in (\{k\} \times \mathbb{N}^H \times \{l\})$ . Then

(i) we have that  $\alpha \boxplus \beta \in (\{k\} \times \mathbb{N}^H \times \{l\}),$ 

(ii) we have that  $\beta \boxplus \gamma \in (\{k\} \times \mathbb{N}^H \times \{l\})$ , and

(iii) we have that  $(\alpha \boxplus \beta) \boxplus \gamma = \alpha \boxplus (\beta \boxplus \gamma)$ .

**Lemma 4.4** (Triangle inequality–[24, Lemma 3.5]). Consider the notations given in Subsection 1.4, assume Setting 4.1, let  $k, l, H \in \mathbb{N}$ ,  $\alpha, \beta \in \{k\} \times \mathbb{N}^H \times \{l\}$ . Then we have that  $|||\alpha \boxplus \beta ||| \le |||\alpha||| + |||\beta|||$ .

**Lemma 4.5** (DNNs for affine transformations–[24, Lemma 3.7]). Assume Setting 1.1 and let  $d, m \in \mathbb{N}, \lambda \in \mathbb{R}$ ,  $b \in \mathbb{R}^d$ ,  $a \in \mathbb{R}^m$ ,  $\Psi \in \mathbb{N}$  satisfy that  $\mathcal{R}(\Psi) \in C(\mathbb{R}^d, \mathbb{R}^m)$ . Then we have that  $\lambda((\mathcal{R}(\Psi))(\cdot + b) + a) \in \mathcal{R}(\{\Phi \in \mathbb{N} : \mathcal{D}(\Phi) = \mathcal{D}(\Psi)\})$ .

**Lemma 4.6** (Composition of functions generated by DNNs–[24, Lemma 3.8]). Assume Setting 4.1 and let  $d_1, d_2, d_3 \in \mathbb{N}, f_1 \in C(\mathbb{R}^{d_2}, \mathbb{R}^{d_3}), f_2 \in C(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}), \alpha, \beta \in \mathbf{D}$  satisfy both that  $f_1 \in \mathcal{R}(\{\Phi \in \mathbf{N} : \mathcal{D}(\Phi) = \alpha\})$  as well as  $f_2 \in \mathcal{R}(\{\Phi \in \mathbf{N} : \mathcal{D}(\Phi) = \beta\})$ . Then we have that  $(f_1 \circ f_2) \in \mathcal{R}(\{\Phi \in \mathbf{N} : \mathcal{D}(\Phi) = \alpha \odot \beta\})$ .

**Lemma 4.7** (Sum of DNNs of the same length–[24, Lemma 3.9]). Consider the notations given in Subsection 1.4, assume Setting 4.1 and let  $p, q, M, H \in \mathbb{N}, \alpha_1, \alpha_2, \ldots, \alpha_M \in \mathbb{R}, k_i \in \mathbf{D}, g_i \in C(\mathbb{R}^p, \mathbb{R}^q), i \in [1, M] \cap \mathbb{N}$ , satisfy for all  $i \in [1, M] \cap \mathbb{N}$  that  $\dim(k_i) = H + 2$  and  $g_i \in \mathcal{R}(\{\Phi \in \mathbf{N} : \mathcal{D}(\Phi) = k_i\})$ . Then we have that  $\sum_{i=1}^{M} \alpha_i g_i \in \mathcal{R}(\{\Phi \in \mathbf{N} : \mathcal{D}(\Phi) = \boxplus_{i=1}^{M} k_i\})$ .

**Lemma 4.8** (Existence of DNNs with *H* hidden layers for  $\mathrm{Id}_{\mathbb{R}^d}$ –[8, Lemma 3.6]). Assume Setting 4.1 and let  $d, H \in \mathbb{N}$ . Then we have that  $\mathrm{Id}_{\mathbb{R}^d} \in \mathcal{R}(\{\Phi \in \mathbb{N} : \mathcal{D}(\Phi) = \mathfrak{n}_{H+2}^d\})$ .

**Lemma 4.9** ([8, Lemma 3.7]). Consider the notations given in Subsection 1.4, assume Setting 1.1, let  $H, p, q \in \mathbb{N}$ , and let  $g \in C(\mathbb{R}^p, \mathbb{R}^q)$  satisfy that  $g \in \mathcal{R}(\{\Phi \in \mathbb{N} : \dim(\mathcal{D}(\Phi)) = H + 2\})$ . Then for all  $n \in \mathbb{N}_0$  we have that  $g \in \mathcal{R}(\{\Phi \in \mathbb{N} : \dim(\mathcal{D}(\Phi)) = H + 2 + n\})$ .

**Lemma 4.10.** Consider the notations given in Subsection 1.4 and assume Setting 4.1. Then for all  $n \in \mathbb{N}$ ,  $d_0, d_1, \ldots, d_n \in \mathbb{N}$ ,  $f_1 \in C(\mathbb{R}^{d_1}, \mathbb{R}^{d_0}), f_2 \in C(\mathbb{R}^{d_2}, \mathbb{R}^{d_1}), \ldots, f_n \in C(\mathbb{R}^{d_n}, \mathbb{R}^{d_{n-1}}), \phi_1, \phi_2, \ldots, \phi_n \in \mathbb{N}$  with  $\forall i \in [1, n] \cap \mathbb{Z}$ :  $f_i = \mathcal{R}(\phi_i)$  we have that

$$\left\| \left\| \bigcup_{i=1}^{n} \mathcal{D}(\phi_{i}) \right\| \le \max \left\{ \left\| \mathcal{D}(\phi_{1}) \right\|, \left\| \mathcal{D}(\phi_{2}) \right\|, \dots, \left\| \mathcal{D}(\phi_{n}) \right\|, 2d_{1}, 2d_{2}, \dots, 2d_{n-1} \right\}$$
(134)

*Proof of Lemma 4.10.* We will prove by induction on  $n \in \mathbb{N}$ . The base case n = 1 is clear. For the induction step  $\mathbb{N} \in n-1 \mapsto n \in \mathbb{N}$  let  $n \in \mathbb{N} \cap [2, \infty)$  satisfy that for all  $d_0, d_1, \ldots, d_{n-1} \in \mathbb{N}$ ,  $f_1 \in C(\mathbb{R}^{d_1}, \mathbb{R}^{d_0})$ ,  $f_2 \in C(\mathbb{R}^{d_2}, \mathbb{R}^{d_1}), \ldots, f_{n-1} \in C(\mathbb{R}^{d_{n-1}}, \mathbb{R}^{d_{n-2}})$ ,  $\phi_1, \phi_2, \ldots, \phi_{n-1} \in \mathbb{N}$  with  $\forall i \in [1, n-1] \cap \mathbb{Z}$ :  $f_i = \mathcal{R}(\phi_i)$  we have that

$$\left\| \stackrel{n-1}{\odot}_{i=1} \mathcal{D}(\phi_i) \right\| \le \max\left\{ \| \mathcal{D}(\phi_1) \|, \| \mathcal{D}(\phi_2) \|, \dots, \| \mathcal{D}(\phi_{n-1}) \|, 2d_1, 2d_2, \dots, 2d_{n-2} \right\}.$$
(135)

This shows that for all  $d_0, d_1, \ldots, d_n \in \mathbb{N}$ ,  $f_1 \in C(\mathbb{R}^{d_1}, \mathbb{R}^{d_0})$ ,  $f_2 \in C(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ ,  $\ldots, f_n \in C(\mathbb{R}^{d_n}, \mathbb{R}^{d_{n-1}})$ ,  $\phi_1, \phi_2, \ldots, \phi_n \in \mathbb{N}$  with  $\forall i \in [1, n-1] \cap \mathbb{Z}$ :  $f_i = \mathcal{R}(\phi_i)$  there exist  $H_1, H_2 \in \mathbb{N}$ ,  $\mathbf{a}_1 \in \mathbb{R}^{H_1}$ ,  $\mathbf{a}_2 \in \mathbb{R}^{H_2}$  such that

$$C(\mathbb{R}^{d_{n-1}},\mathbb{R}^{d_0}) \ni f_1 \circ f_2 \circ \ldots \circ f_{n-1} \in \mathcal{R}\left(\left\{\Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \bigcup_{i=1}^{n-1} \mathcal{D}(\phi_i)\right\}\right),\tag{136}$$

 $\bigcirc_{i=1}^{n-1} \mathcal{D}(\phi_i) = (d_{n-1}, \mathbf{a}_1, d_0), \ \mathcal{D}(\phi_n) = (d_n, \mathbf{a}_2, d_{n-1}), \ (\bigcirc_{i=1}^{n-1} \mathcal{D}(\phi_i)) \odot \mathcal{D}(\phi_n) = (d_n, \mathbf{a}_2, 2d_{n-1}, \mathbf{a}_1, d_0), \ \text{and}$ 

$$\begin{aligned} \left\| \begin{bmatrix} n \\ \odot \\ i=1 \end{bmatrix} \mathcal{D}(\phi_{i}) \right\| &= \left\| \begin{bmatrix} n-1 \\ \odot \\ i=1 \end{bmatrix} \mathcal{D}(\phi_{i}) \right\| \\ &\leq \max \left\{ \max \left\{ \| \mathcal{D}(\phi_{1}) \|, \| \mathcal{D}(\phi_{2}) \|, \dots, \| \mathcal{D}(\phi_{n-1}) \|, 2d_{1}, 2d_{2}, \dots, 2d_{n-2} \right\}, \| \mathcal{D}(\phi_{n}) \|, 2d_{n-1} \right\} \\ &\leq \max \left\{ \| \mathcal{D}(\phi_{1}) \|, \| \mathcal{D}(\phi_{2}) \|, \dots, \| \mathcal{D}(\phi_{n}) \|, 2d_{1}, 2d_{2}, \dots, 2d_{n-2} \right\}, \| \mathcal{D}(\phi_{n}) \|, 2d_{n-1} \right\} \\ &\leq \max \left\{ \| \mathcal{D}(\phi_{1}) \|, \| \mathcal{D}(\phi_{2}) \|, \dots, \| \mathcal{D}(\phi_{n}) \|, 2d_{1}, 2d_{2}, \dots, 2d_{n-1} \right\}. \end{aligned}$$
(137)

This proves the induction step. Induction hence completes the proof of Lemma 4.10.

4.2. **DNN representation of our Euler-Maruyama approximations.** In Lemma 4.11 below we prove that Euler-Maruyama approximations can be represented by DNNs if their coefficients are represented by DNNs.

**Lemma 4.11.** Consider the notations given in Subsection 1.4, assume Setting 1.1, let  $T \in (0, \infty)$ ,  $K \in \mathbb{N}$ , let  $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{R}$  satisfy for all  $t \in \mathbb{R}$  that  $\lfloor t \rfloor = \max(\{0, \frac{T}{K}, \frac{2T}{K}, \dots, T\} \cap ((-\infty, t) \cup \{0\}))$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $v \in \mathbb{R}^d$  let  $\beta_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^{d \times d})$ ,  $\Phi_{\beta_{\varepsilon}^d}, \Phi_{\sigma_{\varepsilon}^d,v} \in \mathbb{N}$  satisfy that  $\beta_{\varepsilon}^d = \mathcal{R}(\Phi_{\beta_{\varepsilon}^d})$ ,  $\sigma_{\varepsilon}^d(\cdot)v = \mathcal{R}(\Phi_{\sigma_{\varepsilon}^d,v})$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $\gamma_{\varepsilon}^d : \mathbb{R}^{2d} \to \mathbb{R}^d$ ,  $F_{\varepsilon}^d : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ ,  $G^d : \mathbb{R}^d \to \mathbb{R}^d$  be measurable and satisfy for all  $y, z \in \mathbb{R}^d$  that  $\gamma_{\varepsilon}^d(y, z) = F_{\varepsilon}^d(y)G^d(z)$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $v \in \mathbb{R}^d$  let  $\Phi_{F_{\varepsilon}^d,v} \in \mathbb{N}$  satisfy  $F_{\varepsilon}^d(\cdot)v = \mathcal{R}(\Phi_{F_{\varepsilon}^d,v})$ , assume for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1]$ ,  $v \in \mathbb{R}^d$  that  $\mathcal{D}(\Phi_{\sigma_{\varepsilon}^d,v}) = \mathcal{D}(\Phi_{\sigma_{\varepsilon}^d,0})$  and  $\mathcal{D}(\Phi_{F_{\varepsilon}^d,v}) = \mathcal{D}(\Phi_{F_{\varepsilon}^d,v})$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$  be a probability space satisfying the usual conditions, let  $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$ , for every  $d \in \mathbb{N}$  let  $W^{d,\theta} : \Omega \times [0,T] \to \mathbb{R}^d$ ,  $\theta \in \Theta$ , be independent  $(\mathbb{F}_t)_{t \in [0,T]}$ -Poisson random measures on  $[0,\infty) \times (\mathbb{R}^d \setminus \{0\})$  with intensity  $\nu^d$ , for every  $d \in \mathbb{N}$ ,  $\theta \in \Theta$  let  $\tilde{N}^{d,\theta}(dt, dz) = N^{d,\theta}(dt, dz) - dt \nu^d(dz)$ , assume for all  $d \in \mathbb{N}$  that  $\mathcal{F}_0, (N^{d,\theta})_{\theta \in \Theta}$  and  $(W^{d,\theta})_{\theta \in \Theta}$ , are independent, for every

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES  $d \in \mathbb{N}, \theta \in \Theta, x \in \mathbb{R}^d, \varepsilon \in (0, 1), t \in [0, T)$  let  $(X_s^{d,\theta,K,\varepsilon,t,x})_{s \in [t,T]}$  satisfy that  $X_t^{d,\theta,K,\varepsilon,t,x} = x$  and

$$X_{s}^{d,\theta,K,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d} (X_{\max\{t,\lfloor u-\rfloor\}}^{d,\theta,K,\varepsilon,t,x}) du + \int_{t}^{s} \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor u-\rfloor\}}^{d,\theta,K,\varepsilon,t,x}) dW_{u}^{d,\theta} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor u-\rfloor\}}^{d,\theta,K,\varepsilon,t,x}, z) \tilde{N}^{d,\theta} (du, dz).$$

$$(138)$$

and let  $\omega \in \Omega$ . Then there exists  $(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})_{d\in\mathbb{N},\theta\in\Theta,\varepsilon\in(0,1),t\in[0,T),s\in(t,T]} \subseteq \mathbf{N}$  such that

- (i) for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T)$ ,  $s \in (t, T]$ ,  $x \in \mathbb{R}^d$  we have that  $\mathcal{R}(\mathcal{X}^{d,\theta,K,\varepsilon}_{t,s}) \in C(\mathbb{R}^d, \mathbb{R}^d)$ and  $(\mathcal{R}(\mathcal{X}^{d,\theta,K,\varepsilon}_{t,s}))(x) = X^{d,\theta,K,\varepsilon,t,x}_s(\omega)$ ,
- (ii) for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t_1 \in [0, T)$ ,  $s_1 \in (t_1, T]$ ,  $t_2 \in [0, T)$ ,  $s_2 \in (t_2, T]$ ,  $x \in \mathbb{R}^d$  we have that  $\mathcal{D}(\mathcal{X}_{t_1, s_1}^{d, \theta_1, K, \varepsilon}) = \mathcal{D}(\mathcal{X}_{t_2, s_2}^{d, \theta_2, K, \varepsilon})$ ,
- (iii) for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0,1)$ ,  $t \in [0,T)$ ,  $s \in (t,T]$  we have that  $\dim(\mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})) = K(\max\{\dim(\mathcal{D}(\Phi_{\beta^d_{\varepsilon}})),\dim(\mathcal{D}(\Phi_{\sigma^d_{\varepsilon},0})),\dim(\mathcal{D}(\Phi_{F^d_{\varepsilon},0}))\}-1)+1$ , and
- (iv) for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0,1)$ ,  $t \in [0,T)$ ,  $s \in (t,T]$  we have that  $\left\| \left| \mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}) \right| \right\| \le 2d + \left\| \left| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| + \left\| \left| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d},0}) \right\| \right\|$ .

*Proof of Lemma 4.11.* Throughout this proof let the notation in Setting 4.1 be given. First, observe that for all  $d \in \mathbb{N}, \theta \in \Theta, x \in \mathbb{R}^d, \varepsilon \in (0, 1), k \in [1, K] \cap \mathbb{Z}, t \in [0, T), s \in [\frac{kT}{K}, \frac{(k+1)T}{K}]$  we have that

$$X_{s}^{d,\theta,K,\varepsilon,t,x}(\omega) = X_{\max\{t,\frac{kT}{K}\}}^{d,\theta,K,\varepsilon,t,x}(\omega) + \beta_{\varepsilon}^{d} \left( X_{\max\{t,\frac{kT}{K}\}}^{d,\theta,K,\varepsilon,t,x}(\omega) \right) \left( s - \max\{t,\frac{kT}{K}\} \right) + \sigma_{\varepsilon}^{d} \left( X_{\max\{t,\frac{kT}{K}\}}^{d,\theta,K,\varepsilon,t,x}(\omega) \right) \left( W_{s}^{d,\theta}(\omega) - W_{\max\{t,\frac{kT}{K}\}}^{d,\theta}(\omega) \right) + F_{\varepsilon}^{d} \left( X_{\max\{t,\frac{kT}{K}\}}^{d,\theta,K,\varepsilon,t,x}(\omega) \right) \int_{\max\{t,\frac{kT}{K}\}}^{s} \int_{\mathbb{R}^{d}\setminus\{0\}} G^{d}(z) \tilde{N}^{d,\theta}(\omega) (du, dz).$$
(139)

Next, for every  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $x \in \mathbb{R}^d$ ,  $\varepsilon \in (0, 1)$ ,  $k \in [1, K] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $s \in (t, T]$  let  $J_k(s) \in \mathbb{R}$ ,  $\phi_{t,s,k}^{d,\theta,K,\varepsilon}(x) \in \mathbb{R}^d$  satisfy that

$$J_{k}(s) = \max\{t, \frac{(k-1)T}{K}\}\mathbb{1}_{[0,\max\{t, \frac{(k-1)T}{K}\}]}(s) + s\mathbb{1}_{\max\{t, \frac{(k-1)T}{K}\},\max\{t, \frac{kT}{K}\}]}(s) + \max\{t, \frac{kT}{K}\}\mathbb{1}_{(\max\{t, \frac{kT}{K}\},T]}(s)$$
(140)

and

$$\phi_{t,s,k}^{d,\theta,K,\varepsilon}(x) = x + \beta_{\varepsilon}^{d}(x) \left( J_{k}(s) - \max\{t, \frac{(k-1)T}{K}\} \right) + \sigma_{\varepsilon}^{d}(x) \left( W_{J_{k}(s)}^{d,\theta}(\omega) - W_{\max\{t, \frac{(k-1)T}{K}\}}^{d,\theta}(\omega) \right) + F_{\varepsilon}^{d}(x) \int_{\max\{t, \frac{(k-1)T}{K}\}}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} G^{d}(z) \tilde{N}^{d,\theta}(\omega) (du, dz)$$
(141)

Next, for every  $d \in \mathbb{N}, \theta \in \Theta, \varepsilon \in (0, 1), k \in [1, K] \cap \mathbb{Z}, t \in [0, T), s \in (t, T]$  let

$$\psi_{t,s,k}^{d,\theta,K,\varepsilon} = \phi_{t,s,k}^{d,\theta,K,\varepsilon} \circ \phi_{t,s,k-1}^{d,\theta,K,\varepsilon} \circ \dots \circ \phi_{t,s,1}^{d,\theta,K,\varepsilon}.$$
(142)

Note that for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0,1)$ ,  $k \in [1, K-1] \cap \mathbb{Z}$ ,  $s \in [0, \max\{t, \frac{(k-1)T}{K}\}]$  we have that  $\phi_{t,s,k}^{d,\theta,K,\varepsilon} = \operatorname{Id}_{\mathbb{R}^d}$ . This ensures for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0,1)$ ,  $k \in [1, K-1] \cap \mathbb{Z}$ ,  $n \in [k+1, K] \cap \mathbb{Z}$ ,  $s \in [0, \max\{t, \frac{kT}{K}\}]$  that  $\psi_{t,s,k}^{d,\theta,K,\varepsilon} = \psi_{t,s,n}^{d,\theta,K,\varepsilon}$  and in particular  $\psi_{t,s,k}^{d,\theta,K,\varepsilon} = \psi_{t,s,K}^{d,\theta,K,\varepsilon}$ . Observe that for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0,1)$ ,  $k \in [1, K] \cap \mathbb{Z}$ ,  $s \in [0, \max\{t, \frac{kT}{K}\}]$ ,  $x \in \mathbb{R}^d$  that  $\psi_{t,s,k}^{d,\theta,K,\varepsilon}(x) = X_s^{d,\theta,K,\varepsilon,t,x}(\omega)$ . Therefore, for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that  $\psi_{t,s,K}^{d,\theta,K,\varepsilon}(x) = X_s^{d,\theta,K,\varepsilon,t,x}$ , i.e.,

$$X_s^{d,\theta,K,\varepsilon,t,x}(\omega) = \phi_{t,s,K}^{d,\theta,K,\varepsilon} \circ \phi_{t,s,K-1}^{d,\theta,K,\varepsilon} \circ \dots \circ \phi_{t,s,1}^{d,\theta,K,\varepsilon}(x).$$
(143)

Next, assume w.l.o.g. (cf. Lemma 4.9) that

$$\dim(\mathcal{D}(\Phi_{\beta^d_{\varepsilon}})) = \dim(\mathcal{D}(\Phi_{\sigma^d_{\varepsilon},0})) = \dim(\mathcal{D}(\Phi_{F^d_{\varepsilon},0})).$$
(144)

$$\mathrm{Id}_{\mathbb{R}^d} \in \mathcal{R}(\{\Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \mathfrak{n}^d_{\dim(\Phi_{\beta^d_{\mathfrak{s}}})}\})$$
(145)

This, (141), (144), and Lemma 4.7 show for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\delta, \varepsilon \in (0, 1)$ ,  $k \in [1, K] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $s \in (t, T]$  that

$$\phi_{t,s,k}^{d,\theta,K,\varepsilon}(\cdot) \in \mathcal{R}\left(\left\{\Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \mathfrak{n}_{\dim(\Phi_{\beta_{\varepsilon}^{d}})}^{d} \boxplus \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \boxplus \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \boxplus \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0})\right\}\right)$$
(146)

This, (143), and Lemma 4.6 show that there exists  $(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})_{d\in\mathbb{N},\theta\in\Theta,\varepsilon\in(0,1),t\in[0,T),s\in(t,T]} \subseteq \mathbb{N}$  such that for all  $d\in\mathbb{N}, \theta\in\Theta, \varepsilon\in(0,1), t\in[0,T), s\in(t,T], x\in\mathbb{R}^d$  we have that

$$\mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}) = \bigotimes_{k=1}^{K} \left[ \mathfrak{n}_{\dim(\Phi_{\beta_{\varepsilon}^{d}})}^{d} \boxplus \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \boxplus \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \boxplus \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right],$$

$$(\mathcal{R}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}))(x) = X_{s}^{d,\theta,K,\varepsilon,t,x}(\omega).$$
(147)

This, the definition of  $\odot$  and an induction argument show that for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T)$ ,  $s \in (t, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\dim(\mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})) = K(\dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^d})) - 1) + 1.$$
(148)

Next, (147), Lemma 4.10, the triangle inequality (cf. Lemma 4.4), and (133) show that for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T)$ ,  $s \in (t, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\begin{aligned} \left\| \left| \mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}) \right\| &= \left\| \left\| \begin{array}{c} {}^{K}_{\odot} \\ {}^{\otimes}_{k=1} \left[ \mathfrak{n}_{\dim(\Phi_{\beta_{\varepsilon}^{d}})}^{d} \boxplus \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \boxplus \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \boxplus \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right] \right\| \\ &\leq \max \left\{ 2d, \left\| \left| \mathfrak{n}_{\dim(\Phi_{\beta_{\varepsilon}^{d}})}^{d} \boxplus \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \boxplus \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \boxplus \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| \right| \right\} \\ &\leq \max \left\{ 2d, \left\| \left| \mathfrak{n}_{\dim(\Phi_{\beta_{\varepsilon}^{d}})}^{d} \right\| + \left\| \left| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| \right\| \right\} \\ &= 2d + \left\| \left| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| + \left\| \left| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| \right\|. \end{aligned}$$
(149)

The proof of Lemma 4.11 is thus completed.

4.3. **DNN representation of our MLP approximations.** In Lemma 4.12 below we prove that the MLP approximations under consideration can be represented by DNNs.

**Lemma 4.12.** Assume the setting of Lemma 4.11, for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $f_{\varepsilon} \in C(\mathbb{R}, \mathbb{R})$ ,  $g_{\varepsilon}^{d} \in C(\mathbb{R}^{d}, \mathbb{R})$ ,  $\Phi_{f_{\varepsilon}}, \Phi_{g_{\varepsilon}^{d}} \in \mathbb{N}$  satisfy that  $\mathcal{R}(\Phi_{f_{\varepsilon}}) = f_{\varepsilon}$  and  $\mathcal{R}(\Phi_{g_{\varepsilon}^{d}}) = g_{\varepsilon}^{d}$ , let  $\mathfrak{t}^{\theta} \colon \Omega \to [0,1]$ ,  $\theta \in \Theta$ , be *i.i.d* random variables which satisfy for all  $t \in (0,1)$  that  $\mathbb{P}(\mathfrak{t}^{0} \leq t) = t$ , for every  $\theta \in \Theta$ ,  $t \in [0,T]$  let  $\mathfrak{T}_{\mathfrak{t}}^{\theta} \colon \Omega \to \mathbb{R}$  satisfy for all  $\theta \in \Theta$  that  $\mathfrak{T}_{\mathfrak{t}}^{\theta} = t + (T-t)\mathfrak{t}^{\theta}$ , assume for all  $d \in \mathbb{N}$  that  $(\mathfrak{t}^{\theta})_{\theta \in \Theta}$ ,  $(N^{d,\theta})_{\theta \in \Theta}$  and  $(W^{d,\theta})_{\theta \in \Theta}$ , are independent, for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $U_{n,m}^{d,\theta,K,\varepsilon} \colon [0,T] \times \mathbb{R}^{d} \times \Omega \to \mathbb{R}$ ,  $\theta \in \Theta$ ,  $n, m \in \mathbb{Z}$ , satisfy for all  $\theta \in \Theta$ ,  $n \in \mathbb{N}_{0}$ ,  $m \in \mathbb{N}$ ,  $t \in [0,T]$ ,  $x \in \mathbb{R}^{d}$  that

$$U_{n,m}^{d,\theta,K,\varepsilon}(t,x) = \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^{n}} \sum_{i=1}^{m^{n}} g_{\varepsilon}^{d} \Big( X_{T}^{d,(\theta,0,-i),K,\varepsilon,t,x} \Big) + \sum_{\ell=0}^{n-1} \frac{(T-t)}{m^{n-\ell}} \sum_{i=1}^{m^{n-\ell}} \Big( f_{\varepsilon} \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} - \mathbb{1}_{\mathbb{N}}(\ell) f_{\varepsilon} \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon} \Big) \Big( \mathfrak{T}_{t}^{(\theta,\ell,i)}, X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}}^{d,(\theta,\ell,i),K,\varepsilon,t,x} \Big),$$

$$(150)$$

and let  $(c_{d,\varepsilon})_{d\in\mathbb{N},\varepsilon\in(0,1)}\subseteq\mathbb{R}$  satisfy for all  $d\in\mathbb{N}, \varepsilon\in(0,1)$  that

$$c_{d,\varepsilon} \ge 2d + \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \right\| + \left\| \left| \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| \right\|.$$
(151)

Then for all  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  there exists  $(\Phi_{n,m,t}^{d,\theta,K,\varepsilon})_{t \in [0,T],\theta \in \Theta} \subseteq \mathbb{N}$  such that

(i) we have for all  $t_1, t_2 \in [0, T]$ ,  $\theta_1, \theta_2 \in \Theta$  that  $\mathcal{D}(\Phi_{n,m,t_1}^{d,\theta_1,K,\varepsilon}) = \mathcal{D}(\Phi_{n,m,t_2}^{d,\theta_2,K,\varepsilon})$ ,

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES (ii) we have for all  $t \in [0, T]$ ,  $\theta \in \Theta$  that

$$\dim(\mathcal{D}(\Phi_{n,m,t}^{d,\theta,K,\varepsilon})) = (n+1) \left[ K \left( \max\left\{ \dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})), \dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d}})) \right\} - 1 \right) + 1 \right] + n(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^{d}})) - 1,$$
(152)

(iii) we have for all  $t \in [0, T]$ ,  $\theta \in \Theta$  that  $\left\| \left| \mathcal{D}(\Phi_{n,m,t}^{d,\theta,K,\varepsilon}) \right| \right\| \leq c_{d,\varepsilon}(3m)^n$ , and (iv) we have for all  $t \in [0, T]$ ,  $\theta \in \Theta$ ,  $x \in \mathbb{R}^d$  that  $U_{n,m}^{d,\theta,K,\varepsilon}(t, x, \omega) = (\mathcal{R}(\Phi_{n,m,t}^{d,\theta,K,\varepsilon}))(x)$ .

Proof of Lemma 4.12. Throughout this proof let the notation in Setting 4.1 be given. First, Lemma 4.11 and (151) show that there exists  $(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})_{d\in\mathbb{N},\theta\in\Theta,\varepsilon\in(0,1),t\in[0,T),s\in(t,T]} \subseteq \mathbb{N}$  such that

for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T)$ ,  $s \in (t, T]$ ,  $x \in \mathbb{R}^d$  we have that  $\mathcal{R}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}) \in C(\mathbb{R}^d, \mathbb{R}^d)$  and

$$(\mathcal{R}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}))(x) = X_s^{d,\theta,K,\varepsilon,t,x}(\omega),$$
(153)

for all  $d \in \mathbb{N}, \theta \in \Theta, \varepsilon \in (0, 1), t_1 \in [0, T), s_1 \in (t_1, T], t_2 \in [0, T), s_2 \in (t_2, T], x \in \mathbb{R}^d$  we have that

$$\mathcal{D}(\mathcal{X}_{t_1,s_1}^{d,\theta_1,K,\varepsilon}) = \mathcal{D}(\mathcal{X}_{t_2,s_2}^{d,\theta_2,K,\varepsilon})$$
(154)

for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T)$ ,  $s \in (t, T]$  we have that

$$\dim(\mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})) = K(\max\{\dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})), \dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0})), \dim(\mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}))\} - 1) + 1,$$
(155)

and

for all  $d \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $\varepsilon \in (0, 1)$ ,  $t \in [0, T)$ ,  $s \in (t, T]$  we have that

$$\left\| \mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon}) \right\| \le 2d + \left\| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| + \left\| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| + \left\| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| \le c_{d,\varepsilon}.$$
(156)

Throughout the rest of this proof let  $m \in \mathbb{N}$ ,  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  be fixed, let Id<sub>R</sub> be the identity on R, and let  $L \in \mathbb{Z}$  satisfy that

$$L = \dim(\mathcal{D}(\mathcal{X}_{t,s}^{d,\theta,K,\varepsilon})) = K\left(\max\left\{\dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})),\dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d}}))\right\} - 1\right) + 1.$$
(157)

(cf. (155)). We will prove the lemma via induction on  $n \in \mathbb{N}_0$ . First, the base case n = 0 is true since the 0function can be represented by DNN function of arbitrary length. For the induction step  $\mathbb{N}_0 \ni n \mapsto n+1 \in \mathbb{N}$ let  $n \in \mathbb{N}_0$  and assume that there exists  $(\Phi_{\ell,m,t}^{d,\theta,K,\varepsilon})_{t\in[0,T],\theta\in\Theta} \subseteq \mathbb{N}, \ell \in [0,n] \cap \mathbb{Z}$ , such that

we have for all  $t_1, t_2 \in [0, T], \theta_1, \theta_2 \in \Theta, \ell \in [0, n] \cap \mathbb{Z}$  that

$$\mathcal{D}(\Phi_{\ell,m,t_1}^{d,\theta_1,K,\varepsilon}) = \mathcal{D}(\Phi_{\ell,m,t_2}^{d,\theta_2,K,\varepsilon}),\tag{158}$$

we have for all  $t \in [0, T], \theta \in \Theta, \ell \in [0, n] \cap \mathbb{Z}$  that

$$\dim(\mathcal{D}(\Phi_{\ell,m,t}^{d,\theta,K,\varepsilon})) = (\ell+1)L + \ell(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^d})) - 1,$$
(159)

we have for all  $t \in [0, T], \theta \in \Theta, \ell \in [0, n] \cap \mathbb{Z}$  that

$$\left\| \left\| \Phi_{\ell,m,t}^{d,\theta,K,\varepsilon} \right\| \right\| \le c_{d,\varepsilon} (3m)^{\ell}, \tag{160}$$

and

we have for all  $t \in [0, T], \theta \in \Theta, x \in \mathbb{R}^d, \ell \in [0, n] \cap \mathbb{Z}$  that

$$U_{\ell,m}^{d,\theta,K,\varepsilon}(t,x,\omega) = (\mathcal{R}(\Phi_{\ell,m,t}^{d,\theta,K,\varepsilon}))(x).$$
(161)

Next, Lemma 4.8, the fact that  $g_{\varepsilon}^d = \mathcal{R}(\Phi_{g_{\varepsilon}^d})$ , (153), (154), and Lemma 4.6 show for all  $\theta \in \Theta$ ,  $i \in [1, m^{n+1}] \cap$  $\mathbb{Z}, t \in [0, T]$  that

$$g_{\varepsilon}^{d} \Big( X_{t,T}^{d,(\theta,0,-i),K,\varepsilon,\cdot}(\omega) \Big) = \mathrm{Id}_{\mathbb{R}} \left( g_{\varepsilon}^{d} \Big( X_{t,T}^{d,(\theta,0,-i),K,\varepsilon,\cdot}(\omega) \Big) \right)$$
  

$$\in \mathcal{R} \left( \left\{ \Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2 + L\right) + 1} \odot \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\} \right).$$
(162)

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES In addition, the definition of  $\odot$ , (133), and (157) show that

$$\dim \left( \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2+L\right)+1} \odot \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right)$$

$$= \dim \left( \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2+L\right)+1} \right) + \dim \left( \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right) + \dim \left( \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right) - 2$$

$$= (n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2 + L\right) + 1 + \dim \left( \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right) + L - 2$$

$$= (n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2\right) + (n+2)L + \dim \left( \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right) - 1.$$
(163)

Next, Lemma 4.8, the fact that  $f_{\varepsilon} = \mathcal{R}(\Phi_{f_{\varepsilon}})$ , (158), (161), (153), (154), and Lemma 4.6 show for all  $\theta \in \Theta$ ,  $i \in [1, m^{n+1}] \cap \mathbb{Z}, t \in [0, T]$  that

$$\left( f_{\varepsilon} \circ U_{n,m}^{d,(\theta,n,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{t,\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,\cdot}(\omega) \right)$$

$$\in \mathcal{R} \left( \left\{ \Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\} \right).$$

$$(164)$$

In addition, the definition of  $\odot$ , (159), and (157) show that

$$\dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right)$$

$$= \dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \right) + \dim \left( \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \right) + \dim \left( \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right) - 2$$

$$= \dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \right) + \left( (n+1)L + n(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^{d}})) - 1 \right) + L - 2$$

$$= (n+1) \left( \dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2 \right) + (n+2)L + \dim \left( \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right) - 1.$$
(165)

Furthermore, the fact that  $f_{\varepsilon} = \mathcal{R}(\Phi_{f_{\varepsilon}})$ , Lemma 4.8, (158), (161), (153), (154), and Lemma 4.6 show for all  $\theta \in \Theta$ ,  $i \in [1, m^{n+1}] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $\ell \in [0, n-1] \cap \mathbb{Z}$  that

$$\left( f_{\varepsilon} \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,\cdot}, \omega \right)$$

$$= \left( f_{\varepsilon} \circ \operatorname{Id}_{\mathbb{R}} \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,\cdot}, \omega \right)$$

$$\in \mathcal{R} \left( \left\{ \Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell)} (\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L)+1} \odot \mathcal{D}(\Phi_{\ell,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\} \right).$$

$$(166)$$

In addition, the definition of  $\odot$ , (133), (159), and (157) show for all  $\ell \in [0, n-1] \cap \mathbb{Z}$  that

$$\dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell) \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) - 2 + L \right) + 1} \odot \mathcal{D}(\Phi_{\ell,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right)$$

$$= \dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \right) + \dim \left( \mathfrak{n}_{(n-\ell) \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) - 2 + L \right) + 1} \right) + \dim \left( \mathcal{D}(\Phi_{\ell,m,0}^{d,0,K,\varepsilon}) \right) + \dim \left( \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right) - 3$$

$$= \dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \right) + \left( (n-\ell) \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) - 2 + L \right) + 1 \right)$$

$$+ \left( (\ell+1)L + \ell(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^{d}})) - 1 \right) + L - 3$$

$$= (n+1) \left( \dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2 \right) + (n+2)L + \dim \left( \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right) - 1.$$
(167)

Furthermore, the fact that  $f_{\varepsilon} = \mathcal{R}(\Phi_{f_{\varepsilon}})$ , Lemma 4.8, (158), (161), (153), (154), and Lemma 4.6 show for all  $\theta \in \Theta$ ,  $i \in [1, m^{n+1}] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $\ell \in [1, n] \cap \mathbb{Z}$  that

$$\left( f_{\varepsilon} \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,\cdot}(\omega), \omega \right)$$

$$= \left( f_{\varepsilon} \circ \operatorname{Id}_{\mathbb{R}} \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,\cdot}(\omega), \omega \right)$$

$$\in \mathcal{R} \left( \left\{ \Phi \in \mathbf{N} \colon \mathcal{D}(\Phi) = \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell+1)\left(\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L\right)+1} \odot \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\} \right).$$

$$(168)$$

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES In addition, the definition of  $\odot$ , (133), (159), and (157) show for all  $\ell \in [1, n] \cap \mathbb{Z}$  that

$$\dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell+1)} (\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L)+1} \odot \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right)$$

$$= \dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \right) + \dim \left( \mathfrak{n}_{(n-\ell+1)} (\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L)+1 \right)$$

$$+ \dim \left( \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \right) + \dim \left( \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right) - 3$$

$$= \dim \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) \right) + \left( (n-\ell+1) (\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L)+1 \right)$$

$$+ \left( \ell L + (\ell-1) (\dim (\mathcal{D}(\Phi_{f_{\varepsilon}}))-2) + \dim (\mathcal{D}(\Phi_{g_{\varepsilon}^{d}}))-1 \right) + L - 3$$

$$= (n+1) \left( \dim (\mathcal{D}(\Phi_{f_{\varepsilon}}))-2 \right) + (n+2)L + \dim \left( \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right) - 1.$$
(169)

Now, (162)–(169) and Lemma 4.7 show that there exists  $(\Phi_{n+1,m,t}^{d,\theta,K,\varepsilon})_{t\in[0,T],\theta\in\Theta}$  such that  $t\in[0,T], \theta\in\Theta$ ,  $x\in\mathbb{R}^d$  we have that

$$\begin{aligned} (\mathcal{R}(\Phi_{n+1,m,t}^{d,\theta,K,\varepsilon}))(x) &= \frac{1}{m^{n+1}} \sum_{i=1}^{m^{n+1}} g_{\varepsilon}^{d} \Big( X_{T}^{d,(\theta,0,-i),K,\varepsilon,t,x}(\omega) \Big) \\ &+ \frac{1}{m} \sum_{i=1}^{m} \Big( f_{\varepsilon} \circ U_{n,m}^{d,(\theta,n,i),K,\varepsilon} \Big) \Big( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,x}(\omega), \omega \Big) \\ &+ \sum_{\ell=0}^{n-1} \frac{(T-t)}{m^{n+1-\ell}} \sum_{i=1}^{m^{n+1-\ell}} \Big( f_{\varepsilon} \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} \Big) \Big( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,x}(\omega), \omega \Big) \\ &- \sum_{\ell=1}^{n} \frac{(T-t)}{m^{n+1-\ell}} \sum_{i=1}^{m^{n+1-\ell}} \Big( f_{\varepsilon} \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon} \Big) \Big( \mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega), X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}(\omega)}^{d,(\theta,\ell,i),K,\varepsilon,t,x}(\omega), \omega \Big) \\ &= U_{n+1,m}^{d,\theta,K,\varepsilon}(t,x), \end{aligned}$$

$$\mathcal{D}(\Phi_{n+1,m,t}^{d,\theta,K,\varepsilon}) = (n+1) \left( \dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2 \right) + (n+2)L + \dim\left( \mathcal{D}(\Phi_{g_{\varepsilon}^d}) \right) - 1,$$
(171)

and

$$\mathcal{D}(\Phi_{n+1,m,t}^{d,\theta,K,\varepsilon}) = \begin{bmatrix} \underset{i=1}{\overset{m}{\cong}} \left[ \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2+L\right)+1} \odot \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right] \right] \\ \boxplus \begin{bmatrix} \underset{i=1}{\overset{m}{\cong}} \left[ \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right] \right] \\ \boxplus \begin{bmatrix} \underset{\ell=0}{\overset{n-1}{\cong}} \underset{i=1}{\overset{m}{\cong}} \left[ \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell)\left(\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L\right)+1} \odot \mathcal{D}(\Phi_{\ell,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right] \right] \\ \boxplus \begin{bmatrix} \underset{\ell=1}{\overset{m}{\cong}} \underset{i=1}{\overset{m}{\cong}} \left[ \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell+1)\left(\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L\right)+1} \odot \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right] \right].$$
(172)

This shows for all  $t_1,t_2\in[0,T],$   $\theta_1,\theta_2\in\Theta$  that

$$\mathcal{D}(\Phi_{n+1,m,t_1}^{d,\theta_1,K,\varepsilon}) = \mathcal{D}(\Phi_{n+1,m,t_2}^{d,\theta_2,K,\varepsilon}).$$
(173)

Furthermore, Lemma 4.10, (133), (151), and (156) show that

$$\left\| \left\| \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2+L\right)+1} \odot \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\|$$

$$\leq \max\left\{ 2d, \left\| \left\| \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2+L\right)+1} \right\|, \left\| \left| \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right\| \right\|, \left\| \left| \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\| \right\} \leq c_{d,\varepsilon}.$$
(174)

Next, Lemma 4.10, (151), (160), and (156) show that

$$\left\| \left\| \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\| \le \max \left\{ 2d, \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \right\| \right|, \left\| \left| \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \right\| \right|, \left\| \left| \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right| \right\} \le c_{d,\varepsilon} (3m)^{n}.$$

$$(175)$$

Furthermore, Lemma 4.10, (151), (160), and (156) show for all  $\ell \in [0, n-1] \cap \mathbb{Z}$  that

$$\left\| \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell) \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) - 2 + L \right) + 1} \odot \mathcal{D}(\Phi_{\ell,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\|$$

$$\leq \max \left\{ 2d, \left\| \mathcal{D}(\Phi_{f_{\varepsilon}}) \right\|, \left\| \mathfrak{n}_{(n-\ell) \left( \mathcal{D}(\Phi_{f_{\varepsilon}}) - 2 + L \right) + 1} \right\|, \left\| \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\} \le c_{d,\varepsilon} (3m)^{\ell},$$

$$(176)$$

In addition, Lemma 4.10, (151), (160), and (156) show for all  $\ell \in [1, n] \cap \mathbb{Z}$  that

$$\left\| \left\| \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell+1)\left(\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L\right)+1} \odot \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\| \\ \leq \max\left\{ 2d, \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \right\| \right\|, \left\| \left| \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \right\| \right\|, \left\| \left| \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\| \right\} \le c_{d,\varepsilon} (3m)^{\ell-1}.$$

$$(177)$$

Now, (172), the triangle inequality, and (174)–(177) show for all  $t \in [0, T]$ ,  $\theta \in \Theta$  that

$$\begin{split} \left\| \left| \mathcal{D}(\Phi_{n+1,m,t}^{d,\theta,K,\varepsilon}) \right\| \\ &= \sum_{i=1}^{m^{n+1}} \left\| \left| \mathfrak{n}_{(n+1)\left(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}}))-2+L\right)+1} \odot \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right| \right\| \\ &+ \sum_{i=1}^{m} \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathcal{D}(\Phi_{n,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right| \right\| \\ &+ \sum_{\ell=0}^{n-1} \sum_{i=1}^{m^{n+1-\ell}} \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell)\left(\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L\right)+1} \odot \mathcal{D}(\Phi_{\ell,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right| \right\| \\ &+ \sum_{\ell=1}^{n} \sum_{i=1}^{m^{n+1-\ell}} \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \odot \mathfrak{n}_{(n-\ell+1)\left(\mathcal{D}(\Phi_{f_{\varepsilon}})-2+L\right)+1} \odot \mathcal{D}(\Phi_{\ell-1,m,0}^{d,0,K,\varepsilon}) \odot \mathcal{D}(\mathcal{X}_{0,T}^{d,0,K,\varepsilon}) \right\| \right\|. \end{split}$$

and

$$\begin{aligned} \left\| \left| \mathcal{D}(\Phi_{n+1,m,t}^{d,\theta,K,\varepsilon}) \right\| \right\| \\ &\leq \left[ \sum_{i=1}^{m^{n+1}} c_{d,\varepsilon} \right] + \left[ \sum_{i=1}^{m} c_{d,\varepsilon} (3m)^{n} \right] + \left[ \sum_{\ell=0}^{n-1} \sum_{i=1}^{m^{n+1-\ell}} c_{d,\varepsilon} (3m)^{\ell} \right] + \left[ \sum_{\ell=1}^{n} \sum_{i=1}^{m^{n+1-\ell}} c_{d,\varepsilon} (3m)^{\ell-1} \right] \\ &= m^{n+1} c_{d,\varepsilon} + m c_{d,\varepsilon} (3m)^{n} + \left[ \sum_{\ell=0}^{n-1} m^{n+1-\ell} c_{d,\varepsilon} (3m)^{\ell} \right] + \left[ \sum_{\ell=1}^{n} m^{n+1-\ell} c_{d,\varepsilon} (3m)^{\ell-1} \right] \\ &= m^{n+1} c_{d,\varepsilon} \left[ 1 + 3^{n} + \sum_{\ell=0}^{n-1} 3^{\ell} + \sum_{\ell=1}^{n} 3^{\ell-1} \right] = m^{n+1} c_{d,\varepsilon} \left[ 1 + \sum_{\ell=0}^{n} 3^{\ell} + \sum_{\ell=1}^{n} 3^{\ell-1} \right] \\ &\leq cm^{n+1} \left[ 1 + 2 \sum_{\ell=0}^{n} 3^{\ell} \right] = cm^{n+1} \left[ 1 + 2 \frac{3^{n+1} - 1}{3 - 1} \right] = c_{d,\varepsilon} (3m)^{n+1}. \end{aligned}$$

This, (173), (171), the definition of L (see (157)), and (170) completes the induction step. The proof of Lemma 4.12 is thus completed.

## 5. DNN APPROXIMATIONS OF PIDES

In Theorem 5.1 below we combine the result of Lemmas 3.3, 3.2, and 2.1 to prove the existence of a DNN that approximates the solution to (190) and whose number of parameters depend only polynomially on the dimension d and the reciprocal of the prescribed accuracy  $\epsilon$ .

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES

**Theorem 5.1.** Consider the notations given in Subsection 1.4, assume Setting 1.1, let  $T \in (0, \infty)$ ,  $b, c \in [2, \infty)$  satisfy that

$$16\left(1 + \left|c^{\frac{1}{2}}\left(4c^{\frac{1}{2}} + 2c^{\frac{1}{2}}T^{-\frac{3}{2}}\right)\right|^{\frac{1}{2}}\right) \le \frac{b}{4},\tag{180}$$

for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $v \in \mathbb{R}^d$  let  $\beta_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R}^{d \times d})$ ,  $\Phi_{\beta_{\varepsilon}^d}, \Phi_{\sigma_{\varepsilon}^d, v} \in \mathbb{N}$  satisfy that  $\beta_{\varepsilon}^d = \mathcal{R}(\Phi_{\beta_{\varepsilon}^d})$ ,  $\sigma_{\varepsilon}^d(\cdot)v = \mathcal{R}(\Phi_{\sigma_{\varepsilon}^d, v})$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $\gamma_{\varepsilon}^d \colon \mathbb{R}^{2d} \to \mathbb{R}^d$ ,  $F_{\varepsilon}^d \colon \mathbb{R}^{d \times d}$ ,  $G^d \colon \mathbb{R}^d \to \mathbb{R}^d$  be measurable and satisfy for all  $y, z \in \mathbb{R}^d$  that  $\gamma_{\varepsilon}^d(y, z) = F_{\varepsilon}^d(y)G^d(z)$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $v \in \mathbb{R}^d$  let  $\Phi_{F_{\varepsilon}^d, v} \in \mathbb{N}$  satisfy  $F_{\varepsilon}^d(\cdot)v = \mathcal{R}(\Phi_{F_{\varepsilon}^d, v})$ , assume for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1]$ ,  $v \in \mathbb{R}^d$  that  $\mathcal{D}(\Phi_{\sigma_{\varepsilon}^d, v}) = \mathcal{D}(\Phi_{\sigma_{\varepsilon}^d, 0})$  and  $\mathcal{D}(\Phi_{F_{\varepsilon}^d, v}) = \mathcal{D}(\Phi_{F_{\varepsilon}^d, 0})$ , for every  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  let  $g_{\varepsilon}^d \in C(\mathbb{R}^d, \mathbb{R})$ ,  $\Phi_{g_{\varepsilon}^d} \in \mathbb{N}$ satisfy that  $\mathcal{R}(\Phi_{g_{\varepsilon}^d}) = g_{\varepsilon}^d$ , for every  $d \in \mathbb{N}$  let  $\beta^d \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma^d \in C(\mathbb{R}^d, \mathbb{R}^{d \times d})$ ,  $g^d \in C(\mathbb{R}^d, \mathbb{R})$ , let  $f \in C(\mathbb{R}, \mathbb{R})$ , for every  $d \in \mathbb{N}$  let  $\nu^d \colon \mathcal{B}(\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$  be a Lévy measure, assume that for all  $d \in \mathbb{N}$ there exists  $C_d \in (0, \infty)$  such that for all  $x, y, z \in \mathbb{R}^d$ ,  $t \in [0, T]$  we have that

$$\left\|\gamma^{d}(x,z)\right\| \le C_{d}\left(1 \wedge \|z\|^{2}\right), \quad \left\|\gamma^{d}(x,z) - \gamma^{d}(y,z)\right\|^{2} \le C_{d}\|x-y\|^{2}\left(1 \wedge \|z\|^{2}\right), \tag{181}$$

assume that for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x, z \in \mathbb{R}^d$  the Jacobian matrix  $(D_x \gamma^d)(x, z)$  exists, assume that for all  $d \in \mathbb{N}$  there exists  $\lambda_d \in (0, \infty)$  such that for all  $t \in [0, T]$ ,  $x, z \in \mathbb{R}^d$ ,  $\delta \in [0, 1]$  we have that

$$\lambda_d \le \left| \det(I_d + \delta(D_x \gamma^d)(x, z)) \right|, \tag{182}$$

where  $I_d$  denotes the  $d \times d$  identity matrix, assume for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}$ ,  $\varepsilon \in (0, 1)$  that

$$\begin{aligned} \left\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(y)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma_{\varepsilon}^{d}(y)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(y,z)\right)\right\|^{2} \nu^{d}(dz) \\ \leq c\|x - y\|^{2}, \end{aligned}$$
(183)

$$|f(w_1) - f(w_2)|^2 \le c|w_1 - w_2|^2, \quad \left|g_{\varepsilon}^d(x) - g_{\varepsilon}^d(y)\right|^2 \le cd^c T^{-1}||x - y||^2, \tag{184}$$

$$\left\|\beta_{\varepsilon}^{d}(0)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(0,z)\right\|^{2} \nu^{d}(dz) + T^{3}(|f(0)|+1)^{2} + T|g_{\varepsilon}^{d}(0)|^{2} \le cd^{c},$$
(185)

$$\begin{aligned} \left\| \beta_{\varepsilon}^{d}(x) - \beta^{d}(x) \right\|^{2} + \left\| \sigma_{\varepsilon}^{d}(x) - \sigma^{d}(x) \right\|_{\mathrm{F}}^{2} \\ + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\| \gamma_{\varepsilon}^{d}(x, z) - \gamma^{d}(x, z) \right\|^{2} \nu^{d}(dz) + \left| g_{\varepsilon}^{d}(x) - g^{d}(x) \right|^{2} \\ \leq \varepsilon c d^{c} (d^{c} + \|x\|^{2}), \end{aligned}$$
(186)

$$\left\| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| + \left\| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| + \left\| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| + \left\| \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right\| \le \frac{bd^{c}\varepsilon^{-c}}{4}, \tag{187}$$

$$\dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})) + \dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0})) + \dim(\mathcal{D}(\Phi_{F_{\varepsilon}^{d},0})) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^{d}})) \le \frac{bd^{c}\varepsilon^{-c}}{4},$$
(188)

and for every  $d \in \mathbb{N}$  let  $\Gamma^d \colon \mathcal{B}(\mathbb{R}^d) \to [0,1]$  be a probability measure satisfying

$$\left(\int_{\mathbb{R}^d} \|x\|^4 \,\Gamma^d(dx)\right)^{\frac{1}{2}} \le cd^c. \tag{189}$$

Then

(i) for every  $d \in \mathbb{N}$  there exists a unique viscosity solution  $u^d : [0,T] \times \mathbb{R}^d \to \mathbb{R}$  to the PIDE

$$\begin{pmatrix}
\left(\frac{\partial}{\partial t}u^{d}\right)(t,x) + \left\langle\beta^{d}(x), (\nabla_{x}u^{d})(t,x)\right\rangle \\
+ \frac{1}{2}\operatorname{trace}\left(\sigma^{d}(t,x)(\sigma^{d}(t,x))^{\top}\operatorname{Hess}_{x}u^{d}(t,x)\right) + f(u^{d}(t,x)) \\
+ \int_{\mathbb{R}^{d}}\left(u^{d}(x+\gamma^{d}(x,z)) - u^{d}(t,x) - \left\langle(\nabla_{x}u^{d})(t,x),\gamma^{d}(x,z)\right\rangle\right)\nu^{d}(dz) = 0, \quad (190) \\
\forall t \in [0,T), x \in \mathbb{R}^{d}, \\
u^{d}(T,x) = g^{d}(x), \forall x \in \mathbb{R}^{d},
\end{cases}$$

satisfying that  $\sup_{s \in [0,T], y \in \mathbb{R}^d} \frac{|u^d(s,y)|}{1+||y||} < \infty$  and (ii) there exists  $(C_{\delta})_{\delta \in (0,1)} \subseteq (0,\infty)$ ,  $\eta \in (0,\infty)$ ,  $(\Psi_{d,\epsilon})_{d \in \mathbb{N}, \epsilon \in (0,1)}$  such that for all  $d \in \mathbb{N}$ ,  $\epsilon \in (0,1)$  we have that  $\mathcal{R}(\Psi_{d,\epsilon}) \in C(\mathbb{R}^d, \mathbb{R}), \ \mathcal{P}(\Psi_{d,\epsilon}) \leq C_{\delta} \eta d^{3c+12c^2+2c(6+\delta)} \epsilon^{-6c-6-\delta}, \ and$ 

$$\left(\int_{\mathbb{R}^d} \left| (\mathcal{R}(\Psi_{d,\epsilon}))(x) - u^d(t,x) \right|^2 \Gamma^d(dx) \right)^{\frac{1}{2}} \le \epsilon.$$
(191)

*Proof of Theorem 5.1.* First, (183)–(186) (with  $\varepsilon \to 0$ ) show for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  that

$$\left\|\beta^{d}(x) - \beta^{d}(y)\right\|^{2} + \left\|\sigma^{d}(x) - \sigma^{d}(y)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma^{d}(x,z) - \gamma^{d}(y,z)\right\|^{2} \nu^{d}(dz) \le c\|x - y\|^{2}, \quad (192)$$

$$|f(w_1) - f(w_2)|^2 \le c|w_1 - w_2|^2, \quad \left|g^d(x) - g^d(y)\right|^2 \le cd^c T^{-1} ||x - y||^2, \tag{193}$$

$$\left\|\beta^{d}(0)\right\|^{2} + \left\|\sigma^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma^{d}(0,z)\right\|^{2} \nu^{d}(dz) + T^{3}|f(0)|^{2} + T|g^{d}(0)|^{2} \le cd^{c},$$
(194)

Next, [24, Corollary 3.13] (applied for all  $\varepsilon \in (0,1)$  with  $L \leftarrow c^{\frac{1}{2}}$ ,  $q \leftarrow 2$ ,  $\epsilon \leftarrow (\varepsilon/2)^{\frac{1}{2}}$  in the notation of [24, Corollary 3.13]), (185), and (180) show that there exist  $f_{\varepsilon} \in C(\mathbb{R}, \mathbb{R}), \Phi_{f_{\varepsilon}} \in \mathbb{N}$  such that for all  $\varepsilon \in (0, 1)$ ,  $w_1, w_2 \in \mathbb{R}$  that

$$\mathcal{R}(\Phi_{f_{\varepsilon}}) = f_{\varepsilon}, \quad |f_{\varepsilon}(w_1) - f_{\varepsilon}(w_1)|^2 \le c|w_1 - w_2|^2, \quad \dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) = 3 \le \frac{bd^c \varepsilon^{-c}}{4}, \tag{195}$$

$$\||\mathcal{D}(\Phi_{f_{\varepsilon}})||| \leq 16 \left(1 + \left|c^{\frac{1}{2}}(4c^{\frac{1}{2}} + 2|f(0)|)\right|^{\frac{1}{2}}\right)\varepsilon^{-2} \leq 16 \left(1 + \left|c^{\frac{1}{2}}(4c^{\frac{1}{2}} + 2c^{\frac{1}{2}}T^{-\frac{3}{2}})\right|^{\frac{1}{2}}\right)\varepsilon^{-2} \leq \frac{b\varepsilon^{-2}}{4},$$
(196)

and

$$|f(w_1) - f_{\varepsilon}(w_1)|^2 \le \frac{\varepsilon^2}{2} (1 + |w_1|^2)^2 \le \varepsilon (1 + |w_1|^4).$$
(197)

This, (187), and (188) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  that

$$2d + \left\| \left| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| + \left\| \left| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right\| \right\| + \left\| \left| \mathcal{D}(\Phi_{f_{\varepsilon}}) \right\| \right\| \le bd^{c}\varepsilon^{-c}$$

$$\tag{198}$$

and

$$\dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})) + \dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0})) + \dim(\mathcal{D}(\Phi_{F_{\varepsilon}^{d},0})) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^{d}})) + \dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) \le bd^{c}\varepsilon^{-c}.$$
 (199)

Furthermore, (197), (185), and the triangle inequality show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$  that  $|f_{\varepsilon}(0)| \leq \sqrt{\varepsilon} + 1$  $|f(0)| \le 1 + |f(0)|$  and hence

$$\left\|\beta_{\varepsilon}^{d}(0)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(0,z)\right\|^{2} \nu^{d}(dz) + T^{3}|f_{\varepsilon}(0)| + T|g_{\varepsilon}^{d}(0)|^{2} \le cd^{c}.$$
 (200)

Next, for every  $K \in \mathbb{N}$  let  $\lfloor \cdot \rfloor_K : \mathbb{R} \to \mathbb{R}$  satisfy for all  $t \in \mathbb{R}$  that  $\lfloor t \rfloor_K = \max(\{0, \frac{T}{K}, \frac{2T}{K}, \dots, T\})$  $((-\infty,t) \cup \{0\}))$ , let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$  be a probability space satisfying the usual conditions, let  $\Theta =$  $\bigcup_{n\in\mathbb{N}}\mathbb{Z}^n$ , for every  $d\in\mathbb{N}$  let  $W^{d,\theta}\colon\Omega\times[0,T]\to\mathbb{R}^d$ ,  $\theta\in\Theta$ , be identically independently distributed standard  $(\mathbb{F}_t)_{t \in [0,T]}$ -Brownian motions, for every  $d \in \mathbb{N}$  let  $N^{d,\theta}$ ,  $\theta \in \Theta$ , be independent  $(\mathbb{F}_t)_{t \in [0,T]}$ -Poisson random measures on  $[0,\infty) \times (\mathbb{R}^d \setminus \{0\})$  with intensity  $\nu^d$ , for every  $d \in \mathbb{N}$ ,  $\theta \in \Theta$  let  $\tilde{N}^{d,\theta}(dt, dz) =$  $N^{d,\theta}(dt, dz) - dt \nu^d(dz)$ , assume for all  $d \in \mathbb{N}$  that  $\mathcal{F}_0$ ,  $(N^{d,\theta})_{\theta \in \Theta}$  and  $(W^{d,\theta})_{\theta \in \Theta}$ , are independent, for every  $d, K \in \mathbb{N}, \theta \in \Theta, x \in \mathbb{R}^d, \varepsilon \in (0, 1), t \in [0, T)$  let  $(X_s^{d, \theta, K, \varepsilon, t, x})_{s \in [t, T]}$  satisfy that

$$X_{s}^{d,\theta,K,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}) dr + \int_{t}^{s} \sigma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}) dW_{r}^{d,\theta} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d} (X_{\max\{t,\lfloor r-\rfloor_{K}\}}^{d,\theta,K,\varepsilon,t,x}, z) \tilde{N}^{d,\theta}(dr,dz),$$

$$(201)$$

let  $\mathfrak{t}^{\theta} \colon \Omega \to [0,1], \theta \in \Theta$ , be i.i.d random variables which satisfy for all  $t \in (0,1)$  that  $\mathbb{P}(\mathfrak{t}^0 \leq t) = t$ , for every  $\theta \in \Theta, t \in [0, T]$  let  $\mathfrak{T}_t^{\theta} \colon \Omega \to \mathbb{R}$  satisfy for all  $\theta \in \Theta$  that  $\mathfrak{T}_t^{\theta} = t + (T - t)\mathfrak{t}^{\theta}$ , assume for all  $d \in \mathbb{N}$  that  $(\mathfrak{t}^{\theta})_{\theta \in \Theta}$ ,

$$U_{n,m}^{d,\theta,K,\varepsilon}(t,x) = \frac{\mathbb{1}_{\mathbb{N}}(n)}{m^{n}} \sum_{i=1}^{m^{n}} g_{\varepsilon}^{d} \left( X_{T}^{d,(\theta,0,-i),K,\varepsilon,t,x} \right) + \sum_{\ell=0}^{n-1} \frac{(T-t)}{m^{n-\ell}} \sum_{i=1}^{m^{n-\ell}} \left( f_{\varepsilon} \circ U_{\ell,m}^{d,(\theta,\ell,i),K,\varepsilon} - \mathbb{1}_{\mathbb{N}}(\ell) f_{\varepsilon} \circ U_{\ell-1,m}^{d,(\theta,-\ell,i),K,\varepsilon} \right) \left( \mathfrak{T}_{t}^{(\theta,\ell,i)}, X_{\mathfrak{T}_{t}^{(\theta,\ell,i)}}^{d,(\theta,\ell,i),K,\varepsilon,t,x} \right).$$

$$(202)$$

A standard result on existence and uniqueness of SDEs with jumps (cf., e.g., [29, Theorem 9.1]), (183), and (185) show that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0,1)$ ,  $t \in [0,T]$ ,  $x \in \mathbb{R}^d$  there exist adapted càdlàg processes  $(X_s^{d,\varepsilon,t,x})_{s\in[t,T]}, (X_s^{d,t,x})_{s\in[t,T]}$  such that for all  $s\in[t,T]$  we have  $\mathbb{P}$ -a.s. that

$$X_{s}^{d,t,x} = x + \int_{t}^{s} \beta^{d}(X_{r-}^{d,t,x}) dr + \int_{t}^{s} \sigma^{d}(X_{r-}^{d,t,x}) dW_{r}^{d,0} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma^{d}(X_{r-}^{d,t,x},z) \tilde{N}^{d,0}(dr,dz)$$
(203)

and

$$X_{s}^{d,\varepsilon,t,x} = x + \int_{t}^{s} \beta_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) dr + \int_{t}^{s} \sigma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x}) dW_{r}^{d,0} + \int_{t}^{s} \int_{\mathbb{R}^{d} \setminus \{0\}} \gamma_{\varepsilon}^{d}(X_{r-}^{d,\varepsilon,t,x},z) \tilde{N}^{d,0}(dr,dz).$$

$$(204)$$

Observe that for all  $d, K \in \mathbb{N}, \varepsilon \in (0, 1), t \in [0, T], x \in \mathbb{R}^d$  we have that

$$\max\left\{\mathbb{E}\left[d^{c} + \left\|X_{s}^{d,K,\varepsilon,t,x}\right\|^{2}\right], \mathbb{E}\left[d^{c} + \left\|X_{s}^{d,\varepsilon,t,x}\right\|^{2}\right], \mathbb{E}\left[d^{c} + \left\|X_{s}^{d,t,x}\right\|^{2}\right]\right\} \le (d^{c} + \|x\|^{2})e^{7c(s-t)}$$
(205)

(cf. Lemmas 2.1 and 3.2). Next, (185) and (184) show for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$ ,  $x \in \mathbb{R}^d$  that

$$\left|g_{\varepsilon}^{d}(x)\right| \leq \left|g_{\varepsilon}^{d}(0)\right| + (cd^{c}T^{-1})^{\frac{1}{2}} \|x\| \leq (cd^{c}T^{-1})^{\frac{1}{2}} + (cd^{c}T^{-1})^{\frac{1}{2}} \|x\| \leq 2(cd^{c}T^{-1})^{\frac{1}{2}} (d^{c} + \|x\|^{2})^{\frac{1}{2}}.$$
 (206)

This and (186) (with  $\varepsilon \to 0$ ) show for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that

$$\left|g^{d}(x)\right| \le 2(cd^{c}T^{-1})^{\frac{1}{2}}(d^{c} + ||x||^{2})^{\frac{1}{2}}.$$
(207)

This, (206), (205), (184), (195), and [23, Proposition 2.2] (applied with  $V \leftarrow ([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto$  $e^{4.5c(T-t)}(d^c + ||x||^2)^{\frac{1}{2}} \in (0,\infty))$  in the notation of [23, Proposition 2.2]) show that for all  $\varepsilon \in (0,1)$ ,  $d, K \in \mathbb{N}$  there exist measurable functions  $u^{d,K,\varepsilon}, u^{d,\varepsilon}, u^d \colon [0,T] \times \mathbb{R}^d$  such that for all  $t \in [0,T]$ ,  $x \in \mathbb{R}^{d} \text{ we have that } \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] + \int_{t}^{T} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,K,\varepsilon}(r,X_{r}^{d,K,\varepsilon,t,x}))\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] + \int_{t}^{T} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(r,X_{r}^{d,\varepsilon,t,x}))\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] + \int_{t}^{T} \mathbb{E}\left[\left|f_{\varepsilon}(u^{d,\varepsilon}(r,X_{r}^{d,\varepsilon,t,x}))\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] + \int_{t}^{T} \mathbb{E}\left[\left|f(u^{d}(r,X_{r}^{d,t,x}))\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] + \int_{t}^{T} \mathbb{E}\left[\left|f(u^{d}(r,X_{r}^{d,t,x}))\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x})\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{d,\varepsilon,t,x}\right|\right] dr + \mathbb{E}\left[\left|g_{\varepsilon}^{d}(X_{T}^{$  $\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}\frac{|u^{d,K,\varepsilon}(s,y)|+|u^{d,\varepsilon}(s,y)|+|u^d(s,y)|}{1+\|y\|}<\infty,$ (208)

$$u^{d,K,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,K,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,K,\varepsilon}(r,X^{d,K,\varepsilon,t,x}_r))\right]dr,$$
(209)

$$u^{d,\varepsilon}(t,x) = \mathbb{E}\left[g^d_{\varepsilon}(X^{d,\varepsilon,t,x}_T)\right] + \int_t^T \mathbb{E}\left[f_{\varepsilon}(u^{d,\varepsilon}(r,X^{d,\varepsilon,t,x}_r))\right]dr,$$
(210)

and

$$u^{d}(t,x) = \mathbb{E}\left[g^{d}(X_{T}^{d,t,x})\right] + \int_{t}^{T} \mathbb{E}\left[f(u^{d}(r,X_{r}^{d,t,x}))\right] dr.$$
(211)

This, a result on existence and uniqueness of viscosity solutions to PIDEs (see [31, Propositions 5.4 and 5.16]), and the assumptions of Theorem 5.1 show (i).

Next, let  $c_1, c_2 \in \mathbb{R}$ ,  $(\varepsilon_{d,\epsilon})_{d \in \mathbb{N}, \epsilon \in (0,1)} \subseteq \mathbb{R}$ ,  $(N_{d,\epsilon})_{d \in \mathbb{N}, \epsilon \in (0,1)} \subseteq \mathbb{N}$ ,  $(C_{\delta})_{\delta \in (0,1)} \subseteq [0,\infty]$  satisfy for all  $d \in \mathbb{N}, \delta, \epsilon \in (0, 1)$  that

$$c_1 = 6(cT^{-1})^{\frac{1}{2}} + 12c^{\frac{3}{2}}(T+2)e^{21cT+5cT^2}T^{\frac{1}{2}} + 2ce^{24cT+5cT^2}, \quad c_2 = 2cc_1,$$
(212)

DNNS OVERCOME CURSE OF DIMENSIONALITY WHEN APPROXIMATING SEMILINEAR PIDES

$$c_2 d^{2c} |\varepsilon_{d,\epsilon}|^{\frac{1}{2}} = \frac{\epsilon}{2}, \quad N_{d,\epsilon} = \min\left\{ n \in \mathbb{N} \cap [2,\infty) \colon c_2 d^{2c} \left( \frac{e^{12cTn + \frac{n}{2}}}{n^{\frac{n}{2}}} + \frac{1}{n^{\frac{n}{2}}} \right) \le \frac{\epsilon}{2} \right\},$$
(213)

and

$$C_{\delta} = \sup_{n \in [2,\infty)} \left[ \left( \frac{e^{(12cT+0.5)(n-1)}}{(n-1)^{\frac{n-1}{2}}} \right)^{6+\delta} (3n)^{3n+1} \right].$$
 (214)

Then the triangle inequality, Lemmas 3.3, 3.2, and 2.1 show for all  $d, K \in \mathbb{N}, \theta \in \Theta, \varepsilon \in (0, 1), n, m \in \mathbb{N}$  that

$$\begin{split} &\left(\mathbb{E}\left[\left|U_{n,m}^{d,\theta,K,\varepsilon}(t,x)-u^{d}(t,x)\right|^{2}\right]\right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}\left[\left|U_{n,m}^{d,\theta,K,\varepsilon}(t,x)-u^{d,K,\varepsilon}(t,x)\right|^{2}\right]\right)^{\frac{1}{2}}+\left|u^{d,K,\varepsilon}(t,x)-u^{d,\varepsilon}(t,x)\right|+\left|u^{d,\varepsilon}(t,x)-u^{d}(t,x)\right| \\ &\leq 6e^{\frac{m}{2}}m^{-\frac{n}{2}}e^{12cTn}(cd^{c}T^{-1})^{\frac{1}{2}}\left(d^{c}+\|x\|^{2}\right)^{\frac{1}{2}} \\ &+12c^{\frac{3}{2}}d^{\frac{c}{2}}(T+2)e^{21cT+5cT^{2}}\left(d^{c}+\|x\|^{2}\right)^{\frac{1}{2}}\frac{T^{\frac{1}{2}}}{K^{\frac{1}{2}}}+2cd^{c}\varepsilon^{\frac{1}{2}}\left(d^{c}+\|x\|^{2}\right)e^{24cT+5cT^{2}} \\ &\leq \left[6(cd^{c}T^{-1})^{\frac{1}{2}}+12c^{\frac{3}{2}}d^{\frac{c}{2}}(T+2)e^{21cT+5cT^{2}}T^{\frac{1}{2}}+2cd^{c}e^{24cT+5cT^{2}}\right]\left(d^{c}+\|x\|^{2}\right)\left(\varepsilon^{\frac{1}{2}}+\frac{e^{12cTn+\frac{m}{2}}}{m^{\frac{n}{2}}}+\frac{1}{K^{\frac{1}{2}}}\right) \\ &\leq \left[6(cT^{-1})^{\frac{1}{2}}+12c^{\frac{3}{2}}(T+2)e^{21cT+5cT^{2}}T^{\frac{1}{2}}+2ce^{24cT+5cT^{2}}\right]d^{c}\left(d^{c}+\|x\|^{2}\right)\left(\varepsilon^{\frac{1}{2}}+\frac{e^{12cTn+\frac{m}{2}}}{m^{\frac{n}{2}}}+\frac{1}{K^{\frac{1}{2}}}\right) \\ &\leq c_{1}d^{c}\left(\varepsilon^{\frac{1}{2}}+\frac{e^{12cTn+\frac{m}{2}}}{m^{\frac{n}{2}}}+\frac{1}{K^{\frac{1}{2}}}\right)\left(d^{c}+\|x\|^{2}\right). \end{split}$$

This, the triangle inequality, (189), and (212) show for all  $d, K \in \mathbb{N}, \theta \in \Theta, \varepsilon \in (0, 1), n, m \in \mathbb{N}$  that

$$\left(\int_{\mathbb{R}^{d}} \mathbb{E}\left[\left|U_{n,m}^{d,\theta,K,\varepsilon}(t,x) - u^{d}(t,x)\right|^{2}\right] \Gamma^{d}(dx)\right)^{\frac{1}{2}} \leq c_{1}d^{c} \left(\varepsilon^{\frac{1}{2}} + \frac{e^{12cTn + \frac{m}{2}}}{m^{\frac{n}{2}}} + \frac{1}{K^{\frac{1}{2}}}\right) \left(\int_{\mathbb{R}^{d}} (d^{c} + ||x||^{2})^{2} \Gamma^{d}(dx)\right)^{\frac{1}{2}} \leq c_{1}d^{c} \left(\varepsilon^{\frac{1}{2}} + \frac{e^{12cTn + \frac{m}{2}}}{m^{\frac{n}{2}}} + \frac{1}{K^{\frac{1}{2}}}\right) \left(d^{c} + \left(\int_{\mathbb{R}^{d}} ||x||^{4} \Gamma^{d}(dx)\right)^{\frac{1}{2}}\right) \leq c_{1}d^{c} \left(\varepsilon^{\frac{1}{2}} + \frac{e^{12cTn + \frac{m}{2}}}{m^{\frac{n}{2}}} + \frac{1}{K^{\frac{1}{2}}}\right) 2cd^{c} = c_{2}d^{2c} \left(\varepsilon^{\frac{1}{2}} + \frac{e^{12cTn + \frac{m}{2}}}{m^{\frac{n}{2}}} + \frac{1}{K^{\frac{1}{2}}}\right).$$
(216)

This, Fubini's theorem, and (213) show for all  $d \in \mathbb{N}, \epsilon \in (0, 1)$  that

$$\left( \mathbb{E} \left[ \int_{\mathbb{R}^d} \left| U_{n,n}^{d,\theta,n^n,\varepsilon}(t,x) - u^d(t,x) \right|^2 \Gamma^d(dx) \right] \right)^{\frac{1}{2}} \Big|_{\substack{n=N_{d,\epsilon}\\\varepsilon=\varepsilon_{d,\epsilon}}} \\
= \left( \int_{\mathbb{R}^d} \mathbb{E} \left[ \left| U_{n,n}^{d,\theta,n^n,\varepsilon}(t,x) - u^d(t,x) \right|^2 \right] \Gamma^d(dx) \right)^{\frac{1}{2}} \Big|_{\substack{n=N_{d,\epsilon}\\\varepsilon=\varepsilon_{d,\epsilon}}} \\
\leq c_2 d^{2c} \left( \varepsilon^{\frac{1}{2}} + \frac{e^{12cTn+\frac{n}{2}}}{n^{\frac{n}{2}}} + \frac{1}{n^{\frac{n}{2}}} \right) \Big|_{\substack{n=N_{d,\epsilon}\\\varepsilon=\varepsilon_{d,\epsilon}}} \\
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
(217)

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Then for all  $d\in\mathbb{N},\epsilon\in(0,1)$  there exists  $\omega_{d,\epsilon}\in\Omega$  such that

$$\int_{\mathbb{R}^d} \left| U_{n,n}^{d,\theta,n^n,\varepsilon}(t,x,\omega_{d,\epsilon}) - u^d(t,x) \right|^2 \Gamma^d(dx) \Big|_{\substack{n=N_{d,\epsilon}\\\varepsilon=\varepsilon_{d,\epsilon}}} \le \epsilon^2.$$
(218)

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Next, Lemma 4.12, (199), and (198) show for all  $d \in \mathbb{N}$ ,  $\epsilon \in (0, 1)$  that there exists  $\Psi_{d,\epsilon} \in \mathbb{N}$  such that (A) we have for all  $t \in [0, T]$ ,  $\theta \in \Theta$  that

$$\dim(\mathcal{D}(\Psi_{d,\epsilon})) = (n+1) \left[ n^n \left( \max\left\{ \dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^d})), \dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^d})) \right\} - 1 \right) + 1 \right] \\ + n(\dim(\mathcal{D}(\Phi_{f_{\varepsilon}})) - 2) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^d})) - 1 \Big|_{n = N_{d,\epsilon}, \varepsilon = \varepsilon_{d,\epsilon}}$$

$$\leq 2nn^n bd^c \varepsilon^{-c} + nbd^c \varepsilon^{-c} + bd^c \varepsilon^{-c} \Big|_{n = N_{d,\epsilon}} \leq 4n^n nbd^c \varepsilon^{-c} \Big|_{n = N_{d,\epsilon}, \varepsilon = \varepsilon_{d,\epsilon}},$$
(219)

(B) we have for all  $t \in [0, T], \theta \in \Theta$  that

$$\begin{aligned} \|\mathcal{D}(\Psi_{d,\epsilon})\| &\leq \left(2d + \left\|\left|\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})\right|\right\| + \left\|\left|\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0})\right|\right\| \\ &+ \left\|\left|\mathcal{D}(\Phi_{F_{\varepsilon}^{d},0})\right|\right\| + \left\|\left|\mathcal{D}(\Phi_{g_{\varepsilon}^{d}})\right|\right\| + \left\|\mathcal{D}(\Phi_{f_{\varepsilon}})\right\|\right)(3n)^{n}\right|_{n=N_{d,\epsilon},\varepsilon=\varepsilon_{d,\epsilon}} \end{aligned}$$

$$\leq bd^{c}\varepsilon^{-c}(3n)^{n}|_{n=N_{d,\epsilon},\varepsilon=\varepsilon_{d,\epsilon}},$$

$$(220)$$

and

(C) we have for all  $t \in [0,T]$ ,  $\theta \in \Theta$ ,  $x \in \mathbb{R}^d$  that  $U_{n,n}^{d,\theta,n^n,\varepsilon}(t,x,\omega_{d,\epsilon})\Big|_{\substack{n=N_{d,\epsilon}\\\varepsilon=\varepsilon_{d,\epsilon}}} = (\mathcal{R}(\Psi_{d,\epsilon}))(x).$ 

This and (218) show for all  $d \in \mathbb{N}$ ,  $\epsilon \in (0, 1)$  that

$$\int_{\mathbb{R}^d} \left| (\mathcal{R}(\Psi_{d,\epsilon}))(x) - u^d(t,x) \right|^2 \Gamma^d(dx) \le \epsilon^2.$$
(221)

Furthermore, the fact that  $\forall \Phi \in \mathbf{N} \colon \mathcal{P}(\Phi) \leq 2 \dim(\mathcal{D}(\Phi)) ||| \mathcal{D}(\Phi) |||^2$ , (219), and (220) show for all  $d \in \mathbb{N}$ ,  $\epsilon \in (0, 1)$  that

$$\mathcal{P}(\Psi_{d,\epsilon}) \leq 2 \dim(\mathcal{D}(\Psi_{d,\epsilon})) \| |\mathcal{D}(\Psi_{d,\epsilon}) \|^2 \leq 2 \cdot 4n^n nbd^c \varepsilon^{-c} \left( bd^c \varepsilon^{-c} (3n)^n \right)^2 \Big|_{n=N_{d,\epsilon}}$$

$$= 8n^n nb^2 d^{3c} |\varepsilon_{d,\epsilon}|^{-3c} (3n)^{2n} \Big|_{n=N_{d,\epsilon},\varepsilon=\varepsilon_{d,\epsilon}}$$
(222)

Recall that in (213) we have for all  $d \in \mathbb{N}$ ,  $\epsilon \in (0, 1)$  that  $c_2 d^{2c} |\varepsilon_{d,\epsilon}|^{\frac{1}{2}} = \frac{\epsilon}{2}$ . Hence, for all  $d \in \mathbb{N}$ ,  $\epsilon \in (0, 1)$ we have that  $\varepsilon_{d,\epsilon} = \frac{\epsilon^2}{4} |c_2|^{-2} d^{-4c}$ . This, (222), and (214) show for all  $d \in \tilde{\mathbb{N}}, \epsilon, \delta \in (0, 1)$  that

$$\begin{aligned} \mathcal{P}(\Psi_{d,\epsilon}) &\leq 8n^{n}nb^{2}d^{3c}|\varepsilon_{d,\epsilon}|^{-3c}(3n)^{2n}\Big|_{n=N_{d,\epsilon}} \\ &\leq 8b^{2}d^{3c}|\varepsilon_{d,\epsilon}|^{-3c}(3n)^{3n+1}\Big|_{n=N_{d,\epsilon}} \\ &\leq 8b^{2}d^{3c}\left[\frac{\epsilon^{2}}{4}|c_{2}|^{-2}d^{-4c}\right]^{-3c}(3n)^{3n+1}\Big|_{n=N_{d,\epsilon}} \\ &\leq 4^{3c+2}b^{2}d^{3c+12c^{2}}|c_{2}|^{6c}\epsilon^{-6c}(3N_{d,\epsilon})^{3N_{d,\epsilon}+1} \\ &\leq 4^{3c+2}b^{2}d^{3c+12c^{2}}|c_{2}|^{6c}\epsilon^{-6c-6-\delta}\epsilon^{6+\delta}(3N_{d,\epsilon})^{3N_{d,\epsilon}+1} \\ &\leq 4^{3c+2}b^{2}d^{3c+12c^{2}}|c_{2}|^{6c}\epsilon^{-6c-6-\delta}\left(\frac{4c_{2}d^{2c}e^{(12cT+0.5)(N_{d,\epsilon}-1)}}{(N_{d,\epsilon}-1)^{\frac{N_{d,\epsilon}-1}{2}}}\right)^{6+\delta}(3N_{d,\epsilon})^{3N_{d,\epsilon}+1} \\ &\leq 4^{3c+8+\delta}b^{2}d^{3c+12c^{2}+2c(6+\delta)}|c_{2}|^{6c+2c(6+\delta)}\epsilon^{-6c-6-\delta}\left(\frac{e^{(12cT+0.5)(N_{d,\epsilon}-1)}}{(N_{d,\epsilon}-1)^{\frac{N_{d,\epsilon}-1}{2}}}\right)^{6+\delta}(3N_{d,\epsilon})^{3N_{d,\epsilon}+1} \\ &\leq 4^{3c+8+\delta}b^{2}d^{3c+12c^{2}+2c(6+\delta)}|c_{2}|^{6c+2c(6+\delta)}\epsilon^{-6c-6-\delta}C_{\delta}. \end{aligned}$$

This, (221), the fact that  $c_2$  does not depend on d (see (212)), and the fact that  $\forall \delta \in (0, 1)$ :  $C_{\delta} < \infty$  (cf. (171) in [8]) complete the proof of Theorem 5.1. 

*Proof of Theorem 1.2.* Let  $b, \tilde{c} \in [2, \infty)$  satisfy that

$$c \le \tilde{c}, \quad Tcd^c \le \tilde{c}d^{\tilde{c}}, \quad T^3(c^{\frac{1}{2}}d^{\frac{c}{2}}+1) \le \tilde{c}d^{\tilde{c}}, \tag{224}$$

and

$$16\left(1 + \left| (3\tilde{c})^{\frac{1}{2}} (4(3\tilde{c})^{\frac{1}{2}} + 2(3\tilde{c})^{\frac{1}{2}} T^{-\frac{3}{2}}) \right|^{\frac{1}{2}} \right) \le \frac{b}{4}.$$
(225)

Then Theorem 1.2 follows from Theorem 5.1 (applied with  $c \leftarrow 3\tilde{c}$ ,  $(\Gamma^d)_{d \in \mathbb{N}} \leftarrow (\int_{(\cdot)\cap[0,1]^d} dx)_{d \in \mathbb{N}}$  in the notation of Theorem 5.1). This can be easily checked as follows. From (183)–(188), (224), and (225) it follows that for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}$ ,  $\varepsilon \in (0, 1)$  we have that

$$\left\|\beta_{\varepsilon}^{d}(x) - \beta_{\varepsilon}^{d}(y)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma_{\varepsilon}^{d}(y)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d} \setminus \{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma_{\varepsilon}^{d}(y,z)\right)\right\|^{2} \nu^{d}(dz) \leq \tilde{c} \|x - y\|^{2}, \quad (226)$$

$$|f(w_1) - f(w_2)|^2 \le \tilde{c}|w_1 - w_2|^2, \quad \left|g_{\varepsilon}^d(x) - g_{\varepsilon}^d(y)\right|^2 \le cd^c ||x - y||^2 \le \tilde{c}d^{\tilde{c}}T^{-1}||x - y||^2, \tag{227}$$

$$\left\|\beta_{\varepsilon}^{d}(0)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(0)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(0,z)\right\|^{2} \nu^{d}(dz) \leq \tilde{c}d^{\tilde{c}},\tag{228}$$

$$T^{3}(|f(0)|+1)^{2} \le T^{3}(c^{\frac{1}{2}}d^{\frac{c}{2}}+1) \le \tilde{c}d^{\tilde{c}},$$
(229)

$$T|g^d_{\varepsilon}(0)|^2 \le Tcd^c \le \tilde{c}d^{\tilde{c}},\tag{230}$$

$$\begin{aligned} \left\|\beta_{\varepsilon}^{d}(x) - \beta^{d}(x)\right\|^{2} + \left\|\sigma_{\varepsilon}^{d}(x) - \sigma^{d}(x)\right\|_{\mathrm{F}}^{2} + \int_{\mathbb{R}^{d}\setminus\{0\}} \left\|\gamma_{\varepsilon}^{d}(x,z) - \gamma^{d}(x,z)\right\|^{2} \nu^{d}(dz) + \left|g_{\varepsilon}^{d}(x) - g^{d}(x)\right|^{2} \\ \leq \varepsilon \tilde{c} d^{\tilde{c}}(d^{\tilde{c}} + \|x\|^{2}), \end{aligned}$$

$$(231)$$

$$\begin{aligned} \left\| \mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}}) \right\| + \left\| \mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0}) \right\| + \left\| \mathcal{D}(\Phi_{F_{\varepsilon}^{d},0}) \right\| + \left\| \mathcal{D}(\Phi_{g_{\varepsilon}^{d}}) \right\| \\ &\leq \mathcal{P}(\Phi_{\beta_{\varepsilon}^{d}}) + \mathcal{P}(\Phi_{\sigma_{\varepsilon}^{d},0}) + \mathcal{P}(\Phi_{F_{\varepsilon}^{d},0}) + \mathcal{P}(\Phi_{g_{\varepsilon}^{d}}) \leq \frac{bd^{c}\varepsilon^{-c}}{4}, \end{aligned}$$

$$(232)$$

and

$$\dim(\mathcal{D}(\Phi_{\beta_{\varepsilon}^{d}})) + \dim(\mathcal{D}(\Phi_{\sigma_{\varepsilon}^{d},0})) + \dim(\mathcal{D}(\Phi_{F_{\varepsilon}^{d},0})) + \dim(\mathcal{D}(\Phi_{g_{\varepsilon}^{d}}))$$

$$\leq \mathcal{P}(\Phi_{\beta_{\varepsilon}^{d}}) + \mathcal{P}(\Phi_{\sigma_{\varepsilon}^{d},0}) + \mathcal{P}(\Phi_{F_{\varepsilon}^{d},0}) + \mathcal{P}(\Phi_{g_{\varepsilon}^{d}}) \leq \frac{bd^{c}\varepsilon^{-c}}{4}.$$
(233)

In addition, for every  $d \in \mathbb{N}$  let  $\Gamma^d \colon \mathcal{B}(\mathbb{R}^d) \to [0,1]$  be a probability measure satisfying  $\Gamma^d(\cdot) = \int_{(\cdot) \cap [0,1]^d} dx$ . Then for all  $d \in \mathbb{N}$  we have that

$$\left(\int_{\mathbb{R}^d} \|x\|^4 \,\Gamma^d(dx)\right)^{\frac{1}{2}} = \left(\int_{[0,1]^d} \|x\|^4 \,dx\right)^{\frac{1}{2}} \le d^2 \le \tilde{c}d^{\tilde{c}}.$$
(234)

The proof of Theorem 1.2 is thus completed.

#### REFERENCES

- ACKERMANN, J., JENTZEN, A., KRUSE, T., KUCKUCK, B., AND PADGETT, J. L. Deep neural networks with ReLU, leaky ReLU, and softplus activation provably overcome the curse of dimensionality for Kolmogorov partial differential equations with Lipschitz nonlinearities in the L<sup>p</sup>-sense. arXiv:2309.13722v1 (2023).
- [2] AL-ARADI, A., CORREIA, A., NAIFF, D. D. F., JARDIM, G., AND SAPORITO, Y. Extensions of the deep Galerkin method. arXiv:1912.01455 (2019).
- [3] BECK, C., GONON, L., AND JENTZEN, A. Overcoming the curse of dimensionality in the numerical approximation of highdimensional semilinear elliptic partial differential equations. arXiv preprint arXiv:2003.00596 (2020).
- [4] BECK, C., HORNUNG, F., HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. Overcoming the curse of dimensionality in the numerical approximation of Allen–Cahn partial differential equations via truncated full-history recursive multilevel Picard approximations. *Journal of Numerical Mathematics* 28, 4 (2020), 197–222.
- [5] BOUSSANGE, V., BECKER, S., JENTZEN, A., KUCKUCK, B., AND PELLISSIER, L. Deep learning approximations for non-local nonlinear PDEs with Neumann boundary conditions. arXiv:2205.03672 (2022).
- [6] CASTRO, J. Deep learning schemes for parabolic nonlocal integro-differential equations. arXiv:2103.15008 (2021).
- [7] CHERIDITO, P., AND ROSSMANNEK, F. Efficient Sobolev approximation of linear parabolic PDEs in high dimensions. arXiv:2306.16811 (2023).
- [8] CIOICA-LICHT, P. A., HUTZENTHALER, M., AND WERNER, P. T. Deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear partial differential equations. arXiv:2205.14398v1 (2022).

- [9] CONT, R., AND TANKOV, P. Financial modelling with jump processes. Chapman & Hall/CRC, Boca Raton, 2004.
- [10] CONT, R., AND VOLTCHKOVA, E. Integro-differential equations for option prices in exponential Lévy models. *Finance and Stochastics* 9, 3 (2005), 299–325.
- [11] DELONG, Ł. Backward stochastic differential equations with jumps and their actuarial and financial applications. Springer, 2013.
- [12] E, W., HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. On multilevel Picard numerical approximations for highdimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. *Journal of Scientific Computing* 79, 3 (2019), 1534–1571.
- [13] FREY, R., AND KÖCK, V. Convergence Analysis of the Deep Splitting Scheme: the Case of Partial Integro-Differential Equations and the associated FBSDEs with Jumps. *arXiv:2206.01597* (2022).
- [14] FREY, R., AND KÖCK, V. Deep neural network algorithms for parabolic PIDEs and applications in insurance mathematics. In *Methods and Applications in Fluorescence* (2022), Springer, pp. 272–277.
- [15] GILES, M. B., JENTZEN, A., AND WELTI, T. Generalised multilevel Picard approximations. arXiv preprint arXiv:1911.03188 (2019).
- [16] GNOATTO, A., PATACCA, M., AND PICARELLI, A. A deep solver for BSDEs with jumps. arXiv:2211.04349 (2022).
- [17] GONON, L., AND SCHWAB, C. Deep ReLU network expression rates for option prices in high-dimensional, exponential Lévy models. *Finance and Stochastics* 25, 4 (2021), 615–657.
- [18] GONON, L., AND SCHWAB, C. Deep ReLU neural networks overcome the curse of dimensionality for partial integrodifferential equations. *Analysis and Applications 21*, 1 (2023), 1–47.
- [19] GROHS, P., HORNUNG, F., JENTZEN, A., AND VON WURSTEMBERGER, P. A Proof that Artificial Neural Networks Overcome the Curse of Dimensionality in the Numerical Approximation of Black–Scholes Partial Differential Equations. *Memoirs of the American Mathematical Society* 284, 1410 (2023).
- [20] GYÖNGY, I., AND WU, S. Itô's formula for jump processes in L<sub>p</sub>-spaces. Stochastic Processes and their Applications 131 (2021), 523–552.
- [21] HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. *Partial Differential Equations and Applications* 2, 6 (2021), 1–31.
- [22] HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. Overcoming the curse of dimensionality in the numerical approximation of parabolic partial differential equations with gradient-dependent nonlinearities. *Foundations of Computational Mathematics* 22, 4 (2022), 905–966.
- [23] HUTZENTHALER, M., JENTZEN, A., KRUSE, T., AND NGUYEN, T. A. Multilevel Picard approximations for high-dimensional semilinear second-order PDEs with Lipschitz nonlinearities. arXiv:2009.02484 (2020).
- [24] HUTZENTHALER, M., JENTZEN, A., KRUSE, T., AND NGUYEN, T. A. A proof that rectified deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear heat equations. SN Partial Differential Equations and Applications 1, 10 (2020).
- [25] HUTZENTHALER, M., JENTZEN, A., KRUSE, T., NGUYEN, T. A., AND VON WURSTEMBERGER, P. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *Proceedings of the Royal* Society A: Mathematical, Physical and Engineering Sciences 476, 2244 (2020), 20190630.
- [26] HUTZENTHALER, M., JENTZEN, A., AND VON WURSTEMBERGER, P. Overcoming the curse of dimensionality in the approximative pricing of financial derivatives with default risks. *Electronic Journal of Probability* 25 (2020), 1–73.
- [27] HUTZENTHALER, M., AND KRUSE, T. Multilevel Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. *SIAM Journal on Numerical Analysis* 58, 2 (2020), 929–961.
- [28] HUTZENTHALER, M., AND NGUYEN, T. A. Multilevel Picard approximations of high-dimensional semilinear partial differential equations with locally monotone coefficient functions. *Applied Numerical Mathematics* 181 (2022), 151–175.
- [29] IKEDA, N., AND WATANABE, S. Stochastic Differential Equations and Diffusion Processes. North-Holland Publishing Company, 1989.
- [30] JENTZEN, A., SALIMOVA, D., AND WELTI, T. A proof that deep artificial neural networks overcome the curse of dimensionality in the numerical approximation of Kolmogorov partial differential equations with constant diffusion and nonlinear drift coefficients. *Communications in Mathematical Sciences 19*, 5 (2021), 1167–1205.
- [31] NEUFELD, A., AND WU, S. Multilevel Picard approximation algorithm for semilinear partial integro-differential equations and its complexity analysis. *arXiv:2205.09639v3* (2022).
- [32] ØKSENDAL, B., AND SULEM, A. Stochastic Control of jump diffusions. Springer, 2005.
- [33] SIRIGNANO, J., AND SPILIOPOULOS, K. DGM: a deep learning algorithm for solving partial differential equations. J. Comput. *Phys.* 375 (2018), 1339–1364.
- [34] YUAN, L., NI, Y.-Q., DENG, X.-Y., AND HAO, S. A-pinn: Auxiliary physics informed neural networks for forward and inverse problems of nonlinear integro-differential equations. *Journal of Computational Physics* (2022), 111260.

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