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KNIGHTIAN UNCERTAINTY IN MATHEMATICAL FINANCE

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To my love.

Abstract

This thesis seeks to gain further insight into different classical problems of mathematical finance under Knightian uncertainty. Random events are called *uncertain* if their exact probabilities are unknown. Knightian uncertainty is formalized by a set \mathfrak{P} of probability measures, where each probability measure $P \in \mathfrak{P}$ stands for a possible scenario of the law of the price process. The results of the thesis are divided into four chapters.

In Chapter II, we establish the duality formula for the superreplication price in a setting of volatility uncertainty. In contrast to previous results, the contingent claim is not assumed to satisfy any continuity conditions (as a functional of the stock price).

Given a càdlàg process X on a filtered measurable space, we construct in Chapter III a version of its semimartingale characteristics which is measurable with respect to the underlying probability law. More precisely, let \mathfrak{P}_{sem} be the set of all probability measures P under which X is a semimartingale. We construct processes (B^P, C, ν^P) which are jointly measurable in time, space, and the probability law P , and are versions of the semimartingale characteristics of X under P for each $P \in \mathfrak{P}_{sem}$. The second characteristic C can be constructed as a single process not depending on P . A similar result is obtained for the differential characteristics.

In Chapter IV, we develop a general construction for nonlinear Lévy processes with given characteristics. More precisely, given a set Θ of Lévy triplets, we construct a sublinear expectation on Skorohod space under which the canonical process has stationary independent increments and a nonlinear generator corresponding to the supremum of all generators of classical Lévy processes with triplets in Θ .

In Chapter V, we study a robust portfolio optimization problem under model uncertainty for an investor with logarithmic or power utility. The uncertainty is specified by a set of possible Lévy triplets; that is, possible instantaneous drift, volatility and jump characteristics of the price process. We show that an optimal investment strategy exists and compute it in semi-closed form. Moreover, we provide a saddle point analysis describing a worst-case model.

Kurzfassung

Diese Dissertation beschäftigt sich mit klassischen Problemen aus der Finanzmathematik unter Knight'scher Unsicherheit. Zufällige Ereignisse werden als *unsicher* bezeichnet, wenn ihre genauen Eintrittswahrscheinlichkeiten nicht bekannt sind. Knight'sche Unsicherheit wird formal dargestellt durch eine Menge \mathfrak{P} von Wahrscheinlichkeitsmassen, wobei jedes Element $P \in \mathfrak{P}$ ein mögliches Szenario für das Wahrscheinlichkeitsmass des Preisprozesses bedeutet. Die Resultate dieser Dissertation sind in vier Kapitel aufgeteilt.

In Kapitel II beweisen wir die Dualitätsformel für den Superreplikationspreis unter Volatilitätsunsicherheit. Im Vergleich zu früheren Ergebnissen setzen wir keine Stetigkeitsannahmen an der Eventualforderung (als Funktional des Preisprozesses) voraus.

Für einen gegebenen Prozess X mit càdlàg Pfaden auf einem filtrierten messbaren Raum konstruieren wir in Kapitel III eine Version von seinen Semimartingal-Charakteristiken, welche messbar ist bezüglich dem zugehörigen Wahrscheinlichkeitsmass. Genauer gesagt, bezeichne \mathfrak{P}_{sem} die Menge aller Wahrscheinlichkeitsmasse unter welchem X ein Semimartingal ist. Wir konstruieren Prozesse (B^P, C, ν^P) welche Produkt-messbar sind in der Zeit, im Raum und dem Wahrscheinlichkeitsmass P , und welche Versionen der Semimartingal-Charakteristiken von X unter jedem Mass $P \in \mathfrak{P}_{sem}$ sind. Die zweite Charakteristik C kann als einen einzigen Prozess unabhängig von P konstruiert werden. Wir erhalten ein ähnliches Resultat auch bezüglich den differentiellen Charakteristiken.

In Kapitel IV entwickeln wir eine allgemeine Konstruktion von nicht-linearen Lévy Prozessen mit gegebenen Charakteristiken. Genauer gesagt, gegeben sei eine Menge Θ von Lévy Tripeln. Wir konstruieren einen sublinearen Erwartungswert auf dem Skorohod-Raum unter welchem der kanonische Prozess stationäre und unabhängige Zuwächse besitzt und welcher einen nichtlinearen Generator hat, der dem Supremum aller Generatoren von klassischen Lévy Prozessen mit Tripeln in Θ entspricht.

In Kapitel V beschäftigen wir uns mit einem robusten Optimierungsproblem unter Modellunsicherheit für einen Investor mit Logarithmischer- oder Power-Nutzenfunktion. Die Unsicherheit ist spezifiziert durch eine Menge von möglichen Lévy Tripeln; sprich möglichen instantanen Drift-, Volatilität- und Sprung-Charakteristiken des Preisprozesses. Wir beweisen die Existenz

einer optimalen Handelsstrategie und berechnen diese in einer semi-expliziten Form. Des Weiteren zeigen wir eine Sattelpunkt-Analyse welche ein Worst-Case-Modell beschreibt.

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Chapter I

Introduction

In classical mathematical finance, a financial market is mostly modeled in the following way: one starts with a constant asset $B = 1$ which plays the role of the *bank account* and d *risky assets*, whose evolution in units of the bank account are described by a stochastic process $S := (S_t^1, \dots, S_t^d)$, called the *price process*. The d assets are called risky as they not only depend on time but also on randomness. The *unique* probability law of the price process, which characterizes its randomness, is given. This means that one assumes that financial agents who are trading in the market have the knowledge of the law of the price process. However, it seems to be much more realistic that probabilities of events in a market are unknown. Already in 1921, Knight argued in [33] the difference between *risk* and *uncertainty*. He distinguished between random events whose randomness are precisely "measurable," calling them *risky*, and events whose randomness are not precisely "measurable," calling them *uncertain*. Knight argued that if uncertainty would not occur in a financial market, then financial institutions like a bank or an insurance should be able to price financial derivatives or insurance policies in such a way that they are able to control their possible losses. Unfortunately, we have seen not only in the recent years with the financial crisis that this is far away from being true. Later in [23], the notion of uncertain and risky events was specified by calling random events risky if their probabilities are known, whereas random events with unknown probabilities are called uncertain. It was shown in [23] that people prefer to take risks where the probabilities are known, rather than on unknown ones, even when the known probabilities of getting a profit are low whereas events with unknown probability might lead to a gain for sure. Such an effect cannot occur on models for a financial market where the probability measure is given. To sum up, to get a better understanding of financial markets, it seems necessary to develop a theory in which uncertainty occurs. There are two ways to model uncertainty. In the *model-free approach*, one does not assign any probabilistic assumptions on the behavior of the price process. Another way to model the lack of

knowledge of probabilities of events is to consider a set of different probability measures rather than fixing a unique law for the price process. In the so-called *Knightian uncertainty approach*, each probability measure of that given set stands for a possible scenario of the law of the price process.

This thesis seeks to gain further insight into different classical problems of mathematical finance under Knightian uncertainty. The results are divided into four chapters which correspond to the articles [37, 38, 39, 40].

Superreplication under Volatility Uncertainty. In a financial market without uncertainty, the superhedging price of a contingent claim is the fair price for selling the claim without having any risk *under the given law* P . Under a no-arbitrage condition, the superhedging price Π_P equals to the supremum of the evaluation of the claim under each linear pricing functional with respect to an equivalent local martingale measure, which is known as the *duality formula*.

Under Knightian uncertainty, the law of the price process is not specified, but a set \mathfrak{P} of possible scenarios for the law is given. To avoid possible losses being caused by uncertainty, the superhedging price under Knightian uncertainty $\Pi_{\mathfrak{P}}$ has to be the fair price for selling the claim without having any risk under *any possible law* $P \in \mathfrak{P}$. If there exists a reference probability measure P_* with respect to which all scenarios $P \in \mathfrak{P}$ are absolutely continuous, the resulting problem can be reduced to the classical one. In the case of volatility uncertainty, this fails, i.e. the set of scenarios \mathfrak{P} is *nondominated*. Considering the duality formula without model uncertainty, one guesses that the superreplication price under Knightian uncertainty must be equal to the supremum of the superreplication prices under each possible law $P \in \mathfrak{P}$, i.e.

$$\Pi_{\mathfrak{P}} = \sup_{P \in \mathfrak{P}} \Pi_P.$$

In Chapter II, we consider a particular set \mathfrak{P} of possible laws which corresponds to volatility uncertainty. Under each scenario $P \in \mathfrak{P}$, the price process is a continuous local martingale, but the volatility may vary in each P . Moreover, under each $P \in \mathfrak{P}$, the corresponding market is complete, i.e. there is no other equivalent local martingale measure for the price process. Therefore, we guess that in our setting, the superreplication price under Knightian uncertainty of a claim ξ satisfies

$$\Pi_{\mathfrak{P}}(\xi) = \sup_{P \in \mathfrak{P}} E^P[\xi].$$

Indeed, we establish the duality formula for the superreplication price in our setting of volatility uncertainty. It has been already established for several cases and through different approaches: [16] used ideas from capacity theory, [53, 69, 73] used an approximation by Markovian control problems,

and [72, 48] used dynamic sublinear expectations. The main difference between our results and the previous ones is that we do not require any continuity assumptions on the claim (as a functional of the stock price). Thus, on the one hand, our result extends the duality formula to traded claims such as digital options or options on realized variance, which are not quasi-continuous (cf. [16]), and cases where the regularity is not known, like an American option evaluated at an optimal exercise time (cf. [50]). On the other hand, our result confirms the general robustness of the duality.

Measurability of Semimartingale Characteristics with Respect to the Law.

For the purpose of this introduction, consider the coordinate-mapping process X on the Skorohod space $\Omega = D[0, \infty)$; that is, the set of right-continuous paths with left limits. If P is a law on Ω such that X is a P -semimartingale, we can consider the corresponding triplet (B^P, C^P, ν^P) of *predictable semimartingale characteristics*. Roughly speaking, B^P describes the drift, C^P the continuous diffusion, and ν^P the jumps of X . We say that X has absolutely continuous characteristics under P if $(dB^P, dC^P, d\nu^P) = (b_t^P dt, c_t^P dt, F_t^P dt)$ and call (b^P, c^P, F^P) *differential characteristics*. The semimartingale characteristics depend on P and are defined P -almost surely; for instance, if P' is equivalent to P , the characteristics under P' are in general different from the ones under P , whereas if P and P' are singular, it is a priori meaningless to compare the characteristics. In standard situations of stochastic analysis, the characteristics are considered under a fixed probability, or one describes their transformation under an absolutely continuous change of measure as in Girsanov's theorem.

There are, however, numerous applications of stochastic analysis and dynamic programming where we work with a large set \mathfrak{P} of semimartingale laws, often mutually singular. For instance, when considering a standard stochastic control problem based on a controlled stochastic differential equation, it is useful to recast the problem on Skorohod space by taking \mathfrak{P} to be the set of all laws of solutions of the controlled equation; see e.g. [45]. This so-called *weak formulation* of the control problem is advantageous because the Skorohod space has a convenient topological structure; in fact, control problems are often stated directly in this form (cf. [20, 22] among many others). A similar weak formulation exists in the context of stochastic differential games; here this choice is even more important as the existence of a value may depend on the formulation; see [56, 67] and the references therein. Or, in the context of a nonlinear expectation $\mathcal{E}(\cdot)$, the set \mathfrak{P} of all measures P such that $E^P[\cdot] \leq \mathcal{E}(\cdot)$ plays an important role; see [49, 53, 55]. For instance, the set of all laws of continuous semimartingales whose drift and diffusion coefficients satisfy given bounds is related to G -Brownian motion. Other examples where sets of semimartingale laws play a role are path-dependent PDEs [19], robust superhedging as in [58] and Chapter II or nonlinear optimal stopping problems as in [50], to name but a few. It is

well known that the dynamic programming principle is delicate as soon as the regularity of the value function is not known a priori; this is often the case when the reward/cost function is discontinuous or in the presence of state constraints. In this situation, the measurability of the set of controls is crucial to establish the dynamic programming and the measurability of the value function; see [21, 49, 79].

As a guiding example, let us consider the set \mathfrak{P} that occurs in Chapter IV in the probabilistic construction of nonlinear Lévy processes and which was our initial motivation. The starting point is a collection Θ of Lévy triplets. In this application, the set \mathfrak{P} of interest consists of all laws of semimartingales whose differential characteristics take values in Θ . The collection Θ plays the role of a generalized Lévy triplet since the case of a singleton corresponds to a classical Lévy process. Since a dynamic programming principle is crucial to the theory, we need to establish the measurability of \mathfrak{P} . Let us mention that the set \mathfrak{P} often fails to be closed (e.g., because pure jump processes can converge to a continuous diffusion), so that it is indeed natural to examine the measurability directly. After a moment's reflection, we see that the fundamental question underlying such issues is *the measurability of the characteristics as a function of the law P* ; indeed, \mathfrak{P} is essentially the preimage of Θ under the mapping which associates to P the characteristics of X under P .

In Chapter III, we show that the set \mathfrak{P}_{sem} of all semimartingale laws is Borel-measurable and we construct processes (B^P, C, ν^P) which are jointly measurable in time, space, and the probability law P , and are versions of the semimartingale characteristics of X under P for each $P \in \mathfrak{P}_{sem}$. The second characteristic C can be constructed as a single process not depending on P . A similar result is obtained for the differential characteristics.

Nonlinear Lévy Processes and their Characteristics. Starting with a set \mathfrak{P} of probability measures, representing the possible scenarios of the law of the price process, one can define the corresponding sublinear expectation $\mathcal{E}(\cdot) := \sup_{P \in \mathfrak{P}} E^P[\cdot]$, which provides a robust way e.g. to measure the risk of possible losses or to price contingent claims under Knightian uncertainty. Conversely, starting with a given sublinear expectation $\mathcal{E}(\cdot)$, there exists (under some weak assumptions on the sublinear expectation) a set of probability measures \mathfrak{P} such that $\mathcal{E}(\cdot) = \sup_{P \in \mathfrak{P}} E^P[\cdot]$. The corresponding set of laws \mathfrak{P} can then be interpreted as the possible laws of the price process. Therefore, the notion of a sublinear expectation has particular interest in model uncertainty. If one is interested in the time development of a financial market under model uncertainty, described by a given sublinear expectation, one would like to construct a corresponding conditional sublinear expectation which satisfies the *time-consistency* property.

One particular example of a sublinear expectation and its corresponding conditional sublinear expectation is the *G-expectation*, introduced in

[51, 52], which describes volatility uncertainty of the price process. The G -expectation $\mathcal{E}^G(\cdot)$ is constructed directly from the solution of the nonlinear PDE related to Brownian motion with uncertain (constant) volatility described by a given set $\Theta_c \subseteq \mathbb{S}_+^d$. However, it is not clear what the stochastic interpretation of the G -expectation is; though

$$\mathcal{E}^G(\cdot) = \sup_{P \in \mathfrak{P}} E^P[\cdot]$$

for some set \mathfrak{P} follows directly from being a sublinear expectation, it is not clear from the construction which set \mathfrak{P} corresponds to the G -expectation. Later in [14, 17], \mathfrak{P} has been characterized as the set of continuous local martingale laws with volatility taking values in Θ_c .

Under a sublinear expectation, one can still define the notion of distributions and independence of random variables. A related notion to the G -expectation is the notion of a *G-Brownian motion*, see [51, 52]. Given a sublinear expectation, a stochastic process is called G -Brownian motion if it has stationary and independent increments (with respect to the given sublinear expectation) and has a nonlinear generator corresponding to the supremum of all generators of classical Brownian motion with constant volatility in a given set. Therefore, one can interpret a G -Brownian motion as a Brownian motion with uncertain volatility. For example, the canonical process on the continuous path space is a G -Brownian motion with respect to the G -expectation.

Given a set Θ of Lévy triplets, a time-consistent sublinear expectation was constructed in [25] similarly to the G -expectation, but where the solution of the nonlinear PIDE corresponding to a Lévy process with uncertain Lévy triplet was used. Therefore, Θ represents the uncertainty simultaneously in drift, volatility and jumps. As in the case of the G -expectation, the construction of the sublinear expectation in [25] does not provide a stochastic interpretation and the answer is left open. Moreover, only a small class of sets of Lévy triplets are allowed for the construction of the sublinear expectation. For example, Lévy triplets with corresponding Lévy measures having infinite variation jumps are excluded.

Nonlinear Lévy processes were introduced in [25]. Given a sublinear expectation $\mathcal{E}(\cdot)$, a process is called a *nonlinear Lévy process* if it has stationary and independent increments with respect to $\mathcal{E}(\cdot)$. Therefore, we see that if the sublinear expectation is an usual expectation, the notion of nonlinear Lévy processes and classical Lévy processes coincide. Moreover, as in the classical case, a G -Brownian motion is an example of a nonlinear Lévy process. It was shown in [25] that the canonical process on the Skorohod space is a nonlinear Lévy process with respect to the sublinear expectation which has been introduced in that paper. Moreover, it has a nonlinear generator corresponding to the supremum of all generators of classical Lévy processes with Lévy triplets in the given set Θ . However, this result is only valid as

long as the construction of the corresponding sublinear expectation is valid, which is very restrictive on the choice of the set of Lévy triplets Θ .

In Chapter IV, we introduce a probabilistic construction of nonlinear Lévy processes. Given an arbitrary set Θ of Lévy triplets, we let $\mathfrak{P} = \mathfrak{P}_\Theta$ be the set of all laws (on the Skorohod space) of semimartingales whose differential characteristics take values in Θ ; that is, their predictable semimartingale characteristics (B, C, ν) are of the form $(b_t dt, c_t dt, F_t dt)$ and the processes (b, c, F) evolve in Θ . We consider the sublinear expectation

$$\mathcal{E}(\cdot) := \sup_{P \in \mathfrak{P}_\Theta} E^P[\cdot]$$

and extend it to a time-consistent (conditional) sublinear expectation. As a consequence, the canonical process X on the Skorohod space is a nonlinear Lévy process under that sublinear expectation. For these results, we only require the weak condition that Θ is measurable. If we assume some additional conditions on Θ , we get that X has a nonlinear generator corresponding to the supremum of all generators of classical Lévy processes with Lévy triplets in the given set Θ , as in [25]. We point out that our conditions on Θ are much weaker than the ones in [25], for example Lévy triplets with corresponding Lévy measures having infinite variation jumps are not excluded in our case.

Summing up, we provide an alternative way to construct nonlinear Lévy processes, which answers the question of the stochastic interpretation of the sublinear expectation introduced in [25]. Moreover, our construction allows us to construct nonlinear Lévy processes for a much bigger class of sets of Lévy triplets Θ . Therefore, our construction can also be identified as an extension of [25].

Robust Utility Maximization with Lévy Processes. The classical utility maximization problem deals with the question of a financial agent on finding an investment strategy $\hat{\pi}$ which maximizes expected utility from terminal wealth, i.e. a strategy $\hat{\pi}$ which satisfies

$$E[U(W_T^{\hat{\pi}})] = \sup_{\pi} E[U(W_T^{\pi})],$$

where W_T^{π} is the wealth at time T resulting from investing in stocks according to the trading strategy π and U is an utility function, an increasing and concave function modeling the preferences of the agent. Within the rich literature on this kind of portfolio optimization problem going back to [36, 64], a branch focuses on obtaining explicit or semi-explicit expressions for optimal portfolios. Essentially, this is possible only for *isoelastic utility* functions, that is either the *logarithmic utility* $U(x) = \log(x)$ or a *power utility* $U(x) = \frac{1}{p}x^p$ for some $p \in (-\infty, 0) \cup (0, 1)$; moreover, a tractable model for the stock prices is required. While [36] provides the closed-form solution in the classical Black–Scholes model, a semi-explicit optimizer is still available

for exponential Lévy processes; see, e.g., [29, 42]. Semi-explicit solutions are also available for certain stochastic volatility models such as Heston's; see, e.g., [30, 74], among many others. The main merit of these solutions is to yield insight into how the presence of a specific phenomenon, such as stochastic volatility or jumps, may influence the choice of an investment strategy in comparison to more classical models. Here, our purpose is to study specifically the influence of model uncertainty.

Given a set \mathfrak{P} of probability measures describing the possible laws of the price process, the robust utility maximization problem is of the form

$$\sup_{\pi} \inf_{P \in \mathfrak{P}} E^P[U(W_T^{\pi})]. \quad (0.1)$$

Much of the literature on robust utility maximization in mathematical finance, starting with [61, 65], assumes that the set \mathfrak{P} of models is dominated by a reference measure P_* . In continuous-time, this assumption leads to a setting where volatilities and jump sizes are perfectly known, only drifts may be uncertain. By contrast, we are interested in uncertainty about all these three components, so that \mathfrak{P} is nondominated. In a general setting, the existence of optimal portfolios is known only in discrete time [47].

For continuous-time models where prices have continuous paths, there are several results related to the robust utility maximization problem. The early contribution [76] studies a class of related model risk management problems and shows that the lower value function (inf sup) solves a non-linear PDE (these problems, however, do not admit a saddle point in general). In [15], a minimax result and the existence of a worst-case measure is established in a setup where prices have continuous paths and the utility function is bounded. In [35], existence of an optimizer is obtained in a problem where U is an isoelastic utility function, volatility is uncertain (within an interval) but the drift is known, by considering an associated second order backward stochastic differential equation. On the other hand, [77] studies the Hamilton–Jacobi–Bellman–Isaacs PDE related to the robust utility maximization problem in a diffusion model with a non-tradable factor and miss-specified drift and volatility coefficients for the traded asset; here a saddle point can be found after a randomization. A model with several uncorrelated stocks, where drift, interest rate and volatility are uncertain within a specific parametrization, is considered in [34]. A saddle point is found and analyzed, again by dynamic programming arguments. Recently, [7] also constructs a saddle point in a setting where the uncertainty in the drift may depend on the realization of the volatility in a specific way. Finally, [24] considers a stochastic volatility model with uncertain correlation (but known drift) and describes an asymptotic closed-form solution.

Continuous-time models with jumps have not been studied in the extant literature. In Chapter V, our main contribution is to exhibit the robust utility maximization problem (0.1) in a continuous-time setting that in-

cludes uncertainty about fairly general models, including jump uncertainty, while remaining very tractable. In (0.1) we choose U to be either the logarithmic utility $U(x) = \log(x)$ or a power utility $U(x) = \frac{1}{p}x^p$ for some $p \in (-\infty, 0) \cup (0, 1)$, whereas the model uncertainty is parametrized by a set Θ of Lévy triplets (b, c, F) and then \mathfrak{P} consists of all semimartingale laws P such that the associated differential characteristics (b_t^P, c_t^P, F_t^P) take values in Θ , $P \times dt$ -a.e. In particular, \mathfrak{P} includes all Lévy processes with triplet in Θ , but unless Θ is a singleton, \mathfrak{P} will also contain many laws for which (b_t^P, c_t^P, F_t^P) are time-dependent and random. Thus, our setup describes uncertainty about drift, volatility and jumps over a class of fairly general models.

In our setting, we show that an optimal trading strategy $\hat{\pi}$ exists for (0.1). This strategy is of the constant-proportion type; that is, a constant fraction of the current wealth is invested in each stock. We compute this fraction in semi-closed form, so that the impact of model uncertainty can be readily read off. Thus, our specification of model uncertainty retains much of the tractability of the classical utility maximization problem for exponential Lévy processes. This is noteworthy for the power utility as \mathfrak{P} contains models P that are not Lévy and in which the classical power utility investor is not myopic. Moreover, while the classical log utility investor is myopic in any given semimartingale model, this property generally fails in robust problems, due to the nonlinearity caused by the infimum—retaining the myopic feature is specific to the setup chosen here, and in particular its nonlinear i.i.d. property (in the sense of Chapter IV).

Moreover, under a compactness condition on Θ , we show the existence of a saddle point $(\hat{P}, \hat{\pi})$ for the problem (0.1). Therefore, $\hat{P} \in \mathfrak{P}$ can be seen as a worst-case model. This model is a Lévy law and the corresponding Lévy triplet $(\hat{b}, \hat{c}, \hat{F})$ is computed in semi-closed form. The strategy $\hat{\pi}$ and the triplet $(\hat{b}, \hat{c}, \hat{F})$ are characterized as a saddle point of a deterministic function. The fact that \hat{P} is a Lévy model may be compared with option pricing in the Uncertain Volatility Model, where in general the worst-case model is a non-Lévy law unless the option is convex or concave.

Chapter II

Superreplication under Volatility Uncertainty for Measurable Claims

In this chapter, which corresponds to the article [37], we establish the duality formula for the superreplication price in a setting of volatility uncertainty. In contrast to previous results, the contingent claim is not assumed to satisfy any continuity conditions.

II.1 Introduction

This chapter is concerned with superreplication-pricing in a setting of volatility uncertainty. We see the canonical process B on the space Ω of continuous paths as the stock price process and formalize this uncertainty via a set \mathfrak{P} of (non-equivalent) martingale laws on Ω . Given a contingent claim ξ measurable at time $T > 0$, we are interested in determining the minimal initial capital $x \in \mathbb{R}$ for which there exists a trading strategy H whose terminal gain $x + \int_0^T H_u dB_u$ exceeds ξ P -a.s., simultaneously for all $P \in \mathfrak{P}$. The aim is to show that under suitable assumptions, this minimal capital is given by $x = \sup_{P \in \mathfrak{P}} E^P[\xi]$. We prove this duality formula for Borel-measurable (and, more generally, upper semianalytic) claims ξ and a model \mathfrak{P} where the possible values of the volatility are determined by a set-valued process. Such a model of a “random G -expectation” was first introduced in [46], as an extension of the “ G -expectation” of [51, 52].

The duality formula under volatility uncertainty has been established for several cases. The main difference between our results and the previous ones is that we do not impose continuity assumptions on the claim ξ (as a functional of the stock price), so that we retrieve the level of generality that is standard in mathematical finance. The main difficulty in our endeavor is to construct the superreplicating strategy H . We adopt the approach of [72]

and [48], which can be outlined as follows:

- (i) Construct the conditional (nonlinear) expectation $\mathcal{E}_t(\xi)$ related to \mathfrak{P} and show the tower property $\mathcal{E}_s(\mathcal{E}_t(\xi)) = \mathcal{E}_s(\xi)$ for $s \leq t$.
- (ii) Check that the right limit $Y_t := \mathcal{E}_{t+}(\xi)$ exists and defines a supermartingale under each $P \in \mathfrak{P}$ (in a suitable filtration).
- (iii) For every $P \in \mathfrak{P}$, show that the martingale part in the Doob–Meyer decomposition of Y is of the form $\int H^P dB$. Using that H^P can be expressed via the quadratic (co)variation processes of Y and B , deduce that there exists a universal process H coinciding with H^P under each P , and check that H is the desired strategy.

Step (iii) can be accomplished by ensuring that each $P \in \mathfrak{P}$ has the predictable representation property. To this end—and for some details of Step (ii) that we shall skip for the moment—[72] introduced the set of Brownian martingale laws with positive volatility, which we shall denote by \mathfrak{P}_S : if \mathfrak{P} is chosen as a subset of \mathfrak{P}_S , then every $P \in \mathfrak{P}$ has the representation property (cf. Lemma II.4.1) and Step (iii) is feasible.

Step (i) is the main reason why previous results required continuity assumptions on ξ . Recently, it was shown in [49] that the theory of analytic sets can be used to carry out Step (i) when ξ is merely Borel-measurable (or only upper semianalytic), provided that the set of measures satisfies a condition of measurability and invariance, called Condition (A) below (cf. Proposition II.2.2). It was also shown in [49] that this condition is satisfied for a model of random G -expectation where the measures are chosen from the set of all (not necessarily Brownian) martingale laws. Thus, to follow the approach outlined above, we formulate a similar model using elements of \mathfrak{P}_S and show that Condition (A) is again satisfied. This essentially boils down to proving that the set \mathfrak{P}_S itself satisfies Condition (A), which is our main technical result (Theorem II.2.4). Using this fact, we can go through the approach outlined above and establish our duality result (Theorem II.2.3 and Corollary II.2.6). As an aside of independent interest, Theorem II.2.4 yields a rigorous proof for a dynamic programming principle with a fairly general reward functional (cf. Remark II.2.7).

The remainder of this chapter is organized as follows. In Section II.2, we first describe our setup and notation in detail and we recall the relevant facts from [49]; then, we state our main results. Theorem II.2.4 is proved in Section II.3, and Section II.4 concludes with the proof of Theorem II.2.3.

II.2 Results

II.2.1 Notation

We fix the dimension $d \in \mathbb{N}$ and let $\Omega = \{\omega \in C([0, \infty); \mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous paths equipped with the topology of locally uniform convergence. Moreover, let $\mathcal{F} = \mathcal{B}(\Omega)$ be its Borel σ -algebra. We denote by $B := (B_t)_{t \geq 0}$ the canonical process $B_t(\omega) = \omega_t$, by P_0 the Wiener measure and by $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ the (raw) filtration generated by B . Furthermore, we denote by $\mathfrak{P}(\Omega)$ the set of all probability measures on Ω , equipped with the topology of weak convergence.

We recall that a subset of a Polish space is called analytic if it is the image of a Borel subset of another Polish space under a Borel map. Moreover, an \mathbb{R} -valued function f is called upper semianalytic if $\{f > c\}$ is analytic for each $c \in \mathbb{R}$; in particular, any Borel-measurable function is upper semianalytic. (See [6, Chapter 7] for background.) Finally, the universal completion of a σ -field \mathcal{A} is given by $\mathcal{A}^* := \bigcap_P \mathcal{A}^{(P)}$, where $\mathcal{A}^{(P)}$ denotes the completion with respect to P and the intersection is taken over all probability measures on \mathcal{A} .

Throughout this chapter, “stopping time” will refer to a finite \mathbb{F} -stopping time. Let τ be a stopping time. Then the concatenation of $\omega, \tilde{\omega} \in \Omega$ at τ is the path

$$(\omega \otimes_{\tau} \tilde{\omega})_u := \omega_u \mathbf{1}_{[0, \tau(\omega))}(u) + (\omega_{\tau(\omega)} + \tilde{\omega}_{u - \tau(\omega)}) \mathbf{1}_{[\tau(\omega), \infty)}(u), \quad u \geq 0.$$

For any probability measure $P \in \mathfrak{P}(\Omega)$, there is a regular conditional probability distribution $\{P_{\tau}^{\omega}\}_{\omega \in \Omega}$ given \mathcal{F}_{τ} satisfying

$$P_{\tau}^{\omega} \{\omega' \in \Omega : \omega' = \omega \text{ on } [0, \tau(\omega)]\} = 1 \quad \text{for all } \omega \in \Omega;$$

cf. [75, p. 34]. We then define $P^{\tau, \omega} \in \mathfrak{P}(\Omega)$ by

$$P^{\tau, \omega}(A) := P_{\tau}^{\omega}(\omega \otimes_{\tau} A), \quad A \in \mathcal{F}, \quad \text{where } \omega \otimes_{\tau} A := \{\omega \otimes_{\tau} \tilde{\omega} : \tilde{\omega} \in A\}.$$

Given a function ξ on Ω and $\omega \in \Omega$, we also define the function $\xi^{\tau, \omega}$ on Ω by

$$\xi^{\tau, \omega}(\tilde{\omega}) := \xi(\omega \otimes_{\tau} \tilde{\omega}), \quad \tilde{\omega} \in \Omega.$$

Then, we have $E^{P^{\tau, \omega}}[\xi^{\tau, \omega}] = E^P[\xi | \mathcal{F}_{\tau}](\omega)$ for P -a.e. $\omega \in \Omega$.

II.2.2 Preliminaries

We formalize volatility uncertainty via a set of local martingale laws with different volatilities. To this end, we denote by \mathbb{S} the set of all symmetric $d \times d$ -matrices and by $\mathbb{S}^{>0}$ its subset of strictly positive definite matrices. The set $\mathfrak{P}_{\mathbb{S}} \subset \mathfrak{P}(\Omega)$ consists of all local martingale laws of the form

$$P^{\alpha} = P_0 \circ \left(\int_0^{\cdot} \alpha_s^{1/2} dB_s \right)^{-1}, \quad (2.1)$$

where α ranges over all \mathbb{F} -progressively measurable processes with values in $\mathbb{S}^{>0}$ satisfying $\int_0^T |\alpha_s| ds < \infty$ P_0 -a.s. for every $T \in \mathbb{R}_+$. (We denote by $|\cdot|$ the Euclidean norm in any dimension.) In other words, these are all laws of stochastic integrals of a Brownian motion, where the integrand is strictly positive and adapted to the Brownian motion. The set \mathfrak{P}_S was introduced in [72] and its elements have several nice properties; in particular, they have the predictable representation property which plays an important role in the proof of the duality result below (see also Section II.4).

We intend to model “uncertainty” via a subset $\mathfrak{P} \subset \mathfrak{P}(\Omega)$ (below, each $P \in \mathfrak{P}$ will be a possible scenario for the volatility). However, for technical reasons, we make a detour and consider an entire family of subsets of $\mathfrak{P}(\Omega)$, indexed by $(s, \omega) \in \mathbb{R}_+ \times \Omega$, whose elements at $s = 0$ coincide with \mathfrak{P} . As illustrated in Example II.2.1 below, this family is of purely auxiliary nature.

Let $\{\mathfrak{P}(s, \omega)\}_{(s, \omega) \in \mathbb{R}_+ \times \Omega}$ be a family of subsets of $\mathfrak{P}(\Omega)$, adapted in the sense that

$$\mathfrak{P}(s, \omega) = \mathfrak{P}(s, \tilde{\omega}) \quad \text{if} \quad \omega|_{[0, s]} = \tilde{\omega}|_{[0, s]},$$

and define $\mathfrak{P}(\tau, \omega) := \mathfrak{P}(\tau(\omega), \omega)$ for any stopping time τ . Note that the set $\mathfrak{P}(0, \omega)$ is independent of ω as all paths start at the origin. Thus, we can define $\mathfrak{P} := \mathfrak{P}(0, \omega)$. Before giving the example, let us state a condition on $\{\mathfrak{P}(s, \omega)\}$ whose purpose will be to construct the conditional sublinear expectation related to \mathfrak{P} .

Condition (A). Let $s \in \mathbb{R}_+$, let τ be a stopping time such that $\tau \geq s$, let $\bar{\omega} \in \Omega$ and $P \in \mathfrak{P}(s, \bar{\omega})$. Set $\theta := \tau^{s, \bar{\omega}} - s$.

(A1) The graph $\{(P', \omega) : \omega \in \Omega, P' \in \mathfrak{P}(\tau, \omega)\} \subseteq \mathfrak{P}(\Omega) \times \Omega$ is analytic.

(A2) We have $P^{\theta, \omega} \in \mathfrak{P}(\tau, \bar{\omega} \otimes_s \omega)$ for P -a.e. $\omega \in \Omega$.

(A3) If $\nu : \Omega \rightarrow \mathfrak{P}(\Omega)$ is an \mathcal{F}_θ -measurable kernel and $\nu(\omega) \in \mathfrak{P}(\tau, \bar{\omega} \otimes_s \omega)$ for P -a.e. $\omega \in \Omega$, then the measure defined by

$$\bar{P}(A) = \iint (\mathbf{1}_A)^{\theta, \omega}(\omega') \nu(d\omega'; \omega) P(d\omega), \quad A \in \mathcal{F} \quad (2.2)$$

is an element of $\mathfrak{P}(s, \bar{\omega})$.

Conditions (A1)–(A3) will ensure that the conditional expectation is measurable and satisfies the “tower property” (see Proposition II.2.2 below), which is the dynamic programming principle in this context (see [6] for background). We remark that (A2) and (A3) imply that the family $\{\mathfrak{P}(s, \omega)\}$ is essentially determined by the set \mathfrak{P} . As mentioned above, in applications, \mathfrak{P} will be the primary object and we shall simply write down a corresponding family $\{\mathfrak{P}(s, \omega)\}$ such that $\mathfrak{P} = \mathfrak{P}(0, \omega)$ and such that Condition (A) is satisfied. To illustrate this, let us state a model where the possible values of the volatility are described by a set-valued process \mathbf{D} and which will be

the main application of our results. This model was first introduced in [46] and further studied in [49]; it generalizes the G -expectation of [51, 52] which corresponds to the case where \mathbf{D} is a (deterministic) compact convex set.

Example II.2.1 (Random G -Expectation). We consider a set-valued process $\mathbf{D} : \Omega \times \mathbb{R}_+ \rightarrow 2^{\mathbb{S}}$; i.e., $\mathbf{D}_t(\omega)$ is a set of matrices for each (t, ω) . We assume that \mathbf{D} is progressively graph-measurable: for every $t \in \mathbb{R}_+$,

$$\{(\omega, s, A) \in \Omega \times [0, t] \times \mathbb{S} : A \in \mathbf{D}_s(\omega)\} \in \mathcal{F}_t \times \mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{S}),$$

where $\mathcal{B}([0, t])$ and $\mathcal{B}(\mathbb{S})$ denote the Borel σ -fields of \mathbb{S} and $[0, t]$.

We want a set \mathfrak{P} consisting of all $P \in \mathfrak{P}_{\mathbb{S}}$ under which the volatility takes values in \mathbf{D} P -a.s. To this end, we introduce the auxiliary family $\{\mathfrak{P}(s, \omega)\}$: given $(s, \omega) \in \mathbb{R}_+ \times \Omega$, we define $\mathfrak{P}(s, \omega)$ to be the collection of all $P \in \mathfrak{P}_{\mathbb{S}}$ such that

$$\frac{d\langle B \rangle_u}{du}(\tilde{\omega}) \in \mathbf{D}_{u+s}(\omega \otimes_s \tilde{\omega}) \quad \text{for } P \times du\text{-a.e. } (\tilde{\omega}, u) \in \Omega \times \mathbb{R}_+. \quad (2.3)$$

In particular, $\mathfrak{P} := \mathfrak{P}(0, \omega)$ then consists, as desired, of all $P \in \mathfrak{P}_{\mathbb{S}}$ such that $d\langle B \rangle_u/du \in \mathbf{D}_u$ holds $P \times du$ -a.e. We shall see in Corollary II.2.6 that Condition (A) is satisfied in this example.

The following is the main result of [49]; it is stated with the conventions $\sup \emptyset = -\infty$ and $E^P[\xi] := -\infty$ if $E^P[\xi^+] = E^P[\xi^-] = +\infty$, and ess sup^P denotes the essential supremum under P .

Proposition II.2.2. *Suppose that $\{\mathfrak{P}(s, \omega)\}$ satisfies Condition (A) and that $\mathfrak{P} \neq \emptyset$. Let $\sigma \leq \tau$ be stopping times and let $\xi : \Omega \rightarrow \overline{\mathbb{R}}$ be an upper semianalytic function. Then the function*

$$\mathcal{E}_\tau(\xi)(\omega) := \sup_{P \in \mathfrak{P}(\tau, \omega)} E^P[\xi^{\tau, \omega}], \quad \omega \in \Omega$$

is \mathcal{F}_τ^* -measurable and upper semianalytic. Moreover,

$$\mathcal{E}_\sigma(\xi)(\omega) = \mathcal{E}_\sigma(\mathcal{E}_\tau(\xi))(\omega) \quad \text{for all } \omega \in \Omega. \quad (2.4)$$

Furthermore, with $\mathfrak{P}(\sigma; P) = \{P' \in \mathfrak{P} : P' = P \text{ on } \mathcal{F}_\sigma\}$, we have

$$\mathcal{E}_\sigma(\xi) = \text{ess sup}_{P' \in \mathfrak{P}(\sigma; P)}^P E^{P'}[\mathcal{E}_\tau(\xi) | \mathcal{F}_\sigma] \quad P\text{-a.s. for all } P \in \mathfrak{P}. \quad (2.5)$$

II.2.3 Main Results

Some more notation is needed to state our duality result. In what follows, the set \mathfrak{P} determined by the family $\{\mathfrak{P}(s, \omega)\}$ will be a subset of $\mathfrak{P}_{\mathbb{S}}$. We shall use a finite time horizon $T \in \mathbb{R}_+$ and the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$, where

$$\mathcal{G}_t := \mathcal{F}_t^* \vee \mathcal{N}^{\mathfrak{P}};$$

here \mathcal{F}_t^* is the universal completion of \mathcal{F}_t and $\mathcal{N}^{\mathfrak{P}}$ is the collection of sets which are (\mathcal{F}_T, P) -null for all $P \in \mathfrak{P}$.

Let H be a \mathbb{G} -predictable process taking values in \mathbb{R}^d and such that $\int_0^T H_u^\top d\langle B \rangle_u H_u < \infty$ P -a.s. for all $P \in \mathfrak{P}$. Then H is called an admissible trading strategy if $\int H dB$ is a P -supermartingale for all $P \in \mathfrak{P}$; as usual, this is to rule out doubling strategies. We denote by \mathcal{H} the set of all admissible trading strategies.

Theorem II.2.3. *Suppose that $\{\mathfrak{P}(s, \omega)\}$ satisfies Condition (A) and that $\emptyset \neq \mathfrak{P} \subset \mathfrak{P}_S$. Moreover, let $\xi : \Omega \rightarrow \mathbb{R}$ be an \mathcal{G}_T -measurable, upper semianalytic function such that $\sup_{P \in \mathfrak{P}} E^P[|\xi|] < \infty$. Then*

$$\begin{aligned} & \sup_{P \in \mathfrak{P}} E^P[\xi] \\ &= \min \left\{ x \in \mathbb{R} : \exists H \in \mathcal{H} \text{ with } x + \int_0^T H_u dB_u \geq \xi \text{ } P\text{-a.s. for all } P \in \mathfrak{P} \right\}. \end{aligned}$$

The assumption that $\mathfrak{P} \subset \mathfrak{P}_S$ will be essential for our proof, which is stated in Section II.4. In order to have nontrivial examples where the previous theorem applies, it is essential to know that the set \mathfrak{P}_S (seen as a constant family $\mathfrak{P}(s, \omega) \equiv \mathfrak{P}_S$) satisfies itself Condition (A). This fact is our main technical result.

Theorem II.2.4. *The set \mathfrak{P}_S satisfies Condition (A).*

The proof is stated Section II.3. It is easy to see that if two families satisfy Condition (A), then so does their intersection. In particular, we have:

Corollary II.2.5. *If $\{\mathfrak{P}(s, \omega)\}$ satisfies Condition (A), so does $\{\mathfrak{P}(s, \omega) \cap \mathfrak{P}_S\}$.*

The following is the main application of our results.

Corollary II.2.6. *The family $\{\mathfrak{P}(s, \omega)\}$ related to the random G -expectation (as defined in Example II.2.1) satisfies Condition (A). In particular, the duality result of Theorem II.2.3 applies in this case.*

Proof. Let $\mathfrak{M}_a \subset \mathfrak{P}(\Omega)$ be the set of all local martingale laws on Ω under which the quadratic variation of B is absolutely continuous with respect to the Lebesgue measure; then $\mathfrak{P}_S \subset \mathfrak{M}_a$. Moreover, let $\tilde{\mathfrak{P}}(s, \omega)$ be the set of all $P \in \mathfrak{M}_a$ such that (2.3) holds. Then, clearly, $\mathfrak{P}(s, \omega) = \tilde{\mathfrak{P}}(s, \omega) \cap \mathfrak{P}_S$, and since we know from [49, Theorem 4.3] that $\{\tilde{\mathfrak{P}}(s, \omega)\}$ satisfies Condition (A), Corollary II.2.5 shows that $\{\mathfrak{P}(s, \omega)\}$ again satisfies Condition (A). \square

¹Here $\int H dB$ is, with some abuse of notation, the usual Itô integral under the fixed measure P . We remark that we could also define the integral simultaneously under all $P \in \mathfrak{P}$ by the construction of [44]. This would yield a cosmetically nicer result, but we shall avoid the additional set-theoretic subtleties as this is not central to our approach.

Remark II.2.7. In view of (2.4), Theorem II.2.4 yields the dynamic programming principle for the optimal control problem $\sup_{\alpha} E^{P_0}[\xi(X^\alpha)]$ with a very general reward functional ξ , where $X^\alpha = \int_0^\cdot \alpha_s^{1/2} dB_s$. We remark that the arguments in the proof of Theorem II.2.4 could be extended to other control problems; for instance, the situation where the state process X^α is defined as the solution of a stochastic functional/differential equation as in [45].

II.3 Proof of Theorem II.2.4

In this section, we prove that \mathfrak{P}_S (i.e., the constant family $\mathfrak{P}(s, \omega) \equiv \mathfrak{P}_S$) satisfies Condition (A). Up to some minor differences in notation, property (A2) was already shown in [72, Lemma 4.1], so we focus on (A1) and (A3).

Let us fix some more notation. We denote by $E[\cdot]$ the expectation under the Wiener measure P_0 ; more generally, any notion related to Ω that implicitly refers to a measure will be understood to refer to P_0 . Unless otherwise stated, any topological space is endowed with its Borel σ -field. As usual, $L^0(\Omega; \mathbb{R})$ denotes the set of equivalence classes of random variables on Ω , endowed with the topology of convergence in measure. Moreover, we denote by $\bar{\Omega} = \Omega \times \mathbb{R}_+$ the product space; here the measure is $P_0 \times dt$ by default, where dt is the Lebesgue measure. The basic space in this section is $L^0(\bar{\Omega}; \mathbb{S})$, the set of equivalence classes of \mathbb{S} -valued processes that are product-measurable. We endow $L^0(\bar{\Omega}; \mathbb{S})$ (and its subspaces) with the topology of local convergence in measure; that is, the metric

$$d(\cdot, \cdot) = \sum_{k \in \mathbb{N}} 2^{-k} \frac{d_k(\cdot, \cdot)}{1 + d_k(\cdot, \cdot)}, \quad \text{where} \quad d_k(X, Y) = E \left[\int_0^k 1 \wedge |X_s - Y_s| ds \right]. \quad (3.1)$$

As a result, $X^n \rightarrow X$ in $L^0(\bar{\Omega}; \mathbb{S})$ if and only if $\lim_n E[\int_0^T 1 \wedge |X_s^n - X_s| ds] = 0$ for all $T \in \mathbb{R}_+$.

II.3.1 Proof of (A1)

The aim of this subsection is to show that $\mathfrak{P}_S \subset \mathfrak{P}(\Omega)$ is analytic. To this end, we shall show that $\mathfrak{P}_S \subset \mathfrak{P}(\Omega)$ is the image of a Borel space (i.e., a Borel subset of a Polish space) under a Borel map; this implies the claim by [6, Proposition 7.40, p. 165]. Indeed, let $L_{prog}^0(\bar{\Omega}; \mathbb{S}) \subset L^0(\bar{\Omega}; \mathbb{S})$ be the subset of \mathbb{F} -progressively measurable processes and

$$L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) = \left\{ \alpha \in L_{prog}^0(\bar{\Omega}; \mathbb{S}^{>0}) : \int_0^T |\alpha_s| ds < \infty \text{ } P_0\text{-a.s. for all } T \in \mathbb{R}_+ \right\}.$$

Moreover, we denote by

$$\Phi : L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \rightarrow \mathfrak{P}(\Omega), \quad \alpha \mapsto P^\alpha = P_0 \circ \left(\int_0^\cdot \alpha_s^{1/2} dB_s \right)^{-1} \quad (3.2)$$

the map which associates to α the corresponding law. Then \mathfrak{P}_S is the image

$$\mathfrak{P}_S = \Phi(L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}));$$

therefore, the claimed property (A1) follows from the two subsequent lemmas.

Lemma II.3.1. *$L_{prog}^0(\bar{\Omega}; \mathbb{S})$ is Polish and $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \subset L_{prog}^0(\bar{\Omega}; \mathbb{S})$ is Borel.*

Proof. We start by noting that $L^0(\bar{\Omega}; \mathbb{S})$ is Polish. Indeed, as \mathbb{R}_+ and Ω are separable metric spaces, we have that $L^2(\Omega \times [0, T]; \mathbb{S})$ is separable for all $T \in \mathbb{N}$ (e.g., [18, Section 6.15, p. 92]). A density argument and the definition (3.1) then show that $L^0(\bar{\Omega}; \mathbb{S})$ is again separable. On the other hand, the completeness of \mathbb{S} is inherited by $L^0(\bar{\Omega}; \mathbb{S})$; see, e.g., [10, Corollary 3]. Since $L_{prog}^0(\bar{\Omega}; \mathbb{S}) \subset L^0(\bar{\Omega}; \mathbb{S})$ is closed, it is again a Polish space.

Next, we show that $L_{loc}^1(\bar{\Omega}; \mathbb{S})$ is a Borel subset of $L_{prog}^0(\bar{\Omega}; \mathbb{S})$. We first observe that

$$L_{loc}^1(\bar{\Omega}; \mathbb{S}) = \bigcap_{T \in \mathbb{N}} \left\{ \alpha \in L_{prog}^0(\bar{\Omega}; \mathbb{S}) : P_0 \left[\arctan \left(\int_0^T |\alpha_s| ds \right) \geq \frac{\pi}{2} \right] = 0 \right\}.$$

Therefore, it suffices to show that for fixed $T \in \mathbb{N}$,

$$\alpha \mapsto P_0 \left[\arctan \left(\int_0^T |\alpha_s| ds \right) \geq \frac{\pi}{2} \right]$$

is Borel. Indeed, this is the composition of the function

$$L^0(\Omega; \mathbb{R}) \rightarrow \mathbb{R}, \quad f \mapsto P_0 [f \geq \pi/2],$$

which is upper semicontinuous by the Portmanteau theorem and thus Borel, with the map

$$L_{prog}^0(\bar{\Omega}; \mathbb{S}) \rightarrow L^0(\Omega; \mathbb{R}), \quad \alpha \mapsto \arctan \left(\int_0^T |\alpha_s| ds \right).$$

The latter is Borel because it is, by monotone convergence, the pointwise limit of the maps

$$\alpha \mapsto \arctan \left(\int_0^T n \wedge |\alpha_s| ds \right),$$

which are continuous for fixed $n \in \mathbb{N}$ due to the elementary estimate

$$\begin{aligned} E \left[1 \wedge \left| \arctan \left(\int_0^T n \wedge |\alpha_s| ds \right) - \arctan \left(\int_0^T n \wedge |\tilde{\alpha}_s| ds \right) \right| \right] \\ \leq E \left[\int_0^T n \wedge |\alpha_s - \tilde{\alpha}_s| ds \right]. \end{aligned}$$

This completes the proof that $L_{loc}^1(\bar{\Omega}; \mathbb{S})$ is a Borel subset of $L_{prog}^0(\bar{\Omega}; \mathbb{S})$. To deduce the same property for $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$, note that

$$L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) = \bigcap_{T \in \mathbb{N}} \{ \alpha \in L_{loc}^1(\bar{\Omega}; \mathbb{S}) : \mu_T[\alpha \in \mathbb{S} \setminus \mathbb{S}^{>0}] = 0 \},$$

where μ_T is the product measure $\mu_T(A) = E[\int_0^T \mathbf{1}_A ds]$. As $\mathbb{S}^{>0} \subset \mathbb{S}$ is open, $\alpha \mapsto \mu_T[\alpha \in \mathbb{S} \setminus \mathbb{S}^{>0}]$ is upper semicontinuous and we conclude that $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ is again Borel. \square

Lemma II.3.2. *The map $\Phi : L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \rightarrow \mathfrak{P}(\Omega)$ defined in (3.2) is Borel.*

Proof. Consider first, for fixed $n \in \mathbb{N}$, the mapping Φ_n defined by

$$\Phi_n(\alpha) = P_0 \circ \left(\int_0^\cdot \pi_n(\alpha_s^{1/2}) dB_s \right)^{-1},$$

where π_n is the projection onto the ball of radius n around the origin in \mathbb{S} . It follows from a direct extension of the dominated convergence theorem for stochastic integrals [60, Theorem IV.32, p. 176] that

$$\alpha \mapsto \int_0^\cdot \pi_n(\alpha_s^{1/2}) dB_s$$

is continuous for the topology of uniform convergence on compacts in probability (“ucp”), and hence that Φ_n is continuous for the topology of weak convergence. In particular, Φ_n is Borel. On the other hand, a second application of dominated convergence shows that

$$\int_0^\cdot \pi_n(\alpha_s^{1/2}) dB_s \rightarrow \int_0^\cdot \alpha_s^{1/2} dB_s \quad \text{ucp as } n \rightarrow \infty$$

for all $\alpha \in L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ and hence that $\Phi(\alpha) = \lim_n \Phi_n(\alpha)$ in $\mathfrak{P}(\Omega)$ for all α . Therefore, Φ is again Borel. \square

II.3.2 Proof of (A3)

Given a stopping time τ , a measure $P \in \mathfrak{P}_S$ and an \mathcal{F}_τ -measurable kernel $\nu : \Omega \rightarrow \mathfrak{P}(\Omega)$ with $\nu(\omega) \in \mathfrak{P}_S$ for P -a.e. $\omega \in \Omega$, our aim is to show that

$$\bar{P}(A) := \iint (\mathbf{1}_A)^{\tau, \omega}(\omega') \nu(\omega, d\omega') P(d\omega), \quad A \in \mathcal{F}$$

defines an element of \mathfrak{P}_S . That is, we need to show that $\bar{P} = P^{\bar{\alpha}}$ for some $\bar{\alpha} \in L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$. In the case where ν has only countably many values, this can be accomplished by explicitly writing down an appropriate process $\bar{\alpha}$, as was shown already in [72]. The present setup requires general kernels and a measurable selection proof. Roughly speaking, up to time τ , $\bar{\alpha}$ is given by

the integrand α determining P , whereas after τ , it is given by the integrand of $\nu(\omega)$, for a suitable ω . In Step 1 below, we state precisely the requirement for $\bar{\alpha}$; in Step 2, we construct a measurable selector for the integrand of $\nu(\omega)$; finally, in Step 3, we show how to construct the required process $\bar{\alpha}$ from this selector.

Step 1. Let $\alpha \in L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ be such that $P = P^\alpha$, let $X^\alpha := \int_0^\cdot \alpha_s^{1/2} dB_s$, and let $\tilde{\tau} := \tau \circ X^\alpha$. Suppose we have found $\hat{\alpha} \in L_{prog}^0(\bar{\Omega}; \mathbb{S})$ such that

$$\hat{\alpha}^\omega := \hat{\alpha}_{\cdot + \tilde{\tau}(\omega)}(\omega \otimes \tilde{\tau} \cdot) \in L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \text{ and } P^{\hat{\alpha}^\omega} = \nu(X^\alpha(\omega)) \text{ for } P_0\text{-a.e. } \omega \in \Omega.$$

Then $\bar{P} = P^{\bar{\alpha}}$ for $\bar{\alpha}$ defined by

$$\bar{\alpha}_s(\omega) = \alpha_s(\omega) \mathbf{1}_{[0, \tilde{\tau}(\omega)]}(s) + \hat{\alpha}_s(\omega) \mathbf{1}_{(\tilde{\tau}(\omega), \infty)}(s).$$

Indeed, we clearly have $\bar{\alpha} \in L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$. Moreover, we note that $\tilde{\tau}$ is again a stopping time by Galmarino's test [12, Theorem IV.100, p. 149]. To see that $\bar{P} = P^{\bar{\alpha}}$, it suffices to show that

$$E^{\bar{P}}[\psi(B_{t_1}, \dots, B_{t_n})] = E^{P_0}[\psi(X_{t_1}^{\bar{\alpha}}, \dots, X_{t_n}^{\bar{\alpha}})]$$

for all $n \in \mathbb{N}$, $0 < t_1 < t_2 < \dots < t_n < \infty$, and any bounded continuous function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$. Recall that B has stationary and independent increments under the Wiener measure P_0 . For P_0 -a.e. $\omega \in \Omega$ such that $\tilde{t} := \tilde{\tau}(\omega) \in [t_j, t_{j+1})$, we have

$$\begin{aligned} & E^{P_0}[\psi(X_{t_1}^{\bar{\alpha}}, \dots, X_{t_n}^{\bar{\alpha}}) | \mathcal{F}_{\tilde{\tau}}](\omega) \\ &= E^{P_0}[\psi(X_{t_1}^{\bar{\alpha}}(\omega \otimes_{\tilde{t}} B), \dots, X_{t_n}^{\bar{\alpha}}(\omega \otimes_{\tilde{t}} B))] \\ &= E^{P_0}\left[\psi\left(X_{t_1}^\alpha(\omega), \dots, X_{t_j}^\alpha(\omega), X_{\tilde{t}}^\alpha(\omega) + \int_{\tilde{t}}^{t_{j+1}} \hat{\alpha}_s^{1/2}(\omega \otimes_{\tilde{t}} B) dB_{s-\tilde{t}}, \dots, \right. \right. \\ & \qquad \qquad \qquad \left. \left. X_{\tilde{t}}^\alpha(\omega) + \int_{\tilde{t}}^{t_n} \hat{\alpha}_s^{1/2}(\omega \otimes_{\tilde{t}} B) dB_{s-\tilde{t}}\right)\right] \end{aligned}$$

and thus, by the definition of α^ω ,

$$\begin{aligned} & E^{P_0}[\psi(X_{t_1}^{\bar{\alpha}}, \dots, X_{t_n}^{\bar{\alpha}}) | \mathcal{F}_{\tilde{\tau}}](\omega) \\ &= E^{P^{\alpha^\omega}}[\psi(X_{t_1}^\alpha(\omega), \dots, X_{t_j}^\alpha(\omega), X_{\tilde{t}}^\alpha(\omega) + B_{t_{j+1}-\tilde{t}}, \dots, X_{\tilde{t}}^\alpha(\omega) + B_{t_n-\tilde{t}})] \\ &= \int \psi(X_{t_1}^\alpha(\omega), \dots, X_{t_j}^\alpha(\omega), X_{\tilde{t}}^\alpha(\omega) + B_{t_{j+1}-\tilde{t}}(\omega'), \dots, \\ & \qquad \qquad \qquad X_{\tilde{t}}^\alpha(\omega) + B_{t_n-\tilde{t}}(\omega')) \nu(X^\alpha(\omega), d\omega'). \end{aligned}$$

Integrating both sides with respect to $P_0(d\omega)$ and noting that $\tilde{t} \in [t_j, t_{j+1})$

implies $t := \tau(\omega) \in [t_j, t_{j+1})$ P -a.s., we conclude that

$$\begin{aligned}
 & E^{P_0} [\psi(X_{t_1}^{\bar{\alpha}}, \dots, X_{t_n}^{\bar{\alpha}})] \\
 &= \iint \psi(X_{t_1}^{\alpha}(\omega), \dots, X_{t_j}^{\alpha}(\omega), X_{\bar{t}}^{\alpha}(\omega) + B_{t_{j+1}-\bar{t}}(\omega'), \dots, \\
 &\quad X_{\bar{t}}^{\alpha}(\omega) + B_{t_n-\bar{t}}(\omega')) \nu(X^{\alpha}(\omega), d\omega') P_0(d\omega) \\
 &= \iint \psi(B_{t_1}(\omega), \dots, B_{t_j}(\omega), B_t(\omega) + B_{t_{j+1}-t}(\omega'), \dots, \\
 &\quad B_t(\omega) + B_{t_n-t}(\omega')) \nu(\omega, d\omega') P(d\omega) \\
 &= \iint \psi^{\tau, \omega}(B_{t_1}, \dots, B_{t_n})(\omega') \nu(\omega, d\omega') P(d\omega) \\
 &= E^{\bar{P}} [\psi(B_{t_1}, \dots, B_{t_n})].
 \end{aligned}$$

This completes the first step of the proof.

Step 2. We show that there exists an \mathcal{F}_τ -measurable function

$$\phi : \Omega \rightarrow L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \quad \text{such that} \quad P^{\phi(\omega)} = \nu(\omega) \quad \text{for } P\text{-a.e. } \omega \in \Omega.$$

To this end, consider the set

$$A = \{(\omega, \alpha) \in \Omega \times L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) : \nu(\omega) = P^\alpha\}.$$

We have seen in Lemma II.3.1 that $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ is a Borel space. On the other hand, we have from Lemma II.3.2 that $\alpha \mapsto P^\alpha$ is Borel, and ν is Borel by assumption. Hence, A is a Borel subset of $\Omega \times L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$. As a result, we can use the Jankov–von Neumann theorem [6, Proposition 7.49, p. 182] to obtain an analytically measurable map ϕ from the Ω -projection of A to $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ whose graph is contained in A ; that is,

$$\phi : \{\omega \in \Omega : \nu(\omega) \in \mathfrak{P}_S\} \rightarrow L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \quad \text{such that} \quad P^{\phi(\cdot)} = \nu(\cdot).$$

Since ϕ is, in particular, universally measurable, and since $\nu(\cdot) \in \mathfrak{P}_S$ P -a.s., we can alter ϕ on a P -nullset to obtain a Borel-measurable map

$$\phi : \Omega \rightarrow L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}) \quad \text{such that} \quad P^{\phi(\cdot)} = \nu(\cdot) \quad P\text{-a.s.}$$

Finally, we can replace ϕ by $\omega \mapsto \phi(\omega_{\wedge \tau(\omega)})$, then we have the required \mathcal{F}_τ -measurability as a consequence of Galmarino's test. Moreover, since $A \in \mathcal{F}_\tau \otimes \mathcal{B}(L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0}))$ due to the \mathcal{F}_τ -measurability of ν , Galmarino's test also shows that we still have $P^{\phi(\cdot)} = \nu(\cdot)$ P -a.s., which completes the second step of the proof.

Step 3. It remains to construct $\hat{\alpha} \in L_{prog}^0(\bar{\Omega}; \mathbb{S})$ as postulated in Step 1. While the map ϕ constructed in Step 2 will eventually yield the processes $\hat{\alpha}^\omega$

defined in Step 1, we note that ϕ is a map into a *space of processes* and so we still have to glue its values into an actual process. This is simple when there are only countably many values; therefore, following a construction of [68], we use an approximation argument.

Since $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ is separable (always for the metric introduced in (3.1)), we can construct for any $n \in \mathbb{N}$ a countable Borel partition $(A^{n,k})_{k \geq 1}$ of $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ such that the diameter of $A^{n,k}$ is smaller than $1/n$. Moreover, we fix $\gamma^{n,k} \in A^{n,k}$ for $k \geq 1$. Then,

$$\phi_n(\omega) := \sum_{k \geq 1} \gamma^{n,k} \mathbf{1}_{A^{n,k}}(\phi(\omega))$$

satisfies

$$\sup_{\omega \in \Omega} d(\phi_n(\omega), \phi(\omega)) \leq \frac{1}{n}; \quad (3.3)$$

that is, ϕ_n converges uniformly to ϕ , as an $L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ -valued map.

Let α and $\tilde{\tau} = \tau \circ X^\alpha$ be as in Step 1. Moreover, for any stopping time σ , denote

$$\omega^\sigma := \omega_{\cdot + \sigma(\omega)} - \omega_{\sigma(\omega)}, \quad \omega \in \Omega.$$

Then, for fixed n , the process

$$\begin{aligned} (\omega, s) &\mapsto \hat{\alpha}_s^n(\omega) := \mathbf{1}_{[\tilde{\tau}(\omega), \infty)}(s) [\phi_n(X^\alpha(\omega))]_{s - \tilde{\tau}(\omega)}(\omega^{\tilde{\tau}(\omega)}) \\ &\equiv \mathbf{1}_{[\tilde{\tau}(\omega), \infty)}(s) \sum_{k \geq 1} \gamma_{s - \tilde{\tau}(\omega)}^{n,k}(\omega^{\tilde{\tau}(\omega)}) \mathbf{1}_{A^{n,k}}(\phi(X^\alpha(\omega))) \end{aligned}$$

is well defined P_0 -a.s., and in fact an element of the Polish space $L_{prog}^0(\bar{\Omega}; \mathbb{S})$. We show that $(\hat{\alpha}^n)$ is a Cauchy sequence and that the limit $\hat{\alpha}$ yields the desired process. Fix $T \in \mathbb{R}_+$ and $m, n \in \mathbb{N}$, then (3.3) implies that

$$\int_{\Omega} \int_0^T \mathbf{1} \wedge |[\phi_m(\omega)]_s(\omega') - [\phi_n(\omega)]_s(\omega')| ds P_0(d\omega') \leq c_T \left(\frac{1}{m} + \frac{1}{n} \right) \quad (3.4)$$

for all $\omega \in \Omega$, where c_T is an unimportant constant coming from the definition of d in (3.1). In particular,

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \int_0^T \mathbf{1} \wedge |[\phi_m(X^\alpha(\omega))]_s(\omega') - [\phi_n(X^\alpha(\omega))]_s(\omega')| ds P_0(d\omega') P_0(d\omega) \\ \leq c_T \left(\frac{1}{m} + \frac{1}{n} \right). \end{aligned} \quad (3.5)$$

Since P_0 is the Wiener measure, we have the formula

$$\int_{\Omega} g(\omega_{\cdot \wedge \tilde{\tau}(\omega)}, \omega^{\tilde{\tau}}) P_0(d\omega) = \int_{\Omega} \int_{\Omega} g(\omega_{\cdot \wedge \tilde{\tau}(\omega)}, \omega') P_0(d\omega') P_0(d\omega)$$

for any bounded, progressively measurable functional g on $\Omega \times \Omega$. As ϕ is \mathcal{F}_τ -measurable, we conclude from (3.5) that

$$\int_{\Omega} \int_0^T 1 \wedge |\hat{\alpha}_s^m(\omega) - \hat{\alpha}_s^n(\omega)| ds P_0(d\omega) \leq c_T \left(\frac{1}{m} + \frac{1}{n} \right). \quad (3.6)$$

This implies that $(\hat{\alpha}^n)$ is Cauchy for the metric d . Let $\hat{\alpha} \in L_{prog}^0(\bar{\Omega}; \mathbb{S})$ be the limit. Then, using again the same formula, we obtain that

$$\phi_n(X^\alpha(\omega)) = \hat{\alpha}_{+\bar{\tau}(\omega)}^n(\omega \otimes_{\bar{\tau}} \cdot) \rightarrow \hat{\alpha}_{+\bar{\tau}(\omega)}(\omega \otimes_{\bar{\tau}} \cdot) \equiv \hat{\alpha}^\omega$$

with respect to d , for P_0 -a.e. $\omega \in \Omega$, after passing to a subsequence. On the other hand, by (3.3), we also have $\phi_n(X^\alpha(\omega)) \rightarrow \phi(X^\alpha(\omega))$ for P_0 -a.e. $\omega \in \Omega$. Hence,

$$\hat{\alpha}^\omega = \phi(X^\alpha(\omega))$$

for P_0 -a.e. $\omega \in \Omega$. In view of Step 2, we have $\phi(X^\alpha(\omega)) \in L_{loc}^1(\bar{\Omega}; \mathbb{S}^{>0})$ and $P\phi(X^\alpha(\omega)) = \nu(X^\alpha(\omega))$ for P_0 -a.e. $\omega \in \Omega$. Hence, $\hat{\alpha}$ satisfies all requirements from Step 1 and the proof is complete.

II.4 Proof of Theorem II.2.3

We note that one inequality in Theorem II.2.3 is trivial: if $x \in \mathbb{R}$ and there exists $H \in \mathcal{H}$ such that $x + \int_0^T H dB \geq \xi$, the supermartingale property stated in the definition of \mathcal{H} implies that $x \geq E^P[\xi]$ for all $P \in \mathfrak{P}$. Hence, our aim in this section is to show that there exists $H \in \mathcal{H}$ such that

$$\sup_{P \in \mathfrak{P}} E^P[\xi] + \int_0^T H_u dB_u \geq \xi \quad P\text{-a.s. for all } P \in \mathfrak{P}. \quad (4.1)$$

The line of argument (see also the Introduction) is similar as in [72] or [48]; hence, we shall be brief.

We first recall the following known result (e.g., [27, Theorem 1.5], [70, Lemma 8.2], [48, Lemma 4.4]) about the P -augmentation \mathbb{F}^P of \mathbb{F} ; it is the main motivation to work with \mathfrak{P}_S as the basic set of scenarios. We denote by $\mathbb{G}_+ = \{\mathcal{G}_{t+}\}_{0 \leq t \leq T}$ the minimal right-continuous filtration containing \mathbb{G} .

Lemma II.4.1. *Let $P \in \mathfrak{P}_S$. Then \mathbb{F}^P is right-continuous and in particular contains \mathbb{G}_+ . Moreover, P has the predictable representation property; i.e., for any right-continuous (\mathbb{F}^P, P) -local martingale M there exists an \mathbb{F}^P -predictable process H such that $M = M_0 + \int H dB$, P -a.s.*

We recall our assumption that $\sup_{P \in \mathfrak{P}} E^P[|\xi|] < \infty$ and that ξ is \mathcal{G}_T -measurable. We also recall from Proposition II.2.2 that the random variable

$$\mathcal{E}_t(\xi)(\omega) := \sup_{P \in \mathfrak{P}(t, \omega)} E^P[\xi^{t, \omega}]$$

is \mathcal{G}_t -measurable for all $t \in \mathbb{R}_+$. Moreover, we note that $\mathcal{E}_T(\xi) = \xi$ P -a.s. for all $P \in \mathfrak{P}$. Indeed, for any fixed $P \in \mathfrak{P}$, Lemma II.4.1 implies that we can find an \mathcal{F}_T -measurable function ξ' which is equal to ξ outside a P -nullset $N \in \mathcal{F}_T$, and now the definition of $\mathcal{E}_T(\xi)$ and Galmarino's test show that $\mathcal{E}_T(\xi) = \mathcal{E}_T(\xi') = \xi' = \xi$ outside N .

Step 1. We fix t and show that $\sup_{P \in \mathfrak{P}} E^P[|\mathcal{E}_t(\xi)|] < \infty$. Note that $|\xi|$ need not be upper semianalytic, so that the claim does not follow directly from (2.4). Hence, we make a small detour and first observe that \mathfrak{P} is stable in the following sense: if $P \in \mathfrak{P}$, $\Lambda \in \mathcal{F}_t$ and $P_1, P_2 \in \mathfrak{P}(t; P)$ (notation from Proposition II.2.2), the measure \bar{P} defined by

$$\bar{P}(A) := E^P[P_1(A|\mathcal{F}_t)\mathbf{1}_\Lambda + P_2(A|\mathcal{F}_t)\mathbf{1}_{\Lambda^c}], \quad A \in \mathcal{F}$$

is again an element of \mathfrak{P} . Indeed, this follows from (A2) and (A3) as

$$\bar{P}(A) = \iint (\mathbf{1}_A)^{t,\omega}(\omega') \nu(\omega, d\omega') P(d\omega)$$

for the kernel $\nu(\omega, d\omega') = P_1^{t,\omega}(d\omega') \mathbf{1}_\Lambda(\omega) + P_2^{t,\omega}(d\omega') \mathbf{1}_{\Lambda^c}(\omega)$. Following a standard argument, this stability implies that for any $P \in \mathfrak{P}$, there exist $P_n \in \mathfrak{P}(t; P)$ such that

$$E^{P_n}[|\xi| |\mathcal{F}_t] \nearrow \operatorname{ess\,sup}_{P' \in \mathfrak{P}(t; P)}^P E^{P'}[|\xi| |\mathcal{F}_t] \quad P\text{-a.s.}$$

Since (2.5), applied with $\tau = T$, yields that

$$E^P[|\mathcal{E}_t(\xi)|] = E^P \left[\left| \operatorname{ess\,sup}_{P' \in \mathfrak{P}(t; P)}^P E^{P'}[\xi | \mathcal{F}_t] \right| \right] \leq E^P \left[\operatorname{ess\,sup}_{P' \in \mathfrak{P}(t; P)}^P E^{P'}[|\xi| | \mathcal{F}_t] \right],$$

monotone convergence then allows us to conclude that

$$E^P[|\mathcal{E}_t(\xi)|] \leq \lim_{n \rightarrow \infty} E^{P_n}[|\xi|] \leq \sup_{P \in \mathfrak{P}} E^P[|\xi|] < \infty.$$

Step 2. We show that the right limit $Y_t := \mathcal{E}_{t+}(\xi)$ defines a (\mathbb{G}_+, P) -supermartingale for all $P \in \mathfrak{P}$. Indeed, Step 1 and (2.5) show that $\mathcal{E}_t(\xi)$ is an (\mathbb{F}^*, P) -supermartingale for all $P \in \mathfrak{P}$. The standard modification theorem for supermartingales [13, Theorem VI.2] then yields that Y is well defined P -a.s. and that Y is a (\mathbb{G}_+, P) -supermartingale for all $P \in \mathfrak{P}$, where the second conclusion uses Lemma II.4.1. We omit the details; they are similar as in the proof of [48, Proposition 4.5].

For later use, let us also establish the inequality

$$Y_0 \leq \sup_{P' \in \mathfrak{P}} E^{P'}[\xi] \quad P\text{-a.s.} \quad \text{for all } P \in \mathfrak{P}. \quad (4.2)$$

Indeed, let $P \in \mathfrak{P}$. Then [13, Theorem VI.2] shows that

$$E^P[Y_0|\mathcal{F}_0] \leq \mathcal{E}_0(\xi) \quad P\text{-a.s.},$$

where, of course, we have $E^P[Y_0|\mathcal{F}_0] = E^P[Y_0]$ P -a.s. since $\mathcal{F}_0 = \{\emptyset, \Omega\}$. However, as Y_0 is \mathcal{G}_{0+} -measurable and \mathcal{G}_{0+} is P -a.s. trivial by Lemma II.4.1, we also have that $Y_0 = E^P[Y_0]$ P -a.s. In view of the definition of $\mathcal{E}_0(\xi)$, the inequality (4.2) follows.

Step 3. Next, we construct the process $H \in \mathcal{H}$. In view of Step 2, we can fix $P \in \mathfrak{P}$ and consider the Doob–Meyer decomposition $Y = Y_0 + M^P - K^P$ under P , in the filtration \mathbb{F}^P . By Lemma II.4.1, the local martingale M^P can be represented as an integral, $M^P = \int H^P dB$, for some \mathbb{F}^P -predictable integrand H^P . The crucial observation (due to [72]) is that this process can be described via $d\langle Y, B \rangle = H^P d\langle B \rangle$, and that, as the quadratic co-variation processes can be constructed pathwise by Bichteler’s integral [8, Theorem 7.14], this relation allows to define a process H such that $H = H^P$ $P \times dt$ -a.e. for all $P \in \mathfrak{P}$. More precisely, since $\langle Y, B \rangle$ is continuous, it is not only adapted to \mathbb{G}_+ , but also to \mathbb{G} , and hence we see by going through the arguments in the proof of [48, Proposition 4.11] that H can be obtained as a \mathbb{G} -predictable process in our setting. To conclude that $H \in \mathcal{H}$, note that for every $P \in \mathfrak{P}$, the local martingale $\int H dB$ is P -a.s. bounded from below by the martingale $E^P[\xi|\mathbb{G}]$; hence, on the compact $[0, T]$, it is a supermartingale as a consequence of Fatou’s lemma. Summing up, we have found $H \in \mathcal{H}$ such that

$$Y_0 + \int_0^T H_u dB_u \geq Y_T = \mathcal{E}_{T+}(\xi) = \xi \quad P\text{-a.s.} \quad \text{for all } P \in \mathfrak{P},$$

and in view of (4.2), this implies (4.1).

Chapter III

Measurability of Semimartingale Characteristics with Respect to the Law

Given a càdlàg process X on a filtered measurable space, we construct in this chapter, which corresponds to the article [38], a version of its semimartingale characteristics which is measurable with respect to the underlying probability law. More precisely, let \mathfrak{P}_{sem} be the set of all probability measures P under which X is a semimartingale. We construct processes (B^P, C, ν^P) which are jointly measurable in time, space, and the probability law P , and are versions of the semimartingale characteristics of X under P for each $P \in \mathfrak{P}_{sem}$. The second characteristic C can be constructed as a single process not depending on P . A similar result is obtained for the differential characteristics.

III.1 Introduction

We study the measurability of semimartingale characteristics with respect to the probability law. Consider a càdlàg process X on a filtered measurable space $(\Omega, \mathcal{F}, \mathbb{F})$, where Ω is a separable metric space, $\mathcal{F} = \mathcal{B}(\Omega)$ the corresponding Borel σ -field and each σ -field \mathcal{F}_t of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is separable. Our main result (Theorem III.2.5) states that the set \mathfrak{P}_{sem} of all semimartingale laws is Borel-measurable and that there exists a Borel-measurable map

$$\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d \times \mathbb{S}_+^d \times \bar{\mathcal{L}}, \quad (P, \omega, t) \mapsto (B_t^P(\omega), C_t(\omega), \nu^P(\omega))$$

such that (B^P, C, ν^P) are P -semimartingale characteristics of X for each $P \in \mathfrak{P}_{sem}$, where $\bar{\mathcal{L}}$ is the space of Lévy measures on $\mathbb{R}_+ \times \mathbb{R}^d$. A similar result is obtained for the differential characteristics. Indeed, Theorem III.2.6 states that the set \mathfrak{P}_{sem}^{ac} of all semimartingale laws with absolutely continuous

characteristics (with respect to the Lebesgue measure) is Borel-measurable and that there exists a Borel-measurable map

$$\mathfrak{P}_{sem}^{ac} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}, \quad (P, \omega, t) \mapsto (b_t^P(\omega), c_t(\omega), F_{\omega,t}^P)$$

such that (b^P, c, F^P) are P -differential characteristics of X for each $P \in \mathfrak{P}_{sem}^{ac}$, where \mathcal{L} is the space of Lévy measures on \mathbb{R}^d . The second characteristic C (and hence also c) can be constructed as a single process not depending on P ; roughly speaking, this is possible because two measures under which X has different diffusion are necessarily singular. By contrast, the first and the third characteristic have to depend on P as they are predictable compensators. We point out that the conditions on X and $(\Omega, \mathcal{F}, \mathbb{F})$ are satisfied in particular when X is the coordinate-mapping process on Skorohod space and \mathbb{F} is the filtration generated by X .

Our construction of the characteristics proceeds through versions of the classical results on the structure of semimartingales, such as the Doob–Meyer theorem, with an additional measurable dependence on the law P . The starting point is that for discrete-time processes, the Doob decomposition can be constructed explicitly and of course all adapted processes are semimartingales. Thus, the passage to the continuous-time limit is the main obstacle, just like in the classical theory of semimartingales. We have found the recent proofs of [4, 5] for the Doob–Meyer and the Bichteler–Dellacherie theorem to be particularly useful as they are built around a compactness argument for which we can provide a measurable version.

The remainder of this chapter is organized as follows. In Section III.2, we describe the setting and terminology in some detail (mainly because we cannot work with the “usual assumptions”) and proceed to state the main results. Section III.3 contains some auxiliary results, in particular a version of Alaoglu’s theorem for $L^2(P)$ which allows to choose convergent subsequences that depend measurably on P . The measurability of the set of all semimartingale laws is proved in Section III.4. In Section III.5, we show that the Doob–Meyer decomposition can be chosen to be measurable with respect to P and deduce corresponding results for the compensator of a process with integrable variation and the canonical decomposition of a bounded semimartingale. Using these tools, the jointly measurable version of the characteristics is constructed in Section III.6, whereas the corresponding results for the differential characteristics are obtained in the concluding Section III.7.

III.2 Main Results

III.2.1 Basic Definitions and Notation

Let (Ω, \mathcal{F}) be a measurable space and let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtration of sub- σ -fields of \mathcal{F} . A process $X = (X_t)$ is called right-continuous if *all* its paths are

right-continuous. In the presence of a probability measure P , we shall say that X is P -a.s. right-continuous if P -almost all paths are right-continuous; the same convention is used for other path properties such as being càdlàg, of finite variation, etc.

We denote by $\mathbb{F}_+ := (\mathcal{F}_{t+})$ the right-continuous version of \mathbb{F} , defined by $\mathcal{F}_{t+} = \bigcap_{u>t} \mathcal{F}_u$. Similarly, the left-continuous version is $\mathbb{F}_- = (\mathcal{F}_{t-})$. For $t = 0$, we use the convention $\mathcal{F}_{0-} = \mathcal{F}_{(0+)-} = \{\emptyset, \Omega\}$. As a result, the predictable σ -field \mathcal{P} of \mathbb{F} on $\Omega \times \mathbb{R}_+$, generated by the \mathbb{F}_- -adapted processes which are left-continuous on $(0, \infty)$, coincides with the predictable σ -field of \mathbb{F}_+ ; this fact will be used repeatedly without further mention. Given a probability measure P , the augmentation $\mathbb{F}_+^P = (\mathcal{F}_{t+}^P)$ of \mathbb{F}_+ , also called the usual augmentation of \mathbb{F} , is obtained by adjoining all P -nullsets of (Ω, \mathcal{F}) to \mathcal{F}_{t+} for all t , including $t = 0-$. The corresponding predictable σ -field will be denoted by \mathcal{P}^P .

Finally, $\mathfrak{P}(\Omega)$ is the set of all probability measures on (Ω, \mathcal{F}) . In most of this chapter, Ω will be a separable metric space and \mathcal{F} its Borel σ -field. In this case, $\mathfrak{P}(\Omega)$ is a separable metric space for the weak convergence of probability measures and its Borel σ -field $\mathcal{B}(\mathfrak{P}(\Omega))$ coincides with the one generated by the maps $P \mapsto P(A)$, $A \in \mathcal{F}$. Unless otherwise mentioned, any metric space is equipped with its Borel σ -field. Similarly, product spaces are always equipped with their product σ -fields and measurability then refers to joint measurability.

It will be convenient to define the integral of any (appropriately measurable) function f taking values in the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$, regardless of its integrability. For instance, the expectation under a probability measure P is defined by $E^P[f] := E^P[f^+] - E^P[f^-]$; here and everywhere else, the convention

$$\infty - \infty = -\infty$$

is used. Similarly, conditional expectations are also defined for $\overline{\mathbb{R}}$ -valued functions.

Definition III.2.1. Let $(\Omega, \mathcal{G}, \mathbb{G}, P)$ be a filtered probability space. A \mathbb{G} -adapted stochastic process $X : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ with càdlàg paths is a P - \mathbb{G} -semimartingale if there exist right-continuous, \mathbb{G} -adapted processes M and A with $M_0 = A_0 = 0$ such that M is a P - \mathbb{G} -local martingale, A has paths of (locally) finite variation P -a.s., and

$$X = X_0 + M + A \quad P\text{-a.s.}$$

The dimension $d \in \mathbb{N}$ is fixed throughout. Fix also a truncation function $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$; that is, a bounded measurable function such that $h(x) = x$ in a neighborhood of the origin. The characteristics of a semimartingale X on $(\Omega, \mathcal{G}, \mathbb{G}, P)$ are a triplet (B, C, ν) of processes defined as follows. First,

consider the càdlàg process

$$\tilde{X}_t := X_t - X_0 - \sum_{0 \leq s \leq t} (\Delta X_s - h(\Delta X_s)),$$

which has bounded jumps. This process has a (P -a.s. unique) canonical decomposition $\tilde{X} = M' + B'$, where M' and B' have the same properties as the processes in Definition III.2.1, but in addition B' is predictable. (See [78, Theorem 7.2.6, p. 160] for the existence of the canonical decomposition in a general filtration.) Moreover, let μ^X be the integer-valued random measure associated with the jumps of X ,

$$\mu^X(\omega, dt, dx) = \sum_{s \geq 0} \mathbf{1}_{\{\Delta X_s(\omega) \neq 0\}} \mathbf{1}_{(s, \Delta X_s(\omega))}(dt, dx).$$

Processes (B, C, ν) with values in \mathbb{R}^d , $\mathbb{R}^{d \times d}$, and the set of measures on $\mathbb{R}_+ \times \mathbb{R}^d$, respectively, will be called characteristics of X (relative to h) if $B = B'$ P -a.s., C equals the predictable covariation process of the continuous local martingale part of M' P -a.s., and ν equals the predictable compensator of μ^X P -a.s. All these notions are relative to the given filtration \mathbb{G} which, in the sequel, will be either the basic filtration \mathbb{F} , its right-continuous version \mathbb{F}_+ , or its usual augmentation \mathbb{F}_+^P . Our first observation is that the characteristics do not depend on this choice.

Proposition III.2.2. *Let X be a càdlàg, \mathbb{R}^d -valued, \mathbb{F} -adapted process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The following are equivalent:*

- (i) X is an \mathbb{F} -semimartingale,
- (ii) X is an \mathbb{F}_+ -semimartingale,
- (iii) X is an \mathbb{F}_+^P -semimartingale.

Moreover, the semimartingale characteristics associated with these filtrations are the same.

The proof is stated in Section III.4. In order to study the measurability of the third characteristic ν , we introduce a σ -field on the set of Lévy measures; namely, we shall use the Borel σ -field associated with a natural metric that we define next. Given a metric space Ω' , let $\mathfrak{M}(\Omega')$ denote the set of all (nonnegative) measures on $(\Omega', \mathcal{B}(\Omega'))$. We introduce the set of Lévy measures on \mathbb{R}^d ,

$$\mathcal{L} = \left\{ \nu \in \mathfrak{M}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |x|^2 \wedge 1 \nu(dx) < \infty \text{ and } \nu(\{0\}) = 0 \right\},$$

as well as their analogues on $\mathbb{R}_+ \times \mathbb{R}^d$,

$$\bar{\mathcal{L}} = \left\{ \nu \in \mathfrak{M}(\mathbb{R}_+ \times \mathbb{R}^d) \mid \int_0^N \int_{\mathbb{R}^d} |x|^2 \wedge 1 \nu(dt, dx) < \infty \forall N \in \mathbb{N}, \right. \\ \left. \nu(\{0\} \times \mathbb{R}^d) = \nu(\mathbb{R}_+ \times \{0\}) = 0 \right\}. \quad (2.1)$$

The space $\mathfrak{M}^f(\mathbb{R}^d)$ of all finite measures on \mathbb{R}^d is a separable metric space under a metric $d_{\mathfrak{M}^f(\mathbb{R}^d)}$ which induces the weak convergence relative to $C_b(\mathbb{R}^d)$; cf. [9, Theorem 8.9.4, p. 213]; this topology is the natural extension of the more customary weak convergence of probability measures. With any $\mu \in \mathcal{L}$, we can associate a finite measure

$$A \mapsto \int_A |x|^2 \wedge 1 \mu(dx), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

denoted by $|x|^2 \wedge 1 \cdot \mu$ for brevity. We can then define a metric $d_{\mathcal{L}}$ on \mathcal{L} via

$$d_{\mathcal{L}}(\mu, \nu) = d_{\mathfrak{M}^f(\mathbb{R}^d)}(|x|^2 \wedge 1 \cdot \mu, |x|^2 \wedge 1 \cdot \nu), \quad \mu, \nu \in \mathcal{L}.$$

We proceed similarly with $\bar{\mathcal{L}}$. First, given $N > 0$, let $\bar{\mathcal{L}}_N$ be the restriction of $\bar{\mathcal{L}}$ to $[0, N] \times \mathbb{R}^d$. For any $\mu \in \bar{\mathcal{L}}_N$, let $|x|^2 \wedge 1 \cdot \mu$ be the finite measure

$$A \mapsto \int_A |x|^2 \wedge 1 \mu(dt, dx), \quad A \in \mathcal{B}([0, N] \times \mathbb{R}^d);$$

then we can again define a metric

$$d_{\bar{\mathcal{L}}_N}(\mu, \nu) = d_{\mathfrak{M}^f([0, N] \times \mathbb{R}^d)}(|x|^2 \wedge 1 \cdot \mu, |x|^2 \wedge 1 \cdot \nu), \quad \mu, \nu \in \bar{\mathcal{L}}_N.$$

Finally, we can metrize $\bar{\mathcal{L}}$ by

$$d_{\bar{\mathcal{L}}}(\mu, \nu) = \sum_{N \in \mathbb{N}} 2^{-N} (1 \wedge d_{\bar{\mathcal{L}}_N}(\mu, \nu)), \quad \mu, \nu \in \bar{\mathcal{L}}.$$

Lemma III.2.3. *The pairs $(\mathcal{L}, d_{\mathcal{L}})$, $(\bar{\mathcal{L}}_N, d_{\bar{\mathcal{L}}_N})$, $(\bar{\mathcal{L}}, d_{\bar{\mathcal{L}}})$ are separable metric spaces.*

This is proved by reducing to the properties of \mathfrak{M}^f ; we omit the details. The above metrics define the Borel structures $\mathcal{B}(\mathcal{L})$, $\mathcal{B}(\bar{\mathcal{L}}_N)$ and $\mathcal{B}(\bar{\mathcal{L}})$. Alternatively, we could have defined the σ -fields through the following result, which will be useful later on.

Lemma III.2.4. *Let (Y, \mathcal{Y}) be a measurable space and consider a function $\kappa : Y \rightarrow \bar{\mathcal{L}}$, $y \mapsto \kappa(y, dt, dx)$. The following are equivalent:*

- (i) $\kappa : (Y, \mathcal{Y}) \rightarrow (\bar{\mathcal{L}}, \mathcal{B}(\bar{\mathcal{L}}))$ is measurable,
- (ii) for all measurable functions $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$,

$$(Y, \mathcal{Y}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})), \quad y \mapsto \int_0^\infty \int_{\mathbb{R}^d} (|x|^2 \wedge 1) f(t, x) \kappa(y, dt, dx)$$

is measurable.

Corresponding assertions hold for \mathcal{L} and $\bar{\mathcal{L}}_N$.

Proof. A similar result is standard, for instance, for the set of probability measures on a Polish space; cf. [6, Proposition 7.25, p. 133]. The arguments in this reference can be adapted to the space $\bar{\mathcal{L}}_N$ by using the facts stated in [9, Chapter 8]. Then, one can lift the result to the space $\bar{\mathcal{L}}$. We omit the details. \square

III.2.2 Main Results

We can now state our main result, the existence of a jointly measurable version $(P, \omega, t) \mapsto (B_t^P(\omega), C_t(\omega), \nu^P(\omega))$ of the characteristics of a process X under a family of measures P . Here the second characteristic C is a single process not depending on P ; roughly speaking, this is possible because two measures under which X has different diffusion are necessarily singular. By contrast, the first and the third characteristic have to depend on P in all nontrivial cases: in general, two equivalent measures will lead to different drifts and compensators, so that the families $(B^P)_P$ and $(\nu^P)_P$ are not consistent with respect to P and cannot be aggregated into single processes. We write \mathbb{S}_+^d for the set of symmetric nonnegative definite $d \times d$ -matrices.

Theorem III.2.5. *Let X be a càdlàg, \mathbb{F} -adapted, \mathbb{R}^d -valued process on a filtered measurable space $(\Omega, \mathcal{F}, \mathbb{F})$, where Ω is a separable metric space, $\mathcal{F} = \mathcal{B}(\Omega)$ and each σ -field \mathcal{F}_t of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is separable. Then the set*

$$\mathfrak{P}_{sem} = \{P \in \mathfrak{P}(\Omega) \mid X \text{ is a semimartingale on } (\Omega, \mathcal{F}, \mathbb{F}, P)\} \subseteq \mathfrak{P}(\Omega)$$

is Borel-measurable and there exists a Borel-measurable map

$$\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d \times \mathbb{S}_+^d \times \bar{\mathcal{L}}, \quad (P, \omega, t) \mapsto (B_t^P(\omega), C_t(\omega), \nu^P(\omega))$$

such that for each $P \in \mathfrak{P}_{sem}$,

- (i) (B^P, C, ν^P) are P -semimartingale characteristics of X ,
- (ii) B^P is \mathbb{F}_+ -adapted, \mathbb{F}_+^P -predictable and has right-continuous, P -a.s. finite variation paths,
- (iii) C is \mathbb{F} -predictable and has P -a.s. continuous, increasing paths¹ in \mathbb{S}_+^d ,
- (iv) ν^P is an \mathbb{F}_+^P -predictable random measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

Moreover, there exists a decomposition

$$\nu^P(\cdot, dt, dx) = K^P(\cdot, t, dx) dA_t^P \quad P\text{-a.s.},$$

where

- (v) $(P, \omega, t) \mapsto A_t^P(\omega)$ is Borel-measurable and for all $P \in \mathfrak{P}_{sem}$, A^P is an \mathbb{F}_+ -adapted, \mathbb{F}_+^P -predictable, P -integrable process with right-continuous and P -a.s. increasing paths,
- (vi) $(P, \omega, t) \mapsto K^P(\omega, t, dx)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+, \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$ and for all $P \in \mathfrak{P}_{sem}$, $(\omega, t) \mapsto K^P(\omega, t, dx)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\Omega \times \mathbb{R}_+, \mathcal{P}^P)$.

¹Alternately, one can construct C such that all paths are continuous and increasing, at the expense of being predictable in a slightly larger filtration. See Proposition III.6.6.

The measurability of \mathfrak{P}_{sem} is proved in Section III.4, whereas the characteristics are constructed in Section III.6. We remark that the conditions of the theorem are satisfied in particular when X is the coordinate-mapping process on Skorohod space and \mathbb{F} is the filtration generated by X . This is by far the most important example—the slightly more general situation in the theorem does not cause additional work.

Of course, we are particularly interested in measures P such that the characteristics are absolutely continuous with respect to the Lebesgue measure dt on \mathbb{R}_+ ; that is, the set

$$\mathfrak{P}_{sem}^{ac} = \{P \in \mathfrak{P}_{sem} \mid (B^P, C, \nu^P) \ll dt, P\text{-a.s.}\}.$$

(Absolute continuity does not depend on the choice of the truncation function h ; cf. [26, Proposition 2.24, p. 81].) Given a triplet of absolutely continuous characteristics, the corresponding derivatives (defined dt -a.e.) are called the differential characteristics of X and denoted by (b^P, c, F^P) .

Theorem III.2.6. *Let X and $(\Omega, \mathcal{F}, \mathbb{F})$ be as in Theorem III.2.5. Then the set*

$$\mathfrak{P}_{sem}^{ac} = \{P \in \mathfrak{P}_{sem} \mid (B^P, C, \nu^P) \ll dt, P\text{-a.s.}\}$$

is Borel-measurable and there exists a Borel-measurable map

$$\mathfrak{P}_{sem}^{ac} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}, \quad (P, \omega, t) \mapsto (b_t^P(\omega), c_t(\omega), F_{\omega,t}^P)$$

such that for each $P \in \mathfrak{P}_{sem}^{ac}$,

- (i) (b^P, c, F^P) are P -differential characteristics of X ,
- (ii) b^P is \mathbb{F} -predictable,
- (iii) c is \mathbb{F} -predictable,
- (iv) $(\omega, t) \mapsto F_{\omega,t}^P(dx)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\Omega \times \mathbb{R}_+, \mathcal{P})$.

In applications, we are interested in constraining the set \mathfrak{P}_{sem}^{ac} via the values of the differential characteristics. Given a collection $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ of Lévy triplets, we let

$$\mathfrak{P}_{sem}^{ac}(\Theta) = \{P \in \mathfrak{P}_{sem}^{ac} \mid (b^P, c, F^P) \in \Theta, P \otimes dt\text{-a.e.}\}.$$

Corollary III.2.7. *Let X and $(\Omega, \mathcal{F}, \mathbb{F})$ be as in Theorem III.2.5. Then $\mathfrak{P}_{sem}^{ac}(\Theta)$ is Borel-measurable whenever $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ is Borel-measurable.*

The proofs for Theorem III.2.6 and Corollary III.2.7 are stated in Section III.7.

Remark III.2.8. The arguments in the subsequent sections yield similar results when X is \mathbb{F}_+ -adapted (instead of \mathbb{F} -adapted), or if X is replaced by an appropriately measurable family $(X^P)_P$ as in Proposition III.5.1 below—we have formulated the main results in the setting which is most appropriate for the applications we have in mind.

III.3 Auxiliary Results

This section is a potpourri of tools that will be used repeatedly later on; they mainly concern the possibility of choosing $L^1(P)$ -convergent subsequences and limits in a measurable way (with respect to P). Another useful result concerns right-continuous modifications of processes.

Throughout this section, we assume the setting of Theorem III.2.5; that is, Ω is a separable metric space, $\mathcal{F} = \mathcal{B}(\Omega)$ and $\mathbb{F} = (\mathcal{F}_t)$ is a filtration such that \mathcal{F}_t is separable for all $t \geq 0$. Moreover, we fix a measurable set $\mathfrak{P} \subseteq \mathfrak{P}(\Omega)$; recall that $\mathfrak{P}(\Omega)$ carries the Borel structure induced by the weak convergence. (The results of this section also hold for a general measurable space (Ω, \mathcal{F}) if $\mathfrak{P}(\Omega)$ is instead endowed with the σ -field generated by the maps $P \mapsto P(A)$, $A \in \mathcal{F}$.)

As P plays the role of a measurable parameter, it is sometimes useful to consider the filtered measurable space

$$(\widehat{\Omega}, \widehat{\mathcal{F}}) := (\mathfrak{P} \times \Omega, \mathcal{B}(\mathfrak{P}) \otimes \mathcal{F}), \quad \widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t \geq 0}, \quad \widehat{\mathcal{F}}_t := \mathcal{B}(\mathfrak{P}) \otimes \mathcal{F}_t \quad (3.1)$$

and its right-continuous filtration $\widehat{\mathbb{F}}_+$; a few facts can be obtained simply by applying standard results in this extended space.

Lemma III.3.1. *Let $t \geq 0$ and let $f : \widehat{\Omega} \rightarrow \overline{\mathbb{R}}$ be measurable. Then the function $\mathfrak{P} \rightarrow \overline{\mathbb{R}}$, $P \mapsto E^P[f(P, \cdot)]$ is measurable. Moreover, there exist versions of the conditional expectations $E^P[f(P, \cdot) | \mathcal{F}_t]$ and $E^P[f(P, \cdot) | \mathcal{F}_{t+}]$ such that*

$$\widehat{\Omega} \rightarrow \overline{\mathbb{R}}, \quad (P, \omega) \mapsto E^P[f(P, \cdot) | \mathcal{F}_t](\omega), \quad (P, \omega) \mapsto E^P[f(P, \cdot) | \mathcal{F}_{t+}](\omega)$$

are measurable with respect to $\widehat{\mathcal{F}}_t$ and $\widehat{\mathcal{F}}_{t+}$, respectively, while for fixed $P \in \mathfrak{P}$,

$$\Omega \rightarrow \overline{\mathbb{R}}, \quad \omega \mapsto E^P[f(P, \cdot) | \mathcal{F}_t](\omega), \quad \omega \mapsto E^P[f(P, \cdot) | \mathcal{F}_{t+}](\omega)$$

are measurable with respect to \mathcal{F}_t and \mathcal{F}_{t+} , respectively.

Proof. It suffices to consider the case where f is bounded. We first show that $P \mapsto E^P[f(P, \cdot)]$ is measurable. By a monotone class argument, it suffices to consider a function f of the form $f(P, \omega) = g(P)h(\omega)$, where g and h are measurable. In this case, $P \mapsto E^P[f(P, \cdot)] = g(P)E^P[h]$, and $P \mapsto E^P[h]$ is measurable due to [6, Proposition 7.25, p. 133].

The construction of the conditional expectation follows the usual scheme. Fix $t \geq 0$, let $(A_n)_{n \in \mathbb{N}}$ be a sequence generating \mathcal{F}_t and let $(A_n^m)_m$ be a finite partition generating $\mathcal{A}_n := \sigma(A_1, \dots, A_n)$. Using the supermartingale convergence theorem as in [13, V.56, p. 50] and the convention $0/0 = 0$, we can define a version of the conditional expectation given \mathcal{F}_t by

$$E^P[f(P, \cdot) | \mathcal{F}_t] := \limsup_{n \rightarrow \infty} \sum_m \frac{E^P[f(P, \cdot) \mathbf{1}_{A_n^m}]}{P[A_n^m]} \mathbf{1}_{A_n^m}.$$

In view of the first part, this function is $\widehat{\mathcal{F}}_t$ -measurable, and \mathcal{F}_t -measurable for fixed P . Finally, using the backward martingale convergence theorem,

$$E^P[f(P, \cdot) | \mathcal{F}_{t+}] := \limsup_{n \rightarrow \infty} E^P[f(P, \cdot) | \mathcal{F}_{t+1/n}]$$

is a version of the conditional expectation given \mathcal{F}_{t+} having the desired properties. \square

In what follows, we shall always use the measurable versions of the conditional expectations as in Lemma III.3.1.

Lemma III.3.2. *Let $f^n : \widehat{\Omega} \rightarrow \overline{\mathbb{R}}^d$ be measurable functions such that $f^n(P, \cdot)$ is a convergent sequence in $L^1(P)$ for every $P \in \mathfrak{P}$. There exists a measurable function $f : \widehat{\Omega} \rightarrow \overline{\mathbb{R}}^d$ such that $f(P, \cdot) = \lim_n f^n(P, \cdot)$ in $L^1(P)$ for every $P \in \mathfrak{P}$. Moreover, there exists an increasing sequence $(n_k^P)_k \subseteq \mathbb{N}$ such that $P \mapsto n_k^P$ is measurable and $\lim_k f^{n_k^P}(P, \cdot) = f(P, \cdot)$ P -a.s. for all $P \in \mathfrak{P}$.*

Proof. For $P \in \mathfrak{P}$, let $n_0^P := 1$ and define recursively

$$\begin{aligned} \tilde{n}_k^P &:= \min \{n \in \mathbb{N} \mid \|f^u(P, \cdot) - f^v(P, \cdot)\|_{L^1(P)} \leq 2^{-k} \text{ for all } u, v \geq n\}, \\ n_k^P &:= \max \{\tilde{n}_k^P, n_{k-1}^P + 1\}. \end{aligned}$$

It follows from Lemma III.3.1 that $P \mapsto n_k^P$ is measurable, and so the composition $(P, \omega) \mapsto f^{n_k^P}(P, \omega)$ is again measurable. Moreover, we have

$$\sum_{k \geq 0} \|f^{n_{k+1}^P}(P, \cdot) - f^{n_k^P}(P, \cdot)\|_{L^1(P)} < \infty, \quad P \in \mathfrak{P}$$

by construction, which implies that $(f^{n_k^P}(P, \cdot))_{k \in \mathbb{N}}$ converges P -a.s. Thus, we can set (componentwise)

$$f(P, \omega) := \limsup_{k \rightarrow \infty} f^{n_k^P}(P, \omega)$$

to obtain a jointly measurable limit. \square

The next result is basically a variant of Alaoglu's theorem in L^2 (or the Dunford–Pettis theorem in L^1 , or Komlos' lemma) which yields measurability with respect to the underlying measure. It will be crucial to obtain measurable versions of the compactness arguments of semimartingale theory in the later sections. We denote by $\text{conv } A$ the convex hull of a set $A \subseteq \mathbb{R}^d$.

Proposition III.3.3. (i) *Let $f^n : \mathfrak{P} \times \Omega \rightarrow \mathbb{R}^d$ be a sequence of measurable functions such that*

$$\sup_{n \in \mathbb{N}} \|f^n(P, \cdot)\|_{L^2(P)} < \infty, \quad P \in \mathfrak{P}. \quad (3.2)$$

Then there exist measurable functions $P \mapsto N_n^P \in \{n, n+1, \dots\}$ and $P \mapsto \lambda_i^{P,n} \in [0, 1]$ satisfying $\sum_{i=n}^{N_n^P} \lambda_i^{P,n} = 1$ and $\lambda_i^{P,n} = 0$ for $i \notin \{n, \dots, N_n^P\}$ such that

$$(P, \omega) \mapsto g^{P,n}(\omega) := \sum_{i=n}^{N_n^P} \lambda_i^{P,n} f^i(P, \omega) \in \text{conv}\{f^n(P, \omega), f^{n+1}(P, \omega), \dots\}$$

is measurable and $(g^{P,n})_{n \in \mathbb{N}}$ converges in $L^2(P)$ for all $P \in \mathfrak{P}$.

(ii) For each $m \in \mathbb{N}$, let $(f_m^n)_{n \in \mathbb{N}}$ be a sequence as in (i). Then there exist N_n^P and $\lambda_i^{P,n}$ as in (i) such that

$$(P, \omega) \mapsto g_m^{P,n}(\omega) := \sum_{i=n}^{N_n^P} \lambda_i^{P,n} f_m^i(P, \omega) \in \text{conv}\{f_m^n(P, \omega), f_m^{n+1}(P, \omega), \dots\}$$

is measurable and $(g_m^{P,n})_{n \in \mathbb{N}}$ converges in $L^2(P)$ for all $P \in \mathfrak{P}$ and $m \in \mathbb{N}$.

(iii) Let $f^n : \mathfrak{P} \times \Omega \rightarrow \mathbb{R}^d$ be measurable functions such that

$$\{f^n(P, \cdot)\}_{n \in \mathbb{N}} \subseteq L^1(P) \quad \text{is uniformly integrable,} \quad P \in \mathfrak{P}.$$

Then the assertion of (i) holds with convergence in $L^1(P)$ instead of $L^2(P)$.

Proof. (i) For $n \in \mathbb{N}$, consider the sets

$$G^{P,n} = \text{conv}\{f^n(P, \cdot), f^{n+1}(P, \cdot), \dots\}, \quad P \in \mathfrak{P}.$$

Moreover, for $k \in \mathbb{N}$, let Λ_k^n be the (finite) set of all $\lambda = (\lambda_1, \lambda_2, \dots) \in [0, 1]^{\mathbb{N}}$ such that $\sum_i \lambda_i = 1$,

$$\lambda_i = \frac{a_i}{b_i} \quad \text{for some } a_i \in \{0, 1, \dots, b_i\}, \quad b_i \in \{1, 2, \dots, k\}$$

and $\lambda_i = 0$ for $i \notin \{n, \dots, n+k\}$. Thus,

$$g^P(\lambda) := \sum_{i \geq 1} \lambda_i f^i(P, \cdot) \in G^{P,n}$$

for all $\lambda \in \Lambda_k^n$. Let

$$\alpha_k^{P,n} = \min \{\|g^P(\lambda)\|_{L^2(P)} \mid \lambda \in \Lambda_k^n\}, \quad \alpha^{P,n} = \inf \{\|g\|_{L^2(P)} \mid g \in G^{P,n}\}$$

and $\alpha^P = \lim_n \alpha^{P,n}$; note that $(\alpha^{P,n})_n$ is increasing. We observe that any sequence $g^{P,n} \in G^{P,n}$ such that $\|g^{P,n}\|_{L^2(P)} \leq \alpha^{P,n} + 1/n$ is a Cauchy sequence in $L^2(P)$. Indeed, if $\varepsilon > 0$ is given and n is large, then $\|(g^{P,k} + g^{P,l})/2\|_{L^2(P)} \geq \alpha^P - \varepsilon$ for all $k, l \geq n$, which by the parallelogram identity yields that

$$\|g^{P,k} - g^{P,l}\|_{L^2(P)}^2 \leq 4(\alpha^{P,n} + 1/n)^2 - 4(\alpha^P - \varepsilon)^2.$$

As $\alpha^{P,n}$ tends to α^P , this shows the Cauchy property. To select such a sequence in a measurable way, we first observe that $(\alpha_k^{P,n})_k$ decreases to $\alpha^{P,n}$, due to (3.2). Thus,

$$k^{P,n} := \min \{k \in \mathbb{N} \mid |\alpha_k^{P,n} - \alpha^{P,n}| \leq 1/n\}$$

is well defined and finite. By Lemma III.3.1, $P \mapsto (\alpha_k^{P,n}, \alpha^{P,n})$ is measurable, and this implies that $P \mapsto k^{P,n}$ is measurable. Applying a selection theorem in the Polish space $[0, 1]^{\mathbb{N}}$ (e.g., [1, Theorem 18.13, p. 600]), we can find for each n a measurable minimizer $P \mapsto \widehat{\lambda}^{P,n}$ in the (finite) set $\Lambda_{k^{P,n}}^n$ such that

$$\|g^P(\widehat{\lambda}^{P,n})\|_{L^2(P)} = \alpha_{k^{P,n}}^{P,n} \equiv \min \{\|g^P(\lambda)\|_{L^2(P)} \mid \lambda \in \Lambda_{k^{P,n}}^n\}.$$

According to the above, $g^P(\widehat{\lambda}^{P,n})$ is Cauchy and so the result follows by setting $N_n^P = n + k^{P,n}$.

(ii) This assertion follows from (i) by a standard “diagonal argument.”

(iii) For $m, n \in \mathbb{N}$, define the function $f_m^n : \mathfrak{P} \times \Omega \rightarrow \mathbb{R}^d$ by

$$f_m^n(P, \omega) := f^n(P, \omega) \mathbf{1}_{\{|f^n(P, \omega)| \leq m\}}.$$

Then $\sup_{n \in \mathbb{N}} \|f_m^n(P, \cdot)\|_{L^2(P)} < \infty$ for each m . Thus, for each m , (ii) yields an $L^2(P)$ -convergent sequence

$$g_m^{P,n} = \sum_{i=n}^{N_n^P} \lambda_i^{P,n} f_m^i(P, \cdot)$$

with suitably measurable coefficients. We use the latter to define

$$g^{P,n} := \sum_{i=n}^{N_n^P} \lambda_i^{P,n} f^i(P, \cdot).$$

By the assumed uniform integrability, we have

$$\lim_{m \rightarrow \infty} \sup_{n \geq 1} \|f_m^n(P, \cdot) - f^n(P, \cdot)\|_{L^1(P)} = 0, \quad P \in \mathfrak{P};$$

thus, the Cauchy property of $(g^{P,n})_n$ in $L^1(P)$ follows from the corresponding property of the sequences $(g_m^{P,n})_n$. \square

The last two lemmas in this section are observations about the measurability of processes and certain right-continuous modifications.

Lemma III.3.4. *Let $f : \mathfrak{P} \times \Omega \times \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}$ be such that $f(\cdot, \cdot, t)$ is $\widehat{\mathcal{F}}_t$ -measurable for all t and $f(P, \omega, \cdot)$ is right-continuous for all (P, ω) . Then f is measurable and $f|_{\mathfrak{P} \times \Omega \times [0, t]}$ is $\widehat{\mathcal{F}}_t \otimes \mathcal{B}([0, t])$ -measurable for all $t \in \mathbb{R}_+$.*

The same assertion holds if $\widehat{\mathcal{F}}_t$ is replaced by $\widehat{\mathcal{F}}_{t+}$ throughout.

Proof. This is simply the standard fact that a right-continuous, adapted process is progressively measurable, applied on the extended space $\widehat{\Omega}$. \square

Finally, we state a variant on a regularization for processes in right-continuous but non-complete filtrations. As usual, the price to pay for the lack of completion is that the resulting paths are not càdlàg in general.

Lemma III.3.5. *Let $f : \mathfrak{P} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be such that $f(\cdot, \cdot, t)$ is $\widehat{\mathcal{F}}_{t+}$ -measurable for all t . There exists a measurable function $\bar{f} : \mathfrak{P} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that \bar{f} is $\widehat{\mathbb{F}}_+$ -optional, $\bar{f}(P, \omega, \cdot)$ is right-continuous for all (P, ω) , and for any $P \in \mathfrak{P}$ such that $f(P, \cdot, \cdot)$ is an \mathbb{F}_+ -adapted P - \mathbb{F}_+ -supermartingale with right-continuous expectation $t \mapsto E^P[f(P, \cdot, t)]$, the process $\bar{f}(P, \cdot, \cdot)$ is an \mathbb{F}_+ -adapted P -modification of $f(P, \cdot, \cdot)$ and in particular a P - \mathbb{F}_+ -supermartingale.*

Proof. Let D be a countable dense subset of \mathbb{R}_+ . For any $a < b \in \mathbb{R}$ and $t \in \mathbb{R}_+$, denote by $M_a^b(D \cap [0, t], P, \omega)$ the number of upcrossings of the restricted path $f(P, \omega, \cdot)|_{D \cap [0, t]}$ over the interval $[a, b]$. Moreover, let

$$\begin{aligned} \tau_a^b(P, \omega) &= \inf \{t \in \mathbb{Q}_+ \mid M_a^b(D \cap [0, t], P, \omega) = \infty\}, \\ \sigma(P, \omega) &= \inf \{t \in \mathbb{Q}_+ \mid \sup_{s \leq t, s \in D} |f(P, \omega, s)| = \infty\}, \\ \rho(P, \omega) &= \sigma(P, \omega) \wedge \inf_{a < b \in \mathbb{Q}} \tau_a^b(P, \omega) \end{aligned}$$

and define the function \bar{f} by

$$\bar{f}(P, \omega, t) := \left(\limsup_{s \in D, s \downarrow t} f(P, \omega, s) \right) \mathbf{1}_{\{t < \rho(P, \omega)\}}.$$

Using the arguments in the proof of [13, Remark VI.5, p. 70], we can verify that \bar{f} has the desired properties. \square

III.4 Semimartingale Property and \mathfrak{P}_{sem}

In this section, we prove Proposition III.2.2 and the measurability of \mathfrak{P}_{sem} .

Proof of Proposition III.2.2. Let X be a càdlàg, \mathbb{F} -adapted process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. We begin with the equivalence of

- (i) X is an \mathbb{F} -semimartingale,
- (ii) X is an \mathbb{F}_+ -semimartingale,
- (iii) X is an \mathbb{F}_+^P -semimartingale.

To see that (i) implies (iii), let $X = X_0 + M + A$ be an \mathbb{F} -semimartingale, where M is a right-continuous \mathbb{F} -local martingale and A is a right-continuous

\mathbb{F} -adapted process with paths of P -a.s. finite variation. The same decomposition is admissible in \mathbb{F}_+^P ; to see this, note that any right-continuous \mathbb{F} -martingale N is also an \mathbb{F}_+^P -martingale: by the backward martingale convergence theorem, $N_s = E^P[N_t | \mathcal{F}_s]$ for $s \leq t$ implies

$$N_s = N_{s+} = E^P[N_t | \mathcal{F}_{s+}] = E^P[N_t | \mathcal{F}_{s+}^P] \quad P\text{-a.s.}, \quad s \leq t.$$

Next, we show that (iii) implies (i). We observe that the process

$$\tilde{X}_t := X_t - X_0 - \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > 1\}},$$

is a semimartingale if and only if X is. Thus, we may assume that $X_0 = 0$ and that X has jumps bounded by one. In particular, X then has a canonical decomposition $X = M + B$, where M is a right-continuous \mathbb{F}_+^P -local martingale and B is a right-continuous \mathbb{F}_+^P -predictable process of finite variation. We can decompose the latter into a difference $B = B^1 - B^2$ of increasing, right-continuous \mathbb{F}_+^P -predictable processes. By [78, Lemma 6.5.10, p. 143], there exist right-continuous, P -a.s. increasing and \mathbb{F} -predictable processes \hat{B}^1 and \hat{B}^2 which are indistinguishable from B^1 and B^2 , respectively. Define $\hat{B} = \hat{B}^1 - \hat{B}^2$; then \hat{B} is \mathbb{F} -predictable, right-continuous and P -a.s. of finite variation, and of course indistinguishable from B .

As a consequence, $\hat{M} := X - \hat{B}$ is right-continuous, \mathbb{F} -adapted and indistinguishable from M ; in particular, it is still an \mathbb{F}_+^P -local martingale. By [11, Theorem 3], there exists an \mathbb{F}_+^P -predictable localizing sequence $(\tilde{\tau}_n)$ for \hat{M} . For any $\tilde{\tau}_n$, there exists an \mathbb{F} -predictable stopping time τ_n such that $\tilde{\tau}_n = \tau_n$ P -a.s.; cf. [12, Theorem IV.78, p. 133]. Thus, the sequence (τ_n) is a localizing sequence of \mathbb{F} -stopping times for the \mathbb{F}_+^P -local martingale \hat{M} . Since \hat{M} is \mathbb{F} -adapted, we deduce from the tower property of the conditional expectation that \hat{M} is an \mathbb{F} -local martingale. As a result, $X = \hat{M} + \hat{B}$ is a decomposition as required and we have shown that (iii) implies (i).

The equivalence between (ii) and (iii) now follows because we can apply the equivalence of (i) and (iii) to the filtration $\mathbb{F}' := \mathbb{F}_+$.

It remains to show the indistinguishability of the characteristics. Let (B, C, ν) be \mathbb{F} -characteristics of X and let (B', C', ν') be \mathbb{F}_+^P -characteristics. The second characteristic is the continuous part of the quadratic variation $[X]$, which can be constructed pathwise P -a.s. (see the proof of Proposition III.6.6) and thus is independent of the filtration. As a result, $C = C'$ P -a.s. To identify the first characteristic, consider the process

$$\tilde{X}_t := X_t - \sum_{0 \leq s \leq t} (\Delta X_s - h(\Delta X_s)).$$

As \tilde{X} has uniformly bounded jumps, it is an \mathbb{F} -special semimartingale. Let $\tilde{X} = X_0 + \tilde{M} + B$ be the canonical decomposition with respect to \mathbb{F} (cf. [78,

Theorem 7.2.6, p. 160]). By the arguments in the first part of the proof, this is also the canonical decomposition with respect to \mathbb{F}_+^P and thus $B = B'$ P -a.s. by the definition of the first characteristic.

Next, we show that $\nu = \nu'$ P -a.s. To this end, we may assume that ν is already the \mathbb{F} -predictable compensator of μ^X . (The existence of the latter follows from [26, Theorem II.1.8, p. 66] and [13, Lemma 7, p. 399].) Let us check that ν is also a predictable random measure with respect to \mathbb{F}_+^P . Let $W^P = W^P(\omega, t, x)$ be a $\mathcal{P}^P \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function; we claim that W^P is indistinguishable from a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function W , in the sense that the set $\{\omega \in \Omega \mid W(\omega, t, x) \neq W^P(\omega, t, x) \text{ for some } (t, x)\}$ is P -null. To see this, consider first the case where $W^P(\omega, t, x) = H^P(\omega, t)J(x)$ with H^P being \mathcal{P}^P -measurable and J being $\mathcal{B}(\mathbb{R}^d)$ -measurable. By [13, Lemma 7, p. 399], there exists a \mathcal{P} -measurable process H indistinguishable from H^P and thus $W(\omega, t, x) = H(\omega, t)J(x)$ has the desired properties. The general case follows by a monotone class argument. Since ν is a predictable random measure with respect to \mathbb{F} , the process defined by

$$(W * \nu)_t := \int_0^t \int_{\mathbb{R}^d} W(s, x) \nu(ds, dx)$$

is \mathcal{P} -measurable. As a result, the indistinguishable process $(W^P * \nu)$ is \mathcal{P}^P -measurable, showing that ν is a predictable random measure with respect to \mathbb{F}_+^P .

To see that ν is the compensator of the jump measure μ^X of X with respect to \mathbb{F}_+^P , suppose that W^P is nonnegative. Then by the indistinguishability of W and W^P and [26, Theorem II.1.8, p. 66],

$$E^P[(W^P * \nu)_\infty] = E^P[(W * \nu)_\infty] = E^P[(W * \mu^X)_\infty] = E^P[(W^P * \mu^X)_\infty].$$

Now the uniqueness of the \mathbb{F}_+^P -compensator as stated in the cited theorem shows that $\nu = \nu'$ P -a.s. This completes the proof that $(B, C, \nu) = (B', C', \nu')$ P -a.s.

Again, the argument for \mathbb{F}_+ is contained in the above as a special case, and so the proof of Proposition III.2.2 is complete. \square

To study the measurability of \mathfrak{P}_{sem} , we need to express the semimartingale property in a way which is more accessible than the mere existence of a semimartingale decomposition. To this end, it will be convenient to use some facts which were developed in [4] to give an alternative proof of the Bichteler–Dellacherie theorem.

We continue to consider a càdlàg, \mathbb{R}^d -valued, \mathbb{F} -adapted process X on an arbitrary filtered space $(\Omega, \mathcal{F}, \mathbb{F})$, but fix a finite time horizon $T > 0$. Let $(\tilde{X}_t)_{t \in [0, T]}$ be the process defined by

$$\tilde{X}_t := X_t - X_0 - \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > 1\}}$$

and consider the sequence of \mathbb{F} -stopping times

$$T_m := \inf \{t \geq 0 \mid |\tilde{X}_t| \geq m \text{ or } |\tilde{X}_{t-}| \geq m\}.$$

Moreover, for any $m \in \mathbb{N}$, define the process $(\tilde{X}_t^m)_{t \in [0, T]}$ by

$$\tilde{X}_t^m := (m+1)^{-1} \tilde{X}_{T_m \wedge t}.$$

Given $P \in \mathfrak{P}(\Omega)$, we can consider the Doob decomposition of \tilde{X}^m sampled on the n -th dyadic partition of $[0, T]$ under P and \mathbb{F}_+ ; namely, $A^{m, P, n} := 0$ and

$$A_{kT/2^n}^{m, P, n} := \sum_{j=1}^k E^P [\tilde{X}_{jT/2^n}^m - \tilde{X}_{(j-1)T/2^n}^m \mid \mathcal{F}_{(j-1)T/2^n+}], \quad 1 \leq k \leq 2^n,$$

$$M_{kT/2^n}^{m, P, n} := \tilde{X}_{kT/2^n}^m - A_{kT/2^n}^{m, P, n}, \quad 0 \leq k \leq 2^n.$$

Furthermore, given $c > 0$, we define the \mathbb{F}_+ -stopping times

$$\sigma_{m, n}(c) := \inf \left\{ \frac{kT}{2^n} \mid \sum_{j=1}^k \left| \tilde{X}_{\frac{jT}{2^n}}^m - \tilde{X}_{\frac{(j-1)T}{2^n}}^m \right|^2 \geq c - 4 \right\},$$

$$\tau_{m, P, n}(c) := \inf \left\{ \frac{kT}{2^n} \mid \sum_{j=1}^k \left| A_{\frac{jT}{2^n}}^{m, P, n} - A_{\frac{(j-1)T}{2^n}}^{m, P, n} \right| \geq c - 2 \right\}.$$

Proposition III.4.1. *Let $P \in \mathfrak{P}(\Omega)$. The process $(X_t)_{t \in [0, T]}$ is a P - \mathbb{F} -semimartingale if and only if for all $m \in \mathbb{N}$ and $\varepsilon > 0$ there exists a constant $c = c(m, \varepsilon) > 0$ such that*

$$P[\sigma_{m, n}(c) < \infty] < \frac{\varepsilon}{2} \quad \text{and} \quad P[\tau_{m, P, n}(c) < \infty] < \frac{\varepsilon}{2} \quad \text{for all } n \geq 1. \quad (4.1)$$

Proof. Clearly X is a P - \mathbb{F} -semimartingale if and only if \tilde{X}^m has this property for all m . Moreover, by Proposition III.2.2, this is equivalent to \tilde{X}^m being a P - \mathbb{F}_+^P -semimartingale.

If \tilde{X}^m is a P - \mathbb{F}_+^P -semimartingale, [4, Theorem 1.6] implies that it satisfies the property “no free lunch with vanishing risk and little investment” introduced in [4, Definition 1.5]. As $\sup_{t \in [0, T]} |\tilde{X}_t^m| \leq 1$, we deduce from [4, Proposition 3.1] that for any $\varepsilon > 0$ there exists a constant $c = c(m, \varepsilon) > 0$ such that (4.1) holds. Conversely, suppose that there exist such constants; then, as $\sup_{t \in [0, T]} |\tilde{X}_t^m| \leq 1$, the proof of [4, Theorem 1.6] shows that \tilde{X}^m is a P - \mathbb{F}_+^P -semimartingale. \square

Corollary III.4.2. *Under the conditions of Theorem III.2.5, the set*

$$\mathfrak{P}_{sem, T} = \{P \in \mathfrak{P}(\Omega) \mid (X_t)_{0 \leq t \leq T} \text{ is a semimartingale on } (\Omega, \mathcal{F}, \mathbb{F}, P)\}$$

is Borel-measurable for every $T > 0$, and so is $\mathfrak{P}_{sem} = \bigcap_{T \in \mathbb{N}} \mathfrak{P}_{sem, T}$.

Proof. Let $T > 0$; then Proposition III.4.1 allows us to write $\mathfrak{P}_{sem,T}$ as

$$\bigcap_{m,k \in \mathbb{N}} \bigcup_{c \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ P \in \mathfrak{P}(\Omega) \mid P[\sigma_{m,n}(c) < \infty] + P[\tau_{m,P,n}(c) < \infty] < 1/k \right\};$$

hence, it suffices to argue that the right-hand side is measurable. Indeed, $\omega \mapsto \sigma_{m,n}(c)(\omega)$ and $(P, \omega) \mapsto \tau_{m,P,n}(c)(\omega)$ are measurable by Lemma III.3.1, so the required measurability follows by another application of the same lemma. \square

III.5 Measurable Doob-Meyer and Canonical Decompositions

In this section, we first obtain a version of the Doob–Meyer decomposition which is measurable with respect to the probability measure P . Then, we apply this result to construct the canonical decomposition of a bounded semimartingale with the same measurability; together with a localization argument, this will provide the first semimartingale characteristic B^P in the subsequent section. The conditions of Theorem III.2.5 are in force; moreover, we fix a measurable set $\mathfrak{P} \subseteq \mathfrak{P}(\Omega)$. As the results of this section can be applied componentwise, we consider scalar processes without compromising the generality.

There are various proofs of the Doob–Meyer theorem, all based on compactness arguments, which use a passage to the limit from the elementary Doob decomposition in discrete time. The latter is measurable with respect to P by Lemma III.3.1. Thus, the main issue is to go through a compactness argument while retaining measurability. Our Proposition III.3.3 is tailored to that purpose, and it combines naturally with the proof of the Doob–Meyer decomposition given in [5].

Proposition III.5.1 (Doob–Meyer). *Let $(P, \omega, t) \mapsto S_t^P(\omega)$ be a measurable function such that for all $P \in \mathfrak{P}$, S^P is a right-continuous, \mathbb{F}_+ -adapted P - \mathbb{F}_+^P -submartingale of class D . There exists a measurable function $(P, \omega, t) \mapsto A_t^P(\omega)$ such that for all $P \in \mathfrak{P}$,*

$$S^P - S_0^P - A^P \text{ is a } P\text{-}\mathbb{F}_+^P\text{-martingale,}$$

A^P is right-continuous, \mathbb{F}_+ -adapted, \mathbb{F}_+^P -predictable and P -a.s. increasing.

Proof. It suffices to consider a finite time horizon $T > 0$; moreover, we may assume that $S_0^P = 0$. For each $P \in \mathfrak{P}$ and $n \in \mathbb{N}$, consider the Doob decomposition of the process $(S_{jT/2^n}^P)_{j=0,\dots,2^n}$, defined by $A_0^{P,n} = 0$ and

$$A_{kT/2^n}^{P,n} = \sum_{j=1}^k E^P \left[S_{jT/2^n}^P - S_{(j-1)T/2^n}^P \mid \mathcal{F}_{(j-1)T/2^n+} \right], \quad 1 \leq k \leq 2^n,$$

$$M_{kT/2^n}^{P,n} = S_{kT/2^n}^P - A_{kT/2^n}^{P,n}, \quad 0 \leq k \leq 2^n.$$

Note that $(A_{jT/2^n}^{P,n})_{j=0,\dots,2^n}$ has P -a.s. increasing paths and that $(P, \omega) \mapsto A_{kT/2^n}^{P,n}(\omega)$ is measurable by Lemma III.3.1. As a consequence, $(P, \omega) \mapsto M_T^{P,n}(\omega)$ is measurable as well. We deduce from [5, Lemma 2.2] that for each $P \in \mathfrak{P}$ the sequence $(M_T^{P,n})_{n \in \mathbb{N}} \subseteq L^1(P)$ is uniformly integrable. Therefore, we can apply Proposition III.3.3 to obtain an $L^1(P)$ -convergent sequence of convex combinations

$$\mathcal{M}_T^{P,n} := \sum_{i=n}^{N_n^P} \lambda_i^{P,n} M_T^{P,i}$$

which are measurable in (P, ω) . By Lemma III.3.2, we can find a version \mathcal{M}_T^P of the limit which is again jointly measurable in (P, ω) .

On the strength of Lemma III.3.1 and Lemma III.3.5, we can find a measurable function $(P, \omega, t) \mapsto M_t^P(\omega)$ such that for each $P \in \mathfrak{P}$, $(M_t^P)_{t \in [0, T]}$ is a right-continuous P - \mathbb{F}_+ -martingale and a P -modification of the process $(E^P[\mathcal{M}_T^P | \mathcal{F}_{t+}])_{0 \leq t \leq T}$. We define A^P by

$$A_t^P := S_t^P - M_t^P;$$

then A^P is right-continuous and \mathbb{F}_+ -adapted and $(P, \omega, t) \mapsto A_t^P(\omega)$ is measurable. Following the arguments in [5, Section 2.3], we see that A^P is P -a.s. increasing and P -indistinguishable from a \mathcal{P} -measurable process, hence predictable with respect to \mathbb{F}_+^P . \square

We can now construct the compensator of a process with integrable variation. We recall the filtration $\widehat{\mathbb{F}}$ on $\mathfrak{P} \times \Omega$ introduced in (3.1).

Corollary III.5.2 (Compensator). *Let $(P, \omega, t) \mapsto S_t^P(\omega)$ be an $\widehat{\mathbb{F}}_+$ -adapted right-continuous process such that for all $P \in \mathfrak{P}$, S^P is an \mathbb{F}_+ -adapted process of P -integrable variation. There exists a measurable function $(P, \omega, t) \mapsto A_t^P(\omega)$ such that for all $P \in \mathfrak{P}$,*

$$S^P - S_0^P - A^P \quad \text{is a } P\text{-}\mathbb{F}_+^P\text{-martingale,}$$

A^P is right-continuous, \mathbb{F}_+ -adapted, \mathbb{F}_+^P -predictable and P -a.s. of finite variation.

Proof. We may assume that $S_0^P = 0$. By Lemma III.3.4, $(P, \omega, t) \mapsto S_t^P(\omega)$ is measurable. Thus, if S^P is P -a.s. increasing for all $P \in \mathfrak{P}$, Proposition III.5.1 immediately yields the result. Therefore, it suffices to show that there exists a decomposition

$$S^P = S^{1,P} - S^{2,P} \quad P\text{-a.s.}$$

into \mathbb{F}_+ -adapted, P -integrable processes having right-continuous and P -a.s. increasing paths such that $(P, \omega, t) \mapsto S_t^{i,P}(\omega)$ is measurable. Let $\text{Var}(S^P)$

denote the total variation process of S^P . By the right-continuity of S^P , we have

$$\text{Var}(S^P)_t(\omega) = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} |S_{kt/2^n}^P(\omega) - S_{(k-1)t/2^n}^P(\omega)| \quad \text{for all } (P, \omega, t).$$

In particular, $\text{Var}(S^P)$ is \mathbb{F}_+ -adapted and $(P, \omega, t) \mapsto \text{Var}(S^P)_t(\omega)$ is $\widehat{\mathbb{F}}_+$ -adapted. For each $P \in \mathfrak{P}$, we define

$$\sigma^P := \inf \{t \geq 0 \mid \text{Var}(S^P)_t = \infty\}.$$

The identity

$$\{\sigma^P < t\} = \bigcup_{q \in \mathbb{Q}, q < t} \{\text{Var}(S^P)_q = \infty\}$$

shows that $(P, \omega) \mapsto \sigma^P(\omega)$ is an $\widehat{\mathbb{F}}_+$ -stopping time and in particular measurable. As S^P is of P -integrable variation, we have $\sigma^P = \infty$ P -a.s. Using Lemma III.3.4 and the fact that $\text{Var}(S^P)\mathbf{1}_{[0, \sigma^P[}$ is right-continuous, it follows that the processes

$$S^{1,P} := \frac{\text{Var}(S^P) + S^P}{2} \mathbf{1}_{[0, \sigma^P[}, \quad S^{2,P} := \frac{\text{Var}(S^P) - S^P}{2} \mathbf{1}_{[0, \sigma^P[}$$

have the required properties. \square

In the second part of this section, we construct the canonical decomposition of a bounded semimartingale. Ultimately, this decomposition can be obtained from the discrete Doob decomposition, a compactness argument and the existence of the compensator for bounded variation processes. Hence, we will combine Proposition III.3.3 and the preceding Corollary III.5.2. The following lemma is an adaptation of the method developed in [4] to our needs; it contains the mentioned compactness argument. We fix a finite time horizon $T > 0$.

Lemma III.5.3. *Let $S = (S_t)_{t \in [0, T]}$ be a càdlàg, \mathbb{F}_+ -adapted process with $S_0 = 0$ and $\sup_{t \in [0, T]} |S_t| \leq 1$ such that S is a P - \mathbb{F}_+^P -semimartingale for all $P \in \mathfrak{P}$. For all $\varepsilon > 0$ and $P \in \mathfrak{P}$ there exist*

- (i) *a $[0, T] \cup \{\infty\}$ -valued \mathbb{F}_+ -stopping time α^P such that $(P, \omega) \mapsto \alpha^P(\omega)$ is an $\widehat{\mathbb{F}}_+$ -stopping time and*

$$P[\alpha^P < \infty] \leq \varepsilon,$$

- (ii) *a constant c^P and right-continuous, \mathbb{F}_+ -adapted processes $\mathcal{A}^P, \mathcal{M}^P$ with $\mathcal{A}_0^P = \mathcal{M}_0^P = 0$ such that $(P, \omega, t) \mapsto (\mathcal{A}_t^P(\omega), \mathcal{M}_t^P(\omega))$ is $\widehat{\mathbb{F}}_+$ -adapted,*

$$\mathcal{M}^P \text{ is a } P\text{-}\mathbb{F}_+\text{-martingale and } \text{Var}(\mathcal{A}^P) \leq c^P \text{ } P\text{-a.s.}$$

such that

$$\mathcal{M}_t^P + \mathcal{A}_t^P = S_{\alpha^P \wedge t}, \quad t \in [0, T].$$

Proof. This lemma is basically a version of [4, Theorem 1.6] with added measurability in P ; we only give a sketch of the proof. The first step is to obtain a version of [4, Proposition 3.1]: For $P \in \mathfrak{P}$ and $n \in \mathbb{N}$, consider the Doob decomposition of the discrete-time process $(S_{jT/2^n})_{j=0, \dots, 2^n}$ with respect to P and \mathbb{F}_+ , defined by $A_0^{P,n} = 0$ and

$$A_{kT/2^n}^{P,n} := \sum_{j=1}^k E^P \left[S_{jT/2^n} - S_{(j-1)T/2^n} \mid \mathcal{F}_{(j-1)T/2^n+} \right], \quad 1 \leq k \leq 2^n,$$

$$M_{kT/2^n}^{P,n} := S_{kT/2^n} - A_{kT/2^n}^{P,n}, \quad 0 \leq k \leq 2^n.$$

By adapting the proof of [4, Proposition 3.1] and using Lemma III.3.1, one shows that for all $\varepsilon > 0$ and $P \in \mathfrak{P}$ there exist a constant $c^P \in \mathbb{N}$ and a sequence of $\{T/2^n, \dots, (2^n - 1)T/2^n, T\} \cup \{\infty\}$ -valued \mathbb{F}_+ -stopping times $(\rho_{P,n})_{n \in \mathbb{N}}$ such that $(P, \omega) \mapsto \rho_{P,n}(\omega)$ is an $\widehat{\mathbb{F}}_+$ -stopping time,

$$P[\rho_{P,n} < \infty] < \varepsilon$$

and

$$\sum_{j=1}^{\frac{2^n}{T}(\rho_{P,n} \wedge T)} |A_{jT/2^n}^{P,n} - A_{(j-1)T/2^n}^{P,n}| \leq c^P, \quad \|M_{T \wedge \rho_{P,n}}^{P,n}\|_{L^2(P)}^2 \leq c^P$$

for all $n \in \mathbb{N}$. The second step is to establish the following assertion: for all $\varepsilon > 0$ and $P \in \mathfrak{P}$ there exist a constant $c^P \in \mathbb{N}$, a $[0, T] \cup \{\infty\}$ -valued \mathbb{F}_+ -stopping time α^P such that $(P, \omega) \mapsto \alpha^P(\omega)$ is an $\widehat{\mathbb{F}}_+$ -stopping time, and a sequence of right-continuous, \mathbb{F}_+ -adapted processes $(\mathcal{A}^{P,k})_{k \in \mathbb{N}}$ and $(\mathcal{M}^{P,k})_{k \in \mathbb{N}}$ on $[0, T]$ which are measurable in (P, ω, t) , such that $(\mathcal{M}_t^{P,k})_{0 \leq t \leq T}$ is a P - \mathbb{F}_+ -martingale and

$$(\mathcal{M}^{P,k})_t^{\alpha^P} + (\mathcal{A}^{P,k})_t^{\alpha^P} = S_t^{\alpha^P}, \quad P[\alpha^P < \infty] \leq \varepsilon,$$

$$\sum_{j=1}^{2^k} \left| (\mathcal{A}^{P,k})_{jT/2^k}^{\alpha^P} - (\mathcal{A}^{P,k})_{(j-1)T/2^k}^{\alpha^P} \right| \leq c^P \quad P\text{-a.s.}, \quad \left\| (\mathcal{M}^{P,k})_t^{\alpha^P} \right\|_{L^2(P)}^2 \leq c^P.$$

Here the first equality holds for all ω rather than P -a.s. and the usual notation for the ‘‘stopped process’’ is used; for instance, $S_t^{\alpha^P} = S_{\alpha^P \wedge t}$. To derive this assertion from the first step, we combine the arguments in the proof of [4, Proposition 3.6] with Lemma III.3.1, Lemma III.3.2, Proposition III.3.3 and Lemma III.3.5.

Finally, to derive Lemma III.5.3 from the preceding step, we adapt the proof of [4, Theorem 1.6], again making crucial use of Lemma III.3.2 and Proposition III.3.3. \square

Proposition III.5.4 (Canonical Decomposition). *Let S be a càdlàg, \mathbb{F}_+ -adapted process with $S_0 = 0$ and $\sup_{t \geq 0} |S_t| \leq 1$ such that S is a P - \mathbb{F}_+^P -semimartingale for all $P \in \mathfrak{P}$. There exists a measurable function $(P, \omega, t) \mapsto B_t^P(\omega)$ such that for all $P \in \mathfrak{P}$,*

$$S - B^P \quad \text{is a } P\text{-}\mathbb{F}_+^P\text{-martingale,}$$

B^P is right-continuous, \mathbb{F}_+ -adapted, \mathbb{F}_+^P -predictable and P -a.s. of finite variation.

Proof. We first fix $T > 0$ and consider the stopped process $Y = S^T$. For each $n \in \mathbb{N}$, let $\alpha^{P,n}$, $\mathcal{M}^{P,n}$ and $\mathcal{A}^{P,n}$ be the stopping times and processes provided by Lemma III.5.3 for the choice $\varepsilon = 2^{-n}$; that is, $P[\alpha^{P,n} < \infty] < 2^{-n}$ and

$$Y^{\alpha^{P,n}} = \mathcal{M}^{P,n} + \mathcal{A}^{P,n}.$$

By Corollary III.5.2, we can construct the compensator of $\mathcal{A}^{P,n}$ with respect to P - \mathbb{F}_+^P , denoted by $\{\mathcal{A}^{P,n}\}^P$, such that $\{\mathcal{A}^{P,n}\}^P$ is right-continuous, \mathbb{F}_+ -adapted and P -a.s. of finite variation, and $(P, \omega, t) \mapsto \{\mathcal{A}^{P,n}\}_t^P(\omega)$ is measurable. We define the process $\overline{\mathcal{M}}^{P,n}$ by

$$\overline{\mathcal{M}}^{P,n} = \mathcal{M}^{P,n} + \mathcal{A}^{P,n} - \{\mathcal{A}^{P,n}\}^P.$$

By construction, $\overline{\mathcal{M}}^{P,n}$ is a right-continuous, \mathbb{F}_+ -adapted P - \mathbb{F}_+^P -martingale and $(P, \omega, t) \mapsto \overline{\mathcal{M}}_t^{P,n}(\omega)$ is measurable. Furthermore,

$$Y^{\alpha^{P,n}} = \overline{\mathcal{M}}^{P,n} + \{\mathcal{A}^{P,n}\}^P$$

is the canonical decomposition of the P - \mathbb{F}_+^P -semimartingale $Y^{\alpha^{P,n}}$.

We have $\sum_{n \in \mathbb{N}} P\{\alpha^{P,n} < \infty\} < \infty$ for each $P \in \mathfrak{P}$. By the Borel–Cantelli Lemma, this implies that $\lim_{n \rightarrow \infty} \alpha^{P,n} = \infty$ P -a.s. Let

$$\beta^{P,n} := \inf_{k \geq n} \alpha^{P,k}.$$

Then $\beta^{P,n}$ are \mathbb{F}_+ -stopping times increasing to infinity P -a.s. for each P and $(P, \omega) \mapsto \beta^{P,n}(\omega)$ is an $\widehat{\mathbb{F}}_+$ -stopping time for each n . As $\beta^{P,n+1} \wedge \alpha^{P,n} = \beta^{P,n}$, we have

$$Y^{\beta^{P,n}} = (Y^{\alpha^{P,n}})^{\beta^{P,n+1}} = (\overline{\mathcal{M}}^{P,n})^{\beta^{P,n+1}} + (\{\mathcal{A}^{P,n}\}^P)^{\beta^{P,n+1}},$$

which is the canonical decomposition of $Y^{\beta^{P,n}}$. Thus, by uniqueness of the canonical decomposition,

$$Y = \sum_{n=1}^{\infty} (\overline{\mathcal{M}}^{P,n})^{\beta^{P,n+1}} \mathbf{1}_{\llbracket \beta^{P,n-1}, \beta^{P,n} \rrbracket} + \sum_{n=1}^{\infty} (\{\mathcal{A}^{P,n}\}^P)^{\beta^{P,n+1}} \mathbf{1}_{\llbracket \beta^{P,n-1}, \beta^{P,n} \rrbracket}$$

is the canonical decomposition of Y , where we have set $\beta^{P,0} := 0$. Denote the two sums on the right-hand side by $M^{P,T}$ and $B^{P,T}$, respectively, and recall that $Y = S^T$. The decomposition of the full process S is then given by

$$S = M^P + B^P := \sum_{T=1}^{\infty} M^{P,T} \mathbf{1}_{[T-1, T[} + \sum_{T=1}^{\infty} B^{P,T} \mathbf{1}_{[T-1, T[}.$$

By construction, these processes have the required properties. \square

III.6 Measurable Semimartingale Characteristics

In this section, we construct a measurable version of the characteristics (B^P, C, ν^P) of X as stated in Theorem III.2.5. The conditions of that theorem are in force throughout; in particular, X is a càdlàg, \mathbb{F} -adapted process. We recall that the set \mathfrak{P}_{sem} of all $P \in \mathfrak{P}(\Omega)$ under which X is a semimartingale is measurable (Corollary III.4.2) and that a truncation function h has been fixed. When we refer to the results of Section III.3, they are to be understood with the choice $\mathfrak{P} = \mathfrak{P}_{sem}$.

As mentioned in the preceding section, the existence of the first characteristic B^P is a consequence of Proposition III.5.4.

Corollary III.6.1. *There exists a measurable function $\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$, $(P, \omega, t) \mapsto B_t^P(\omega)$ such that for all $P \in \mathfrak{P}_{sem}$, B^P is an \mathbb{F}_+ -adapted, \mathbb{F}_+ -predictable process with right-continuous, P -a.s. finite variation paths, and B^P is a version of the first characteristic of X with respect to P .*

Proof. We may assume that $X_0 = 0$. Let

$$\tilde{X}_t := X_t - \sum_{0 \leq s \leq t} (\Delta X_s - h(\Delta X_s)),$$

$T_0 = 0$ and $T_m = \inf\{t \geq 0 \mid |\tilde{X}_t| > m\}$. As \tilde{X} has càdlàg paths, each T_m is an \mathbb{F}_+ -stopping time and $T_m \rightarrow \infty$. Define

$$\tilde{X}^m = \tilde{X} \cdot \mathbf{1}_{\cdot \wedge T_m};$$

then \tilde{X}^m is a càdlàg, \mathbb{F}_+ -adapted P - \mathbb{F}_+^P semimartingale for each $P \in \mathfrak{P}_{sem}$ and $|\tilde{X}^m| \leq m + \|h\|_{\infty}$. We use Proposition III.5.4 to obtain the corresponding predictable finite variation process $B^{m,P}$ of the canonical decomposition of \tilde{X}^m , and then

$$B^P = \sum_{m \geq 1} B^{m,P} \mathbf{1}_{[T_{m-1}, T_m[}$$

has the desired properties. \square

The next goal is to construct the third characteristic of X , the compensator ν^P of the jump measure of X , and its decomposition as stated in Theorem III.2.5. (The second characteristic is somewhat less related to the preceding results and thus treated later on.) To this end, we first provide a variant of the disintegration theorem for measures on product spaces. As it will be used for the decomposition of ν^P , we require a version where the objects depend measurably on an additional parameter (the measure P). We call a kernel stochastic if its values are probability measures, whereas finite kernel refers to the values being finite measures. A Borel space is (isomorphic to) a Borel subset of a Polish space.

Lemma III.6.2. *Let (G, \mathcal{G}) be a measurable space, (Y, \mathcal{Y}) a separable measurable space and $(Z, \mathcal{B}(Z))$ a Borel space. Moreover, let $\kappa(g, d(y, z))$ be a finite kernel on $(Y \times Z, \mathcal{Y} \otimes \mathcal{B}(Z))$ given (G, \mathcal{G}) and let $\widehat{\kappa}(g, dy)$ be its marginal on Y ,*

$$\widehat{\kappa}(g, A) := \kappa(g, A \times Z), \quad A \in \mathcal{Y}.$$

There exists a stochastic kernel $\alpha((g, y), dz)$ on $(Z, \mathcal{B}(Z))$ given $(G \times Y, \mathcal{G} \otimes \mathcal{Y})$ such that

$$\kappa(g, A \times B) = \int_A \alpha((g, y), B) \widehat{\kappa}(g, dy), \quad A \in \mathcal{Y}, \quad B \in \mathcal{B}(Z), \quad g \in G.$$

Proof. This result can be found e.g. in [6, Proposition 7.27, p. 135], in the special case where Y is a Borel space (and $\kappa(g, d(y, z))$ is a stochastic kernel). In that case, one can identify Y with an interval and the proof of [6, Proposition 7.27, p. 135] makes use of dyadic partitions generating \mathcal{Y} . In the present case, we can give a similar proof where we use directly the separability of \mathcal{Y} ; namely, we can find a refining sequence of finite partitions of Y which generates \mathcal{Y} and apply martingale convergence arguments to the corresponding sequence of finite σ -fields. The details are omitted. \square

In order to apply the disintegration result with $(Y, \mathcal{Y}) = (\Omega \times \mathbb{R}_+, \mathcal{P})$, we need the following observation.

Lemma III.6.3. *The predictable σ -field \mathcal{P} is separable.*

Proof. The σ -field \mathcal{P} is generated by the sets

$$\{0\} \times A, \quad A \in \mathcal{F}_{0-} \quad \text{and} \quad (s, t] \times A, \quad A \in \mathcal{F}_{s-}, \quad 0 < s < t \in \mathbb{Q};$$

cf. [12, Theorem IV.67, p. 125]. Since \mathcal{F}_{0-} is trivial and \mathcal{F}_s is separable for $s \geq 0$, it follows that each \mathcal{F}_{s-} is separable as well. Let $(A_s^n)_{n \geq 1}$ be a generator for \mathcal{F}_{s-} ; then

$$\{0\} \times A_0^n \quad \text{and} \quad (s, t] \times A_s^n, \quad 0 < s < t \in \mathbb{Q}, \quad n \geq 1$$

yield a countable generator for \mathcal{P} . \square

We can now construct the third characteristic and its decomposition. For the following statement, recall the set $\bar{\mathcal{L}}$ from (2.1) and that it has been endowed with its Borel σ -field.

Proposition III.6.4. *There exists a measurable function*

$$\mathfrak{P}_{sem} \times \Omega \rightarrow \bar{\mathcal{L}}, \quad (P, \omega) \mapsto \nu^P(\omega, dt, dx)$$

such that for all $P \in \mathfrak{P}_{sem}$, the \mathbb{F}_+^P -predictable random measure $\nu^P(\cdot, dt, dx)$ is the P - \mathbb{F}_+^P -compensator of μ^X . Moreover, there exists a decomposition

$$\nu^P(\cdot, dt, dx) = K^P(\cdot, t, dx) dA_t^P \quad P\text{-a.s.}$$

where

- (i) $(P, \omega, t) \mapsto A_t^P(\omega)$ is measurable and for all $P \in \mathfrak{P}_{sem}$, A^P is an \mathbb{F}_+ -adapted, \mathbb{F}_+^P -predictable, P -integrable process with right-continuous and P -a.s. increasing paths,
- (ii) $(P, \omega, t) \mapsto K^P(\omega, t, dx)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+, \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$ and for all $P \in \mathfrak{P}_{sem}$, $(\omega, t) \mapsto K^P(\omega, t, dx)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\Omega \times \mathbb{R}_+, \mathcal{P}^P)$.

Proof. We use the preceding results to adapt the usual construction of the compensator, with $P \in \mathfrak{P}_{sem}$ as an additional parameter. By a standard fact recalled in Lemma III.6.5 below, there is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function $V > 0$ such that $0 \leq V * \mu^X \leq 1$; recall the notation $V * \mu^X := \int_0^\cdot \int_{\mathbb{R}^d} V(s, x) \mu^X(ds, dx)$. Define $A := V * \mu^X$. We observe that A is a càdlàg, \mathbb{F}_+ -adapted process, uniformly bounded and increasing; thus, it is a P - \mathbb{F}_+^P -submartingale of class D for any $P \in \mathfrak{P}_{sem}$. By Proposition III.5.1, we can construct the predictable process of the Doob-Meyer decomposition of A with respect to P and \mathbb{F}_+^P , denoted by A^P , such that A^P is P -integrable, \mathbb{F}_+^P -predictable, \mathbb{F}_+ -adapted with right-continuous, P -a.s. increasing paths and $(P, \omega, t) \mapsto A_t^P(\omega)$ is measurable. Define a kernel on $(\Omega \times \mathbb{R}_+ \times \mathbb{R}^d, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d))$ given $(\mathfrak{P}_{sem}, \mathcal{B}(\mathfrak{P}_{sem}))$ by

$$m^P(C) := E^P[(V \mathbf{1}_C * \mu^X)_\infty], \quad C \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d).$$

Note that each measure $m^P(\cdot)$ is a sub-probability. Consider the set

$$\begin{aligned} G &:= \{(P, \omega) \in \mathfrak{P}_{sem} \times \Omega \mid t \mapsto A_t^P(\omega) \text{ is increasing}\} \\ &= \bigcap_{s < t \in \mathbb{Q}} \{(P, \omega) \in \mathfrak{P}_{sem} \times \Omega \mid A_s^P(\omega) < A_t^P(\omega)\}; \end{aligned}$$

the second equality is due to the right-continuity of A^P and shows that $G \in \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{F}$. Moreover, the sections of G satisfy

$$P\{\omega \in \Omega \mid (P, \omega) \in G\} = 1, \quad P \in \mathfrak{P}_{sem}.$$

Thus, the (everywhere increasing, but not \mathbb{F}_+ -adapted) process

$$\bar{A}_t^P(\omega) := A_t^P(\omega) \mathbf{1}_G(P, \omega)$$

is P -indistinguishable from A^P and in particular \mathbb{F}_+^P -predictable, while the map $(P, \omega, t) \mapsto \bar{A}_t^P(\omega)$ is again measurable. We define another finite kernel on $(\Omega \times \mathbb{R}_+, \mathcal{P})$ given $(\mathfrak{P}_{sem}, \mathcal{B}(\mathfrak{P}_{sem}))$ by

$$\hat{m}^P(D) = E^P \left[\int_0^\infty \mathbf{1}_D(t, \omega) d\bar{A}_t^P(\omega) \right], \quad D \in \mathcal{P}.$$

As in the proof of [26, Theorem II.1.8, p. 67], we have $\hat{m}^P(D) = m^P(D \times \mathbb{R}^d)$ for any $D \in \mathcal{P}$; that is, $\hat{m}^P(d\omega, dt)$ is the marginal of $m^P(d\omega, dt, dx)$ on $(\Omega \times \mathbb{R}_+, \mathcal{P})$.

Since $(\Omega \times \mathbb{R}_+, \mathcal{P})$ is separable by Lemma III.6.3, we may apply the disintegration result of Lemma III.6.2 to obtain a stochastic kernel $\alpha^P(\omega, t, dx)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+, \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{P})$ such that

$$m^P(d\omega, dt, dx) = \alpha^P(\omega, t, dx) \hat{m}^P(d\omega, dt).$$

Define a kernel $\tilde{K}^P(\omega, t, dx)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+, \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{P})$ by

$$\tilde{K}^P(\omega, t, E) := \int_E V(\omega, t, x)^{-1} \alpha^P(\omega, t, dx), \quad E \in \mathcal{B}(\mathbb{R}^d). \quad (6.1)$$

Moreover, let $\tilde{\nu}^P(\omega, dt, dx) := \tilde{K}^P(\omega, t, dx) d\bar{A}_t^P(\omega)$ and define the set

$$G' := \left\{ (P, \omega) \in G \mid \int_0^N \int_{\mathbb{R}^d} |x|^2 \wedge 1 \tilde{\nu}^P(\omega, dt, dx) < \infty \forall N \in \mathbb{N}, \right. \\ \left. \tilde{\nu}^P(\omega, \mathbb{R}_+, \{0\}) = 0 = \tilde{\nu}^P(\omega, \{0\}, \mathbb{R}^d) \right\}.$$

We observe that $G' \in \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{F}$. Moreover, by [26, Theorem II.1.8, p. 66] and its proof,

$$P\{\omega \in \Omega \mid (P, \omega) \in G'\} = 1, \quad P \in \mathfrak{P}_{sem}. \quad (6.2)$$

Define the kernel $K^P(\omega, t, dx)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+, \mathcal{B}(\mathfrak{P}_{sem}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$ by

$$K^P(\omega, t, E) := \tilde{K}^P(\omega, t, E) \mathbf{1}_{G'}(P, \omega), \quad E \in \mathcal{B}(\mathbb{R}^d).$$

We see from (6.2) that for fixed $P \in \mathfrak{P}_{sem}$, $K^P(\omega, t, dx)$ is also a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\Omega \times \mathbb{R}_+, \mathcal{P}^P)$. Finally, we set

$$\nu^P(\omega, dt, dx) := K^P(\omega, t, dx) d\bar{A}_t^P(\omega),$$

which clearly entails that $\nu^P(\cdot, dt, dx) = K^P(\cdot, t, dx) dA_t^P$ P -a.s. By construction, $\nu^P(\omega, dt, dx) \in \bar{\mathcal{L}}$ for each $(P, \omega) \in \mathfrak{P}_{sem} \times \Omega$. Moreover, we deduce from [26, Theorem II.1.8, p. 66] that $\nu^P(\omega, dt, dx)$ is the P - \mathbb{F}_+^P -compensator of μ^X for each $P \in \mathfrak{P}_{sem}$. It remains to show that $(P, \omega) \mapsto \nu^P(\omega, dt, dx)$ is measurable. By Lemma III.2.4, it suffices to show that given a Borel function f on $\mathbb{R}_+ \times \mathbb{R}^d$, the map

$$(P, \omega) \mapsto f(t, x) * \nu^P(\omega, dt, dx)$$

is measurable. Suppose first that f is of the form $f(t, x) = g(t)h(x)$, where g and h are measurable functions. Then

$$\begin{aligned} f(t, x) * \nu^P(\omega, dt, dx) &= \int_0^\infty \int_{\mathbb{R}^d} f(t, x) K^P(\omega, t, dx) d\bar{A}_t^P(\omega) \\ &= \int_0^\infty g(t) \int_{\mathbb{R}^d} h(x) K^P(\omega, t, dx) d\bar{A}_t^P(\omega) \end{aligned}$$

is measurable in (P, ω) . The case of a general function f follows by a monotone class argument, which completes the proof. \square

The following standard fact was used in the preceding proof.

Lemma III.6.5. *Let S be a càdlàg, \mathbb{F}_+ -adapted process. There exists a strictly positive $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function V such that $0 \leq V * \mu^S \leq 1$.*

Proof. Let $H_n := \{x \in \mathbb{R}^d \mid |x| > 2^{-n}\}$ for $n \in \mathbb{N}$; then $\cup_n H_n = \mathbb{R}^d \setminus \{0\}$. Define $T_{n,0} = 0$ and

$$T_{n,m} := \inf \{t \geq T_{n,m-1} \mid |S_t - S_{T_{n,m-1}}| > 2^{-(n+1)}\}.$$

As S is càdlàg, each $T_{n,m}$ is an \mathbb{F}_+ -stopping time. Set $G_{n,0} := \Omega \times \mathbb{R}_+ \times \{0\}$ and

$$G_{n,m} := \llbracket 0, T_{n,m} \rrbracket \times H_n \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d);$$

recall that the predictable σ -field associated with \mathbb{F}_+ coincides with \mathcal{P} . Then, $\cup_{n,m} G_{n,m} = \Omega \times \mathbb{R}_+ \times \mathbb{R}^d$ and

$$V(\omega, t, x) := \sum_{n \geq 1} 2^{-n} \left(\mathbf{1}_{G_{n,0}}(\omega, t, x) + \sum_{m \geq 1} \frac{2^{-m}}{m} \mathbf{1}_{G_{n,m}}(\omega, t, x) \right)$$

has the required properties. \square

The final goal of this section is to establish an aggregated version of the second characteristic; that is, a single process C rather than a family $(C^P)_{P \in \mathfrak{P}_{sem}}$. By its definition, C is the quadratic variation of the continuous local martingale part of X under each $P \in \mathfrak{P}_{sem}$; however, the martingale part depends heavily on P and thus would not lead to an aggregated process

C . Instead, we shall obtain C as the continuous part of the (optional) quadratic variation $[X]$ which is essentially measure-independent. For future applications, we establish two versions of C : one is \mathbb{F} -predictable but its paths are irregular on an exceptional set; the other one, denoted \bar{C} , has regular paths and is predictable for the augmentation of \mathbb{F} by the collection of \mathcal{F} -measurable \mathfrak{P}_{sem} -polar sets. More precisely, we let

$$\mathcal{N}_{sem} = \{A \in \mathcal{F} \mid P(A) = 0 \text{ for all } P \in \mathfrak{P}_{sem}\}$$

and consider the filtration $\mathbb{F} \vee \mathcal{N}_{sem} = (\mathcal{F}_t \vee \mathcal{N}_{sem})_{t \geq 0}$. Note that this is still much smaller than the augmentation with all \mathfrak{P}_{sem} -polar sets (or even the P -augmentation for some $P \in \mathfrak{P}_{sem}$), because we are only adding sets already included in \mathcal{F} . In particular, all elements of $\mathcal{F}_t \vee \mathcal{N}_{sem}$ are Borel sets and an $\mathbb{F} \vee \mathcal{N}_{sem}$ -progressively measurable process is automatically $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. For the purposes of the present chapter, both versions are sufficient.

Proposition III.6.6. (i) *There exists an \mathbb{F} -predictable, \mathbb{S}_+^d -valued process C such that*

$$C = \langle X^{c,P} \rangle^{(P)} \quad P\text{-a.s. for all } P \in \mathfrak{P}_{sem},$$

where $X^{c,P}$ denotes the continuous local martingale part of X under P and $\langle X^{c,P} \rangle^{(P)}$ is its predictable quadratic variation under P . In particular, the paths of C are P -a.s. increasing and continuous for all $P \in \mathfrak{P}_{sem}$.

(ii) *There exists an $\mathbb{F} \vee \mathcal{N}_{sem}$ -predictable, \mathbb{S}_+^d -valued process \bar{C} with continuous increasing paths such that*

$$\bar{C} = \langle X^{c,P} \rangle^{(P)} \quad P\text{-a.s. for all } P \in \mathfrak{P}_{sem}.$$

Proof. We begin with (ii). As a first step, we show that there exists an $\mathbb{F} \vee \mathcal{N}_{sem}$ -optional process $[X]$ with values in \mathbb{S}_+^d , having all paths càdlàg and of finite variation, such that

$$[X] = [X]^{(P)} \quad P\text{-a.s. for all } P \in \mathfrak{P}_{sem},$$

where $[X]^{(P)}$ is the usual quadratic covariation process of X under P . To this end, we first apply Bichteler's pathwise integration [8, Theorem 7.14], see also [31] for the same result in modern notation, to $\int X_-^i dX^j$, for fixed $1 \leq i, j \leq d$. This integration was also used in [48, 71, 72] in the context of continuous martingales; however, we have to elaborate on the construction to find a Borel-measurable version.

Define for each $n \geq 1$ the sequence $\tau_0^n := 0$,

$$\tau_{l+1}^n := \inf \{t \geq \tau_l^n \mid |X_t^i - X_{\tau_l^n}^i| \geq 2^{-n} \text{ or } |X_{t-}^i - X_{\tau_l^n}^i| \geq 2^{-n}\}, \quad l \geq 0.$$

Since X is càdlàg, each τ_l^n is an \mathbb{F} -stopping time and $\lim_l \tau_l^n(\omega) = \infty$ holds for all $\omega \in \Omega$. In particular, the processes defined by

$$I_t^n := X_{\tau_k^n}^i (X_t^j - X_{\tau_k^n}^j) + \sum_{l=0}^{k-1} X_{\tau_l^n}^i (X_{\tau_{l+1}^n}^j - X_{\tau_l^n}^j) \quad \text{for } \tau_k^n < t \leq \tau_{k+1}^n, \quad k \geq 0$$

are \mathbb{F} -adapted and càdlàg, thus \mathbb{F} -optional. Finally, we define

$$I_t(\omega) := \limsup_{n \rightarrow \infty} I_t^n(\omega);$$

then I is again \mathbb{F} -optional. Moreover, it is a consequence of the Burkholder-Davis-Gundy inequalities that

$$\sup_{0 \leq t \leq N} \left| I_t^n - \int_0^t X_{s-}^i dX_s^j \right| \rightarrow 0 \quad P\text{-a.s.}, \quad N \geq 1 \quad (6.3)$$

for each $P \in \mathfrak{P}_{sem}$, where the integral is the usual Itô integral under P . For two càdlàg functions f, g on \mathbb{R}_+ , let

$$d(f, g) = \sum_{N \geq 1} 2^{-N} (1 \wedge \|f - g\|_N),$$

where $\|\cdot\|_N$ is the uniform norm on $[0, N]$. Then d metrizes locally uniform convergence and a sequence of càdlàg functions is d -convergent if and only if it is d -Cauchy. Let

$$G = \{\omega \in \Omega \mid I^n(\omega) \text{ is } d\text{-Cauchy}\}.$$

It is elementary to see that $G \in \mathcal{F}$. Since (6.3) implies that $P(G) = 1$ for all $P \in \mathfrak{P}_{sem}$, we conclude that the complement of G is in \mathcal{N}_{sem} . On the other hand, we note that the d -limit of a sequence of càdlàg functions is necessarily càdlàg. Hence,

$$J^{ij} := I \mathbf{1}_G$$

defines an $\mathbb{F} \vee \mathcal{N}_{sem}$ -optional process with càdlàg paths. Define the $\mathbb{R}^{d \times d}$ -valued process $Q = (Q^{ij})$ by

$$Q^{ij} := X^i X^j - J^{ij} - J^{ji}.$$

Then $Q^{ij} = X^i X^j - {}^{(P)}\int X_-^i dX^j - {}^{(P)}\int X_-^j dX^i = ([X]^{(P)})^{ij}$ holds P -a.s. for all $P \in \mathfrak{P}_{sem}$; this is simply the integration-by-parts formula for the Itô integral. In particular, Q has increasing paths in \mathbb{S}_+^d P -a.s. for all $P \in \mathfrak{P}_{sem}$. Since Q is càdlàg, the set $G' = \{\omega \in \Omega \mid Q(\omega) \text{ is increasing in } \mathbb{S}_+^d\}$ is \mathcal{F} -measurable and we conclude that

$$[X] := Q \mathbf{1}_{G'}$$

is an $\mathbb{F} \vee \mathcal{N}_{sem}$ -optional process having càdlàg, increasing paths and satisfying $[X] = [X]^{(P)}$ P -a.s. for all $P \in \mathfrak{P}_{sem}$.

The second step is to construct \tilde{C} from $[X]$. Recall that a càdlàg function f of finite variation can be (uniquely) decomposed into the sum of a continuous part f^c and a discontinuous part f^d ; namely,

$$f_t^d := \sum_{0 \leq s \leq t} (f_s - f_{s-}), \quad f_t^c := f_t - f_t^d,$$

where $f_{0-} := 0$. Since all paths of $[X]$ are càdlàg and of finite variation, we can define $\bar{C} := [X]^c$. Then \bar{C} is $\mathbb{F} \vee \mathcal{N}_{sem}$ -optional (e.g., by [26, Proposition 1.16, p. 69]), $\bar{C}_0 = 0$ and all paths of \bar{C} are increasing in \mathbb{S}_+^d and continuous. Hence, \bar{C} is also $\mathbb{F} \vee \mathcal{N}_{sem}$ -predictable. Let $P \in \mathfrak{P}_{sem}$ and recall (see [26, Theorem 4.52, p. 55]) that

$$[X]^{(P)} = \langle X^{c,P} \rangle^{(P)} + \sum_{0 \leq s \leq \cdot} (\Delta X_s)^2 \quad P\text{-a.s.}$$

By the uniqueness of this decomposition, we have that $\bar{C} = \langle X^{c,P} \rangle^{(P)}$ P -a.s., showing that \bar{C} is indeed a second characteristic of X under P .

For the \mathbb{F} -predictable version (i), we construct $[X]$ as above but with $G = G' = \Omega$; then $[X]$ is \mathbb{F} -optional (instead of $\mathbb{F} \vee \mathcal{N}_{sem}$ -optional) while lacking the path properties. On the other hand, all paths of X are càdlàg and hence the process

$$C' := [X] - \sum_{0 \leq s \leq \cdot} (\Delta X_s)^2$$

is well-defined and \mathbb{F} -optional. Next, define $C'_0 := 0$ and (componentwise)

$$C''_t := \limsup_{n \rightarrow \infty} C'_{t-1/n}, \quad t > 0;$$

then C'' is \mathbb{F} -predictable. Finally, the process $C := C'' \mathbf{1}_{C'' \in \mathbb{S}_+^d}$ has the required properties, because for given $P \in \mathfrak{P}_{sem}$ the paths of C' are already continuous P -a.s. and thus $C = C' = C'' = \langle X^{c,P} \rangle^{(P)}$ P -a.s. \square

III.7 Differential Characteristics

In this section, we prove Theorem III.2.6 and its corollary. The conditions of Theorem III.2.6 (which are the ones of Theorem III.2.5) are in force. We recall the set of semimartingale measures under which X has absolutely continuous characteristics,

$$\mathfrak{P}_{sem}^{ac} = \{P \in \mathfrak{P}_{sem} \mid (B^P, C, \nu^P) \ll dt, P\text{-a.s.}\}.$$

Lemma III.7.1. *The set $\mathfrak{P}_{sem}^{ac} \subseteq \mathfrak{P}(\Omega)$ is measurable.*

Proof. Let (B^P, C, ν^P) and A^P be as stated in Theorem III.2.5. For all $P \in \mathfrak{P}_{sem}$, let R^P be the $[0, \infty]$ -valued process

$$R^P := \sum_{1 \leq i \leq d} \text{Var}(B^{P,i}) + \sum_{1 \leq i, j \leq d} \text{Var}(C^{ij}) + |A^P|,$$

where the indices i, j refer to the components of the \mathbb{R}^d - and $\mathbb{R}^{d \times d}$ -valued processes B^P , C and

$$\text{Var}(f)_t := \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} |f_{kt/2^n} - f_{(k-1)t/2^n}|$$

for any real function f on \mathbb{R}_+ . (If f is right-continuous, this is indeed the total variation up to time t , as the notation suggests.) This definition and the properties stated in Theorem III.2.5 imply that $(P, \omega, t) \mapsto R_t^P(\omega)$ is measurable and that for each $P \in \mathfrak{P}_{sem}$, R^P is finite valued P -a.s. and has P -a.s. right-continuous paths. Moreover, we have P -a.s. that (componentwise)

$$dA^P \ll dR^P, \quad dB^P \ll dR^P \quad \text{and} \quad dC \ll dR^P.$$

Let

$$\varphi_t^{P,n}(\omega) := \sum_{k \geq 0} 2^n (R_{(k+1)2^{-n}}^P(\omega) - R_{k2^{-n}}^P(\omega)) \mathbf{1}_{(k2^{-n}, (k+1)2^{-n}]}(t)$$

for all $(P, \omega, t) \in \mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+$ and

$$\varphi_t^P(\omega) := \limsup_{n \rightarrow \infty} \varphi_t^{P,n}(\omega), \quad (P, \omega, t) \in \mathfrak{P}_{sem} \times \Omega \times \mathbb{R}_+.$$

Clearly $(P, \omega, t) \mapsto \varphi_t^P(\omega)$ is measurable. Moreover, φ^P is P -a.s. the density of the absolutely continuous part of R^P with respect to the Lebesgue measure; cf. [13, Theorem V.58, p. 52] and the subsequent remark. That is, there is a decomposition $R_t^P(\omega) = \int_0^t \varphi_s^P(\omega) ds + \psi_t^P(\omega)$, $t \in \mathbb{R}_+$ for P -a.e. $\omega \in \Omega$, with a function $t \mapsto \psi_t^P(\omega)$ that is singular with respect to the Lebesgue measure. In particular, \mathfrak{P}_{sem}^{ac} can be characterized as

$$\mathfrak{P}_{sem}^{ac} = \{P \in \mathfrak{P}_{sem} \mid E^P[\mathbf{1}_G(P, \cdot)] = 1\}$$

with the set

$$G := \left\{ (P, \omega) \in \mathfrak{P}_{sem} \times \Omega \mid R_t^P(\omega) = \int_0^t \varphi_s^P(\omega) ds \text{ for all } t \in \mathbb{Q}_+ \right\}.$$

As G is measurable, we conclude by Lemma III.3.1 that \mathfrak{P}_{sem}^{ac} is measurable. \square

Next, we prove the remaining statements of Theorem III.2.6.

Proof of Theorem III.2.6. Let B^P, C, ν^P, K^P, A^P be as in Theorem III.2.5 and let $P \in \mathfrak{P}_{sem}^{ac}$. Let

$$\widehat{A}_t^P := \limsup_{n \rightarrow \infty} A_{(t-1/n) \vee 0}^P;$$

then \widehat{A}^P is \mathbb{F}_- -adapted and hence \mathbb{F} -predictable. Moreover, since we know a priori that A^P has continuous paths P -a.s., we have $\widehat{A}^P = A^P$ P -a.s. Consider

$$\widetilde{a}_t^P := \limsup_{n \rightarrow \infty} n(\widehat{A}_t^P - \widehat{A}_{(t-1/n) \vee 0}^P).$$

If we define $a^P := \widetilde{a}^P \mathbf{1}_{\mathbb{R}_+}(\widetilde{a}^P)$, then $(P, \omega, t) \mapsto a_t^P(\omega)$ is measurable and a^P is an \mathbb{F} -predictable process for every $P \in \mathfrak{P}_{sem}^{ac}$. Moreover, since $A_t^P \ll dt$

P -a.s., we also have $a_t^P dt = dA_t^P$ P -a.s. We proceed similarly with B^P and C to define processes b^P and c with values in \mathbb{R}^d and \mathbb{S}_+^d , respectively, having the properties stated in Theorem III.2.6.

Let $\tilde{K}^P(\omega, t, dx)$ be the $\mathcal{B}(\mathfrak{P}_{sem}^{ac}) \otimes \mathcal{P}$ -measurable kernel from (6.1) and let $\tilde{F}_{\omega,t}^P(dx)$ be the kernel on \mathbb{R}^d given $\mathfrak{P}_{sem}^{ac} \times \Omega \times \mathbb{R}_+$ defined by

$$\tilde{F}_{\omega,t}^P(dx) := \tilde{K}^P(\omega, t, dx) a_t^P(\omega).$$

It follows from Fubini's theorem that $\tilde{F}_{\omega,t}^P(dx) \in \mathcal{L}$ holds $P \times dt$ -a.e. for all $P \in \mathfrak{P}_{sem}^{ac}$. To make this hold everywhere, let

$$G = \left\{ (P, \omega, t) \in \mathfrak{P}_{sem}^{ac} \times \Omega \times \mathbb{R}_+ \mid \int_{\mathbb{R}^d} |x|^2 \wedge 1 \tilde{F}_{\omega,t}^P(dx) < \infty \text{ and } \tilde{F}_{\omega,t}^P(\{0\}) = 0 \right\}.$$

Then $G \in \mathcal{B}(\mathfrak{P}_{sem}^{ac}) \otimes \mathcal{F} \otimes B(\mathbb{R}_+)$ and the complements of its sections,

$$G^P := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid (P, \omega, t) \notin G\},$$

satisfy

$$G^P \in \mathcal{P} \quad \text{and} \quad (P \otimes dt)(G^P) = 0, \quad P \in \mathfrak{P}_{sem}^{ac}.$$

Thus, if we define the kernel $F_{\omega,t}^P(dx)$ on \mathbb{R}^d given $\mathfrak{P}_{sem}^{ac} \times \Omega \times \mathbb{R}_+$ by

$$F_{\omega,t}^P(E) := \mathbf{1}_G(P, \omega, t) \tilde{F}_{\omega,t}^P(E), \quad E \in \mathcal{B}(\mathbb{R}^d);$$

then $F_{\omega,t}^P(dx) \in \mathcal{L}$ for all $(P, \omega, t) \in \mathfrak{P}_{sem}^{ac} \times \Omega \times \mathbb{R}_+$, while $(\omega, t) \mapsto F_{\omega,t}^P(dx)$ is a kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given $(\Omega \times \mathbb{R}_+, \mathcal{P})$ for all $P \in \mathfrak{P}_{sem}^{ac}$ and

$$F_{\omega,t}^P(dx) dt = \tilde{K}^P(\omega, t, dx) dA_t^P(\omega) = K^P(\omega, t, dx) dA_t^P(\omega) = \nu^P(\omega, dt, dx)$$

P -a.s. for all $P \in \mathfrak{P}_{sem}^{ac}$. Moreover, $(P, \omega, t) \mapsto \int_E |x|^2 \wedge 1 F_{\omega,t}^P(dx)$ is measurable for any $E \in \mathcal{B}(\mathbb{R}^d)$. Thus, by Lemma III.2.4, the map $(P, \omega, t) \mapsto F_{\omega,t}^P(dx)$ is measurable with respect to $\mathcal{B}(\mathcal{L})$. Finally, it is clear from the construction that (b^P, c, F^P) are indeed differential characteristics of X under P for all $P \in \mathfrak{P}_{sem}^{ac}$. \square

It remains to prove the measurability of the sets $\mathfrak{P}_{sem}^{ac}(\Theta)$.

Proof of Corollary III.2.7. Let $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be a Borel set and let (b^P, c, F^P) be a measurable version of the differential characteristics for $P \in \mathfrak{P}_{sem}^{ac}$ as in Theorem III.2.6; then

$$G := \{(P, \omega, t) \mid (b_t^P, c_t, F_t^P)(\omega) \notin \Theta\} \in \mathcal{B}(\mathfrak{P}_{sem}^{ac}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+).$$

Thus, by Fubini's theorem, $G' := \{(P, \omega) \mid \int_0^\infty \mathbf{1}_G(P, \omega, t) dt = 0\}$ is again measurable. Since G' consists of all (P, ω) such that $(b_t^P, c_t, F_t^P)(\omega) \in \Theta$ holds $P \otimes dt$ -a.e., we have

$$\mathfrak{P}_{sem}^{ac}(\Theta) = \{P \in \mathfrak{P}_{sem}^{ac} \mid E^P[\mathbf{1}_{G'}(P, \cdot)] = 1\},$$

and the set on the right-hand side is measurable due to Lemma III.3.1. \square

Chapter IV

Nonlinear Lévy Processes and their Characteristics

In this chapter, which corresponds to the article [39], we develop a general construction for nonlinear Lévy processes with given characteristics. More precisely, given a set Θ of Lévy triplets, we construct a sublinear expectation on Skorohod space under which the canonical process has stationary independent increments and a nonlinear generator corresponding to the supremum of all generators of classical Lévy processes with triplets in Θ .

IV.1 Introduction

The main goal of this chapter is to construct nonlinear Lévy processes with prescribed local characteristics. This is achieved by a probabilistic construction involving an optimal control problem on Skorohod space where the controls are laws of semimartingales with suitable characteristics.

Let $X = (X_t)_{t \in \mathbb{R}_+}$ be an \mathbb{R}^d -valued process with càdlàg paths and $X_0 = 0$, defined on a measurable space (Ω, \mathcal{F}) which is equipped with a nonlinear expectation $\mathcal{E}(\cdot)$. For our purposes, this will be a sublinear operator

$$\xi \mapsto \mathcal{E}(\xi) := \sup_{P \in \mathfrak{P}} E^P[\xi], \quad (1.1)$$

where \mathfrak{P} is a set of probability measures on (Ω, \mathcal{F}) and $E^P[\cdot]$ is the usual expectation under the measure P . In this setting, if Y and Z are random vectors, Y is said to be independent of Z if

$$\mathcal{E}(\varphi(Y, Z)) = \mathcal{E}(\mathcal{E}(\varphi(Y, z))|_{z=Z})$$

for all bounded Borel functions φ , and if Y and Z are of the same dimension, they are said to be identically distributed if

$$\mathcal{E}(\varphi(Y)) = \mathcal{E}(\varphi(Z))$$

for all bounded Borel functions φ . We note that both definitions coincide with the classical probabilistic notions if \mathfrak{P} is a singleton. Following [25, Definition 19], the process X is a *nonlinear Lévy process* under $\mathcal{E}(\cdot)$ if it has stationary and independent increments; that is, $X_t - X_s$ and X_{t-s} are identically distributed for all $0 \leq s \leq t$, and $X_t - X_s$ is independent of $(X_{s_1}, \dots, X_{s_n})$ for all $0 \leq s_1 \leq \dots \leq s_n \leq s \leq t$. The particular case of a classical Lévy process is recovered when \mathfrak{P} is a singleton.

Let Θ be a set of Lévy triplets (b, c, F) . We recall that each Lévy triplet characterizes the distributional properties and in particular the infinitesimal generator of a classical Lévy process. More precisely, the associated Kolmogorov equation is

$$v_t(t, x) - \left\{ bv_x(t, x) + \frac{1}{2} \text{tr}[cv_{xx}(t, x)] + \int [v(t, x+z) - v(t, x) - v_x(t, x)h(z)] F(dz) \right\} = 0,$$

where, e.g., $h(z) = z\mathbf{1}_{|z| \leq 1}$. Our goal is to construct a nonlinear Lévy process whose local characteristics are described by the set Θ , in the sense that the analogue of the Kolmogorov equation will be the fully nonlinear partial integro-differential equation

$$v_t(t, x) - \sup_{(b,c,F) \in \Theta} \left\{ bv_x(t, x) + \frac{1}{2} \text{tr}[cv_{xx}(t, x)] + \int [v(t, x+z) - v(t, x) - v_x(t, x)h(z)] F(dz) \right\} = 0. \quad (1.2)$$

In fact, our probabilistic construction of the process justifies the name characteristic in a rather direct way.

In our construction, we take X to be the canonical process on Skorohod space and hence $\mathcal{E}(\cdot)$ is the main object of consideration, or more precisely, the set \mathfrak{P} of probability measures appearing in (1.1). Given an arbitrary set Θ of Lévy triplets, we let $\mathfrak{P} = \mathfrak{P}_{sem}^{ac}(\Theta)$ be the set of all laws of semimartingales whose differential characteristics take values in Θ , as introduced in Chapter III. Assuming merely that Θ is measurable, we then show that X is a nonlinear Lévy process under $\mathcal{E}(\cdot)$; this is based on the more general fact that $\mathcal{E}(\cdot)$ satisfies a certain semigroup property (Theorem IV.2.1). The proofs require an analysis of semimartingale characteristics which will be useful for other control problems as well. Under the conditions

$$\sup_{(b,c,F) \in \Theta} \left\{ \int |z| \wedge |z|^2 F(dz) + |b| + |c| \right\} < \infty \quad (1.3)$$

and

$$\lim_{\varepsilon \rightarrow 0} \sup_{(b,c,F) \in \Theta} \int_{|z| \leq \varepsilon} |z|^2 F(dz) = 0 \quad (1.4)$$

on Θ , we show that functionals of the form $v(t, x) = \mathcal{E}(\psi(x + X_t))$ can be characterized as the unique viscosity solution of the nonlinear Kolmogorov equation (1.2) with initial condition ψ (Theorem IV.2.5).

The remainder of this chapter is organized as follows. Section IV.2 details the setup and contains the main results: the probabilistic construction is summarized in Theorem IV.2.1 and the PIDE characterization in Theorem IV.2.5. Moreover, we give two examples of nonlinear Lévy processes. Sections IV.3 and IV.4 provide an analysis of semimartingale laws and the associated characteristics under conditioning and products which forms the main part of the proof of Theorem IV.2.1. In Section IV.5, we show the existence and comparison results for the PIDE (1.2). Related literature is discussed in the concluding Section IV.6.

IV.2 Main Results

Fix $d \in \mathbb{N}$ and let $\Omega = D_0(\mathbb{R}_+, \mathbb{R}^d)$ be the space of all càdlàg paths $\omega = (\omega_t)_{t \geq 0}$ in \mathbb{R}^d with $\omega_0 = 0$. We equip Ω with the Skorohod topology and the corresponding Borel σ -field \mathcal{F} . Moreover, we denote by $X = (X_t)_{t \geq 0}$ the canonical process $X_t(\omega) = \omega_t$ and by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the (raw) filtration generated by X .

Our starting point is a subset of $\mathfrak{P}(\Omega)$, the Polish space of all probability measures on Ω , determined by the semimartingale characteristics as follows. First, let

$$\mathfrak{P}_{sem} = \{P \in \mathfrak{P}(\Omega) \mid X \text{ is a semimartingale on } (\Omega, \mathcal{F}, \mathbb{F}, P)\} \subseteq \mathfrak{P}(\Omega) \quad (2.1)$$

be the set of all semimartingale laws. To be specific, let us agree that if \mathbb{G} is a given filtration, a \mathbb{G} -adapted process Y with càdlàg paths will be called a P - \mathbb{G} -semimartingale if there exist right-continuous, \mathbb{G} -adapted processes M and A with $M_0 = A_0 = 0$ such that M is a P - \mathbb{G} -local martingale, A has paths of (locally) finite variation P -a.s., and $Y = Y_0 + M + A$ P -a.s.; cf. Definition III.2.1. We remark that X is a P -semimartingale for \mathbb{F} if and only if it has this property for the right-continuous filtration \mathbb{F}_+ or the usual augmentation \mathbb{F}_+^P ; cf. Proposition III.2.2. In other words, the precise choice of the filtration in the definition (2.1) is not crucial.

Fix a truncation function $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$; that is, a bounded measurable function such that $h(x) = x$ in a neighborhood of the origin, and let (B^P, C^P, ν^P) be semimartingale characteristics of X under $P \in \mathfrak{P}_{sem}$ and \mathbb{F} , relative to h . To be specific, this means that (B^P, C^P, ν^P) is a triplet of processes such that P -a.s., B^P is the finite variation part in the canonical decomposition of $X - \sum_{0 \leq s \leq \cdot} (\Delta X_s - h(\Delta X_s))$ under P , C^P is the quadratic covariation of the continuous local martingale part of X under P , and ν^P is the P -compensator of μ^X , the integer-valued random measure associated with the jumps of X . (Again, the precise choice of the filtration does not

matter for the present section; see Proposition III.2.2.) We shall mainly work with the subset

$$\mathfrak{P}_{sem}^{ac} = \{P \in \mathfrak{P}_{sem} \mid (B^P, C^P, \nu^P) \ll dt, P\text{-a.s.}\}$$

of semimartingales with absolutely continuous characteristics (with respect to the Lebesgue measure dt). Given $P \in \mathfrak{P}_{sem}^{ac}$, we can consider the associated differential characteristics (b^P, c^P, F^P) , defined via $(dB^P, dC^P, d\nu^P) = (b_t^P dt, c_t^P dt, F_t^P dt)$. The differential characteristics take values in $\mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$, where \mathbb{S}_+^d is the set of symmetric nonnegative definite $d \times d$ -matrices and

$$\mathcal{L} = \left\{ F \text{ measure on } \mathbb{R}^d \mid \int_{\mathbb{R}^d} |x|^2 \wedge 1 F(dx) < \infty \text{ and } F(\{0\}) = 0 \right\}$$

is the set of all Lévy measures, a separable metric space under a suitable version of the weak convergence topology (cf. Section 2 of Chapter III). Any element $(b, c, F) \in \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ is called a Lévy triplet and indeed, there exists a Lévy process having (b, c, F) as its differential characteristics.

IV.2.1 Nonlinear Lévy Processes with given Characteristics

Let $\emptyset \neq \Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be any (Borel) measurable subset. Our aim is to construct a nonlinear Lévy process corresponding to Θ ; of course, the case of a classical Lévy process will correspond to Θ being a singleton. An important object in our construction is the set of all semimartingale laws whose differential characteristics take values in Θ ,

$$\mathfrak{P}_\Theta := \mathfrak{P}_{sem}^{ac}(\Theta) = \{P \in \mathfrak{P}_{sem}^{ac} \mid (b^P, c^P, F^P) \in \Theta, P \otimes dt\text{-a.e.}\},$$

and a key step will be to show that \mathfrak{P}_Θ is amenable to dynamic programming, as formalized by Condition (A) below. To state this condition, we need to introduce some more notation. Let τ be a finite \mathbb{F} -stopping time. Then the concatenation of $\omega, \tilde{\omega} \in \Omega$ at τ is the path

$$(\omega \otimes_\tau \tilde{\omega})_u := \omega_u \mathbf{1}_{[0, \tau(\omega))}(u) + (\omega_{\tau(\omega)} + \tilde{\omega}_{u-\tau(\omega)}) \mathbf{1}_{[\tau(\omega), \infty)}(u), \quad u \geq 0.$$

For any probability measure $P \in \mathfrak{P}(\Omega)$, there is a regular conditional probability distribution $\{P_\tau^\omega\}_{\omega \in \Omega}$ given \mathcal{F}_τ satisfying

$$P_\tau^\omega \{\omega' \in \Omega \mid \omega' = \omega \text{ on } [0, \tau(\omega)]\} = 1 \quad \text{for all } \omega \in \Omega.$$

We then define $P^{\tau, \omega} \in \mathfrak{P}(\Omega)$ by

$$P^{\tau, \omega}(D) := P_\tau^\omega(\omega \otimes_\tau D), \quad D \in \mathcal{F}, \quad \text{where } \omega \otimes_\tau D := \{\omega \otimes_\tau \tilde{\omega} \mid \tilde{\omega} \in D\}.$$

Given a function ξ on Ω and $\omega \in \Omega$, we also define the function $\xi^{\tau, \omega}$ on Ω by

$$\xi^{\tau, \omega}(\tilde{\omega}) := \xi(\omega \otimes_\tau \tilde{\omega}), \quad \tilde{\omega} \in \Omega.$$

If ξ is measurable, we then have $E^{P^{\tau,\omega}}[\xi^{\tau,\omega}] = E^P[\xi|\mathcal{F}_\tau](\omega)$ for P -a.e. $\omega \in \Omega$. (The convention $\infty - \infty = -\infty$ is used throughout; for instance, in defining $E^P[\xi|\mathcal{F}_\tau] := E^P[\xi^+|\mathcal{F}_\tau] - E^P[\xi^-|\mathcal{F}_\tau]$.) Finally, a subset of a Polish space is called analytic if it is the image of a Borel subset of another Polish space under a Borel-measurable mapping; in particular, any Borel set is analytic.

We can now state the mentioned condition for a given set $\mathfrak{P} \subseteq \mathfrak{P}(\Omega)$.

Condition (A). Let τ be a finite \mathbb{F} -stopping time and let $P \in \mathfrak{P}$.

(A1) The set $\mathfrak{P} \subseteq \mathfrak{P}(\Omega)$ is analytic.

(A2) We have $P^{\tau,\omega} \in \mathfrak{P}$ for P -a.e. $\omega \in \Omega$.

(A3) If $\kappa : \Omega \rightarrow \mathfrak{P}(\Omega)$ is an \mathcal{F}_τ -measurable kernel and $\kappa(\omega) \in \mathfrak{P}$ for P -a.e. $\omega \in \Omega$, then the measure defined by

$$\bar{P}(D) = \iint (\mathbf{1}_D)^{\tau,\omega}(\omega') \kappa(\omega, d\omega') P(d\omega), \quad D \in \mathcal{F}$$

is an element of \mathfrak{P} .

Some more notation is needed for the first main result. Given a σ -field \mathcal{G} , the universal completion of \mathcal{G} is the σ -field $\mathcal{G}^* = \bigcap_P \mathcal{G}^{(P)}$, where P ranges over all probability measures on \mathcal{G} and $\mathcal{G}^{(P)}$ is the completion of \mathcal{G} under P . Moreover, an \mathbb{R} -valued function f is called upper semianalytic if $\{f > a\}$ is analytic for each $a \in \mathbb{R}$. Any Borel-measurable function is upper semianalytic and any upper semianalytic function is universally measurable.

Theorem IV.2.1. *Let $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be a measurable set of Lévy triplets, $\mathfrak{P}_\Theta = \{P \in \mathfrak{P}_{sem}^{ac} \mid (b^P, c^P, F^P) \in \Theta, P \otimes dt\text{-a.e.}\}$ and consider the associated sublinear expectation $\mathcal{E}(\cdot) = \sup_{P \in \mathfrak{P}_\Theta} E^P[\cdot]$ on the Skorohod space Ω .*

(i) *The set \mathfrak{P}_Θ satisfies Condition (A)*

(ii) *Let $\sigma \leq \tau$ be finite \mathbb{F} -stopping times and let $\xi : \Omega \rightarrow \bar{\mathbb{R}}$ be upper semianalytic. Then the function $\omega \mapsto \mathcal{E}_\tau(\xi)(\omega) := \mathcal{E}(\xi^{\tau,\omega})$ is \mathcal{F}_τ^* -measurable and upper semianalytic, and*

$$\mathcal{E}_\sigma(\xi)(\omega) = \mathcal{E}_\sigma(\mathcal{E}_\tau(\xi))(\omega) \quad \text{for all } \omega \in \Omega. \quad (2.2)$$

(iii) *The canonical process X is a nonlinear Lévy process under $\mathcal{E}(\cdot)$.*

Thus, this results yields the existence of nonlinear Lévy processes with general characteristic Θ as well as their interpretation in terms of classical stochastic analysis; namely, as a control problem over laws of semimartingales. The semigroup property stated in (2.2) will be the starting point for the PIDE result reported below.

Proof. (i) The verification of (A1) was shown in Corollary III.2.7, which was our initial motivation for Chapter III. Properties (A2) and (A3) will be established in Corollary IV.3.2 and Proposition IV.4.2, respectively. They follow from the analysis of semimartingale characteristics under conditioning and products of semimartingale laws that will be carried out in Sections IV.3 and IV.4.

(ii) Once Condition (A) is established, the validity of (ii) is a consequence of the dynamic programming principle in the form of [49, Theorem 2.3]. (That result is stated for the space of continuous paths, but carries over to Skorohod space with the same proof.)

(iii) We first show that X has stationary increments. Let $s, t \geq 0$ and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be bounded and Borel. Using the identity

$$X_{t+s}^{t,\omega} - X_t^{t,\omega} = X_s, \quad \omega \in \Omega,$$

the tower property (2.2) yields that

$$\mathcal{E}(\varphi(X_{t+s} - X_t)) = \mathcal{E}(\mathcal{E}_t(\varphi(X_{t+s} - X_t))) = \mathcal{E}(\mathcal{E}(\varphi(X_s))) = \mathcal{E}(\varphi(X_s)).$$

Similarly, to see the independence of the increments, let $0 \leq t_1 \leq \dots \leq t_n \leq t$ and let φ be defined on $\mathbb{R}^{(n+1)d}$ instead of \mathbb{R}^d . Then

$$(X_{t+s}^{t,\omega} - X_t^{t,\omega}, X_{t_1}^{t,\omega}, \dots, X_{t_n}^{t,\omega}) = (X_s, X_{t_1}(\omega), \dots, X_{t_n}(\omega)), \quad \omega \in \Omega$$

and (2.2) imply that

$$\begin{aligned} & \mathcal{E}(\varphi(X_{t+s} - X_t, X_{t_1}, \dots, X_{t_n})) \\ &= \mathcal{E}(\mathcal{E}_t(\varphi(X_{t+s} - X_t, X_{t_1}, \dots, X_{t_n}))) \\ &= \mathcal{E}(\mathcal{E}(\varphi(X_s, x_1, \dots, x_n))|_{x_1=X_{t_1}, \dots, x_n=X_{t_n}}) \\ &= \mathcal{E}(\mathcal{E}(\varphi(X_{t+s} - X_t, x_1, \dots, x_n))|_{x_1=X_{t_1}, \dots, x_n=X_{t_n}}), \end{aligned}$$

where the last equality is due to the stationarity of the increments applied to the test function $\varphi(\cdot, x_1, \dots, x_n)$. \square

Remark IV.2.2. The nonlinear Lévy property of X corresponds to the fact that the set $\mathfrak{P} = \mathfrak{P}_\Theta$ is independent of (t, ω) . More precisely, recall that [49] considered more generally a family $\{\mathfrak{P}(t, \omega)\}$ indexed by $t \geq 0$ and $\omega \in \Omega$. In this situation, the conditional nonlinear expectation is given by

$$\mathcal{E}_\tau(\xi)(\omega) := \sup_{P \in \mathfrak{P}(\tau(\omega), \omega)} E^P[\xi^{\tau, \omega}], \quad \omega \in \Omega;$$

this coincides with the above definition when $\mathfrak{P}(t, \omega)$ is independent of (t, ω) . As can be seen from the above proof, the temporal and spatial homogeneity of \mathfrak{P} is essentially in one-to-one correspondence with the independence and stationarity of the increments of X under $\mathcal{E}(\cdot)$.

In classical stochastic analysis, Lévy processes can be characterized as semimartingales with constant differential characteristics. The following shows that the nonlinear case allows for a richer structure.

Remark IV.2.3. The assertion of Theorem IV.2.1 holds more generally for any set $\mathfrak{P} \subseteq \mathfrak{P}(\Omega)$ satisfying Condition (A); this is clear from the proof. According to Theorem IV.3.1, Proposition IV.4.1 and Theorem III.2.5, the collection \mathfrak{P}_{sem} of all semimartingale laws (not necessarily with absolutely continuous characteristics) is another example of such a set. In particular, we see that nonlinear Lévy processes are not constrained to the time scale given by the Lebesgue measure. It is well known that classical Lévy processes have this property and one may say that this is due to the fact that the Lebesgue measure is, up to a normalization, the only homogeneous (shift-invariant) measure on the line. By contrast, there are many sublinear expectations on the line that are homogeneous—for instance, the one determined by the supremum of all measures, which may be seen as the time scale corresponding to \mathfrak{P}_{sem} .

Another property of classical Lévy processes is that they are necessarily semimartingales. A trivial example satisfying Condition (A) is the set $\mathfrak{P} = \mathfrak{P}(\Omega)$ of all probability measures on Ω . Thus, we also see that the semimartingale property, considered under a given $P \in \mathfrak{P}$, does not hold automatically.

One may also note that such (degenerate) examples are far outside the scope of the PIDE-based construction of [25].

Remark IV.2.4. The present setup could be extended to a case where the set Θ is replaced by a set-valued process $(t, \omega) \mapsto \Theta(t, \omega)$, in the spirit of the random G -expectations [46]. Of course, this situation is no longer homogeneous and so the resulting process would be a “nonlinear semimartingale” rather than a Lévy process. We shall see in the subsequent sections that the techniques of the present chapter still yield the desired dynamic programming properties, exactly as it was done in [49] for the case of continuous martingales.

IV.2.2 Nonlinear Lévy Processes and PIDE

For the second main result of this chapter, consider a nonempty measurable set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ satisfying the following two additional assumptions. The first one is

$$\sup_{(b,c,F) \in \Theta} \left\{ \int_{\mathbb{R}^d} |z| \wedge |z|^2 F(dz) + |b| + |c| \right\} < \infty, \quad (2.3)$$

where $|\cdot|$ is the Euclidean norm; this implies that the control problem defining $\mathcal{E}(\cdot)$ is non-singular and that the jumps are integrable. Moreover, we require

that

$$\lim_{\varepsilon \rightarrow 0} \sup_{(b,c,F) \in \Theta} \int_{|z| \leq \varepsilon} |z|^2 F(dz) = 0. \quad (2.4)$$

While this condition does not exclude any particular Lévy measure, it bounds the contribution of small jumps across Θ . In particular, it prevents \mathfrak{P}_Θ from containing a sequence of pure-jump processes which converges weakly to, say, a Brownian motion. Thus, both conditions are necessary to ensure that the PIDE below is indeed the correct dynamic programming equation for our problem.

Namely, we fix $\psi \in C_{b,Lip}(\mathbb{R}^d)$, the space of bounded Lipschitz functions on \mathbb{R}^d , and consider the fully nonlinear PIDE

$$\begin{cases} \partial_t v(t, x) - G(D_x v(t, x), D_{xx}^2 v(t, x), v(t, x + \cdot)) = 0 & \text{on } (0, \infty) \times \mathbb{R}^d, \\ v(0, \cdot) = \psi(\cdot), \end{cases} \quad (2.5)$$

where $G : \mathbb{R}^d \times \mathbb{S}^d \times C_b^2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} G(p, q, f(\cdot)) \\ := \sup_{(b,c,F) \in \Theta} \left\{ pb + \frac{1}{2} \text{tr}[qc] + \int_{\mathbb{R}^d} [f(z) - f(0) - D_x f(0)h(z)] F(dz) \right\}. \end{aligned} \quad (2.6)$$

We remark that this PIDE is nonstandard due to the supremum over a set of Lévy measures; see also [25]. Specifically, since this set is typically large (nondominated), (2.6) does not satisfy a dominated convergence theorem with respect to f , which leads to a discontinuous operator G .

We write $C_b^{2,3}((0, \infty) \times \mathbb{R}^d)$ for the set of functions on $(0, \infty) \times \mathbb{R}^d$ having bounded continuous derivatives up to the second and third order in t and x , respectively. A bounded upper semicontinuous function u on $[0, \infty) \times \mathbb{R}^d$ will be called a viscosity subsolution of (2.5) if $u(0, \cdot) \leq \psi(\cdot)$ and

$$\partial_t \varphi(t, x) - G(D_x \varphi(t, x), D_{xx}^2 \varphi(t, x), \varphi(t, x + \cdot)) \leq 0$$

whenever $\varphi \in C_b^{2,3}((0, \infty) \times \mathbb{R}^d)$ is such that $\varphi \geq u$ on $(0, \infty) \times \mathbb{R}^d$ and $\varphi(t, x) = u(t, x)$ for some $(t, x) \in (0, \infty) \times \mathbb{R}^d$. The definition of a viscosity supersolution is obtained by reversing the inequalities and the semicontinuity. Finally, a bounded continuous function is a viscosity solution if it is both sub- and supersolution. We recall that $\mathcal{E}(\cdot) = \sup_{P \in \mathfrak{P}_\Theta} E^P[\cdot]$ and X is the canonical process.

Theorem IV.2.5. *Let $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be a measurable set satisfying (2.3) and (2.4) and let $\psi \in C_{b,Lip}(\mathbb{R}^d)$. Then*

$$v(t, x) := \mathcal{E}(\psi(x + X_t)), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d \quad (2.7)$$

is the unique viscosity solution of (2.5).

The existence part will be proved in Proposition IV.5.4, whereas the validity of a comparison principle (and thus the uniqueness) is obtained in Proposition IV.5.5. As mentioned in the Introduction, this result allows us to rigorously identify our construction as an extension of [25]. A quite different application is given in Example IV.2.7 below.

IV.2.3 Examples

We conclude this section with two examples of nonlinear Lévy processes in dimension $d = 1$. The first one, called Poisson process with uncertain intensity, is the simplest example of interest and was already introduced in [25] under slightly more restrictive assumptions.

Example IV.2.6. Fix a measurable set $\Lambda \subseteq \mathbb{R}_+$ and consider

$$\Theta := \{(0, 0, \lambda \delta_1(dx)) \mid \lambda \in \Lambda\}.$$

Each triplet in Θ corresponds to a Poisson process with some intensity $\lambda \in \Lambda$, so that Λ can be called the set of possible intensities. To see that Θ is measurable, note that Θ is the image of Λ under $\lambda \mapsto \lambda \delta_1(dx)$. This is a measurable one-to-one mapping from \mathbb{R}_+ into \mathcal{L} , and as \mathcal{L} is a separable metric space according to Lemma III.2.3, it follows by Kuratowski's theorem [6, Proposition 7.15, p. 121] that Θ is indeed measurable.

As a result, Theorem IV.2.1 shows that the canonical process X is a nonlinear Lévy process with respect to $\mathcal{E}(\cdot) = \sup_{P \in \mathfrak{P}_\Theta} E^P[\cdot]$. Moreover, if Λ is bounded, Conditions (2.3) and (2.4) hold and Theorem IV.2.5 yields that $v(t, x) := \mathcal{E}(\psi(x + X_t))$ is the unique viscosity solution of the PIDE (2.5) with nonlinearity

$$G(p, q, f(\cdot)) = \sup_{\lambda \in \Lambda} \lambda (f(1) - f(0) - D_x f(0)h(1)),$$

for all $\psi \in C_{b,Lip}(\mathbb{R})$.

The second example represents uncertainty over a family of stable triplets, which does not fall within the framework of [25] because of the infinite variation jumps. We shall exploit the PIDE result to infer a nontrivial distributional property. In view of the central limit theorem of [54] for the nonlinear Gaussian distribution and classical results for α -stable distributions, one may suspect that this example also yields the limiting distribution in a nonstandard limit theorem¹.

Example IV.2.7. Let $\alpha \in (0, 2)$, fix measurable sets $B \subseteq \mathbb{R}$ and $K_\pm \subseteq \mathbb{R}_+$, and consider

$$\Theta := \{(b, 0, F_{k_\pm}) \mid b \in B, k_\pm \in K_\pm\},$$

¹Such a result was indeed obtained in follow-up work [3].

where $F_{k_{\pm}}$ denotes the α -stable Lévy measure

$$F_{k_{\pm}}(dx) = (k_- \mathbf{1}_{(-\infty, 0)} + k_+ \mathbf{1}_{(0, \infty)})(x) |x|^{-\alpha-1} dx.$$

If f is a bounded measurable function on \mathbb{R} , then $(k_+, k_-) \mapsto \int f(x) F_{k_{\pm}}(dx)$ is measurable by Fubini's theorem. In view of Lemma III.2.4, this means that $(k_+, k_-) \mapsto F_{k_{\pm}}$ is a measurable one-to-one mapping into \mathcal{L} , and thus Kuratowski's theorem again yields that Θ is measurable.

As a result, Theorem IV.2.1 once more shows that the canonical process X is a nonlinear Lévy process with respect to $\mathcal{E}(\cdot) = \sup_{P \in \mathfrak{P}_{\Theta}} E^P[\cdot]$. If B, K_{\pm} are bounded and $\alpha \in (1, 2)$, Conditions (2.3) and (2.4) hold and Theorem IV.2.5 yields that $v(t, x) := \mathcal{E}(\psi(x + X_t))$ is the unique viscosity solution of the PIDE (2.5) with

$$G(p, q, f(\cdot)) = \sup_{b \in B, k_{\pm} \in K_{\pm}} \left\{ pb + \int_{\mathbb{R}} (f(z) - f(0) - D_x f(0)h(z)) F_{k_{\pm}}(dz) \right\},$$

for all $\psi \in C_{b, Lip}(\mathbb{R})$.

With these conditions still in force, we now use the PIDE to see that X indeed satisfies a scaling property like the classical stable processes; namely, that $X_{\lambda t}$ and $\lambda^{1/\alpha} X_t$ have the same distribution in the sense that

$$\mathcal{E}(\psi(X_{\lambda t})) = \mathcal{E}(\psi(\lambda^{1/\alpha} X_t)), \quad \psi \in C_{b, Lip}(\mathbb{R})$$

for all $\lambda > 0$ and $t \geq 0$, provided that X_t is centered. More precisely, as $\alpha \in (1, 2)$, we may state the characteristics with respect to $h(x) = x$. In this parametrization, we suppose that $B = \{0\}$, since clearly no scaling property can exist in the situation with drift uncertainty. Given $\psi \in C_{b, Lip}(\mathbb{R})$, Theorem IV.2.5 yields that $\mathcal{E}(\psi(X_{\lambda t})) = v(\lambda t, 0)$, where v is the unique solution of the PIDE with initial condition ψ . If we define $\tilde{v}(t, x) := v(\lambda t, \lambda^{1/\alpha} x)$, it follows from

$$G(p, q, f(\lambda^{1/\alpha} \cdot)) = \lambda G(p, q, f(\cdot)), \quad f \in C_b^2(\mathbb{R})$$

that \tilde{v} is the (unique) viscosity solution to the same PIDE with initial condition $\tilde{\psi}(x) := \psi(\lambda^{1/\alpha} x)$. In particular, $\tilde{v}(t, 0) = \mathcal{E}(\tilde{\psi}(X_t))$ by Theorem IV.2.5. As a result, we have

$$\mathcal{E}(\psi(X_{\lambda t})) = v(\lambda t, 0) = \tilde{v}(t, 0) = \mathcal{E}(\tilde{\psi}(X_t)) = \mathcal{E}(\psi(\lambda^{1/\alpha} X_t))$$

as claimed. Note that \mathfrak{P}_{Θ} contains many semimartingale laws which do not satisfy the scaling property, so that this identity is indeed not trivial.

IV.3 Conditioned Semimartingale Laws and (A2)

In this section, we show that given $P \in \mathfrak{P}_{sem}$, the measures of the form $P^{\tau, \omega}$ are again semimartingale laws, and we establish the corresponding transformation of the semimartingale characteristics. In particular, this will yield the property (A2) for the set \mathfrak{P}_{Θ} as required by the main results.

We remark that the use of the raw filtration \mathbb{F} has some importance in this section; for instance, we shall frequently apply Galmarino's test and related properties. The following notation will be used. Let $P \in \mathfrak{P}_{sem}$ and let $\nu(\cdot, dt, dz)$ be the P - \mathbb{F} -compensator of μ^X ; that is, the third characteristic under P . Then there exists a decomposition

$$\nu(\cdot, dt, dx) = F_{\cdot,t}(dz) dA_t(\cdot) \quad P\text{-a.s.}, \quad (3.1)$$

where $F_{\cdot,t}(dz)$ is a kernel from $(\Omega \times [0, \infty), \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and A is an \mathbb{F} -predictable process with $A_0 = 0$ and P -a.s. non-decreasing, P -a.s. right-continuous paths; cf. [26, Theorem II.1.8, p. 66] and [13, Lemma 7, p. 399]. We often write $F_t(dz)$ instead of $F_{\cdot,t}(dz)$. Moreover, if Y is a stochastic process and σ, τ are finite stopping times, we simply write $Y_{\sigma+}^{\tau, \omega}$ for $(Y_{\sigma+})^{\tau, \omega}$; that is, the process $(\tilde{\omega}, t) \mapsto Y_{\sigma(\omega \otimes_{\tau} \tilde{\omega})+t}(\omega \otimes_{\tau} \tilde{\omega})$.

Theorem IV.3.1. *Let $P \in \mathfrak{P}_{sem}$, let τ be a finite \mathbb{F} -stopping time, and let $(B, C, F(dz) dA)$ be P - \mathbb{F} -characteristics of X . For P -a.e. $\omega \in \Omega$, we have $P^{\tau, \omega} \in \mathfrak{P}_{sem}$ and the processes*

$$B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega), \quad C_{\tau+}^{\tau, \omega} - C_{\tau(\omega)}(\omega), \quad F_{\tau+}^{\tau, \omega}(dz) d(A_{\tau+}^{\tau, \omega} - A_{\tau(\omega)}(\omega))$$

define a triplet of $P^{\tau, \omega}$ - \mathbb{F} -characteristics of X . Moreover, if $P \in \mathfrak{P}_{sem}^{ac}$ and (b, c, F) are differential P - \mathbb{F} -characteristics, then for P -a.e. $\omega \in \Omega$, we have $P^{\tau, \omega} \in \mathfrak{P}_{sem}^{ac}$ and

$$b_{\tau+}^{\tau, \omega}, \quad c_{\tau+}^{\tau, \omega}, \quad F_{\tau+}^{\tau, \omega}(dz)$$

define differential $P^{\tau, \omega}$ - \mathbb{F} -characteristics of X .

The proof will be given in the course of this section. Before that, let us state a consequence which forms part of Theorem IV.2.1.

Corollary IV.3.2. *Let $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be measurable, let $P \in \mathfrak{P}_{\Theta}$ and let τ be a finite \mathbb{F} -stopping time. Then $P^{\tau, \omega} \in \mathfrak{P}_{\Theta}$ for P -a.e. $\omega \in \Omega$; that is, \mathfrak{P}_{Θ} satisfies (A2).*

Proof. This is a direct consequence of the formula for the differential characteristics under $P^{\tau, \omega}$ from Theorem IV.3.1 and the definition of \mathfrak{P}_{Θ} . \square

As a first step towards the proof of Theorem IV.3.1, we establish two facts about the conditioning of (local) martingales.

Lemma IV.3.3. *Let $P \in \mathfrak{P}(\Omega)$, let M be a P - \mathbb{F} -uniformly integrable martingale with right-continuous paths and let τ be a finite \mathbb{F} -stopping time. Then $M_{\tau+}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - \mathbb{F} -martingale for P -a.e. $\omega \in \Omega$.*

Proof. By Galmarino's test, $M_{\tau+}^{\tau, \omega}$ is \mathbb{F} -adapted. Moreover,

$$E^{P^{\tau, \omega}}[|M_{\tau+t}^{\tau, \omega}|] = E^P[|M_{\tau+t}| | \mathcal{F}_{\tau}](\omega) < \infty \quad \text{for } P\text{-a.e. } \omega \in \Omega$$

and all $t \geq 0$. Let $0 \leq u \leq v < \infty$ and let f be a bounded \mathcal{F}_u -measurable function. Define the function \hat{f} by

$$\hat{f}(\omega) := f(\omega_{\tau(\omega)+} - \omega_{\tau}(\omega)), \quad \omega \in \Omega;$$

then \hat{f} is $\mathcal{F}_{\tau+u}$ -measurable and $\hat{f}^{\tau, \omega} = f$. Applying the optional sampling theorem to the right-continuous, P - \mathbb{F} -uniformly integrable martingale M , we obtain that

$$E^{P^{\tau, \omega}}[(M_{\tau+v}^{\tau, \omega} - M_{\tau+u}^{\tau, \omega}) f] = E^P[(M_{\tau+v} - M_{\tau+u}) \hat{f} | \mathcal{F}_{\tau}](\omega) = 0$$

for P -a.e. $\omega \in \Omega$. This implies the martingale property of $M_{\tau+}^{\tau, \omega}$ as claimed. \square

Lemma IV.3.4. *Let $P \in \mathfrak{P}(\Omega)$, let M be a right-continuous P - \mathbb{F} -local martingale having P -a.s. càdlàg paths and uniformly bounded jumps, and let τ be a finite \mathbb{F} -stopping time. Then $M_{\tau+}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - \mathbb{F} -local martingale for P -a.e. $\omega \in \Omega$.*

Proof. Again, $M_{\tau+}^{\tau, \omega}$ is \mathbb{F} -adapted and has right-continuous paths for any $\omega \in \Omega$. Let $(T_m)_{m \in \mathbb{N}}$ be a localizing sequence of the P - \mathbb{F} -local martingale M such that $T_m \leq m$. Since $T_m \rightarrow \infty$ P -a.s., we have that $T_m^{\tau, \omega} \rightarrow \infty$ $P^{\tau, \omega}$ -a.s. for P -a.e. $\omega \in \Omega$. Moreover, by Lemma IV.3.3, each process $M_{T_m \wedge (\tau+)}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - \mathbb{F} -martingale for P -a.e. $\omega \in \Omega$.

Thus, there exists a subset $\Omega' \subseteq \Omega$ of full P -measure such that for all $\omega \in \Omega'$, we have the following three properties: $M_{\tau+}^{\tau, \omega}$ has càdlàg paths with uniformly bounded jumps $P^{\tau, \omega}$ -a.s., the process $M_{T_m \wedge (\tau+)}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - \mathbb{F} -martingale for all $m \in \mathbb{N}$, and $T_m^{\tau, \omega} \rightarrow \infty$ $P^{\tau, \omega}$ -a.s. In what follows, we fix $\omega \in \Omega'$ and show that $M_{\tau+}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - \mathbb{F} -local martingale. Define

$$\rho_n := \inf \{t \geq 0 \mid |M_{\tau+t}| \geq n \text{ or } |M_{\tau+t-}| \geq n\} \wedge n.$$

Using that $M_{\tau+}^{\tau, \omega}$ has càdlàg paths $P^{\tau, \omega}$ -a.s., we see that $\rho_n^{\tau, \omega}$ is a stopping time of $\mathbb{F}^{P^{\tau, \omega}}$, the augmentation of \mathbb{F} under $P^{\tau, \omega}$, and that $\rho_n^{\tau, \omega} \rightarrow \infty$ $P^{\tau, \omega}$ -a.s. Since $M_{\tau+}^{\tau, \omega}$ has uniformly bounded jumps $P^{\tau, \omega}$ -a.s., we have that

$$E^{P^{\tau, \omega}} \left[\sup_{t \geq 0} |M_{\tau+(\rho_n \wedge t)}^{\tau, \omega}| \right] \leq n + E^{P^{\tau, \omega}} \left[|\Delta M_{\tau+\rho_n}^{\tau, \omega}| \right] < \infty$$

for all n . Therefore, given $0 \leq u \leq v < \infty$, the dominated convergence theorem and the optional sampling theorem applied to the martingale $M_{T_m \wedge (\tau+)}^{\tau, \omega}$ and the stopping time $\rho_n^{\tau, \omega}$ yield that

$$\begin{aligned} E^{P^{\tau, \omega}} \left[M_{\tau+(\rho_n \wedge v)}^{\tau, \omega} \mid \mathcal{F}_u^{P^{\tau, \omega}} \right] &= E^{P^{\tau, \omega}} \left[\lim_{m \rightarrow \infty} M_{T_m \wedge (\tau+(\rho_n \wedge v))}^{\tau, \omega} \mid \mathcal{F}_u^{P^{\tau, \omega}} \right] \\ &= \lim_{m \rightarrow \infty} E^{P^{\tau, \omega}} \left[M_{T_m \wedge (\tau+(\rho_n \wedge v))}^{\tau, \omega} \mid \mathcal{F}_u^{P^{\tau, \omega}} \right] \\ &= \lim_{m \rightarrow \infty} M_{T_m \wedge (\tau+(\rho_n \wedge u))}^{\tau, \omega} \\ &= M_{\tau+(\rho_n \wedge u)}^{\tau, \omega} \quad P^{\tau, \omega}\text{-a.s.} \end{aligned}$$

Thus, $M_{\tau+(\rho_n \wedge \cdot)}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - $\mathbb{F}^{P^{\tau, \omega}}$ -uniformly integrable martingale for each $n \in \mathbb{N}$, meaning that $M_{\tau+}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - $\mathbb{F}^{P^{\tau, \omega}}$ -local martingale, localized by the $\mathbb{F}^{P^{\tau, \omega}}$ -stopping times $\rho_n^{\tau, \omega}$.

It remains to return to the original filtration \mathbb{F} . Indeed, we first note that by a standard backward martingale convergence argument, $M_{\tau+}^{\tau, \omega}$ is also a $P^{\tau, \omega}$ - $\mathbb{F}_+^{P^{\tau, \omega}}$ -local martingale; cf. [63, Lemma II.67.10, p. 173]. It then follows from [11, Theorem 3] that there exists an $\mathbb{F}_+^{P^{\tau, \omega}}$ -predictable localizing sequence for this process, and this sequence can be further modified into an \mathbb{F} -localizing sequence by an application of [12, Theorem IV.78, p. 133]. Thus, $M_{\tau+}^{\tau, \omega}$ is an \mathbb{F} -adapted $P^{\tau, \omega}$ - $\mathbb{F}_+^{P^{\tau, \omega}}$ -local martingale with a localizing sequence of \mathbb{F} -stopping times. By the tower property of the conditional expectation, this actually means that $M_{\tau+}^{\tau, \omega}$ is a $P^{\tau, \omega}$ - \mathbb{F} -local martingale, with the same localizing sequence. As $\omega \in \Omega'$ was arbitrary, the proof is complete. \square

For the rest of this section, we will be concerned with the process

$$\tilde{X}_t := X_t - \sum_{0 \leq s \leq t} [\Delta X_s - h(\Delta X_s)], \quad t \geq 0;$$

recall that h is a fixed truncation function. The process \tilde{X} has uniformly bounded jumps and differs from X by a finite variation process; in particular, X is a P - \mathbb{F} -semimartingale if and only if \tilde{X} is a P - \mathbb{F} -semimartingale, for any $P \in \mathfrak{P}(\Omega)$. In fact, if $P \in \mathfrak{P}_{sem}$, then as \tilde{X} has bounded jumps, it is a special semimartingale with a canonical decomposition $\tilde{X} = M + B$; here M is a right-continuous P - \mathbb{F} -local martingale and B is an \mathbb{F} -predictable process with paths which are right-continuous and P -a.s. of finite variation.

Proposition IV.3.5. *Let τ be a finite \mathbb{F} -stopping time, let $P \in \mathfrak{P}_{sem}$ and let $\tilde{X} = M + B$ be the P - \mathbb{F} -canonical decomposition of \tilde{X} . For P -a.e. $\omega \in \Omega$, we have $P^{\tau, \omega} \in \mathfrak{P}_{sem}$ and the canonical decomposition of \tilde{X} under $P^{\tau, \omega}$ is given by*

$$\tilde{X} = (M_{\tau+}^{\tau, \omega} - M_{\tau(\omega)}(\omega)) + (B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega)). \quad (3.2)$$

Proof. The right-continuous processes $B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega)$ and $M_{\tau+}^{\tau, \omega} - M_{\tau(\omega)}(\omega)$ are \mathbb{F} -adapted by Galmarino's test. As \tilde{X} is a P - \mathbb{F} -semimartingale with uniformly bounded jumps and B has P -a.s. càdlàg paths, it follows that M has P -a.s. càdlàg paths with uniformly bounded jumps; cf. [26, Proposition I.4.24, p. 44]. Thus, we conclude from Lemma IV.3.4 that for P -a.e. $\omega \in \Omega$, $M_{\tau+}^{\tau, \omega} - M_{\tau(\omega)}(\omega)$ is a $P^{\tau, \omega}$ - \mathbb{F} -local martingale. Moreover, we see that $B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega)$ has $P^{\tau, \omega}$ -a.s. finite variation paths for P -a.e. $\omega \in \Omega$; finally, it is adapted to the left-continuous filtration $\mathbb{F}_- = (\mathcal{F}_{t-})_{t \geq 0}$ and therefore \mathbb{F} -predictable as a consequence of [12, Theorem IV.97, p. 147]. We observe that (3.2) holds identically, due to the definition of \tilde{X} and the fact that X is the canonical process. As remarked above, this decomposition also implies that X is a semimartingale under P . \square

Next, we focus on the third characteristic. For ease of reference, we first state two simple lemmas.

Lemma IV.3.6. *Let W be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function, let τ be a finite \mathbb{F} -stopping time and $\omega \in \Omega$. There exists a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function \widetilde{W} such that*

$$\widetilde{W}(\omega \otimes_{\tau} \tilde{\omega}, \tau(\omega) + s, z) = W(\tilde{\omega}, s, z), \quad (\tilde{\omega}, s, z) \in \Omega \times (0, \infty) \times \mathbb{R}^d. \quad (3.3)$$

Moreover, if $W \geq 0$, one can choose $\widetilde{W} \geq 0$.

Proof. Consider the function

$$\widetilde{W}(\tilde{\omega}, s, z) := W(\tilde{\omega}_{\cdot + \tau(\omega)} - \tilde{\omega}_{\tau(\omega)}, s - \tau(\omega), z) \mathbf{1}_{s > \tau(\omega)};$$

then (3.3) holds by definition. To show that \widetilde{W} is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, we may first use the monotone class theorem to reduce to the case where W is a product $W(\omega, t, x) = g(\omega, t)f(x)$. Using again the fact that a process is \mathbb{F} -predictable if and only if it is measurable and adapted to \mathbb{F}_- , cf. [12, Theorem IV.97, p. 147], we then see that the predictability of W implies the predictability of \widetilde{W} . \square

Lemma IV.3.7. *Let $P \in \mathfrak{P}(\Omega)$, let τ be a finite \mathbb{F} -stopping time and let ν be the P - \mathbb{F} -compensator of μ^X . Then, for any $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function $W \geq 0$, we have P -a.s. that*

$$E^P \left[\int_{\tau}^{\infty} W(\cdot, s, z) \mu^X(\cdot, ds, dz) \middle| \mathcal{F}_{\tau} \right] = E^P \left[\int_{\tau}^{\infty} W(\cdot, s, z) \nu(\cdot, ds, dz) \middle| \mathcal{F}_{\tau} \right].$$

Proof. By the definition of the compensator, we have

$$E^P \left[\int_0^{\infty} \overline{W}(\cdot, s, z) \mu^X(\cdot, ds, dz) \right] = E^P \left[\int_0^{\infty} \overline{W}(\cdot, s, z) \nu(\cdot, ds, dz) \right]$$

for any $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function $\overline{W} \geq 0$. Let $A \in \mathcal{F}_{\tau}$ and define the \mathbb{F} -stopping time $\tau_A := \tau \mathbf{1}_A + \infty \mathbf{1}_{A^c}$; then the claim follows by applying this equality to the function $\overline{W} := W \mathbf{1}_{\llbracket \tau_A, \infty \rrbracket}$. \square

Proposition IV.3.8. *Let $P \in \mathfrak{P}(\Omega)$, let ν be the P - \mathbb{F} -compensator of μ^X and let τ be a finite \mathbb{F} -stopping time. Then, for P -a.e. $\omega \in \Omega$, the $P^{\tau, \omega}$ - \mathbb{F} -compensator of μ^X is given by the random measure*

$$D \mapsto \int_{\tau(\omega)}^{\infty} \int_{\mathbb{R}^d} \mathbf{1}_D(s - \tau(\omega), z) \nu(\omega \otimes_{\tau} \cdot, ds, dz), \quad D \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d).$$

Proof. Denote by $\nu^\omega(\cdot, ds, dz)$ the above random measure. To see that it is \mathbb{F} -predictable, let W be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function. If \widetilde{W} is as in Lemma IV.3.6, then

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} W(\tilde{\omega}, s, z) \nu^\omega(\tilde{\omega}, ds, dz) \\ &= \int_{\tau(\omega)}^{\tau(\omega)+t} \int_{\mathbb{R}^d} \widetilde{W}(\omega \otimes_\tau \tilde{\omega}, s, z) \nu(\omega \otimes_\tau \tilde{\omega}, ds, dz), \quad (\tilde{\omega}, t) \in \Omega \times (0, \infty) \end{aligned}$$

and the latter process is \mathbb{F} -predictable as a consequence of [12, Theorem IV.97, p.147] and the fact that ν is an \mathbb{F} -predictable random measure. Thus, $\nu^\omega(\cdot, ds, dz)$ is \mathbb{F} -predictable for every $\omega \in \Omega$.

Let $W \geq 0$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function and let $\widetilde{W} \geq 0$ be as in Lemma IV.3.6. Using the identity $X_\cdot = X_{\tau+}^{\tau, \omega} - X_\tau(\omega)$ and Lemma IV.3.7, we obtain for P -a.e. $\omega \in \Omega$ that

$$\begin{aligned} & E^{P^{\tau, \omega}} \left[\int_0^\infty \int_{\mathbb{R}^d} W(\cdot, s, z) \mu^X(\cdot, ds, dz) \right] \\ &= E^{P^{\tau, \omega}} \left[\int_0^\infty \int_{\mathbb{R}^d} \widetilde{W}(\omega \otimes_\tau \cdot, \tau(\omega) + s, z) \mu^{X_{\tau+}^{\tau, \omega}}(\cdot, ds, dz) \right] \\ &= E^P \left[\int_\tau^\infty \int_{\mathbb{R}^d} \widetilde{W}(\cdot, s, z) \mu^X(\cdot, ds, dz) \Big| \mathcal{F}_\tau \right] (\omega) \\ &= E^P \left[\int_\tau^\infty \int_{\mathbb{R}^d} \widetilde{W}(\cdot, s, z) \nu(\cdot, ds, dz) \Big| \mathcal{F}_\tau \right] (\omega) \\ &= E^{P^{\tau, \omega}} \left[\int_{\tau(\omega)}^\infty \int_{\mathbb{R}^d} \widetilde{W}(\omega \otimes_\tau \cdot, s, z), \nu(\omega \otimes_\tau \cdot, ds, dz) \right] \\ &= E^{P^{\tau, \omega}} \left[\int_0^\infty \int_{\mathbb{R}^d} W(\cdot, s, z) \nu^\omega(\cdot, ds, dz) \right]. \end{aligned}$$

As $W \geq 0$ was arbitrary, it follows that $\nu^\omega(\cdot, ds, dz)$ is the $P^{\tau, \omega}$ - \mathbb{F} -compensator of μ^X for P -a.e. $\omega \in \Omega$; cf. [26, Theorem II.1.8, p. 66]. \square

We can now complete the proof of the main result of this section.

Proof of Theorem IV.3.1. Let $P \in \mathfrak{P}_{sem}$ and let τ be a finite \mathbb{F} -stopping time. The formula $B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega)$ for the first characteristic follows from Proposition IV.3.5 and the very definition of the first characteristic, whereas the formula for the third characteristic follows from Proposition IV.3.8 and the decomposition (3.1). Turning to the second characteristic, we recall from Theorem III.2.5 that there exists an \mathbb{F} -predictable process \hat{C} with the property that for any $P' \in \mathfrak{P}_{sem}$, \hat{C} coincides P' -a.s. with the quadratic variation of the continuous local martingale part of X under P' . The process \hat{C} is constructed by subtracting the squared jumps of X from the quadratic covariation $[X]$ which, in turn, is constructed in a purely pathwise fashion; moreover,

the increments of \hat{C} depend only on the increments of X . More precisely, it follows from the construction of \hat{C} in the proof of Proposition III.6.6 that for P -a.e. $\omega \in \Omega$,

$$\hat{C} = \hat{C}_{\tau+}^{\tau, \omega} - \hat{C}_{\tau(\omega)}(\omega) \quad P^{\tau, \omega}\text{-a.s.}$$

Since the above holds in particular for $P' = P$ and $P' = P^{\tau, \omega}$, and since \hat{C} is \mathbb{F} -predictable, this already yields the formula for the second characteristic under $P^{\tau, \omega}$. Finally, we observe that the assertion about $P \in \mathfrak{P}_{sem}^{ac}$ is a consequence of the general case that we have just established. \square

IV.4 Products of Semimartingale Laws and (A3)

In this section, we first show that the product \bar{P} of a semimartingale law $P \in \mathfrak{P}_{sem}$ and a \mathfrak{P}_{sem} -valued kernel κ is again a semimartingale law. Then, we describe the associated characteristics and deduce the validity of Condition (A3) for \mathfrak{P}_{Θ} . While the naive way to proceed would be to construct directly the semimartingale decomposition under \bar{P} , some technical issues arise as soon as κ has uncountably many values. For that reason, the first step will be achieved in a more abstract way using the Bichteler–Dellacherie criterion. Once the semimartingale property for \bar{P} is established, we know that the associated decomposition and characteristics exist and we can study them using the results of the previous section.

Proposition IV.4.1. *Let τ be a finite \mathbb{F} -stopping time and let $P \in \mathfrak{P}_{sem}$. Moreover, let $\kappa : \Omega \rightarrow \mathfrak{P}(\Omega)$ be an \mathcal{F}_{τ} -measurable kernel with $\kappa(\omega) \in \mathfrak{P}_{sem}$ for P -a.e. $\omega \in \Omega$. Then, the measure \bar{P} defined by*

$$\bar{P}(D) := \iint \mathbf{1}_D^{\tau, \omega}(\omega') \kappa(\omega, d\omega') P(d\omega), \quad D \in \mathcal{F}$$

is an element of \mathfrak{P}_{sem} .

Proof. We recall that the \bar{P} -semimartingale property in \mathbb{F} is equivalent to the one in the usual augmentation $\mathbb{F}_{+}^{\bar{P}}$; cf. Proposition III.2.2. We shall use the Bichteler–Dellacherie criterion [13, Theorem VIII.80, p. 387] to establish the latter; namely, we show that if

$$H^n = \sum_{i=1}^{k^n} h_i^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}, \quad n \geq 1$$

is a sequence of $\mathbb{F}_{+}^{\bar{P}}$ -elementary processes such that $H^n(t, \omega) \rightarrow 0$ uniformly in (t, ω) , then

$$\lim_{n \rightarrow \infty} E^{\bar{P}} \left[\left| \sum_{i=1}^{k^n} h_i^n (X_{t_i^n \wedge t} - X_{t_{i-1}^n \wedge t}) \right| \wedge 1 \right] = 0, \quad t \geq 0.$$

In fact, as $\bar{P} = P$ on \mathcal{F}_τ , it is clear that $X \mathbf{1}_{\llbracket 0, \tau \llbracket}$ is a semimartingale and so it suffices to verify the above property for $X \mathbf{1}_{\llbracket \tau, \infty \llbracket}$ instead of X . To that end, by dominated convergence, it suffices to show that for \bar{P} -a.e. $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} E^{\bar{P}} \left[\left| \sum_{i=1}^{k^n} h_i^n (X_{t_i^n \wedge t} \mathbf{1}_{\{\tau \leq t_i^n \wedge t\}} - X_{t_{i-1}^n \wedge t} \mathbf{1}_{\{\tau \leq t_{i-1}^n \wedge t\}}) \right| \wedge 1 \middle| \mathcal{F}_\tau \right] (\omega) = 0,$$

where $t \geq 0$ is fixed. Define the \mathcal{F}_τ -measurable random variable j^n by

$$j^n := \inf\{0 \leq j \leq k^n \mid t_j^n \wedge t \geq \tau(\omega)\} \wedge k^n.$$

Writing the above limit as a sum of two terms, it then suffices to show that for \bar{P} -a.e. $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} E^{\bar{P}} \left[\left| \sum_{i=j^n+1}^{k^n} h_i^n (X_{t_i^n \wedge t} - X_{t_{i-1}^n \wedge t}) \right| \wedge 1 \middle| \mathcal{F}_\tau \right] (\omega) = 0, \quad (4.1)$$

$$\lim_{n \rightarrow \infty} E^{\bar{P}} \left[\left| h_{j^n}^n X_{t_{j^n}^n \wedge t} \right| \wedge 1 \middle| \mathcal{F}_\tau \right] (\omega) = 0. \quad (4.2)$$

Indeed, as $h_{j^n}^n \rightarrow 0$ uniformly, we have $|h_{j^n}^n X_{t_{j^n}^n \wedge t}| \rightarrow 0$ \bar{P} -a.s. and hence (4.2) follows by dominated convergence.

To show (4.1), we may choose a $\mathcal{F}_{t_{i-1}^n}^n$ -measurable version of each h_i^n . Then, as $\bar{P}^{\tau, \omega} = \kappa(\omega) \in \mathfrak{P}_{sem}$ for \bar{P} -a.e. $\omega \in \Omega$ (cf. [49, Lemma 2.7]), the reverse implication of the Bichteler–Dellacherie theorem applied to $\kappa(\omega)$ yields that

$$\begin{aligned} & \lim_{n \rightarrow \infty} E^{\bar{P}} \left[\left| \sum_{i=j^n+1}^{k^n} h_i^n (X_{t_i^n \wedge t} - X_{t_{i-1}^n \wedge t}) \right| \wedge 1 \middle| \mathcal{F}_\tau \right] (\omega) \\ &= \lim_{n \rightarrow \infty} E^{\bar{P}^{\tau, \omega}} \left[\left| \sum_{i=j^n(\omega)+1}^{k^n} (h_i^n)^{\tau, \omega} (X_{t_i^n \wedge t}^{\tau, \omega} - X_{t_{i-1}^n \wedge t}^{\tau, \omega}) \right| \wedge 1 \right] \\ &= \lim_{n \rightarrow \infty} E^{\kappa(\omega)} \left[\left| \sum_{i=j^n(\omega)+1}^{k^n} (h_i^n)^{\tau, \omega} (X_{(t_i^n \wedge t) - \tau(\omega)} - X_{(t_{i-1}^n \wedge t) - \tau(\omega)}) \right| \wedge 1 \right] = 0 \end{aligned}$$

for \bar{P} -a.e. $\omega \in \Omega$, because $(H^n)^{\tau, \omega}$ defines a sequence of elementary processes converging uniformly to zero. This completes the proof. \square

As announced, we can now proceed to establish (A3) for \mathfrak{P}_Θ .

Proposition IV.4.2. *Let $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be measurable and $P \in \mathfrak{P}_\Theta$. Moreover, let τ be a finite \mathbb{F} -stopping time and let $\kappa : \Omega \rightarrow \mathfrak{P}(\Omega)$ be an \mathcal{F}_τ -measurable kernel with $\kappa(\omega) \in \mathfrak{P}_\Theta$ for P -a.e. $\omega \in \Omega$. Then, the measure \bar{P} defined by*

$$\bar{P}(D) := \iint \mathbf{1}_D^{\tau, \omega}(\omega') \kappa(\omega, d\omega') P(d\omega), \quad D \in \mathcal{F}$$

is an element of \mathfrak{P}_Θ .

Proof. As a first step, we consider the special case $\Theta = \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$; then \mathfrak{P}_Θ is the entire set \mathfrak{P}_{sem}^{ac} . In view of Proposition IV.4.1, we already know that $\bar{P} \in \mathfrak{P}_{sem}$. Thus, the characteristics $(B, C, F(dz) dA)$ of X under \bar{P} and \mathbb{F} are well defined; we show that they are absolutely continuous \bar{P} -a.s. As B has paths of finite variation \bar{P} -a.s., we can write for \bar{P} -a.e. $\omega \in \Omega$ a decomposition

$$B_t(\omega) = \int_0^t \varphi_s(\omega) ds + \psi_t(\omega),$$

where φ, ψ are measurable functions and ψ is \bar{P} -a.s. singular with respect to the Lebesgue measure. Since $\bar{P} = P$ on \mathcal{F}_τ and $P \in \mathfrak{P}_{sem}^{ac}$, we have $dB \ll du$ on $\llbracket 0, \tau \rrbracket$ \bar{P} -a.s. Therefore, it suffices to show that $dB \ll du$ on $\llbracket \tau, \infty \rrbracket$ \bar{P} -a.s., or equivalently, that

$$D := \left\{ B_{\tau+} - B_\tau \neq \int_\tau^{\tau+} \varphi_s ds \right\}$$

is a \bar{P} -nullset. Indeed, it follows from Theorem IV.3.1 that for \bar{P} -a.e. $\omega \in \Omega$, the first characteristic of X under $\bar{P}^{\tau, \omega}$ is given by

$$B_{\tau(\omega)+}^{\tau, \omega} - B_{\tau(\omega)}(\omega) = \int_{\tau(\omega)}^{\tau(\omega)+} \varphi_s^{\tau, \omega} ds + \psi_{\tau+}^{\tau, \omega} - \psi_{\tau(\omega)}(\omega),$$

and $\psi_{\tau+}^{\tau, \omega} - \psi_{\tau(\omega)}(\omega)$ is singular with respect to the Lebesgue measure. Moreover, for \bar{P} -a.e. $\omega \in \Omega$, we have $\bar{P}^{\tau, \omega} = \kappa(\omega) \in \mathfrak{P}_{sem}^{ac}$ and thus

$$\kappa(\omega) \left\{ B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega) \neq \int_{\tau(\omega)}^{\tau(\omega)+} \varphi_s^{\tau, \omega} ds \right\} = 0. \quad (4.3)$$

Define the set

$$D^{\tau, \omega} := \left\{ B_{\tau+}^{\tau, \omega} - B_{\tau(\omega)}(\omega) \neq \int_{\tau(\omega)}^{\tau(\omega)+} \varphi_s^{\tau, \omega} ds \right\}, \quad \omega \in \Omega;$$

then (4.3) states that

$$\kappa(\omega)(D^{\tau, \omega}) = 0 \quad \text{for } \bar{P}\text{-a.e. } \omega \in \Omega.$$

As κ is \mathcal{F}_τ -measurable and $\bar{P} = P$ on \mathcal{F}_τ , this equality holds also for P -a.e. $\omega \in \Omega$. Using Fubini's theorem and the fact that $\mathbf{1}_D^{\tau, \omega} = \mathbf{1}_{D^{\tau, \omega}}$, we conclude that

$$\bar{P}(D) = \int_\Omega \int_\Omega \mathbf{1}_D^{\tau, \omega}(\omega') \kappa(\omega, d\omega') P(d\omega) = \int_\Omega \kappa(\omega)(D^{\tau, \omega}) P(d\omega) = 0$$

as claimed. The proof of absolute continuity for the processes C and A is similar; we use the corresponding formulas from Theorem IV.3.1. This completes the proof for the special case $\Theta = \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$.

Next, we consider the case of a general subset $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$. By the above, $\bar{P} \in \mathfrak{P}_{sem}^{ac}$; we write $(\int b_s ds, \int c_s ds, F_s ds)$ for the characteristics of X under \bar{P} . Since $\bar{P} = P$ on \mathcal{F}_τ and $P \in \mathfrak{P}_\Theta$, we have $(b, c, F) \in \Theta$ on $\llbracket 0, \tau \rrbracket$, $du \times \bar{P}$ -a.s., and it suffices to show that $(b, c, F) \in \Theta$ on $\llbracket \tau, \infty \rrbracket$, $du \times \bar{P}$ -a.s. That is, we need to show that

$$R := \left\{ (u, \omega) \in \llbracket \tau(\omega), \infty \rrbracket \mid (b_u(\omega), c_u(\omega), F_{\omega, u}) \notin \Theta \right\}$$

is a $du \times \bar{P}$ -nullset. By Theorem IV.3.1, $\bar{P}^{\tau, \omega} \in \mathfrak{P}_{sem}^{ac}$ for \bar{P} -a.e. $\omega \in \Omega$ and the differential characteristics of X under $\bar{P}^{\tau, \omega}$ are

$$(b_{\tau+}, c_{\tau+}, F_{\tau+}^{\tau, \omega}).$$

Similarly as in (4.3), this formula and the fact that $\bar{P}^{\tau, \omega} = \kappa(\omega) \in \mathfrak{P}_\Theta$ for \bar{P} -a.e. $\omega \in \Omega$ imply that

$$(du \times \kappa(\omega)) \left\{ (u, \omega') \in \llbracket 0, \infty \rrbracket \mid (b_{\tau+u}^{\tau, \omega}(\omega'), c_{\tau+u}^{\tau, \omega}(\omega'), F_{\omega', \tau+u}^{\tau, \omega}) \notin \Theta \right\} = 0$$

for \bar{P} -a.e. $\omega \in \Omega$. If we define

$$R^{\tau, \omega} := \left\{ (u, \omega') \in \llbracket \tau(\omega), \infty \rrbracket \mid (b_u^{\tau, \omega}(\omega'), c_u^{\tau, \omega}(\omega'), F_{\omega', u}^{\tau, \omega}) \notin \Theta \right\},$$

then this implies that

$$(du \times \kappa(\omega))(R^{\tau, \omega}) = 0 \text{ for } \bar{P}\text{-a.e. } \omega \in \Omega.$$

Again, this holds also for P -a.e. $\omega \in \Omega$ and as $\mathbf{1}_R^{\tau, \omega} = \mathbf{1}_{R^{\tau, \omega}}$, Fubini's theorem yields that

$$\begin{aligned} (du \times \bar{P})(R) &= \int_{\Omega} \int_{\Omega} \int_0^{\infty} \mathbf{1}_R^{\tau, \omega}(u, \omega') du \kappa(\omega, d\omega') P(d\omega) \\ &= \int_{\Omega} (du \times \kappa(\omega))(R^{\tau, \omega}) P(d\omega) = 0. \end{aligned}$$

This completes the proof. \square

IV.5 Connection to PIDE

In this section, we relate the nonlinear Lévy process to a PIDE. Throughout, we fix a measurable set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ satisfying the conditions (2.3) and (2.4) which, for convenience, we state again as

$$\mathcal{K} := \sup_{(b, c, F) \in \Theta} \left\{ \int_{\mathbb{R}^d} |z| \wedge |z|^2 F(dz) + |b| + |c| \right\} < \infty, \quad (5.1)$$

$$\lim_{\varepsilon \rightarrow 0} \mathcal{K}_\varepsilon = 0 \text{ for } \mathcal{K}_\varepsilon := \sup_{F \in \Theta_3} \int_{|z| \leq \varepsilon} |z|^2 F(dz), \quad (5.2)$$

where $\Theta_3 = \text{proj}_3 \Theta$ is the canonical projection of Θ onto \mathcal{L} . Our aim is to show that for given boundary condition $\psi \in C_{b,Lip}(\mathbb{R}^d)$, the value function

$$v(t, x) := \mathcal{E}(\psi(x + X_t)) \equiv \sup_{P \in \mathfrak{P}_\Theta} E^P[\psi(x + X_t)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^d$$

is the unique viscosity solution of the PIDE (2.5).

The existence part relies on the following dynamic programming principle for v ; it is essentially a special case of the semigroup property stated in Theorem IV.2.1(ii).

Lemma IV.5.1. *For all $0 \leq u \leq t < \infty$ and $x \in \mathbb{R}^d$, we have*

$$v(t, x) = \mathcal{E}(v(t - u, x + X_u)).$$

Proof. Let $0 \leq u \leq t < \infty$. As X is the canonical process, we have that

$$\mathcal{E}_u(\psi(x + X_t))(\omega) = \mathcal{E}(\psi(x + X_u(\omega) + X_{t-u})) = v(t - u, x + X_u(\omega)), \quad \omega \in \Omega.$$

Applying $\mathcal{E}(\cdot)$ on both sides, Theorem IV.2.1(ii) yields that

$$v(t, x) = \mathcal{E}(\mathcal{E}_u(\psi(x + X_t))) = \mathcal{E}(v(t - u, x + X_u))$$

as claimed. \square

During most of this section, we will be concerned with a fixed law $P \in \mathfrak{P}_{sem}^{ac}$ and we may use the usual augmentation \mathbb{F}_+^P to avoid any subtleties related to stochastic analysis. This is possible because, as mentioned in Section IV.2, the characteristics associated with \mathbb{F} and \mathbb{F}_+^P coincide P -a.s. To fix some notation, recall that under $P \in \mathfrak{P}_{sem}^{ac}$, the process X has the canonical representation

$$X_t = \int_0^t b_s^P ds + X_t^{c,P} + X_t^{d,P} + \int_0^t \int_{\mathbb{R}^d} [z - h(z)] \mu^X(ds, dz), \quad (5.3)$$

where $X^{c,P}$ is the continuous local martingale part of X with respect to P - \mathbb{F}_+^P , $F_s^P(dz) ds$ is the compensator of $\mu^X(ds, dz)$ and

$$X_t^{d,P} := \int_0^t \int_{\mathbb{R}^d} h(z) (\mu^X(ds, dz) - F_s^P(dz) ds)$$

is a purely discontinuous P - \mathbb{F}_+^P -local martingale; cf. [26, Theorem 2.34, p. 84]. In the subsequent proofs, C is a constant whose value may change from line to line.

The following simple estimate will be used repeatedly.

Lemma IV.5.2. *There exists a constant C_K such that*

$$E^P \left[\sup_{0 \leq u \leq t} |X_u| \right] \leq C_K (t + t^{1/2}), \quad t \geq 0 \quad \text{for all } P \in \mathfrak{P}_\Theta. \quad (5.4)$$

Proof. Let $P \in \mathfrak{P}_\Theta$; then Jensen's inequality and (5.1) imply that

$$\begin{aligned} E^P \left[|[X^{d,P}]_t|^{1/2} \right] &\leq E^P \left[\int_0^t \int_{\mathbb{R}^d} |h(z)|^2 \mu^X(ds, dz) \right]^{1/2} \\ &\leq C E^P \left[\int_0^t \int_{\mathbb{R}^d} |z|^2 \wedge 1 F_s(dz) ds \right]^{1/2} \\ &\leq C \mathcal{K}^{1/2} t^{1/2} \end{aligned}$$

and so the Burkholder–Davis–Gundy (BDG) inequalities yield that

$$E^P \left[\sup_{0 \leq u \leq t} |X_u^{d,P}| \right] \leq C E^P \left[|[X^{d,P}]_t|^{1/2} \right] \leq C \mathcal{K} t^{1/2}. \quad (5.5)$$

Similarly, (5.1) also implies that

$$E^P \left[\sup_{0 \leq u \leq t} |X_u^{c,P}| \right] \leq C \mathcal{K} t^{1/2}, \quad E^P \left[\sup_{0 \leq u \leq t} \left| \int_0^u b_s ds \right| \right] \leq C \mathcal{K} t \quad (5.6)$$

and

$$E^P \left[\sup_{0 \leq u \leq t} \left| \int_0^u \int_{\mathbb{R}^d} [z - h(z)] \mu^X(ds, dz) \right| \right] \leq C \mathcal{K} t.$$

The result now follows from the decomposition (5.3). \square

We deduce the following regularity properties of v .

Lemma IV.5.3. *The value function v is uniformly bounded by $\|\psi\|_\infty$ and jointly continuous. More precisely, $v(t, \cdot)$ is Lipschitz continuous with constant $\text{Lip}(\psi)$ and $v(\cdot, x)$ is locally $1/2$ -Hölder continuous with a constant depending only on $\text{Lip}(\psi)$ and \mathcal{K} .*

Proof. The boundedness and the Lipschitz property follow directly from the definition of v . Let $0 \leq u \leq t$, then Lemma IV.5.1, the Lipschitz continuity of $v(t, \cdot)$ and the estimate (5.4) show that

$$\begin{aligned} |v(t, x) - v(t - u, x)| &= |\mathcal{E}(v(t - u, x + X_u) - v(t - u, x))| \\ &\leq C \mathcal{E}(|X_u|) \\ &\leq C(u + u^{1/2}). \end{aligned}$$

The Hölder continuity from the right is obtained analogously. \square

IV.5.1 Existence

Consider the PIDE introduced in (2.5); namely,

$$\partial_t v(t, x) - G(D_x v(t, x), D_{xx}^2 v(t, x), v(t, x + \cdot)) = 0, \quad v(0, x) = \psi(x)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^d$, where the nonlinearity $G(p, q, f(\cdot))$ is given by

$$\sup_{(b,c,F) \in \Theta} \left\{ pb + \frac{1}{2} \text{tr}[qc] + \int_{\mathbb{R}^d} [f(z) - f(0) - D_x f(0)h(z)] F(dz) \right\}.$$

We recall that $\psi \in C_{b,Lip}(\mathbb{R}^d)$ and $v(t, x) = \mathcal{E}(\psi(x + X_t))$.

Proposition IV.5.4. *The value function v of (2.7) is a viscosity solution of the PIDE (2.5).*

Proof. The basic line of argument is standard in stochastic control. We detail the proof because the presence of small jumps necessitates additional arguments; this is where the condition (5.2) comes into play.

By Lemma IV.5.3, v is continuous on $[0, \infty) \times \mathbb{R}^d$, and we have $v(0, \cdot) = \psi$ by the definition of v . We show that v is a viscosity subsolution of (2.5); the supersolution property is proved similarly.

Let $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and let $\varphi \in C_b^{2,3}((0, \infty) \times \mathbb{R}^d)$ be such that $\varphi \geq v$ and $\varphi(t, x) = v(t, x)$. For $0 < u < t$, Lemma IV.5.1 shows that

$$0 = \sup_{P \in \mathfrak{P}_\Theta} E^P[v(t-u, x+X_u) - v(t, x)] \leq \sup_{P \in \mathfrak{P}_\Theta} E^P[\varphi(t-u, x+X_u) - \varphi(t, x)]. \quad (5.7)$$

We fix $P \in \mathfrak{P}_\Theta$ and recall that (b^P, c^P, F^P) are the differential characteristics of X under P . Applying Itô's formula, we obtain that P -a.s.,

$$\begin{aligned} & \varphi(t-u, x+X_u) - \varphi(t, x) \\ &= \int_0^u \int_{\mathbb{R}^d} D_x \varphi(t-s, x+X_{s-}) d(X_s^{c,P} + X_s^{d,P}) \\ & \quad + \int_0^u -\partial_t \varphi(t-s, x+X_{s-}) ds + \int_0^u D_x \varphi(t-s, x+X_{s-}) b_s^P ds \\ & \quad + \frac{1}{2} \int_0^u \text{tr}[D_{xx}^2 \varphi(t-s, x+X_{s-}) c_s^P] ds \\ & \quad + \int_0^u \int_{\mathbb{R}^d} \left[\varphi(t-s, x+X_{s-} + z) - \varphi(t-s, x+X_{s-}) \right. \\ & \quad \quad \left. - D_x \varphi(t-s, x+X_{s-}) h(z) \right] \mu^X(ds, dz). \end{aligned} \quad (5.8)$$

Since $\varphi \in C_b^{2,3}$, it follows from (5.5) and (5.6) that the first integral in (5.8) is a true martingale; in particular,

$$E^P \left[\int_0^u \int_{\mathbb{R}^d} D_x \varphi(t-s, x+X_{s-}) d(X_s^{c,P} + X_s^{d,P}) \right] = 0, \quad u \geq 0. \quad (5.9)$$

Using (5.1) and (5.4), we can estimate the expectations of the other terms in (5.8). Namely, we have

$$\begin{aligned}
& E^P \left[\int_0^u D_x \varphi(t-s, x + X_{s-}) b_s^P ds \right] \\
& \leq \int_0^u E^P \left[|D_x \varphi(t-s, x + X_{s-}) - D_x \varphi(t, x)| |b_s^P| + D_x \varphi(t, x) b_s^P \right] ds \\
& \leq \int_0^u E^P \left[C(s + |X_{s-}|) \right] + E^P \left[D_x \varphi(t, x) b_s^P \right] ds \\
& \leq C(u^2 + u^{3/2}) + \int_0^u E^P \left[D_x \varphi(t, x) b_s^P \right] ds, \tag{5.10}
\end{aligned}$$

and similarly

$$E^P \left[\int_0^u -\partial_t \varphi(t-s, x + X_{s-}) ds \right] \leq \int_0^u -\partial_t \varphi(t, x) ds + C(u^2 + u^{3/2}) \tag{5.11}$$

as well as

$$\begin{aligned}
& E^P \left[\int_0^u \text{tr} [D_{xx}^2 \varphi(t-s, x + X_{s-}) c_s^P] ds \right] \\
& \leq \int_0^u E^P \left[\text{tr} [D_{xx}^2 \varphi(t, x) c_s^P] \right] ds + C(u^2 + u^{3/2}). \tag{5.12}
\end{aligned}$$

For the last term in (5.8), we shall distinguish between jumps smaller and larger than a given $\varepsilon > 0$, where ε is such that $h(z) = z$ on $\{|z| \leq \varepsilon\}$. Indeed, a Taylor expansion shows that there exist $\xi_z \in \mathbb{R}^d$ such that P -a.s., the integral can be written as the sum

$$\begin{aligned}
& \int_0^u \int_{|z| > \varepsilon} \left[\varphi(t-s, x + X_{s-} + z) - \varphi(t-s, x + X_{s-}) \right. \\
& \quad \left. - D_x \varphi(t-s, x + X_{s-}) h(z) \right] \mu^X(ds, dz) \\
& + \int_0^u \int_{|z| \leq \varepsilon} \frac{1}{2} \text{tr} [D_{xx}^2 \varphi(t-s, x + X_{s-} + \xi_z) z z^\top] \mu^X(ds, dz). \tag{5.13}
\end{aligned}$$

By (5.1), both of these expressions are P -integrable. Using the same arguments as in (5.10), the first integral satisfies

$$\begin{aligned}
& E^P \left[\int_0^u \int_{|z| > \varepsilon} \left[\varphi(t-s, x + X_{s-} + z) - \varphi(t-s, x + X_{s-}) \right. \right. \\
& \quad \left. \left. - D_x \varphi(t-s, x + X_{s-}) h(z) \right] F_s^P(dz) ds \right] \\
& \leq E^P \left[\int_0^u \int_{|z| > \varepsilon} \left[\varphi(t, x + z) - \varphi(t, x) - D_x \varphi(t, x) h(z) \right] F_s^P(dz) ds \right] \\
& \quad + C C_\varepsilon (u^2 + u^{3/2}), \tag{5.14}
\end{aligned}$$

where

$$C_\varepsilon := \sup_{F \in \Theta_3} \int_{|z| > \varepsilon} 1 F(dz)$$

is finite for every fixed $\varepsilon > 0$ due to (5.1). For the second integral in (5.13), we have

$$\begin{aligned} & E^P \left[\int_0^u \int_{|z| \leq \varepsilon} \frac{1}{2} \operatorname{tr} [D_{xx}^2 \varphi(t-s, x + X_{s-} + \xi_z) z z^\top] \mu^X(ds, dz) \right] \\ &= E^P \left[\int_0^u \int_{|z| \leq \varepsilon} \frac{1}{2} \operatorname{tr} [D_{xx}^2 \varphi(t-s, x + X_{s-} + \xi_z) z z^\top] F_s^P(dz) ds \right] \\ &\leq C \mathcal{K}_\varepsilon u; \end{aligned} \quad (5.15)$$

recall (5.2). Thus, taking expectations in (5.8) and using (5.9)–(5.15), we obtain for small $\varepsilon > 0$ that

$$\begin{aligned} & E^P [\varphi(t-u, x + X_u) - \varphi(t, x)] \\ &\leq \int_0^u E^P \left[-\partial_t \varphi(t, x) + D_x \varphi(t, x) b_s^P + \frac{1}{2} \operatorname{tr} [D_{xx}^2 \varphi(t, x) c_s^P] \right. \\ &\quad \left. + \int_{|z| > \varepsilon} [\varphi(t, x+z) - \varphi(t, x) - D_x \varphi(t, x) h(z)] F_s^P(dz) \right] ds \\ &\quad + C \mathcal{K}_\varepsilon u + C C_\varepsilon (u^2 + u^{3/2}) \\ &\leq -u \partial_t \varphi(t, x) + u \sup_{(b, c, F) \in \Theta} \left\{ D_x \varphi(t, x) b + \frac{1}{2} \operatorname{tr} [D_{xx}^2 \varphi(t, x) c] \right. \\ &\quad \left. + \int_{|z| > \varepsilon} [\varphi(t, x+z) - \varphi(t, x) - D_x \varphi(t, x) h(z)] F(dz) \right\} \\ &\quad + C \mathcal{K}_\varepsilon u + C C_\varepsilon (u^2 + u^{3/2}). \end{aligned} \quad (5.16)$$

Regarding the integral in this expression, we note that for each $F \in \Theta_3$,

$$\begin{aligned} & \int_{|z| > \varepsilon} [\varphi(t, x+z) - \varphi(t, x) - D_x \varphi(t, x) h(z)] F(dz) \\ &\leq \int_{\mathbb{R}^d} [\varphi(t, x+z) - \varphi(t, x) - D_x \varphi(t, x) h(z)] F(dz) \\ &\quad + \left| \int_{|z| \leq \varepsilon} [\varphi(t, x+z) - \varphi(t, x) - D_x \varphi(t, x) h(z)] F(dz) \right| \\ &\leq \int_{\mathbb{R}^d} [\varphi(t, x+z) - \varphi(t, x) - D_x \varphi(t, x) h(z)] F(dz) + C \mathcal{K}_\varepsilon \end{aligned} \quad (5.17)$$

by a Taylor expansion as above. We deduce from (5.16), (5.17) and the definition of G that

$$\begin{aligned} & E^P [\varphi(t-u, x + X_u) - \varphi(t, x)] \\ &\leq -u \partial_t \varphi(t, x) + u G(D_x \varphi(t, x), D_{xx}^2 \varphi(t, x), \varphi(t, x + \cdot)) \\ &\quad + C \mathcal{K}_\varepsilon u + C C_\varepsilon (u^2 + u^{3/2}). \end{aligned}$$

By (5.7), it follows that

$$0 \leq -u\partial_t\varphi(t, x) + uG(D_x\varphi(t, x), D_{xx}^2\varphi(t, x), \varphi(t, x + \cdot)) \\ + C\mathcal{K}_\varepsilon u + C C_\varepsilon (u^2 + u^{3/2}).$$

Now divide by u and let first u and then ε tend to zero. As $\mathcal{K}_\varepsilon \rightarrow 0$ by (5.2), we obtain that

$$0 \leq -\partial_t\varphi(t, x) + G(D_x\varphi(t, x), D_{xx}^2\varphi(t, x), \varphi(t, x + \cdot))$$

as desired. \square

IV.5.2 Uniqueness

The aim of this subsection is to show that a comparison principle holds for the PIDE (2.5); in particular, this will establish the uniqueness of the solution. We denote by $\text{USC}_b((0, \infty) \times \mathbb{R}^d)$ the set of all bounded upper semicontinuous functions on $(0, \infty) \times \mathbb{R}^d$. Similarly, LSC_b stands for the bounded lower semicontinuous functions, and $\text{SC}_b := \text{USC}_b \cup \text{LSC}_b$.

Proposition IV.5.5. *Let $u \in \text{USC}_b([0, \infty) \times \mathbb{R}^d)$ be a viscosity subsolution and let $v \in \text{LSC}_b([0, \infty) \times \mathbb{R}^d)$ be a viscosity supersolution of (2.5). If $u(0, \cdot), v(0, \cdot) \in C_{b, \text{Lip}}(\mathbb{R}^d)$ and $u(0, \cdot) \leq v(0, \cdot)$, then $u \leq v$.*

The proof proceeds through the following general result, essentially due to [25] (which, in turn, draws from [2, 28]).

Lemma IV.5.6. *Let $G : \mathbb{R}^d \times \mathbb{S}^d \times C_b^2(\mathbb{R}^d) \rightarrow \mathbb{R}$ and suppose there exist functions $G^\kappa : \mathbb{R}^d \times \mathbb{S}^d \times \text{SC}_b(\mathbb{R}^d) \times C^2(\mathbb{R}^d) \rightarrow \mathbb{R}$, $\kappa \in (0, 1)$ such that Conditions (C1)–(C9) below are satisfied. Then the assertion of Proposition IV.5.5 holds for*

$$\partial_t v(t, x) - G(D_x v(t, x), D_{xx}^2 v(t, x), v(t, x + \cdot)) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Proof. This is essentially the result of [25, Corollary 53]. The only difference is that our Condition (C8) below is slightly weaker than its analogue [25, Theorem 51, Condition (i)]. An inspection of the proof of [25, Theorem 51] shows that the result remains true under the weaker condition. \square

The conditions mentioned in the preceding lemma run as follows.

(C1) Let $(t_k, x_k, p_k, q_k) \rightarrow (t, x, p, q)$ in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$. Moreover, let $f_k, f \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$ be such that $f_k(t_k, x_k + \cdot) \rightarrow f(t, x + \cdot)$ locally uniformly on \mathbb{R}^d , $D_x f_k \rightarrow D_x f$ and $D_{xx}^2 f_k \rightarrow D_{xx}^2 f$ locally uniformly on $(0, \infty) \times \mathbb{R}^d$, and $(f_k)_{k \in \mathbb{N}}$ is uniformly bounded. Then

$$G(p_k, q_k, f_k(t_k, x_k + \cdot)) \rightarrow G(p, q, f(t, x + \cdot)).$$

(C2) Let $(t, x, p, q_1, q_2) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{S}^d$ be such that $q_1 \geq q_2$ and let $f_1, f_2 \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$ be such that $(f_1 - f_2)(t, \cdot)$ has a global minimum at x . Then

$$G(p, q_1, f_1(t, x + \cdot)) \geq G(p, q_2, f_2(t, x + \cdot)).$$

(C3) Let $(t, x, p, q) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$ and $f \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$. Then

$$G(p, q, f(t, x + \cdot) + c) = G(p, q, f(t, x + \cdot)), \quad c \in \mathbb{R}.$$

(C4) Let $(t, x, p, q) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$ and let $f \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$. Then

$$G^\kappa(p, q, f(t, x + \cdot), f(t, x + \cdot)) = G(p, q, f(t, x + \cdot)), \quad \kappa \in (0, 1).$$

(C5) Let $(t, x, p, q_1, q_2) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{S}^d$ be such that $q_1 \geq q_2$, let $f_1 \in \text{LSC}_b((0, \infty) \times \mathbb{R}^d)$ and $f_2 \in \text{USC}_b((0, \infty) \times \mathbb{R}^d)$ be such that $(f_1 - f_2)(t, \cdot)$ has a global minimum at x and let $g_1, g_2 \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$ be such that $(g_1 - g_2)(t, \cdot)$ has a global minimum at x . Then, for all $\kappa \in (0, 1)$,

$$G^\kappa(p, q_1, f_1(t, x + \cdot), g_1(t, x + \cdot)) \geq G^\kappa(p, q_2, f_2(t, x + \cdot), g_2(t, x + \cdot)).$$

(C6) Let $(t, x, p, q) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$, $f \in \text{SC}_b((0, \infty) \times \mathbb{R}^d)$ and $g \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$. Then, for all $\kappa \in (0, 1)$ and $c_1, c_2 \in \mathbb{R}$,

$$G^\kappa(p, q, f(t, x + \cdot) + c_1, g(t, x + \cdot) + c_2) = G^\kappa(p, q, f(t, x + \cdot), g(t, x + \cdot)).$$

(C7) Let $(t, x, p, q) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$, let $f \in \text{SC}_b((0, \infty) \times \mathbb{R}^d)$ and let $f_n, g \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$ be such that $f_n(t, \cdot) \rightarrow f(t, \cdot)$ locally uniformly on \mathbb{R}^d and $(f_n)_{n \in \mathbb{N}}$ is uniformly bounded. Then, for all $\kappa \in (0, 1)$,

$$G^\kappa(p, q, f_n(t, x + \cdot), g(t, x + \cdot)) \rightarrow G^\kappa(p, q, f(t, x + \cdot), g(t, x + \cdot)).$$

(C8) There exists a constant $C > 0$ such that

$$\begin{aligned} & |G^\kappa(p_1, q_1, f(t, \cdot) + \psi(\cdot), g(t, \cdot) + \psi(\cdot)) - G^\kappa(p_2, q_2, f(t, \cdot), g(t, \cdot))| \\ & \leq C(|p_1 - p_2| + |q_1 - q_2| + \|D_x \psi\|_\infty + \|D_{xx}^2 \psi\|_\infty) \end{aligned}$$

for all $\kappa \in (0, 1)$, $t \in (0, \infty)$, $p_1, p_2 \in \mathbb{R}^d$, $q_1, q_2 \in \mathbb{S}^d$, $f \in \text{SC}_b((0, \infty) \times \mathbb{R}^d)$, $g \in C_b^{1,2}((0, \infty) \times \mathbb{R}^d)$ and $\psi \in C_b^2(\mathbb{R}^d)$.

(C9) Let $(t, x, p, q) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$, let $f \in \text{SC}_b((0, \infty) \times \mathbb{R}^d)$ and let $g_1, g_2 \in C^{1,2}((0, \infty) \times \mathbb{R}^d)$ satisfy $D_x g_1(t, x) = D_x g_2(t, x)$. Then

$$\lim_{\kappa \rightarrow 0} |G^\kappa(p, q, f(t, x + \cdot), g_1(t, x + \cdot)) - G^\kappa(p, q, f(t, x + \cdot), g_2(t, x + \cdot))| = 0.$$

In order to deduce Proposition IV.5.5 from Lemma IV.5.6, we define the auxiliary functions $G^\kappa : \mathbb{R}^d \times \mathbb{S}^d \times \text{SC}_b(\mathbb{R}^d) \times C^2(\mathbb{R}^d) \rightarrow \mathbb{R}$, $\kappa \in (0, 1)$ by

$$\begin{aligned} G^\kappa(p, q, f(\cdot), g(\cdot)) &:= \sup_{(b, c, F) \in \Theta} \left\{ \int_{|z| > \kappa} [f(z) - f(0) - D_x g(0)h(z)] F(dz) \right. \\ &\quad \left. + \int_{|z| \leq \kappa} [g(z) - g(0) - D_x g(0)h(z)] F(dz) + pb + \frac{1}{2} \text{tr}[qc] \right\}. \end{aligned} \quad (5.18)$$

In the remainder of this section, we verify that (C1)–(C9) hold for this choice of G^κ and G as in (2.6), which will complete the proof of Proposition IV.5.5. To simplify the notation, we assume that h is the canonical truncation function

$$h(z) = z \mathbf{1}_{|z| \leq 1}.$$

This entails no loss of generality because the PIDE (2.5) does not depend on the choice of h .

Lemma IV.5.7. *The function G of (2.6) satisfies (C1)–(C3).*

Proof. Conditions (C2) and (C3) follow directly from the definitions; we focus on (C1). In view of (5.1), we may fix $N > 1$ and estimate

$$|G(p_k, q_k, f_k(t_k, x_k + \cdot)) - G(p, q, f(t, x + \cdot))| \leq I_k^1 + I_k^2 + I_k^3 + I_{k,N}^4 + I_{k,N}^5,$$

where

$$I_k^1 = \sup_{(b, c, F) \in \Theta} |b| |p_k - p|, \quad I_k^2 = \frac{1}{2} \sup_{(b, c, F) \in \Theta} |c| |q_k - q|,$$

$$\begin{aligned} I_k^3 &= \sup_{(b, c, F) \in \Theta} \left\{ \int_{|z| \leq 1} \left| (f_k(t_k, x_k + z) - f_k(t_k, x_k) - D_x f_k(t_k, x_k)z) \right. \right. \\ &\quad \left. \left. - (f(t, x + z) - f(t, x) - D_x f(t, x)z) \right| F(dz) \right\}, \end{aligned}$$

$$\begin{aligned} I_{k,N}^4 &= \sup_{(b, c, F) \in \Theta} \left\{ \int_{1 \leq |z| \leq N} \left| (f_k(t_k, x_k + z) - f(t, x + z)) \right. \right. \\ &\quad \left. \left. - (f_k(t_k, x_k) - f(t, x)) \right| F(dz) \right\}, \end{aligned}$$

$$I_{k,N}^5 = \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| > N} |(f_k(t_k, x_k + z) - f(t, x + z)) - (f_k(t_k, x_k) - f(t, x))| F(dz) \right\}.$$

In view of the assumptions made in (C1) and (5.1), we see that $I_k^1 + I_k^2 \rightarrow 0$ as $k \rightarrow \infty$. By a Taylor expansion, there are $\xi_{k,z}, \xi_z \in \{|z| \leq 1\}$ such that

$$I_k^3 = \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| \leq 1} \frac{1}{2} \text{tr}[(D_{xx}^2 f_k(t_k, x_k + \xi_{k,z}) - D_{xx}^2 f(t, x + \xi_z)) z z^\top] F(dz) \right\}.$$

Using (5.1) and the locally uniform convergence of $D_{xx}^2 f_k$ to $D_{xx}^2 f$, it follows that $I_k^3 \rightarrow 0$. Similarly, there exist $\xi_{k,z}, \xi_z \in \{|z| \leq N\}$ such that

$$I_{k,N}^4 = \sup_{(b,c,F) \in \Theta} \left\{ \int_{1 \leq |z| \leq N} |(D_x f_k(t_k, x_k + \xi_{k,z}) - D_x f(t, x + \xi_z)) z| F(dz) \right\},$$

and the locally uniform convergence of $D_x f_k$ to $D_x f$ yields that $I_{k,N}^4 \rightarrow 0$ for any fixed N . Using the uniform bound on $(f_k)_k$ assumed in (C1), we also see that

$$I_{k,N}^5 \leq C \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| > N} 1 F(dz) \right\} \leq \frac{C}{N} \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| > 1} |z| F(dz) \right\};$$

note that the right-hand side is independent of k and finite by (5.1). Summarizing the above, we have

$$\limsup_{k \rightarrow \infty} |G(p_k, q_k, f_k(t_k, x_k + \cdot)) - G(p, q, f(t, x + \cdot))| \leq C/N$$

for every $N > 1$ and the result follows. \square

Lemma IV.5.8. *The functions G of (2.6) and $(G^\kappa)_{\kappa \in (0,1)}$ of (5.18) satisfy (C4)–(C7).*

Proof. Conditions (C4)–(C6) follow directly from the definitions of G , G^κ and (5.1). The proof of (C7) is similar to the verification of (C1) and therefore omitted. \square

Lemma IV.5.9. *The functions $(G^\kappa)_{\kappa \in (0,1)}$ of (5.18) satisfy (C8) and (C9).*

Proof. We first show (C8). By definition, we have

$$\begin{aligned} & |G^\kappa(p_1, q_1, f(t, \cdot) + \psi(\cdot), g(t, \cdot) + \psi(\cdot)) - G^\kappa(p_2, q_2, f(t, \cdot), g(t, \cdot))| \\ & \leq \sup_{(b,c,F) \in \Theta} |b| |p_1 - p_2| + \frac{1}{2} \sup_{(b,c,F) \in \Theta} |c| |q_1 - q_2| + I_1 + I_2, \end{aligned}$$

where

$$I_1 = \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| \leq 1} |\psi(z) - \psi(0) - D_x \psi(0)z| F(dz) \right\},$$

$$I_2 = \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| > 1} |\psi(z) - \psi(0)| F(dz) \right\}.$$

By a Taylor expansion, we see that there are $\xi_z \in \mathbb{R}^d$ such that

$$I_1 = \frac{1}{2} \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| \leq 1} |\text{tr}[D_{xx}^2 \psi(\xi_z) z z^\top]| F(dz) \right\}$$

$$\leq \frac{1}{2} \|D_{xx}^2 \psi\|_\infty \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| \leq 1} |z|^2 F(dz) \right\}$$

and the integral on the right-hand side is bounded by \mathcal{K} due to (5.1). Similarly,

$$I_2 = \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| > 1} |D_x \psi(\xi_z)z| F(dz) \right\}$$

$$\leq \|D_x \psi\|_\infty \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| > 1} |z| F(dz) \right\}$$

and again the integral is bounded by \mathcal{K} . Property (C8) follows, with the constant being \mathcal{K} up to a numerical factor.

The assumptions in (C9) imply that

$$|G^\kappa(p, q, f(t, x + \cdot), g_1(t, x + \cdot)) - G^\kappa(p, q, f(t, x + \cdot), g_2(t, x + \cdot))|$$

$$\leq \sup_{(b,c,F) \in \Theta} \left\{ \left| \int_{|z| \leq \kappa} g_1(t, x + z) - g_1(t, x) - D_x g_1(t, x)z F(dz) \right. \right.$$

$$\left. \left. - \int_{|z| \leq \kappa} g_2(t, x + z) - g_2(t, x) - D_x g_2(t, x)z F(dz) \right| \right\}.$$

If $K \subset \mathbb{R}^d$ is the closed ball of unit radius around x , a Taylor expansion shows that the above expression is bounded by

$$\frac{1}{2} (\|D_{xx}^2 g_1(t, \cdot)\|_K + \|D_{xx}^2 g_2(t, \cdot)\|_K) \sup_{(b,c,F) \in \Theta} \left\{ \int_{|z| \leq \kappa} |z|^2 F(dz) \right\},$$

where $\|\cdot\|_K$ is the uniform norm on K . Thus, the claim follows from (5.2). \square

IV.6 Related Literature

Nonlinear Lévy processes were introduced in [25]. First, the authors consider a given pair (X^c, X^d) of processes with stationary and independent

increments under a given sublinear expectation $\mathcal{E}(\cdot)$. The continuous process X^c is assumed to satisfy $\mathcal{E}(|X_t^c|^3)/t \rightarrow 0$ as $t \rightarrow 0$ which implies that it is a G -Brownian motion, whereas the jump part X^d is assumed to satisfy $\mathcal{E}(|X_t^d|) \leq Ct$ for some constant C . The sum $X := X^c + X^d$ is then called a G -Lévy process. It is shown that $G_X[f(\cdot)] := \lim_{t \rightarrow 0} \mathcal{E}(f(X_t))/t$ is well-defined for a suitable class of functions f and has a representation in terms of a set Θ of Lévy triplets satisfying

$$\sup_{(b,c,F) \in \Theta} \left\{ \int |z| F(dz) + |b| + |c| \right\} < \infty, \quad (6.1)$$

meaning that functions $v(t, x) = \mathcal{E}(\psi(x + X_t))$ solve the PIDE (1.2) with initial condition $\psi \in C_{b,Lip}(\mathbb{R}^d)$. We note that (6.1) implies both (1.3) and (1.4), and is in fact significantly stronger because it excludes all triplets with infinite variation jumps—the extension of the representation result to such jumps remains an important open problem. Second, given a set Θ satisfying (6.1), a corresponding nonlinear Lévy process X is constructed directly from the PIDE (1.2), in the following sense². If X is the canonical process, expectations of the form $\mathcal{E}(\psi(X_t))$ with $\psi \in C_{b,Lip}(\mathbb{R}^d)$ can be defined through the solution v by simply setting $\mathcal{E}(\psi(X_t)) := v(t, 0)$. More general expectations of the form $\mathcal{E}(\psi(X_{t_1}, \dots, X_{t_n}))$ can be defined similarly by a recursive application of the PIDE. Thus, one can construct $\mathcal{E}(\xi)$ for all functions ξ in the completion L_G^1 of the space of all functions of the form $\psi(X_{t_1}, \dots, X_{t_n})$ with $\psi \in C_{b,Lip}(\mathbb{R}^{d \times n})$ for some n under the norm $\mathcal{E}(|\cdot|)$. We remark that this space is significantly smaller than the set of measurable functions, see [62], and so it is left open in [25] how to define $\mathcal{E}(\xi)$ for general random variables ξ . A second remark is that while this construction is very direct, it leaves open how to interpret a nonlinear Lévy process from the point of view of classical probability theory.

Summing up, our contribution is twofold. First, we construct nonlinear Lévy processes for arbitrary (measurable) characteristics Θ , possibly with unbounded diffusion and infinite variation jumps, and the distribution is defined for all measurable functions. Our probabilistic construction allows us to understand the PIDE (1.2) as the Hamilton–Jacobi–Bellman equation resulting from the nonstandard control problem

$$\sup_{P \in \mathfrak{P}_\Theta} E^P[\cdot]$$

over the class of all semimartingales with Θ -valued differential characteristics. This control representation gives a global interpretation to the distribution of a nonlinear Lévy process as the worst-case expectation over \mathfrak{P}_Θ ; in particular, this allows for applications in robust control under model uncertainty. Second, under Conditions (1.3) and (1.4), we provide a rigorous

²It seems that such a construction could also be carried out under the weaker conditions (1.3) and (1.4).

link with the PIDE (1.2). On the one hand, this implies that expectations of Markovian functionals can be calculated by means of a differential equation, which is important for applications. On the other hand, it allows us to identify our construction as an extension of [25].

Chapter V

Robust Utility Maximization with Lévy Processes

In this chapter, which corresponds to the article [40], we study a robust portfolio optimization problem under model uncertainty for an investor with logarithmic or power utility. The uncertainty is specified by a set of possible Lévy triplets; that is, possible instantaneous drift, volatility and jump characteristics of the price process. We show that an optimal investment strategy exists and compute it in semi-closed form. Moreover, we provide a saddle point analysis describing a worst-case model.

V.1 Introduction

We study a robust utility maximization problem of the form

$$\sup_{\pi} \inf_P E^P[U(W_T^\pi)] \quad (1.1)$$

in a continuous-time financial market with jumps. Here W_T^π is the wealth at time T resulting from investing in d stocks according to the trading strategy π and U is either the logarithmic utility $U(x) = \log(x)$ or a power utility $U(x) = \frac{1}{p}x^p$ for some $p \in (-\infty, 0) \cup (0, 1)$. The infimum is taken over a class \mathfrak{P} of possible models P for the dynamics of the log-price processes of the stocks. More precisely, the model uncertainty is parametrized by a set Θ of Lévy triplets (b, c, F) and then $\mathfrak{P} := \mathfrak{P}_{sem}^{ac}(\Theta)$ consists of all semimartingale laws P such that the associated differential characteristics (b_t^P, c_t^P, F_t^P) take values in Θ , $P \times dt$ -a.e., as introduced in Chapter III. Thus, our setup describes uncertainty about drift, volatility and jumps over a class of fairly general models.

Our first main result shows that an optimal trading strategy $\hat{\pi}$ exists for (1.1). This strategy is of the constant-proportion type; that is, a constant fraction of the current wealth is invested in each stock. We compute this

fraction in semi-closed form, so that the impact of model uncertainty can be readily read off; cf. Theorem V.2.4.

Under a compactness condition on Θ , we also show the existence of a worst-case model $\hat{P} \in \mathfrak{P}$. This model is a Lévy law and the corresponding Lévy triplet $(\hat{b}, \hat{c}, \hat{F})$ is computed in semi-closed form. More precisely, our second main result yields a saddle point $(\hat{P}, \hat{\pi})$ for the problem (1.1) which may be seen as a two player zero-sum game. The strategy $\hat{\pi}$ and the triplet $(\hat{b}, \hat{c}, \hat{F})$ are characterized as a saddle point of a deterministic function; cf. Theorem V.2.5.

Mathematically, our method of proof follows the local-to-global paradigm. That is, we first derive versions of our main results for a “local” optimization problem that plays the role of a Bellman-Isaacs operator. The passage to the global results is relatively direct in the logarithmic case, because the log investor is myopic in every model $P \in \mathfrak{P}$. For the power utility, this fails and thus the optimal strategy and expected utility for a fixed P cannot be expressed in a simple way. However, we shall see that the worst case over all Lévy laws already corresponds to the worst case over all $P \in \mathfrak{P}$. The key tool for this is a martingale argument; cf. Lemma V.5.1.

The remainder of this chapter is organized as follows. In Section V.2, we specify our model and the optimization problem in detail, and we state our main results. Section V.3 contains the analysis of the local optimization problem. In Section V.4, we give the proofs of the main results for the logarithmic utility, whereas Section V.5 presents the proofs for power utility.

V.2 The Optimization Problem

V.2.1 Setup for Model Uncertainty

We fix the dimension $d \in \mathbb{N}$ and let $\Omega = D_0(\mathbb{R}_+, \mathbb{R}^d)$ be the space of all càdlàg paths $\omega = (\omega_t)_{t \geq 0}$ starting at $0 \in \mathbb{R}^d$. We equip Ω with the Skorohod topology and the corresponding Borel σ -field \mathcal{F} . Moreover, we denote by $X = (X_t)_{t \geq 0}$ the canonical process $X_t(\omega) = \omega_t$, by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the (raw) filtration generated by X , and by $\mathfrak{P}(\Omega)$ the Polish space of all probability measures on Ω . We also fix the time horizon $T \in (0, \infty)$.

The uncertainty about drift, volatility and jumps is parametrized by a nonempty set

$$\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L},$$

where \mathcal{L} is the set of Lévy measures; i.e., the set of all measures F on \mathbb{R}^d that satisfy $\int_{\mathbb{R}^d} |z|^2 \wedge 1 F(dz) < \infty$ and $F(\{0\}) = 0$. We write

$$\mathcal{L}_\Theta = \{F \in \mathcal{L} \mid (b, c, F) \in \Theta\}$$

for the projection of Θ onto \mathcal{L} . The class of models to be considered is represented by the set \mathfrak{P} of semimartingale laws such that the differential

characteristics of the canonical process X take values in Θ , as introduced in Chapter III. More precisely, let

$$\mathfrak{P}_{sem} = \{P \in \mathfrak{P}(\Omega) \mid X \text{ is a semimartingale on } (\Omega, \mathcal{F}, \mathbb{F}, P)\}$$

be the set of all semimartingale laws, denote by (B^P, C^P, ν^P) the predictable characteristics of X under P with respect to a fixed truncation function h , and let

$$\mathfrak{P}_{sem}^{ac} = \{P \in \mathfrak{P}_{sem} \mid (B^P, C^P, \nu^P) \ll dt, P\text{-a.s.}\}$$

be the set of semimartingale laws with absolutely continuous characteristics (with respect to the Lebesgue measure dt). Given such a triplet (B^P, C^P, ν^P) , the corresponding derivatives (defined dt -a.e.) are called the differential characteristics of X and denoted by (b^P, c^P, F^P) . Our set \mathfrak{P} of possible laws is then given by

$$\mathfrak{P} := \mathfrak{P}_{sem}^{ac}(\Theta) = \{P \in \mathfrak{P}_{sem}^{ac} \mid (b^P, c^P, F^P) \in \Theta, P \otimes dt\text{-a.e.}\}.$$

The canonical process X , considered under the set \mathfrak{P} , can be seen as a nonlinear Lévy process in the sense of Chapter IV. Finally, let us denote by

$$\mathfrak{P}_L = \{P \in \mathfrak{P} \mid X \text{ is a Lévy process under } P\}$$

the set of all Lévy laws in \mathfrak{P} . Thus, there is a one-to-one correspondence between \mathfrak{P}_L and the set Θ of Lévy triplets, whereas the set \mathfrak{P} is in general much larger than \mathfrak{P}_L .

V.2.2 Utility and Constraints

To model the preferences of the investor, we consider the logarithmic and the power utility functions on $(0, \infty)$; i.e.,

$$U(x) = \log(x) \quad \text{and} \quad U(x) = \frac{1}{p}x^p \quad \text{for } p \in (-\infty, 0) \cup (0, 1).$$

As usual, we set $U(0) := \lim_{x \rightarrow 0} U(x)$ and $U(\infty) := \lim_{x \rightarrow \infty} U(x)$.

Our investor is endowed with a deterministic initial capital $x_0 > 0$ and chooses a trading strategy π ; that is, a predictable \mathbb{R}^d -valued process which is X -integrable under all $P \in \mathfrak{P}$. Here the canonical process X represents the returns of the (discounted) stock prices and thus the i th component of π is interpreted as the *proportion* of current wealth invested in the i th stock. Under any $P \in \mathfrak{P}$, the corresponding wealth process W^π is given by the stochastic exponential

$$W^\pi = x_0 \mathcal{E}\left(\int \pi dX\right).$$

The portfolio is subject to a no-bankruptcy constraint that can be described by the set of *natural constraints*,

$$\mathcal{C}^0 := \bigcap_{F \in \mathcal{L}_\Theta} \{y \in \mathbb{R}^d \mid F[z \in \mathbb{R}^d \mid y^\top z < -1] = 0\}.$$

Indeed, a strategy π with values in \mathcal{C}^0 satisfies $\pi^\top \Delta X \geq -1$ P -a.s. for all $P \in \mathfrak{P}$ and this is in turn equivalent to $W^\pi \geq 0$ P -a.s. for all $P \in \mathfrak{P}$. For later use, we note that \mathcal{C}^0 is a closed, convex subset of \mathbb{R}^d that contains the origin.

In addition to the natural constraints, we may impose further constraints such as no-shortselling on the investor. These constraints are modeled by an arbitrary closed, convex set $\mathcal{C} \subseteq \mathbb{R}^d$ containing the origin.

The set \mathcal{A} of *admissible strategies* is the collection of all strategies π such that $\pi_t(\omega) \in \mathcal{C} \cap \mathcal{C}^0$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$ and $U(W_T^\pi) > -\infty$ P -a.s. for all $P \in \mathfrak{P}$. The second condition is for notational convenience: if $U(W_T^\pi) = -\infty$ with positive probability for some $P \in \mathfrak{P}$, then π is not relevant for our optimization problem. Note that nothing is being excluded for the power utility with $p \in (0, 1)$, whereas in the other cases we have $U(0) = -\infty$ and thus $\pi \in \mathcal{A}$ implies $W^\pi > 0$ P -a.s. for all $P \in \mathfrak{P}$; cf. [26, Theorem I.4.61, p. 59]. The value function of our robust utility maximization problem is

$$u(x_0) := \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)]. \quad (2.1)$$

Here and below, we define the expectation for any measurable function with values in $\overline{\mathbb{R}} = [-\infty, \infty]$, using the convention $\infty - \infty = -\infty$. We say that the robust utility maximization problem is *finite* if $u(x_0) < \infty$. Under this condition, we call $\pi \in \mathcal{A}$ *optimal* if it attains the supremum in (2.1).

V.2.3 Main Results

We recall that U stands for either $U(x) = \log(x)$ or $U(x) = \frac{1}{p}x^p$ with $p \in (-\infty, 0) \cup (0, 1)$. For convenience of notation, $p = 0$ will refer to the logarithmic case in what follows. The subsequent conditions are in force for the remainder of the chapter.

Assumption V.2.1.

- (i) The set $\mathcal{C} \cap \mathcal{C}^0 \subseteq \mathbb{R}^d$ is compact.
- (ii) The set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ is convex and satisfies $\mathcal{K} < \infty$, where

$$\mathcal{K} := \sup_{(b,c,F) \in \Theta} \begin{cases} |b| + |c| + \int |z|^2 \wedge \log(1 + |z|) F(dz) & \text{if } p = 0, \\ |b| + |c| + \int |z|^2 \wedge |z|^{p(1+\varepsilon)} F(dz) & \text{if } p \in (0, 1), \\ |b| + |c| + \int |z|^2 \wedge 1 F(dz) & \text{if } p < 0. \end{cases}$$

In the case $p \in (0, 1)$, we have fixed an arbitrarily small constant $\varepsilon > 0$ in the above definition of \mathcal{K} .

Remark V.2.2. (i) The compactness assumption on $\mathcal{C} \cap \mathcal{C}^0$ is *not* very restrictive for the cases of our interest: in the presence of jumps, the set \mathcal{C}^0 is typically compact, and then the assumption holds even in the unconstrained case $\mathcal{C} = \mathbb{R}^d$. Indeed, let $d = 1$ for simplicity of notation. As soon as the jumps of X are unbounded from above, for at least one $P \in \mathfrak{P}$, and not bounded away from -1 , for at least one $P \in \mathfrak{P}$, then $\mathcal{C}^0 \subseteq [0, 1]$ and $\mathcal{C} \cap \mathcal{C}^0$ is necessarily compact.

The non-compact case can also be analyzed but leads to technical complications that are not of specific interest to our robust problem. These complications are well-studied in the classical case; cf. [32, 42].

(ii) The second condition in Assumption V.2.1 will guarantee, in particular, the finiteness of the robust utility maximization problem. When $p < 0$, no specific Lévy triplet is excluded as any Lévy measure satisfies $\int |z|^2 \wedge 1 F(dz) < \infty$. When $p \geq 0$, a sufficient condition is that the Lévy process has integrable jumps, which is equivalent to $\int |z|^2 \wedge |z| F(dz) < \infty$.

Definition V.2.3. Let $(b, c, F) \in \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}_\Theta$. For $y \in \mathcal{C}^0$, we define

$$g^{(b,c,F)}(y) := y^\top b + \frac{p-1}{2} y^\top c y + \int_{\mathbb{R}^d} I_y(z) F(dz), \quad (2.2)$$

where

$$I_y(z) := \begin{cases} \log(1 + y^\top z) - y^\top h(z) & \text{if } p = 0, \\ p^{-1}(1 + y^\top z)^p - p^{-1} - y^\top h(z) & \text{if } p \neq 0. \end{cases}$$

We shall see later that $g^{(b,c,F)}$ is a well-defined concave function with values in $[-\infty, \infty)$.

Our first main result states that an optimal strategy exists; moreover, it is given by a constant proportion that can be described in terms of the function g . We recall that Assumption V.2.1 is in force.

Theorem V.2.4 (Optimal Strategy).

(i) *The robust utility maximization problem is finite and*

$$\sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P [U(W_T^\pi)] = \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P [U(W_T^\pi)]. \quad (2.3)$$

(ii) *There exists an optimal strategy which is constant. More precisely, the finite-dimensional problem*

$$\arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{(b,c,F) \in \Theta} g^{(b,c,F)}(y)$$

has at least one solution. Any solution \hat{y} , seen as constant process, is in \mathcal{A} and defines an optimal strategy; i.e.,

$$\inf_{P \in \mathfrak{P}} E^P [U(W_T^{\hat{y}})] = \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P [U(W_T^\pi)],$$

and this value is equal to

$$\begin{cases} \log(x_0) + T \inf_{(b,c,F) \in \Theta} g^{(b,c,F)}(\hat{y}) & \text{if } p = 0, \\ \frac{1}{p} x_0^p \exp\left(pT \inf_{(b,c,F) \in \Theta} g^{(b,c,F)}(\hat{y})\right) & \text{if } p \neq 0. \end{cases}$$

(iii) Conversely, any constant optimal strategy $\tilde{\pi} \in \mathcal{A}$ is an element of

$$\arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{(b,c,F) \in \Theta} g^{(b,c,F)}(y).$$

The robust utility maximization problem can be seen as a two player zero-sum game. Indeed, the minimax identity (2.3) then states the existence of the value. Our second main result is a saddle point analysis of the game. For reference, let us recall that a point $(\hat{x}, \hat{y}) \in \mathcal{X} \times \mathcal{Y}$ in some product set is called a saddle point of the function $f : \mathcal{X} \times \mathcal{Y} \rightarrow [-\infty, \infty]$ if

$$f(\hat{x}, y) \leq f(\hat{x}, \hat{y}) \leq f(x, \hat{y}) \quad \text{for all } x \in \mathcal{X}, y \in \mathcal{Y}.$$

Thus, \hat{x} is the optimal response when the second player chooses \hat{y} , and vice versa. This is equivalent to the three assertions

- (i) $\sup_{y \in \mathcal{Y}} f(\hat{x}, y) = \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y)$,
- (ii) $\inf_{x \in \mathcal{X}} f(x, \hat{y}) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y)$,
- (iii) $\sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y) = \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y)$;

that is, \hat{x} and \hat{y} solve the respective robust optimization problems, and the minimax identity holds.

To provide a saddle point analysis of the game, we need to introduce a topology on the set Θ . Recall first that the space \mathfrak{M}^f of all finite measures on \mathbb{R}^d is a Polish space under a metric $d_{\mathfrak{M}^f}$ which induces the weak convergence relative to $C_b(\mathbb{R}^d)$; cf. [9, Theorem 8.9.4, p. 213]. This topology is the natural extension of the more customary weak convergence of probability measures. With any Lévy measure $\mu \in \mathcal{L}$ we can associate the finite measure

$$A \mapsto \int_A |x|^2 \wedge 1 \mu(dx), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

denoted by $|x|^2 \wedge 1 \cdot \mu$. We can now define a metric $d_{\mathcal{L}}$ via

$$d_{\mathcal{L}}(\mu, \nu) = d_{\mathfrak{M}^f}(|x|^2 \wedge 1 \cdot \mu, |x|^2 \wedge 1 \cdot \nu), \quad \mu, \nu \in \mathcal{L},$$

and then $(\mathcal{L}, d_{\mathcal{L}})$ is a separable metric space; cf. Lemma III.2.3. Moreover, the following version of Prohorov's theorem holds: a set $\mathcal{S} \subseteq \mathcal{L}$ is relatively compact if and only if

- (i) $\sup_{F \in \mathcal{S}} \int_{\mathbb{R}^d} |z|^2 \wedge 1 F(dz) < \infty$ and
- (ii) for any $\delta > 0$ there exists a compact set $K_\delta \subseteq \mathbb{R}^d$ such that

$$\sup_{F \in \mathcal{S}} \int_{K_\delta^c} |z|^2 \wedge 1 F(dz) \leq \delta.$$

This is a consequence of [59, Theorem 1.12].

Having defined a topology on \mathcal{L} , we can equip $\mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ with the corresponding product topology and state our second main result; recall that \mathfrak{P}_L denotes the set of all Lévy laws in \mathfrak{P} .

Theorem V.2.5 (Saddle Point). *Let $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ be compact.*

- (i) *The function $(P, \pi) \mapsto E^P[U(W_T^\pi)]$ has a saddle point on $\mathfrak{P} \times \mathcal{A}$.*

More precisely, the function $g^{(b,c,F)}(y)$ defined in (2.2) has a saddle point on $\Theta \times \mathcal{C} \cap \mathcal{C}^0$. If $((\hat{b}, \hat{c}, \hat{F}), \hat{y})$ is any such saddle point and $\hat{P} \in \mathfrak{P}_L$ denotes the Lévy law with triplet $(\hat{b}, \hat{c}, \hat{F})$, then $\hat{y} \in \mathcal{A}$ and (\hat{P}, \hat{y}) is a saddle point of $(P, \pi) \mapsto E^P[U(W_T^\pi)]$ on $\mathfrak{P} \times \mathcal{A}$, and its value is

$$E^{\hat{P}}[U(W_T^{\hat{y}})] = \begin{cases} \log(x_0) + T g^{(\hat{b}, \hat{c}, \hat{F})}(\hat{y}) & \text{if } p = 0, \\ \frac{1}{p} x_0^p \exp(p T g^{(\hat{b}, \hat{c}, \hat{F})}(\hat{y})) & \text{if } p \neq 0. \end{cases}$$

- (ii) *Conversely, if $(\tilde{P}, \tilde{\pi})$ is a saddle point of $(P, \pi) \mapsto E^P[U(W_T^\pi)]$ on $\mathfrak{P} \times \mathcal{A}$, and $\tilde{P} \in \mathfrak{P}_L$ and $\tilde{\pi}$ is constant, then $((\tilde{b}, \tilde{c}, \tilde{F}), \tilde{\pi})$ is a saddle point of the function $g^{(b,c,F)}(y)$ on $\Theta \times \mathcal{C} \cap \mathcal{C}^0$, where $(\tilde{b}, \tilde{c}, \tilde{F})$ is the Lévy triplet of \tilde{P} .*

We remark that the worst-case model \hat{P} is not unique in any meaningful way. For instance, if $\mathcal{C} = [0, 1]$ and $\Theta \subseteq \mathbb{R}_- \times [0, \infty) \times \{0\}$, then $(P, 0)$ is a saddle point for any $P \in \mathfrak{P}_L$. On the other hand, $\hat{\pi}$ is unique in the sense that $W^{\hat{\pi}}$ is uniquely determined \hat{P} -a.s.

Remark V.2.6. We may compare the situation of uncertainty over the set \mathfrak{P} of semimartingale laws and the (much smaller) set \mathfrak{P}_L of Lévy laws. It follows from the proofs below that the value function, the optimal strategies and the saddle points in the main results are the same in both cases. This is in contrast, for example, to the situation of option pricing in the Uncertain Volatility Model, where the worst-case can be a non-Lévy law.

V.3 The Local Analysis

In this section we analyze the function $g^{(b,c,F)}$ defined in (2.2); the results will be fundamental for the proofs of our main theorems. We set

$$g(y) := \inf_{(b,c,F) \in \Theta} g^{(b,c,F)}(y), \quad y \in \mathcal{C}^0 \quad (3.1)$$

and recall that Assumption V.2.1 is in force.

Lemma V.3.1. *Let $\theta = (b, c, F) \in \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}_\Theta$. The function g^θ of (2.2) is well-defined, proper, concave and upper semicontinuous on \mathcal{C}^0 , with values in $[-\infty, \infty)$. The same holds for the function g of (3.1). As a consequence, g^θ and g attain their maxima on $\mathcal{C} \cap \mathcal{C}^0$.*

Proof. The first assertion follows directly from Assumption V.2.1 and the literature on classical utility maximization; cf. [32, Section 5.1, p. 182] for $p = 0$ and [42, Lemma 5.3] for $p \neq 0$. The remaining assertions are direct consequences. \square

It will be useful to avoid the singularity of $g^{(b,c,F)}$ by considering the closed, convex sets

$$\mathcal{C}_n^0 := \bigcap_{F \in \mathcal{L}_\Theta} \left\{ y \in \mathbb{R}^d \mid F[z \in \mathbb{R}^d \mid y^\top z < -1 + \frac{1}{n}] = 0 \right\} \subseteq \mathcal{C}^0$$

for $n \in \mathbb{N}$. We have the following approximation result.

Lemma V.3.2. *Let $\theta \in \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}_\Theta$ and let \hat{y}^θ be a maximizer of $y \mapsto g^\theta(y)$ on $\mathcal{C} \cap \mathcal{C}^0$; then*

$$\sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) = \lim_{n \rightarrow \infty} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g^\theta(y) = \lim_{n \rightarrow \infty} g^\theta(\hat{y}_n^\theta) \quad \text{for } \hat{y}_n^\theta := (1 - \frac{1}{n})\hat{y}^\theta.$$

Similarly, let \hat{y} be a maximizer of $y \mapsto g(y)$ on $\mathcal{C} \cap \mathcal{C}^0$; then

$$\sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g(y) = \lim_{n \rightarrow \infty} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g(y) = \lim_{n \rightarrow \infty} g(\hat{y}_n) \quad \text{for } \hat{y}_n := (1 - \frac{1}{n})\hat{y}.$$

Proof. Since \mathcal{C} is convex and contains the origin, $\hat{y}_n^\theta \in \mathcal{C} \cap \mathcal{C}_n^0$. Moreover,

$$\sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) \geq \lim_{n \rightarrow \infty} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g^\theta(y) \geq \limsup_{n \rightarrow \infty} g^\theta(\hat{y}_n^\theta)$$

as $\mathcal{C}_n^0 \subseteq \mathcal{C}_{n+1}^0 \subseteq \mathcal{C}^0$. For the converse inequality, note that g^θ is concave and $g^\theta(0) = 0$, so that

$$g^\theta(\hat{y}_n^\theta) = g^\theta\left(\left(1 - \frac{1}{n}\right)\hat{y}^\theta\right) \geq \left(1 - \frac{1}{n}\right)g^\theta(\hat{y}^\theta).$$

Thus, we conclude that

$$\liminf_{n \rightarrow \infty} g^\theta(\hat{y}_n^\theta) \geq g^\theta(\hat{y}^\theta) = \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y)$$

and the first claim follows. The proof of the second claim is analogous. \square

Lemma V.3.3. *The map $\theta \mapsto \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y)$ is real-valued and lower semicontinuous on Θ .*

Proof. We note that $\sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) > -\infty$ as $0 \in \mathcal{C} \cap \mathcal{C}^0$. On the other hand, the conditions in Assumption V.2.1 yield that $\sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) < \infty$.

We turn to the semicontinuity. Without loss of generality, we may assume that the truncation function h is continuous; cf. [26, Proposition 2.24, p. 81]. Using the form of g^θ and the compactness of $\mathcal{C} \cap \mathcal{C}^0$, it suffices to show that for fixed $(b, c) \in \mathbb{R}^d \times \mathbb{S}_+^d$, the map

$$F \mapsto \mathfrak{g}(F) := \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{(b,c,F)}(y)$$

is lower semicontinuous on \mathcal{L}_Θ . Consider the map

$$F \mapsto \mathfrak{g}_n(F) := \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g^{(b,c,F)}(y)$$

for $n \in \mathbb{N}$. We deduce from Lemma V.3.2 that $\mathfrak{g}_n(F)$ increases to $\mathfrak{g}(F)$ as $n \rightarrow \infty$. Therefore, it is sufficient to show that \mathfrak{g}_n is lower semicontinuous for fixed n , and for this, in turn, it suffices to show that $F \mapsto g^{(b,c,F)}(y)$ is lower semicontinuous on \mathcal{L}_Θ for fixed $y \in \mathcal{C} \cap \mathcal{C}_n^0$.

To see this, let

$$I_y(z) = \begin{cases} \log(1 + y^\top z) - y^\top h(z) & \text{if } p = 0, \\ p^{-1}(1 + y^\top z)^p - p^{-1} - y^\top h(z) & \text{if } p \neq 0 \end{cases}$$

denote the integrand in the definition of $g^{(b,c,F)}(y)$. Fix a continuous function $\psi_n : \mathbb{R} \rightarrow [0, 1]$ which satisfies $\psi_n(u) = 1$ for $u \geq -1 + \frac{1}{n}$ and $\psi_n(u) = 0$ for $u < -1 + \frac{1}{2n}$. As $F[z \in \mathbb{R}^d \mid y^\top z < -1 + \frac{1}{n}] = 0$ and $\psi_n(y^\top z) = 1$ on $\{z \in \mathbb{R}^d \mid y^\top z \geq -1 + \frac{1}{n}\}$, we see that

$$\int_{\mathbb{R}^d} I_y(z) F(dz) = \int_{\mathbb{R}^d} I_y(z) \psi_n(y^\top z) F(dz) = \int_{\mathbb{R}^d} I_{y,n}(z) F(dz),$$

where we have set $I_{y,n}(z) := I_y(z) \psi_n(y^\top z)$. Thus, it suffices to show that

$$F \mapsto \int_{\mathbb{R}^d} I_{y,n}(z) F(dz)$$

is lower semicontinuous on \mathcal{L}_Θ . Let $F^k \rightarrow F$ be a convergent sequence in \mathcal{L}_Θ . As $h(z) = z$ in a neighborhood of 0 and by the property of ψ_n ,

$$z \mapsto \frac{I_{y,n}(z)}{|z|^2 \wedge 1}$$

is continuous on $\mathbb{R}^d \setminus \{0\}$ and uniformly bounded from below by a constant K . Define on \mathbb{R}^d the function

$$\tilde{I}_{y,n}(z) = \begin{cases} \frac{I_{y,n}(z)}{|z|^2 \wedge 1} & \text{if } z \neq 0, \\ K & \text{if } z = 0. \end{cases}$$

By construction, $z \mapsto \tilde{I}_{y,n}(z)$ is lower semicontinuous and uniformly bounded from below on \mathbb{R}^d . Thus, there exist bounded continuous functions $\tilde{I}_{y,n}^m(z)$ which increase to $\tilde{I}_{y,n}(z)$; cf. [6, Lemma 7.14, p. 147]. For any $F(dz) \in \mathcal{L}$, let $\tilde{F}(dz) := |z|^2 \wedge 1 \cdot F(dz)$ be the finite measure $A \mapsto \int_A |z|^2 \wedge 1 F(dz)$. By the definition of the topology on \mathcal{L} , we have that $F^k \rightarrow F$ if and only if $\tilde{F}^k \rightarrow \tilde{F}$ in the sense of weak convergence. Using that $F(\{0\}) = \tilde{F}(\{0\}) = 0$ for any $F \in \mathcal{L}$ and Fatou's Lemma, we obtain that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^d} I_{y,n}(z) F^k(dz) &= \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^d} \tilde{I}_{y,n}(z) \tilde{F}^k(dz) \\ &\geq \lim_{m \rightarrow \infty} \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^d} \tilde{I}_{y,n}^m(z) \tilde{F}^k(dz) \\ &= \lim_{m \rightarrow \infty} \int_{\mathbb{R}^d} \tilde{I}_{y,n}^m(z) \tilde{F}(dz) \\ &\geq \int_{\mathbb{R}^d} \tilde{I}_{y,n}(z) \tilde{F}(dz) \\ &= \int_{\mathbb{R}^d} I_{y,n}(z) F(dz). \end{aligned}$$

This completes the proof. \square

We can now show the relevant properties of the function $g^\theta(y)$ defined in (2.2).

Proposition V.3.4. *There exists $\hat{y} \in \mathcal{C} \cap \mathcal{C}^0$ such that*

$$\inf_{\theta \in \Theta} g^\theta(\hat{y}) = \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) = \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y).$$

If $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d \times \mathcal{L}$ is compact, there exists $\hat{\theta} \in \Theta$ such that $(\hat{\theta}, \hat{y})$ is a saddle point for the function $g^\theta(y)$ on $\Theta \times \mathcal{C} \cap \mathcal{C}^0$.

Proof. Recall that $\mathcal{C} \cap \mathcal{C}^0$ and Θ are non-empty convex sets and that $\mathcal{C} \cap \mathcal{C}^0$ is compact. For fixed $\theta \in \Theta$, the function $y \mapsto g^\theta(y)$ is concave and upper semicontinuous (Lemma V.3.1), whereas for fixed $y \in \mathcal{C} \cap \mathcal{C}^0$, the function $\theta \mapsto g^\theta(y)$ is convex. Thus, we deduce from Sion's minimax theorem [66, Theorem 4.2] that

$$\inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) = \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y).$$

To be precise, we require an extension of that theorem to functions taking values in $[-\infty, \infty)$; see, e.g., [57, Appendix E.2]. As $y \mapsto \inf_{\theta \in \Theta} g^\theta(y)$ is upper semicontinuous (Lemma V.3.1), we also obtain $\hat{y} \in \mathcal{C} \cap \mathcal{C}^0$ such that

$$\inf_{\theta \in \Theta} g^\theta(\hat{y}) = \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y).$$

Assume that Θ is compact. Then, since $\theta \mapsto \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y)$ is lower semicontinuous (Lemma V.3.3), there exists $\hat{\theta} \in \Theta$ such that

$$\sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\hat{\theta}}(y) = \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y).$$

In view of the above minimax identity, $(\hat{\theta}, \hat{y})$ is a saddle point. \square

Remark V.3.5. For later use, we note that Proposition V.3.4 also holds true with respect to $\mathcal{C} \cap \mathcal{C}_n^0$ instead of $\mathcal{C} \cap \mathcal{C}^0$. Indeed, we may apply the proposition to the modified constraint $\tilde{\mathcal{C}} = \mathcal{C} \cap \mathcal{C}_n^0$ instead of \mathcal{C} .

V.4 Proofs for Logarithmic Utility

In this section we focus on the logarithmic case $p = 0$ and prove Theorems V.2.4 and V.2.5. By scaling, we may assume that the initial capital is $x_0 = 1$, and we recall that Assumption V.2.1 is in force. Because the logarithmic utility turns out to be myopic under our specific setting of model uncertainty, the passage from the local results in the preceding section to the global ones is relatively direct.

Lemma V.4.1. *Let $P \in \mathfrak{P}$ have differential characteristics $\theta^P = (b^P, c^P, F^P)$ and let $\pi \in \mathcal{A}$. Then*

$$E^P[\log(W_T^\pi)] = E^P\left[\int_0^T g^{\theta^P}(\pi_s) ds\right] \in [-\infty, \infty).$$

Proof. Let μ^X be the integer-valued random measure associated with the jumps of X . Under P , the stochastic integral $\int \pi dX$ has the canonical representation

$$\int_0^\cdot \pi_s dX_s = M^c + M^d + \int_0^\cdot \pi_s^\top b_s^P ds + \int_0^\cdot \pi_s dJ_s$$

where M^c and M^d are continuous and purely discontinuous local martingales, respectively, and

$$\begin{aligned} \langle M^c \rangle &= \int_0^\cdot \pi_s^\top c_s^P \pi_s ds, \\ [M^d] &= \int_0^\cdot \int_{\mathbb{R}^d} (\pi_s^\top h(z))^2 \mu^X(dz, ds), \\ J &= \int_0^\cdot \int_{\mathbb{R}^d} (z - h(z)) \mu^X(dz, ds); \end{aligned}$$

cf. [26, Theorem 2.34, p.84] and Proposition III.2.2. We claim that M^c and M^d are true martingales. Indeed, Jensen's inequality and Assumption V.2.1 imply that

$$\begin{aligned} E^P \left[|[M^d]_T|^{1/2} \right] &\leq E^P \left[\int_0^T \int_{\mathbb{R}^d} |\pi_s|^2 |h(z)|^2 \mu^X(dz, ds) \right]^{1/2} \\ &\leq C E^P \left[\int_0^T \int_{\mathbb{R}^d} |z|^2 \wedge 1 F_s^P(dz) ds \right]^{1/2} \\ &\leq C \mathcal{K}^{1/2} T^{1/2}. \end{aligned}$$

for a constant C . Thus, the Burkholder–Davis–Gundy inequalities yield

$$E^P \left[\sup_{0 \leq u \leq T} |M_u^d| \right] \leq C E^P \left[|[M^d]_T|^{1/2} \right] \leq C_{\mathcal{K}} T^{1/2}.$$

Similarly, Assumption V.2.1 also implies that

$$E^P \left[\sup_{0 \leq u \leq T} |M_u^c| \right] \leq C_{\mathcal{K}} T^{1/2}$$

and we conclude that M^c and M^d are true martingales. Recall that $W^\pi > 0$ and $W_-^\pi > 0$ P -a.s. since $\pi \in \mathcal{A}$; cf. [26, Theorem I.4.61, p. 59]. Thus, Itô's formula yields that

$$\begin{aligned} \log(W_T^\pi) &= M_T^c + M_T^d + \int_0^T \pi_s^\top b_s^P ds - \frac{1}{2} \int_0^T \pi_s^\top c_s^P \pi_s ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} [\log(1 + \pi_s^\top z) - \pi_s^\top h(z)] \mu^X(dz, ds). \end{aligned}$$

In view of Assumption V.2.1 and [26, Theorem II.1.8, p. 66], taking expected values yields the result. \square

The next three lemmas constitute the proof of Theorem V.2.4.

Lemma V.4.2. *Let $\hat{y} \in \arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y)$. Then \hat{y} , seen as a constant process, is an element of \mathcal{A} .*

Proof. We need to show that $W^{\hat{y}} > 0$ P -a.s. for all $P \in \mathfrak{P}$, which by [26, Theorem I.4.61, p. 59] is equivalent to $\hat{y} \in \mathcal{C}^{0,*}$, where

$$\mathcal{C}^{0,*} := \bigcap_{F \in \mathcal{L}_\Theta} \{y \in \mathbb{R}^d \mid F[z \in \mathbb{R}^d \mid y^\top z \leq -1] = 0\}.$$

Since $0 \in \mathcal{C} \cap \mathcal{C}^0$ and $g^\theta(0) = 0$ for all $\theta \in \Theta$, we have

$$\inf_{\theta \in \Theta} g^\theta(\hat{y}) \geq \inf_{\theta \in \Theta} g^\theta(0) = 0$$

and in particular $g^\theta(\hat{y}) > -\infty$ for all $\theta \in \Theta$. The claim now follows from the definition of g^θ . \square

Lemma V.4.3. *We have $u(1) < \infty$. Any $\hat{y} \in \arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y)$ satisfies*

$$\inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\hat{y}})] = \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] = \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[\log(W_T^\pi)]$$

and this value is given by $T \inf_{\theta \in \Theta} g^\theta(\hat{y})$.

Proof. Let $\pi \in \mathcal{A}$ and let θ^P denote the differential characteristics of P . Using Lemma V.4.1, we have that

$$\begin{aligned} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] &= \inf_{P \in \mathfrak{P}} E^P \left[\int_0^T g^{\theta_s^P}(\pi_s) ds \right] \\ &\leq \inf_{P \in \mathfrak{P}_L} E^P \left[\int_0^T g^{\theta^P}(\pi_s) ds \right], \\ &\leq \inf_{P \in \mathfrak{P}_L} E^P \left[\int_0^T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\theta^P}(y) ds \right] \\ &= T \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y). \end{aligned} \quad (4.1)$$

By Proposition V.3.4, we have $\inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) = \inf_{\theta \in \Theta} g^\theta(\hat{y})$ and thus

$$\begin{aligned} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] &\leq T \inf_{\theta \in \Theta} g^\theta(\hat{y}) \\ &= \inf_{P \in \mathfrak{P}} E^P \left[\int_0^T \inf_{\theta \in \Theta} g^\theta(\hat{y}) ds \right] \\ &\leq \inf_{P \in \mathfrak{P}} E^P \left[\int_0^T g^{\theta_s^P}(\hat{y}) ds \right]. \\ &= \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\hat{y}})], \end{aligned}$$

where Lemma V.4.1 was again used. As $\pi \in \mathcal{A}$ was arbitrary, we conclude that

$$\sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] \leq T \inf_{\theta \in \Theta} g^\theta(\hat{y}) \leq \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\hat{y}})].$$

But then these inequalities must be equalities, as claimed. In particular, we have that $u(1) = T \inf_{\theta \in \Theta} g^\theta(\hat{y}) < \infty$; cf. Lemma V.3.1.

It remains to prove the minimax identity. To this end, note that

$$\begin{aligned} T \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) &= \inf_{P \in \mathfrak{P}_L} E^P \left[\int_0^T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\theta_s^P}(y) ds \right] \\ &\geq \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}} E^P \left[\int_0^T g^{\theta^P}(\pi_s) ds \right] \\ &\geq \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P \left[\int_0^T g^{\theta_s^P}(\pi_s) ds \right]. \end{aligned}$$

Using also Proposition V.3.4 and Lemma V.4.1, we conclude that

$$\begin{aligned} \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] &= T \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) \\ &\geq \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P \left[\int_0^T g^{\theta_s^P}(\pi_s) ds \right] \\ &= \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[\log(W_T^\pi)]. \end{aligned}$$

The converse inequality is trivial, so the proof is complete. \square

It remains to prove the third assertion of Theorem V.2.4.

Lemma V.4.4. *Any constant $\tilde{\pi} \in \mathcal{A}$ satisfying*

$$\inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\tilde{\pi}})] = \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] \quad (4.2)$$

is an element of $\arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y)$.

Proof. We deduce from Lemma V.4.3, (4.1) and Proposition V.3.4 that

$$\begin{aligned} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\tilde{\pi}})] &\leq \inf_{P \in \mathfrak{P}_L} E^P[\log(W_T^{\tilde{\pi}})] \\ &\leq T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \\ &= \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\tilde{\pi}})]. \end{aligned}$$

Thus, the above inequalities are in fact equalities; in particular,

$$\inf_{P \in \mathfrak{P}_L} E^P[\log(W_T^{\tilde{\pi}})] = T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y).$$

On the other hand, using Lemma V.4.1 and the fact that $\tilde{\pi} \in \mathcal{A}$ is constant,

$$\inf_{P \in \mathfrak{P}_L} E^P[\log(W_T^{\tilde{\pi}})] = \inf_{P \in \mathfrak{P}_L} E^P \left[\int_0^T g^{\theta_s^P}(\tilde{\pi}) ds \right] = T \inf_{\theta \in \Theta} g^\theta(\tilde{\pi}),$$

so it follows that $\tilde{\pi} \in \arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y)$. \square

The remaining two lemmas constitute the proof of the saddle point result, Theorem V.2.5.

Lemma V.4.5. *Assume that Θ is compact. Let $(\hat{\theta}, \hat{y}) \in \Theta \times \mathcal{C} \cap \mathcal{C}^0$ be a saddle point of the function $g^\theta(y)$ and let $\hat{P} \in \mathfrak{P}_L$ be the Lévy law with triplet $\hat{\theta}$. Then $(\hat{P}, \hat{y}) \in \mathfrak{P}_L \times \mathcal{A}$ is a saddle point of $(P, \pi) \mapsto E^P[\log(W_T^\pi)]$ on $\mathfrak{P} \times \mathcal{A}$ and*

$$\sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)] = T g^{\hat{\theta}}(\hat{y}).$$

Proof. We recall that $\hat{y} \in \mathcal{A}$; cf. Lemma V.4.2. Let $\pi \in \mathcal{A}$; then Lemma V.4.1 yields that

$$E^{\hat{P}}[\log(W_T^\pi)] = E^{\hat{P}}\left[\int_0^T g^{\hat{\theta}}(\pi_s) ds\right] \leq T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\hat{\theta}}(y).$$

Using the same lemma again, $T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\hat{\theta}}(y) = T g^{\hat{\theta}}(\hat{y}) = E^{\hat{P}}[\log(W_T^{\hat{y}})]$, and as $\pi \in \mathcal{A}$ was arbitrary, we deduce that

$$\inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[\log(W_T^\pi)] \leq \sup_{\pi \in \mathcal{A}} E^{\hat{P}}[\log(W_T^\pi)] \leq E^{\hat{P}}[\log(W_T^{\hat{y}})]. \quad (4.3)$$

Since $(\hat{\theta}, \hat{y})$ is a saddle point of the function $g^\theta(y)$,

$$E^{\hat{P}}[\log(W_T^{\hat{y}})] = T g^{\hat{\theta}}(\hat{y}) = T \inf_{\theta \in \Theta} g^\theta(\hat{y}).$$

Moreover, using Lemma V.4.1, we have

$$\begin{aligned} T \inf_{\theta \in \Theta} g^\theta(\hat{y}) &= \inf_{P \in \mathfrak{P}} E^P\left[\int_0^T \inf_{\theta \in \Theta} g^\theta(\hat{y}) ds\right] \\ &\leq \inf_{P \in \mathfrak{P}} E^P\left[\int_0^T g^{\theta_s^P}(\hat{y}) ds\right] \\ &= \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\hat{y}})]. \end{aligned}$$

Thus, we obtain that

$$E^{\hat{P}}[\log(W_T^{\hat{y}})] \leq \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\hat{y}})] \leq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[\log(W_T^\pi)]. \quad (4.4)$$

It follows that the inequalities in (4.3) and (4.4) are in fact equalities, and the proof is complete. \square

Lemma V.4.6. *Assume that Θ is compact and let $(\tilde{P}, \tilde{\pi})$ be a saddle point of $(P, \pi) \mapsto E^P[\log(W^\pi)]$ on $\mathfrak{P} \times \mathcal{A}$ with $\tilde{P} \in \mathfrak{P}_L$ and $\tilde{\pi}$ constant. Then $(\tilde{\theta}, \tilde{\pi})$ is a saddle point of the function $g^\theta(y)$ on $\Theta \times \mathcal{C} \cap \mathcal{C}^0$, where $\tilde{\theta}$ is the Lévy triplet of \tilde{P} .*

Proof. As $\tilde{P} \in \mathfrak{P}_L$, we have

$$E^{\tilde{P}}[\log(W_T^{\tilde{\pi}})] = \inf_{P \in \mathfrak{P}} E^P[\log(W_T^{\tilde{\pi}})] \leq \inf_{P \in \mathfrak{P}_L} E^P[\log(W_T^{\tilde{\pi}})] \leq E^{\tilde{P}}[\log(W_T^{\tilde{\pi}})],$$

which implies that the above inequalities are equalities. By Lemma V.4.1, as $\tilde{\pi}$ is constant, we obtain that

$$T g^{\tilde{\theta}}(\tilde{\pi}) = E^{\tilde{P}}[\log(W_T^{\tilde{\pi}})] = \inf_{P \in \mathfrak{P}_L} E^P[\log(W_T^{\tilde{\pi}})] = T \inf_{\theta \in \Theta} g^\theta(\tilde{\pi}). \quad (4.5)$$

Recall from the proof of Lemma V.4.2 that a constant process y is in \mathcal{A} if and only if $y \in \mathcal{C} \cap \mathcal{C}^{0,*}$. As $(\tilde{P}, \tilde{\pi})$ is a saddle point, we deduce that

$$\begin{aligned} E^{\tilde{P}}[\log(W_T^{\tilde{\pi}})] &= \sup_{\pi \in \mathcal{A}} E^{\tilde{P}}[\log(W_T^{\pi})] = \sup_{\pi \in \mathcal{A}} E^{\tilde{P}}\left[\int_0^T g^{\tilde{\theta}}(\pi_s) ds\right] \\ &\geq \sup_{y \in \mathcal{C} \cap \mathcal{C}^{0,*}} E^{\tilde{P}}\left[\int_0^T g^{\tilde{\theta}}(y) ds\right] \\ &\geq E^{\tilde{P}}\left[\int_0^T g^{\tilde{\theta}}(\tilde{\pi}) ds\right] \\ &= E^{\tilde{P}}[\log(W_T^{\tilde{\pi}})]. \end{aligned}$$

Therefore, again, the above inequalities are in fact equalities. In particular,

$$T g^{\tilde{\theta}}(\tilde{\pi}) = E^{\tilde{P}}[\log(W_T^{\tilde{\pi}})] = \sup_{y \in \mathcal{C} \cap \mathcal{C}^{0,*}} E^{\tilde{P}}\left[\int_0^T g^{\tilde{\theta}}(y) ds\right] = T \sup_{y \in \mathcal{C} \cap \mathcal{C}^{0,*}} g^{\tilde{\theta}}(y),$$

and in the last expression we may replace $\mathcal{C} \cap \mathcal{C}^{0,*}$ by its closure $\mathcal{C} \cap \mathcal{C}^0$ since $g^{\tilde{\theta}}$ is concave and proper. Together with (4.5), this shows that $(\tilde{\theta}, \tilde{\pi})$ is a saddle point of the function $g^{\tilde{\theta}}(y)$. \square

V.5 Proofs for Power Utility

In this section, we focus on the case $U(x) = \frac{1}{p}x^p$, where $p \in (-\infty, 0) \cup (0, 1)$. We recall that Assumption V.2.1 is in force and that the initial capital is $x_0 = 1$, without loss of generality.

The arguments for power utility are less direct than in the logarithmic case, because the power utility investor is typically not myopic. Thus, the optimal strategy and expected utility for a fixed P cannot be expressed by the corresponding function g^{θ} (see [41, 43] for the structure in the general case). However, the power utility problem remains tractable when P is a Lévy law, and we shall see that the worst case over all Lévy laws $P \in \mathfrak{P}_L$ already corresponds to the worst case over all $P \in \mathfrak{P}$. The crucial tool to prove that is a martingale argument, contained in Lemma V.5.1 below.

For some of the arguments we need to avoid the singularity of U at zero and the corresponding singularity of g at the boundary of \mathcal{C}^0 . To this end, recall that

$$\mathcal{C}_n^0 = \bigcap_{F \in \mathcal{L}_{\Theta}} \left\{ y \in \mathbb{R}^d \mid F[z \in \mathbb{R}^d \mid y^{\top} z < -1 + \frac{1}{n}] = 0 \right\} \subseteq \mathcal{C}^0$$

for $n \in \mathbb{N}$ and define \mathcal{A}_n as the set of all predictable processes π such that $\pi_t(\omega) \in \mathcal{C} \cap \mathcal{C}_n^0$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$. This implies that $W^{\pi} > 0$ P -a.s. for all $P \in \mathfrak{P}$ and in particular $\pi \in \mathcal{A}$.

Lemma V.5.1. *Let $P \in \mathfrak{P}$ and let $\theta^P = (b^P, c^P, F^P)$ be the corresponding differential characteristics. If $\pi \in \mathcal{A}_n$ for some $n \in \mathbb{N}$, then*

$$M_t := \frac{(W_t^\pi)^p}{\exp\left(p \int_0^t g^{\theta^P}(\pi_s) ds\right)}$$

is real-valued and $M = (M_t)_{t \leq T}$ is a martingale with unit expectation.

If $P \in \mathfrak{P}_L$ and $\pi \in \mathcal{C} \cap \mathcal{C}^0$ is constant, then

$$E^P[U(W_T^\pi)] = \frac{1}{p} \exp\left(pT g^{\theta^P}(\pi)\right) \in [-\infty, \infty). \quad (5.1)$$

Proof. Let $\pi \in \mathcal{A}_n$; then the function $g^{\theta^P}(\pi_s)$ and its integral are finite. Moreover, both W^π and W_-^π are strictly positive; cf. [26, Theorem I.4.61, p. 59]. Thus, the process M is a semimartingale with values in $(0, \infty)$. In particular, its drift rate

$$a^M := b^M + \int (z - h(z)) F^M(dz)$$

is well-defined with values in $(-\infty, \infty]$; cf. [43, Remark 2.3]. Moreover, M is a σ -martingale and true supermartingale as soon as $a^M = 0$; see, e.g., [43, Lemma 2.4].

Set $(b, c, F) = (b^P, c^P, F^P)$ and $Y = (W^\pi)^p$. An application of Itô's formula shows that the drift rate a^Y of Y satisfies

$$\begin{aligned} \frac{a^Y}{Y_-} &= p\pi^\top b + \frac{p(p-1)}{2} \pi^\top c\pi + \int [(1 + \pi^\top z)^p - 1 - p\pi^\top h(z)] F(dz) \\ &= pg^\theta(\pi). \end{aligned}$$

See, e.g., [43, Lemma 3.4] for a similar calculation. Noting that the process $G_t = \exp\left(p \int_0^t g^{\theta^P}(\pi_s) ds\right)$ is continuous and of finite variation, we have

$$dM = G^{-1} dY + Y_- d(G^{-1}) = G^{-1} dY - Y_- G^{-1} pg^\theta(\pi) dt$$

and it follows that $a^M = G^{-1} a^Y - Y_- G^{-1} pg^\theta(\pi) = 0$. As a result, M is a σ -martingale and a supermartingale. To establish that M is a true martingale, it remains to show that M is of class (D).

Consider first the case $p \in (0, 1)$. Let $\varepsilon > 0$ be as in Assumption V.2.1 and let $\tau \leq T$ be a stopping time; we estimate

$$E[|M_\tau|^{1+\varepsilon}] = E\left[\frac{(W_\tau^\pi)^{p(1+\varepsilon)}}{\exp\left(p(1+\varepsilon) \int_0^\tau g^{\theta^P}(\pi_s) ds\right)}\right].$$

Set $\tilde{p} := p(1+\varepsilon)$ and let $g^\theta(\tilde{p}, \pi)$ be defined like $g^\theta(\pi)$ but with p replaced by \tilde{p} . Using the supermartingale property of M with respect to \tilde{p} (which holds

by the same arguments), we obtain that

$$\begin{aligned}
& E \left[\frac{(W_\tau^\pi)^{p(1+\varepsilon)}}{\exp(p(1+\varepsilon) \int_0^\tau g^{\theta_s}(\pi_s) ds)} \right] \\
&= E \left[\frac{(W_\tau^\pi)^{\tilde{p}}}{\exp(\tilde{p} \int_0^\tau g^{\theta_s}(\tilde{p}, \pi_s) ds)} \frac{\exp(\tilde{p} \int_0^\tau g^{\theta_s}(\tilde{p}, \pi_s) ds)}{\exp(\tilde{p} \int_0^\tau g^{\theta_s}(\pi_s) ds)} \right] \\
&\leq \frac{\exp\left(\tilde{p} T C_1 \sup_{(b,c,F) \in \Theta} \left\{ |b| + \frac{|\tilde{p}-1|}{2} |c| + \int |z|^2 \wedge |z|^{\tilde{p}} F(dz) \right\}\right)}{\exp\left(-\tilde{p} T C_2 \sup_{(b,c,F) \in \Theta} \left\{ |b| + \frac{|p-1|}{2} |c| + \int |z|^2 \wedge |z|^p F(dz) \right\}\right)}
\end{aligned}$$

where C_1, C_2 are finite constants depending only on p, \tilde{p} and the diameter of $\mathcal{C} \cap \mathcal{C}^0$. The last line is finite due to Assumption V.2.1 and does not depend on τ . We have shown that

$$\sup_{\tau \leq T} E[|M_\tau|^{1+\varepsilon}] < \infty,$$

so the de la Vallée-Poussin theorem implies that M is of class (D) and in particular a true martingale.

For the case $p < 0$, choose an arbitrary $\varepsilon > 0$ and recall that $\pi \in \mathcal{A}_n$. A similar estimate as above holds for $\tilde{p} = p(1+\varepsilon) < 0$, except that the signs are reversed, the constants C_1, C_2 now depend on the fixed n , and $|z|^2 \wedge |z|^p, |z|^2 \wedge |z|^{\tilde{p}}$ are replaced by $|z|^2 \wedge 1$. The conclusion remains the same.

Finally, let $P \in \mathfrak{P}_L$ and let $\pi \in \mathcal{C} \cap \mathcal{C}^0$ be constant. We observe that $g^\theta(\pi) \in [-\infty, \infty)$, and the value $-\infty$ can occur only if $p < 0$. If $g^\theta(\pi)$ is finite and $W^\pi > 0$ P -a.s., the above arguments still apply and the identity (5.1) follows.

Let $p \in (0, 1)$. We have just seen that (5.1) holds when $\pi \in \mathcal{A}_n$, and then the general case follows by passing to the limit on both sides; cf. Lemma V.5.5 below.

Let $p < 0$. If $P[W_T^\pi = 0] > 0$, then $F[z \in \mathbb{R}^d | y^\top z = -1] > 0$ and both sides of (5.1) equal $-\infty$. If $W^\pi > 0$ P -a.s. but $g^\theta(\pi) = -\infty$, we need to argue that $E^P[Y_T] = \infty$ for $Y = (W^\pi)^p$. Suppose that $E^P[Y_T] < \infty$. Then as Y is the exponential of a Lévy process [29, Lemma 4.2], Y is of class (D) on $[0, T]$ and a special semimartingale [29, Lemma 4.4]. In particular, its drift rate $a^Y = Y_- p g^\theta(\pi)$ has to be finite [26, Proposition II.2.29, p. 82]. This contradicts $g^\theta(\pi) = -\infty$ and completes the proof. \square

In the next two lemmas, we prove our main results for the set \mathcal{A}_n of strategies, where n is fixed. We shall pass to the desired set \mathcal{A} in a later step.

Lemma V.5.2. *Let $\hat{y} \in \arg \max_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \inf_{\theta \in \Theta} g^\theta(y)$; then*

$$\inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})] = \sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] = \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)]$$

and this value is given by $\frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y}))$.

Proof. Let $\pi \in \mathcal{A}_n$. The classical result for power utility maximization in the Lévy setting, see [42, Theorem 3.2], yields that

$$\inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] \leq \inf_{P \in \mathfrak{P}_L} E^P[U(W_T^\pi)] \leq \inf_{P \in \mathfrak{P}_L} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \frac{1}{p} \exp(pT g^{\theta^P}(y)).$$

In view of Proposition V.3.4 and the definition of \hat{y} ,

$$\inf_{P \in \mathfrak{P}_L} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \frac{1}{p} \exp(pT g^{\theta^P}(y)) = \frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})).$$

Moreover, by Lemma V.5.1, we have for any $P \in \mathfrak{P}$ that

$$\begin{aligned} \frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})) &= E^P \left[\frac{U(W_T^{\hat{y}})}{\frac{1}{p} \exp(p \int_0^T g^{\theta^P}(s, \hat{y}) ds)} \right] \frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})) \\ &\leq E^P[U(W_T^{\hat{y}})]. \end{aligned}$$

Since $\pi \in \mathcal{A}_n$ and $P \in \mathfrak{P}$ were arbitrary, we conclude that

$$\sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] \leq \frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})) \leq \inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})].$$

As $\hat{y} \in \mathcal{A}_n$, these inequalities must be equalities.

It remains to prove the minimax identity. By the definition of \hat{y} and the classical result in [42, Theorem 3.2],

$$\begin{aligned} \frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})) &= \inf_{P \in \mathfrak{P}_L} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \frac{1}{p} \exp(pT g^{\theta^P}(y)) \\ &= \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)] \\ &\geq \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)]. \end{aligned}$$

Together with the above, we have

$$\sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] = \frac{1}{p} \exp(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})) \geq \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)],$$

and the converse inequality is clear. \square

Lemma V.5.3. *Assume that Θ is compact. Let $(\hat{\theta}, \hat{y})$ be a saddle point of the function $g^\theta(y)$ on $\Theta \times \mathcal{C} \cap \mathcal{C}_n^0$ and let $\hat{P} \in \mathfrak{P}_L$ be the Lévy law with triplet $\hat{\theta}$. Then $(\hat{P}, \hat{y}) \in \mathfrak{P}_L \times \mathcal{A}_n$ is a saddle point of $(P, \pi) \mapsto E^P[U(W_T^\pi)]$ on $\mathfrak{P} \times \mathcal{A}_n$ and*

$$\sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] = \frac{1}{p} \exp(pT g^{\hat{\theta}}(\hat{y})).$$

Proof. The line of argument is the same as in Lemma V.4.5 for the logarithmic case, except that the use of Lemma V.4.1 needs to be substituted by Lemma V.5.1 and a martingale argument, much like in the preceding proof. We omit the details. \square

Remark V.5.4. For later reference, we record that Lemmas V.5.2 and V.5.3 remain true if \mathfrak{P} is replaced by \mathfrak{P}_L in the assertion.

Our next goal is to obtain the preceding two results for \mathcal{A} and \mathcal{C}^0 rather than the auxiliary sets \mathcal{A}_n and \mathcal{C}_n^0 . This will be achieved by passing to the limit as $n \rightarrow \infty$, for which some preparations are necessary.

Lemma V.5.5. *Let $P \in \mathfrak{P}$ and $\pi \in \mathcal{A}$. Then $\pi_n := (1 - \frac{1}{n})\pi \in \mathcal{A}_n$ and*

$$\limsup_{n \rightarrow \infty} E^P[U(W_T^{\pi_n})] \leq E^P[U(W_T^\pi)].$$

Moreover, if $p \in (0, 1)$, then $U(W_T^{\pi_n}) \rightarrow U(W_T^\pi)$ in $L^1(P)$.

Proof. It is clear that $\pi_n \in \mathcal{A}_n$. Using that $W_T^\pi = \mathcal{E}(\int \pi dX)_T$, standard arguments show that $W_T^{\pi_n}$ converges P -a.s. to W_T^π , and then $U(W_T^{\pi_n})$ converges P -a.s. to $U(W_T^\pi)$. When $p < 0$, we have $U \leq 0$ and the result follows from Fatou's Lemma. For $p \in (0, 1)$, let $\varepsilon > 0$ be as in Assumption V.2.1 and set $\tilde{p} := p(1 + \varepsilon)$. An estimate as in the proof of Lemma V.5.1 yields that

$$E^P[|U(W_T^{\pi_n})|^{1+\varepsilon}] \leq K < \infty$$

for all n , where K is a constant depending on $\tilde{p} := p(1 + \varepsilon)$, the diameter of $\mathcal{C} \cap \mathcal{C}^0$ and \mathcal{K} . Thus, $(U(W_T^{\pi_n}))_{n \in \mathbb{N}}$ is uniformly integrable and the convergence in $L^1(P)$ follows. \square

As we will be using results from the classical utility maximization problem [42], let us comment on a subtlety regarding the class of strategies. Let $P \in \mathfrak{P}$ and denote by \mathcal{A}^P the set of all predictable processes taking values in $\mathcal{C} \cap \mathcal{C}^0$ such that $W^\pi > 0$ P -a.s.; this is the class of admissible strategies in [42] if $\mathcal{C} \cap \mathcal{C}^0$ is used as the constraint set (which is necessarily contained in the natural constraints with respect to P). In the case $p > 0$, we have $\mathcal{A} \supseteq \mathcal{A}^P$ as we did not enforce strict positivity in the definition of \mathcal{A} . On the other hand, in the case $p < 0$, we have required positivity under *all* models in \mathfrak{P} , which results in an inclusion $\mathcal{A} \subseteq \mathcal{A}^P$. For the set $\mathcal{C} \cap \mathcal{C}_n^0$ that has been used above, no such subtleties exist as the wealth process is automatically strictly positive under all models.

Lemma V.5.6. *Let $P \in \mathfrak{P}_L$; then*

$$\sup_{\pi \in \mathcal{A}^P} E^P[U(W_T^\pi)] \geq \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)].$$

Moreover, if $p \in (0, 1)$, we have equality.

Proof. If $p < 0$, the claim is clear as $\mathcal{A} \subseteq \mathcal{A}^P$. Let $p \in (0, 1)$; then $\mathcal{A}^P \subseteq \mathcal{A}$, so it suffices to show the stated inequality. Let $\pi_n = (1 - \frac{1}{n})\pi \in \mathcal{A}_n$ for $\pi \in \mathcal{A}$. Lemma V.5.5 yields that

$$\sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] = \sup_{\pi \in \mathcal{A}} \lim_{n \rightarrow \infty} E^P[U(W_T^{\pi_n})] \leq \lim_{n \rightarrow \infty} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)].$$

Let θ be the Lévy triplet of P . We deduce from [42, Theorem 3.2] and Lemma V.3.2 that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)] &= \lim_{n \rightarrow \infty} \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g^\theta(y)\right) \\ &= \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y)\right) \\ &= \sup_{\pi \in \mathcal{A}^P} E^P[U(W_T^\pi)]. \end{aligned}$$

□

We can now prove the main lemma for the passage from \mathcal{A}_n to \mathcal{A} .

Lemma V.5.7. *We have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] &= \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y)\right) \\ &= \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)]. \end{aligned}$$

In particular, $u(1) < \infty$.

Proof. Since $\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \subseteq \mathcal{A}$, the limit exists and

$$\lim_{n \rightarrow \infty} \sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] \leq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)].$$

On the other hand, for each $n \in \mathbb{N}$, the minimax result of Lemma V.5.2 and Remark V.5.4 yield that

$$\sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] = \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)] = \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)].$$

Applying the classical result of [42, Theorem 3.2] for each $P \in \mathfrak{P}_L$, we have

$$\inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}_n} E^P[U(W_T^\pi)] = \inf_{P \in \mathfrak{P}_L} \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g^{\theta^P}(y)\right).$$

Using the local minimax result of Proposition V.3.4 with respect to $\mathcal{C} \cap \mathcal{C}_n^0$, cf. Remark V.3.5,

$$\inf_{P \in \mathfrak{P}_L} \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} g^{\theta^P}(y)\right) = \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \inf_{\theta \in \Theta} g^\theta(y)\right).$$

By Lemma V.3.2 and, once again, Proposition V.3.4,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{p} \exp \left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \inf_{\theta \in \Theta} g^\theta(y) \right) &= \frac{1}{p} \exp \left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \right) \\ &= \frac{1}{p} \exp \left(pT \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) \right). \end{aligned}$$

We deduce from [42, Theorem 3.2] and Lemma V.5.6 that

$$\begin{aligned} \frac{1}{p} \exp \left(pT \inf_{\theta \in \Theta} \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^\theta(y) \right) &= \inf_{P \in \mathfrak{P}_L} \frac{1}{p} \exp \left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\theta^P}(y) \right) \\ &= \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}^P} E^P[U(W_T^\pi)] \\ &\geq \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)]. \end{aligned}$$

Noting also the trivial inequalities

$$\inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \geq \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \geq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)],$$

we have established that

$$\begin{aligned} \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] &\geq \lim_{n \rightarrow \infty} \sup_{\pi \in \mathcal{A}_n} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] \\ &= \frac{1}{p} \exp \left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \right) \\ &\geq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] \end{aligned}$$

and hence all these expressions are equal. \square

We are now ready to finish the proof of parts (i) and (ii) of Theorem V.2.4.

Lemma V.5.8. *Let $\hat{y} \in \arg \max_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y)$; then*

$$\inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})] = \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] = \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)].$$

Proof. We first note that $\hat{y} \in \mathcal{A}$. This is obvious from the definition of \mathcal{A} for $p > 0$, whereas for $p < 0$ the proof is identical to Lemma V.4.2. As a result,

$$\inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})] \leq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] \leq \inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)].$$

We first prove the converse to the first inequality. By Lemma V.5.7, it suffices to show that

$$\inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})] \geq \frac{1}{p} \exp \left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \right).$$

Indeed, Lemma V.5.5 shows that $\hat{y}_n := (1 - \frac{1}{n})\hat{y} \in \mathcal{A}_n$ satisfies

$$\inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})] \geq \inf_{P \in \mathfrak{P}} \limsup_{n \rightarrow \infty} E^P[U(W_T^{\hat{y}_n})],$$

while Lemma V.5.1 yields

$$\begin{aligned} & \inf_{P \in \mathfrak{P}} \limsup_{n \rightarrow \infty} E^P[U(W_T^{\hat{y}_n})] \\ &= \inf_{P \in \mathfrak{P}} \limsup_{n \rightarrow \infty} E^P \left[\frac{U(W_T^{\hat{y}_n})}{\frac{1}{p} \exp \left(p \int_0^T g^{\theta^P}(\hat{y}_n) ds \right)} \frac{1}{p} \exp \left(p \int_0^T g^{\theta^P}(\hat{y}_n) ds \right) \right] \\ &\geq \limsup_{n \rightarrow \infty} \inf_{\theta \in \Theta} \frac{1}{p} \exp \left(p T g^\theta(\hat{y}_n) \right) \end{aligned}$$

and finally Lemma V.3.2 shows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \inf_{\theta \in \Theta} \frac{1}{p} \exp \left(p T g^\theta(\hat{y}_n) \right) &= \frac{1}{p} \exp \left(p T \limsup_{n \rightarrow \infty} \sup_{y \in \mathcal{C} \cap \mathcal{C}_n^0} \inf_{\theta \in \Theta} g^\theta(y) \right) \\ &= \frac{1}{p} \exp \left(p T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \right), \end{aligned}$$

which proves the desired inequality. It remains to prove that

$$\inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \leq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)].$$

Indeed, by Lemma V.5.7, it suffices to show that

$$\inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \leq \frac{1}{p} \exp \left(p T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \right).$$

We first notice that Lemma V.5.6 implies

$$\inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \leq \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \leq \inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}^P} E^P[U(W_T^\pi)].$$

Using [42, Theorem 3.2], we see that the right-hand side satisfies

$$\inf_{P \in \mathfrak{P}_L} \sup_{\pi \in \mathcal{A}^P} E^P[U(W_T^\pi)] = \inf_{P \in \mathfrak{P}_L} \frac{1}{p} \exp \left(p T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\theta^P}(y) \right),$$

while the local minimax result of Proposition V.3.4 and the definition of \mathfrak{P}_L yield that

$$\inf_{P \in \mathfrak{P}_L} \frac{1}{p} \exp \left(p T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\theta^P}(y) \right) = \frac{1}{p} \exp \left(p T \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} \inf_{\theta \in \Theta} g^\theta(y) \right).$$

This completes the proof. \square

The proof for part (iii) of Theorem V.2.4 is analogous to Lemma V.4.4 for the logarithmic case. We omit the details and proceed with part (i) of Theorem V.2.5.

Lemma V.5.9. *Assume that Θ is compact. Let $(\hat{\theta}, \hat{y}) \in \Theta \times \mathcal{C} \cap \mathcal{C}^0$ be a saddle point of the function $g^\theta(y)$ and let $\hat{P} \in \mathfrak{P}_L$ be the Lévy law with triplet $\hat{\theta}$. Then $(\hat{P}, \hat{y}) \in \mathfrak{P}_L \times \mathcal{A}$ is a saddle point of $(P, \pi) \mapsto E^P[U(W_T^\pi)]$ on $\mathfrak{P} \times \mathcal{A}$ and*

$$\sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)] = \frac{1}{p} \exp\left(pT g^{\hat{\theta}}(\hat{y})\right).$$

Proof. By Lemma V.5.6 and [42, Theorem 3.2], we have

$$\inf_{P \in \mathfrak{P}} \sup_{\pi \in \mathcal{A}} E^P[U(W_T^\pi)] \leq \sup_{\pi \in \mathcal{A}^{\hat{P}}} E^{\hat{P}}[U(W_T^\pi)] = \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\hat{\theta}}(y)\right).$$

Setting $\hat{y}_n = (1 - \frac{1}{n})\hat{y} \in \mathcal{A}_n$, Lemma V.3.2 yields that

$$\begin{aligned} \frac{1}{p} \exp\left(pT \sup_{y \in \mathcal{C} \cap \mathcal{C}^0} g^{\hat{\theta}}(y)\right) &= \frac{1}{p} \exp\left(pT \inf_{\theta \in \Theta} g^\theta(\hat{y})\right) \\ &= \frac{1}{p} \exp\left(pT \liminf_{n \rightarrow \infty} \inf_{\theta \in \Theta} g^\theta(\hat{y}_n)\right) \\ &= \liminf_{n \rightarrow \infty} \inf_{\theta \in \Theta} \frac{1}{p} \exp\left(pT g^\theta(\hat{y}_n)\right), \end{aligned}$$

and we deduce from Lemma V.5.1 that

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \inf_{\theta \in \Theta} \frac{1}{p} \exp\left(pT g^\theta(\hat{y}_n)\right) \\ &= \liminf_{n \rightarrow \infty} \inf_{P \in \mathfrak{P}} \left(E^P \left[\frac{U(W_T^{\hat{y}_n})}{\frac{1}{p} \exp\left(p \int_0^T g^{\theta_s^P}(\hat{y}_n) ds\right)} \right] \inf_{\theta \in \Theta} \frac{1}{p} \exp\left(pT g^\theta(\hat{y}_n)\right) \right) \\ &\leq \liminf_{n \rightarrow \infty} \inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}_n})] \\ &\leq \inf_{P \in \mathfrak{P}} \limsup_{n \rightarrow \infty} E^P[U(W_T^{\hat{y}_n})]. \end{aligned}$$

Since Lemma V.5.5 shows that

$$\inf_{P \in \mathfrak{P}} \limsup_{n \rightarrow \infty} E^P[U(W_T^{\hat{y}_n})] \leq \inf_{P \in \mathfrak{P}} E^P[U(W_T^{\hat{y}})] \leq \sup_{\pi \in \mathcal{A}} \inf_{P \in \mathfrak{P}} E^P[U(W_T^\pi)],$$

all the above inequalities are equalities, and the result follows. \square

Finally, the argument for part (ii) of Theorem V.2.5 is quite similar to Lemma V.4.6 and therefore omitted. This completes the proofs of Theorems V.2.4 and V.2.5 for power utility.

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