Indicator Function

• Let Ω be any space and let $A \subseteq \Omega$ be any subset.

Definition 1. The indicator function $\mathbb{1}_A$ with respect to A is defined by

$$\Omega \ni \omega \mapsto \mathbb{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

• Let $\mathbb{P}$ be a probability measure on Ω and let $\mathbb{E}_{\mathbb{P}}[\cdot]$ denote the expectation.
Then, for any set $A \subseteq \Omega$, we have

$$\mathbb{P}[A] = \mathbb{E}\big[\mathbb{1}_A\big].$$

• Application of indicator function for example for the proof of Markov inequality:

Proposition 2 (Markov inequality)**.** *Let X be a random variable that takes only non-negative values, and let $a > 0$ be a constant. Then*

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Proof. By monotonicity and homogeneity of the expectation, we have that

$$\mathbb{P}[X \geq a] = \mathbb{E}[\mathbb{1}_{\{X \geq a\}}] \leq \mathbb{E}\big[\mathbb{1}_{\{X \geq a\}} \tfrac{X}{a}\big] \leq \mathbb{E}\big[\tfrac{X}{a}\big] = \tfrac{\mathbb{E}[X]}{a}.$$

□