

# MULTILEVEL PICARD APPROXIMATIONS OVERCOME THE CURSE OF DIMENSIONALITY IN THE NUMERICAL APPROXIMATION OF GENERAL SEMILINEAR PDES WITH GRADIENT-DEPENDENT NONLINEARITIES

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ABSTRACT. Neufeld and Wu (arXiv:2310.12545) developed a multilevel Picard (MLP) algorithm which can approximately solve *general* semilinear parabolic PDEs with gradient-dependent nonlinearities, allowing also for coefficient functions of the corresponding PDE to be non-constant. By introducing a particular stochastic fixed-point equation (SFPE) motivated by the Feynman-Kac representation and the Bismut-Elworthy-Li formula and identifying the first and second component of the unique fixed-point of the SFPE with the unique viscosity solution of the PDE and its gradient, they proved convergence of their algorithm. However, it remained an open question whether the proposed MLP schema in arXiv:2310.12545 does not suffer from the curse of dimensionality. In this paper, we prove that the MLP algorithm in arXiv:2310.12545 indeed can overcome the curse of dimensionality, i.e. that its computational complexity only grows polynomially in the dimension  $d \in \mathbb{N}$  and the reciprocal of the accuracy  $\varepsilon$ , under some suitable assumptions on the nonlinear part of the corresponding PDE.

## 1. INTRODUCTION

Partial differential equations (PDEs) are important tools to analyze many real world phenomena, e.g., in financial engineering, economics, quantum mechanics, or statistical physics to name but a few. In most of the cases such high-dimensional nonlinear PDEs cannot be solved explicitly. It is one of the most challenging problems in applied mathematics to approximately solve high-dimensional nonlinear PDEs. In particular, it is very difficult to find approximation schemata for nonlinear PDEs for which one can rigorously prove that they do overcome the so-called *curse of dimensionality* in the sense that the computational complexity only grows polynomially in the space dimension  $d$  of the PDE and the reciprocal  $\frac{1}{\varepsilon}$  of the accuracy  $\varepsilon$ .

In recent years, there are two types of approximation methods which are quite successful in the numerical approximation of solutions of high-dimensional nonlinear PDEs: neural network based approximation methods for PDEs, cf., [1, 2, 3, 4, 7, 9, 10, 11, 13, 14, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 34, 41, 42, 43, 44, 47, 51, 52, 53, 55, 56, 57, 58] and multilevel Monte-Carlo based approximation methods for PDEs, cf., [5, 6, 8, 15, 16, 21, 31, 32, 33, 35, 36, 37, 38, 50].

Neural networks based algorithms are very efficient in practice. However, a rigorous convergence analysis for them is often missing because when training neural networks the corresponding optimization problems are typically non-convex. On the other hand for multilevel Monte-Carlo based algorithms it is often possible to provide a complete convergence and complexity analysis. It has been proven that under some suitable assumptions, e.g., Lipschitz continuity on the linear part, the nonlinear part, and the initial (or terminal) condition function of the PDE under consideration, the multilevel Picard approximation algorithms can overcome the curse of dimensionality in the sense that the number of computational operations of the proposed Monte-Carlo based approximation method grows at most polynomially in both the reciprocal  $\frac{1}{\varepsilon}$  of the prescribed approximation accuracy  $\varepsilon \in (0, 1)$  and the PDE dimension  $d \in \mathbb{N}$ , see [5, 6, 15, 16, 21, 31, 32, 33, 35, 36, 37, 38, 48, 50].

Nevertheless, for semilinear PDEs whose nonlinear part depends also on the gradient the development of numerical schemes as well as their complexity analysis are still at their infancy. In [31, 36] multilevel Picard (MLP) approximation algorithms together with their convergence

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and complexity analysis have been developed for semilinear heat equations with nonlinear parts depending on the gradients of the solutions. Recently, [49] developed an MLP algorithm for *general* semilinear PDEs with gradient-dependent nonlinearities, allowing also for coefficient functions of the corresponding PDE to be non-constant. The main idea of the algorithm in [49] is to introduce a particular stochastic fixed-point equation (SFPE) motivated by the Feynman-Kac representation and the Bismut-Elworthy-Li formula and to identify the first and second component of the unique fixed-point of the SFPE with the unique viscosity solution of the PDE and its gradient, allowing to prove convergence of their algorithm. However, it remained an open question whether the proposed MLP schema in [49] does not suffer from the curse of dimensionality. The main goal of this paper is to prove that the MLP schema in [49] indeed can overcome the curse of dimensionality under some suitable assumptions on the nonlinear part, namely, (1) and (8) in Theorem 1.1 below, which is the main result of our paper.

**1.1. Notations.** Throughout the paper we use the following notations. Let  $\mathbb{R}$  denote the set of all real numbers. Let  $\mathbb{Z}, \mathbb{N}_0, \mathbb{N}$  denote the sets which satisfy that  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ ,  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Let  $D$  denote the total derivative,  $\nabla$  denote the gradient, and  $\text{Hess}$  denote the Hessian matrix. For every matrix  $A$  let  $A^\top$  denote the transpose of  $A$  and let  $\text{trace}(A)$  denote the trace of  $A$  when  $A$  is a square matrix. Let  $\langle \cdot, \cdot \rangle$  denote the standard scalar product on  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ . Let  $B(\cdot, \cdot)$  denote the Beta function. For  $c \in \mathbb{R}$ , for a set  $X$ , and for a real-valued function  $f: X \rightarrow \mathbb{R}$  we write  $c \leq f$  if  $c \leq f(x)$  for all  $x \in X$ . When applying a result we often use a phrase like ‘‘Lemma 3.8 with  $d \curvearrowright (d-1)$ ’’ that should be read as ‘‘Lemma 3.8 applied with  $d$  (in the notation of Lemma 3.8) replaced by  $(d-1)$  (in the current notation)’’ and we often omit a trivial replacement to lighten the notation, e.g., we rarely write, e.g., ‘‘Lemma 3.28 with  $d \curvearrowright d$ ’’.

## 1.2. Main result.

**Theorem 1.1.** *Let  $\Theta = \cup_{n \in \mathbb{N}} \mathbb{Z}^n$ ,  $T \in (0, \infty)$ ,  $\mathbf{k} \in [0, \infty)$ ,  $c \in [1, \infty)$ . Let  $\|\cdot\|: \cup_{k, \ell \in \mathbb{N}} \mathbb{R}^{k \times \ell} \rightarrow [0, \infty)$  satisfy for all  $k, \ell \in \mathbb{N}$ ,  $s = (s_{ij})_{i \in [1, k] \cap \mathbb{N}, j \in [1, \ell] \cap \mathbb{N}} \in \mathbb{R}^{k \times \ell}$  that  $\|s\|^2 = \sum_{i=1}^k \sum_{j=1}^{\ell} |s_{ij}|^2$ . For every  $d \in \mathbb{N}$  let  $(L_i^d)_{i \in [0, d] \cap \mathbb{Z}} \in \mathbb{R}^{d+1}$  satisfy that*

$$\sum_{i=0}^d L_i^d \leq c. \quad (1)$$

*For every  $K \in \mathbb{N}$  let  $[\cdot]_K: \mathbb{R} \rightarrow \mathbb{R}$  satisfy for all  $t \in \mathbb{R}$  that  $[t]_K = \max(\{0, \frac{T}{K}, \frac{2T}{K}, \dots, T\} \cap ((-\infty, t) \cup \{0\}))$ . For every  $d \in \mathbb{N}$  let  $\Lambda^d = (\Lambda_\nu^d)_{\nu \in [0, d] \cap \mathbb{Z}}: [0, T] \rightarrow \mathbb{R}^{1+d}$  satisfy for all  $t \in [0, T]$  that  $\Lambda^d(t) = (1, \sqrt{t}, \dots, \sqrt{t})$ . For every  $d \in \mathbb{N}$  let  $\text{pr}^d = (\text{pr}_\nu^d)_{\nu \in [0, d] \cap \mathbb{Z}}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  satisfy for all  $w = (w_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}$ ,  $i \in [0, d] \cap \mathbb{Z}$  that  $\text{pr}_i^d(w) = w_i$ . For every  $d \in \mathbb{N}$ ,  $k \in [1, d] \cap \mathbb{Z}$  let  $e_k^d \in \mathbb{R}^d$  denote the  $d$ -dimensional vector with a 1 in the  $k$ -th coordinate and 0's elsewhere. For every  $d \in \mathbb{N}$  let  $f_d \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1}, \mathbb{R})$ ,  $g_d \in C(\mathbb{R}^d, \mathbb{R})$ ,  $\mu_d \in C^3(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_d \in C^3(\mathbb{R}^d, \mathbb{R}^{d \times d})$ . To shorten the notation we write for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $w: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  that*

$$(F_d(w))(t, x) = f_d(t, x, w(t, x)). \quad (2)$$

*Assume for all  $d \in \mathbb{N}$ ,  $i \in [0, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $r \in (t, T]$ ,  $x, y, h \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$  that  $\sigma_d$  is invertible,*

$$\|\mu_d(0)\| + \|\sigma_d(0)\| \leq cd^c, \quad (3)$$

$$\max \{ \|((D\mu_d)(x))(h)\|, \|((D\sigma_d)(x))(h)\| \} \leq c\|h\|, \quad (4)$$

$$y^\top \sigma_d(x) (\sigma_d(x))^\top y \geq \frac{1}{c} \|y\|^2, \quad (5)$$

$$\max \{ \|((D\mu_d)(x) - (D\mu_d)(y))(h)\|, \|((D\sigma_d)(x) - (D\sigma_d)(y))(h)\| \} \leq cd^c \|x - y\| \|h\|, \quad (6)$$

$$|g_d(x)| + |T f_d(t, x, 0)| \leq [(cd^c)^2 + c^2 \|x\|^2]^{\frac{1}{2}}, \quad (7)$$

$$|f_d(t, x, w_1) - f_d(t, y, w_2)| \leq \sum_{\nu=0}^d [L_\nu^d \Lambda_\nu^d(T) |\text{pr}_\nu^d(w_1 - w_2)|] + \frac{1}{T} c \frac{\|x - y\|}{\sqrt{T}}, \quad (8)$$

and

$$|g_d(x) - g_d(y)| \leq c \frac{\|x - y\|}{\sqrt{T}}. \quad (9)$$

Let  $\varrho: \{(\tau, \sigma) \in [0, T]^2: \tau < \sigma\} \rightarrow \mathbb{R}$  satisfy for all  $t \in [0, T)$ ,  $s \in (t, T)$  that

$$\varrho(t, s) = \frac{1}{\text{B}(\frac{1}{2}, \frac{1}{2})} \frac{1}{\sqrt{(T-s)(s-t)}}. \quad (10)$$

Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$  be a filtered probability space which satisfies the usual conditions. For every random variable  $\mathfrak{X}: \Omega \rightarrow \mathbb{R}$  let  $\|\mathfrak{X}\|_2 \in [0, \infty]$  satisfy that  $\|\mathfrak{X}\|_2^2 = \mathbb{E}[\|\mathfrak{X}\|^2]$ . Let  $\mathfrak{r}^\theta: \Omega \rightarrow (0, 1)$ ,  $\theta \in \Theta$ , be independent and identically distributed random variables and satisfy for all  $b \in (0, 1)$  that

$$\mathbb{P}(\mathfrak{r}^\theta \leq b) = \frac{1}{\text{B}(\frac{1}{2}, \frac{1}{2})} \int_0^b \frac{dr}{\sqrt{r(1-r)}}. \quad (11)$$

For every  $d \in \mathbb{N}$  let  $W^{d, \theta}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $\theta \in \Theta$ , be independent  $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motions with continuous sample paths. Assume for every  $d \in \mathbb{N}$  that  $(W^{d, \theta})_{\theta \in \Theta}$  and  $(\mathfrak{r}^\theta)_{\theta \in \Theta}$  are independent. For every  $\theta \in \Theta$ ,  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $k \in [1, d] \cap \mathbb{Z}$  let  $(\mathcal{X}_t^{d, \theta, K, s, x})_{t \in [s, T]}$ ,  $(\mathcal{D}_t^{d, \theta, K, s, x, k})_{t \in [s, T]}: [s, T] \times \Omega \rightarrow \mathbb{R}^d$  be  $(\mathbb{F}_t)_{t \in [s, T]}$ -adapted stochastic processes with continuous sample paths which satisfy for all  $t \in [s, T]$  that  $\mathbb{P}$ -a.s. we have that

$$\mathcal{X}_t^{d, \theta, K, s, x} = x + \int_s^t \mu_d(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) dr + \int_s^t \sigma_d(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) dW_r^{d, \theta} \quad (12)$$

and

$$\begin{aligned} \mathcal{D}_t^{d, \theta, K, s, x, k} &= e_k^d + \int_s^t \left( (D\mu_d)(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) \right) \left( \mathcal{D}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x, k} \right) dr \\ &\quad + \int_s^t \left( (D\sigma_d)(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) \right) \left( \mathcal{D}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x, k} \right) dW_r^{d, \theta}. \end{aligned} \quad (13)$$

For every  $\theta \in \Theta$ ,  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$  let  $\mathcal{V}_t^{d, \theta, K, s, x} = (\mathcal{V}_t^{d, \theta, K, s, x, k})_{k \in [1, d] \cap \mathbb{Z}}: \Omega \rightarrow \mathbb{R}^d$ ,  $\mathcal{Z}_t^{d, \theta, K, s, x} = (\mathcal{Z}_t^{d, \theta, K, s, x, k})_{k \in [0, d] \cap \mathbb{Z}}: \Omega \rightarrow \mathbb{R}^{d+1}$  satisfy that

$$\mathcal{V}_t^{d, \theta, K, s, x} = \frac{1}{t-s} \int_s^t \left( \sigma_d^{-1}(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) \mathcal{D}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x} \right)^\top dW_r^{d, \theta} \quad (14)$$

and  $\mathcal{Z}_t^{d, \theta, K, s, x} = (1, \mathcal{V}_t^{d, \theta, K, s, x})$ . Let  $U_{n, m, K}^{d, \theta}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ ,  $d, K \in \mathbb{N}$ ,  $n, m \in \mathbb{Z}$ ,  $\theta \in \Theta$ , satisfy for all  $d, n, m, K \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that  $U_{-1, m, K}^{d, \theta}(t, x) = U_{0, m, K}^{d, \theta}(t, x) = 0$  and

$$\begin{aligned} U_{n, m, K}^{d, \theta}(t, x) &= (g_d(x), 0) + \sum_{i=1}^{m^n} \frac{g_d(\mathcal{X}_T^{d, (\theta, 0, -i), K, t, x}) - g_d(x)}{m^n} \mathcal{Z}_T^{d, (\theta, 0, -i), K, t, x} \\ &\quad + \sum_{\ell=0}^{n-1} \sum_{i=1}^{m^{n-\ell}} \frac{\left( F_d(U_{\ell, m, K}^{d, (\theta, \ell, i)}) - \mathbb{1}_{\mathbb{N}}(\ell) F_d(U_{\ell-1, m, K}^{d, (\theta, \ell, -i)}) \right) \left( t + (T-t)\mathfrak{r}^{(\theta, \ell, i)}, \mathcal{X}_{t+(T-t)\mathfrak{r}^{(\theta, \ell, i)}}^{d, (\theta, \ell, i), K, t, x} \right) \mathcal{Z}_{t+(T-t)\mathfrak{r}^{(\theta, \ell, i)}}^{d, (\theta, \ell, i), K, t, x}}{m^{n-\ell} \varrho(t, t + (T-t)\mathfrak{r}^{(\theta, \ell, i)})}. \end{aligned} \quad (15)$$

For every  $d \in \mathbb{N}$  let  $\mathfrak{e}_d, \mathfrak{f}_d, \mathfrak{g}_d \in [0, \infty)$  satisfy that

$$\mathfrak{e}_d + \mathfrak{f}_d + \mathfrak{g}_d \leq cd^c. \quad (16)$$

Let  $\mathfrak{C}_{n, m, K}^d \in [0, \infty)$ ,  $n, m \in \mathbb{Z}$ ,  $d, K \in \mathbb{N}$ , satisfy for all  $n \in \mathbb{Z}$ ,  $d, m, K \in \mathbb{N}$  that

$$\mathfrak{C}_{n, m, K}^d \leq m^n (K \mathfrak{e}_d + \mathfrak{g}_d) \mathbb{1}_{\mathbb{N}}(n) + \sum_{\ell=0}^{n-1} \left[ m^{n-\ell} (K \mathfrak{e}_d + \mathfrak{f}_d + \mathfrak{C}_{\ell, m, K}^d + \mathfrak{C}_{\ell-1, m, K}^d) \right]. \quad (17)$$

Then the following items hold.

- (i) For all  $d \in \mathbb{N}$  there exists a unique continuous function  $u_d: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  such that  $v_d := \text{pr}_0^d(u_d)$  is the unique viscosity solution to the following semilinear PDE of parabolic type:

$$\frac{\partial v_d}{\partial t}(t, x) + \langle (\nabla_x v_d)(t, x), \mu_d(x) \rangle + \frac{1}{2} \text{trace}(\sigma_d(x) [\sigma_d(x)]^\top (\text{Hess}_x v_d)(t, x)) + f_d(t, x, v_d(t, x), (\nabla_x v_d)(t, x)) = 0 \quad \forall t \in (0, T), x \in \mathbb{R}^d, \quad (18)$$

$$v_d(T, x) = g_d(x) \quad \forall x \in \mathbb{R}^d \quad (19)$$

and such that  $\nabla_x v_d = (\text{pr}_1^d(u_d), \text{pr}_2^d(u_d), \dots, \text{pr}_d^d(u_d))$ .

- (ii) For all  $d \in \mathbb{N}$  we have that

$$\limsup_{n \rightarrow \infty} \sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, n^{\frac{1}{3}}, n^{\frac{2}{3}}}^{d,0}(t, x) - u_d(t, x) \right) \right\|_2 \right] = 0. \quad (20)$$

- (iii) There exist  $(C_\delta)_{\delta \in (0,1)} \subseteq (0, \infty)$ ,  $\eta \in (0, \infty)$ ,  $(N_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1)} \subseteq \mathbb{N}$  such that for all  $d \in \mathbb{N}$ ,  $\delta, \varepsilon \in (0, 1)$  we have that

$$\sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{2}{3}}}^{d,0}(t, x) - u_d(t, x) \right) \right\|_2 \right] < \varepsilon \quad (21)$$

and

$$\mathfrak{C}_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{2}{3}}}^d \leq C_\delta \varepsilon^{-(4+\delta)} \eta d^\eta. \quad (22)$$

**Remark 1.2.** The conditions  $\mu_d \in C^3(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_d \in C^3(\mathbb{R}^d, \mathbb{R}^{d \times d})$  are only needed to ensure the existence and uniqueness of the viscosity solution of the PDE (18)-(19) satisfying a stochastic representation of Feynman-Kac and Bismut-Elworthy-Li type, see [49, Proposition 5.1, Proposition 5.2]. Later, in the recent work [40] (see also their accompanying paper [54]), the authors only assume that  $\mu_d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_d \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$  in order to obtain the analog result as [49, Proposition 5.1, Proposition 5.2], see [40, Theorem 1.1]. However it comes with the price that in [40, Theorem 1.1] they need instead to additionally assume that the nonlinear term  $f_d$  and the terminal condition  $g_d$  satisfy that  $f_d \in L^2([0, T] \times \mathbb{R}^{2d+1}, \mathbb{R})$  and  $g_d \in L^2(\mathbb{R}^d, \mathbb{R})$ , which is not satisfied, for example, by PDEs with (piece-wise) linear  $f_d$  and/or  $g_d$ .

Hence, while we have decided to impose our conditions  $\mu_d \in C^3(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_d \in C^3(\mathbb{R}^d, \mathbb{R}^{d \times d})$  so that we can directly apply [49, Proposition 5.1, Proposition 5.2], one could instead use the conditions  $\mu_d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_d \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$  together with additionally assuming that  $f_d \in L^2([0, T] \times \mathbb{R}^{2d+1}, \mathbb{R})$  and  $g_d \in L^2(\mathbb{R}^d, \mathbb{R})$  and then use [40, Theorem 1.1] instead. Of course in that case, all the remaining conditions imposed in our Theorem 1.1, such as, e.g., (8) combined with (1), would need to be kept in order to guarantee that our Theorem 1.1 and its proof remain valid.

**Remark 1.3.** Note that in the MLP approximation (15) we employ random variables  $\mathfrak{r}^\theta$  with Beta distribution, whereas uniform random variables are usually used in the MLP approximations for PDEs without gradient-dependent nonlinearities. (see, e.g., [32, 33, 35, 38, 50].) The MLP approximation (15) is based on the Bismut-Elworthy-Li formula (see [49, Equation (5.3)]), and the key point of approximating the time-dependent integral is to deal with the singular integral arising in the second term on the right hand side of [49, Equation (5.3)]. More precisely, we see, e.g., in (177) and (178) that we would lose integrability of the corresponding integrals if we had chosen random variables  $\mathfrak{r}^\theta$  being uniformly distributed. Similarly, we see for the choice of  $\mathfrak{r}^\theta$  satisfying  $\mathbb{P}(\mathfrak{r}^\theta \leq b) = \sqrt{b}$  for all  $b \in (0, 1)$  as in [31] that we would lose the integrability, e.g., in (178).

Theorem 1.1 follows directly from Theorem 6.1 (see the proof of Theorem 1.1 after the proof of Theorem 6.1 in Section 6). Let us make some comments on the mathematical objects in

**Theorem 1.1.** For every  $d \in \mathbb{N}$  we want to approximately solve the PDE (18) with terminal condition (19) where  $(\mu_d, \sigma_d)$  is the linear part,  $f_d$  is the nonlinear part, and  $g_d$  is the terminal condition. To make sure that for all  $d \in \mathbb{N}$ , (18)–(19) have a unique viscosity solution we need (3)–(9) which are Lipschitz and linear growth conditions. Here, (3) tells us that the initial values of  $\mu_d, \sigma_d$  grow at most polynomially in  $d$ . Next, (4) assumes that the operator norms of  $(D\mu_d)(x)$  and  $(D\sigma_d)(x)$  are uniformly bounded by  $c$ . Moreover, (5) gives a lower bound condition on  $\sigma_d$  and (6) is a Lipschitz condition of  $D\mu_d$  and  $D\sigma_d$  with respect to the operator norm. In addition, (7) is a growth condition on  $g_d$  and the initial values of  $f_d$  and (8) as well as (9) are Lipschitz condition of  $f_d$  and  $g_d$ . Note that the factors  $T$ ,  $\sqrt{T}$ , and  $\Lambda_\nu^d(T)$ ,  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $d \in \mathbb{N}$ , in (8) as well as (9) are only constants depending on  $T$  which are introduced to later simplify the calculations. To approximate the exact solutions and its derivatives we introduce the MLP approximation in (15) based on Euler-Maruyama schemata (11)–(14) that approximate the forward processes. The motivation for (15) as well as (11)–(14) is the so-called Bismut-Elworthy-Li formula (see the discussion in the introduction of [49]). To describe the computational complexity, for each  $d \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$ ,  $M \in \mathbb{N}$  we introduce  $\mathfrak{c}_{n,m,K}^d \in \mathbb{N}$  to denote the sum of: the number of function evaluations of  $g_d$ , the number of function evaluations of  $(\mu_d, (D\mu_d))$ , the number of function evaluations of  $(\sigma_d, (D\sigma_d), \sigma_d^{-1})$ , and the number of realizations of scalar random variables used to obtain one realization of the MLP approximation algorithm in (15). Moreover, for each  $d \in \mathbb{N}$  we use  $\mathfrak{g}_d$  to denote the number of function evaluations of  $g_d$ , we use  $\mathfrak{f}_d$  to denote the number of function evaluations of  $f_d$ , and we use  $\mathfrak{e}_d$  to denote the sum of: the number of realizations of scalar random variables generated, the number of function evaluations of  $(\mu_d, (D\mu_d))$ , and the number of function evaluations of  $(\sigma_d, (D\sigma_d), \sigma_d^{-1})$ . Here, we count the number of evaluations of  $(\sigma_d)^{-1}$  since in practice we often calculate and save  $(\sigma_d)^{-1}$  first before we start with multilevel Picard approximations. Assumption (16) implies that the computational effort to evaluate the input functions to (18)–(19) grows only polynomially in the dimension  $d$  which is a reasonable assumption for MLP approximation (15) to overcome the curse of dimensionality when approximately solving (18)–(19). We highlight that condition (8) combined with (1) is stronger than condition (2.2) in [49], which allows the numerical schema to overcome the curse of dimensionality.

Our paper is organized as follows. In Section 2 we study existence, uniqueness, and the spatial and temporal regularity of solutions to SFPEs. Section 3 is a perturbation result that estimates the difference between two fixed points. Section 4 provides an abstract framework for the study of MLP approximations. Section 5 contains some results for solutions to stochastic differential equations (SDEs) and their discretizations with explicit constants independent of the dimension  $d$ . Section 6 combines the results in Sections 3–5 to prove the main result, Theorem 1.1 above.

## 2. STOCHASTIC FIXED POINT EQUATIONS

Lemmata 2.1–2.3 are some simple but useful auxiliary results.

### 2.1. A Grönwall-type inequality.

**Lemma 2.1.** *Let  $T \in (0, \infty)$ . Then for all  $t \in [0, T)$  we have that  $\int_t^T \frac{dr}{\sqrt{(T-r)(r-t)}} = B(\frac{1}{2}, \frac{1}{2}) \leq 4$ .*

*Proof of Lemma 2.1.* The substitution  $s = \frac{r-t}{T-t}$ ,  $ds = \frac{dr}{T-t}$  shows for all  $t \in [0, T)$  that

$$\begin{aligned} \int_t^T \frac{dr}{\sqrt{(T-r)(r-t)}} &= \int_t^T \frac{dr}{(T-r)^{\frac{1}{2}}(r-t)^{\frac{1}{2}}} = \int_0^1 \frac{(T-t)ds}{[(1-s)(T-t)]^{\frac{1}{2}}[s(T-t)]^{\frac{1}{2}}} \\ &= \int_0^1 \frac{ds}{[s(1-s)]^{\frac{1}{2}}} = B(\frac{1}{2}, \frac{1}{2}) \leq 4. \end{aligned} \tag{23}$$

This completes the proof of Lemma 2.1. □

**Lemma 2.2.** Let  $T \in (0, \infty)$ ,  $p \in (1, 2)$ ,  $q \in (2, \infty)$  satisfy that  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $H: [0, T) \rightarrow [0, \infty)$  be measurable. Then for all  $t \in [0, T)$  we have that

$$T \int_t^T \frac{H(r) dr}{\sqrt{(T-r)(r-t)}} \leq T^{\frac{1}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{1}{p}} \left( \int_t^T |H(r)|^q dr \right)^{\frac{1}{q}}. \quad (24)$$

*Proof of Lemma 2.2.* Hölder's inequality and the substitution  $s = \frac{r-t}{T-t}$ ,  $ds = \frac{dr}{T-t}$  imply that for all  $t \in [0, T)$  we have that

$$\begin{aligned} T \int_t^T \frac{H(r) dr}{\sqrt{(T-r)(r-t)}} &\leq T \left( \int_t^T \frac{dr}{(T-r)^{\frac{p}{2}}(r-t)^{\frac{p}{2}}} \right)^{\frac{1}{p}} \left( \int_t^T |H(r)|^q dr \right)^{\frac{1}{q}} \\ &= T \left( \int_0^1 \frac{(T-t)ds}{[(1-s)(T-t)]^{\frac{p}{2}}[s(T-t)]^{\frac{p}{2}}} \right)^{\frac{1}{p}} \left( \int_t^T |H(r)|^q dr \right)^{\frac{1}{q}} \\ &\leq T(T-t)^{\frac{1}{p}-1} \left( \int_0^1 \frac{ds}{[s(1-s)]^{\frac{p}{2}}} \right)^{\frac{1}{p}} \left( \int_t^T |H(r)|^q dr \right)^{\frac{1}{q}} \\ &\leq T^{\frac{1}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{1}{p}} \left( \int_t^T |H(r)|^q dr \right)^{\frac{1}{q}}. \end{aligned} \quad (25)$$

This completes the proof of Lemma 2.2.  $\square$

**Lemma 2.3** (Grönwall-type inequality). Let  $T \in (0, \infty)$ ,  $a, b \in [0, \infty)$ ,  $p \in (1, 2)$ ,  $q \in (2, \infty)$  satisfy that  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $H: [0, T) \rightarrow [0, \infty)$  be measurable. Assume for all  $t \in [0, T)$  that  $\int_0^T |H(s)|^q ds < \infty$  and

$$H(t) \leq a + bT \int_t^T \frac{H(r) dr}{\sqrt{(T-r)(r-t)}}. \quad (26)$$

Then for all  $t \in [0, T)$  we have that

$$H(t) \leq 2^{\frac{q-1}{q}} a \exp \left( \frac{2^{q-1} |b|^q T^{\frac{q}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{q}{p}}}{q} (T-t) \right). \quad (27)$$

*Proof of Lemma 2.3.* Lemma 2.2 and (26) prove for all  $t \in [0, T)$  that

$$H(t) \leq a + bT^{\frac{1}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{1}{p}} \left( \int_t^T |H(r)|^q dr \right)^{\frac{1}{q}}. \quad (28)$$

This and the fact that  $\forall x, y \in \mathbb{R}: |x+y|^q \leq 2^{q-1}|x|^q + 2^{q-1}|y|^q$  show for all  $t \in [0, T)$  that

$$|H(t)|^q \leq 2^{q-1}|a|^q + 2^{q-1}|b|^q T^{\frac{q}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{q}{p}} \int_t^T |H(r)|^q dr. \quad (29)$$

Hence, the fact that  $\int_0^T |H(s)|^q ds < \infty$  and Grönwall's lemma imply for all  $t \in [0, T)$  that

$$|H(t)|^q \leq 2^{q-1}|a|^q \exp \left( 2^{q-1} |b|^q T^{\frac{q}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{q}{p}} (T-t) \right). \quad (30)$$

This proves for all  $t \in [0, T)$  that

$$H(t) \leq 2^{\frac{q-1}{q}} a \exp \left( \frac{2^{q-1} |b|^q T^{\frac{q}{p}} \left( B(1 - \frac{p}{2}, 1 - \frac{p}{2}) \right)^{\frac{q}{p}}}{q} (T-t) \right). \quad (31)$$

The proof of Lemma 2.3 is thus completed.  $\square$

**Corollary 2.4.** Let  $T \in (0, \infty)$ ,  $a, b \in [0, \infty)$ . Let  $H: [0, T) \rightarrow [0, \infty)$  be measurable. Assume for all  $t \in [0, T)$  that  $\sup_{s \in [0, T)} |H(s)| < \infty$  and  $H(t) \leq a + bT \int_t^T \frac{H(r) dr}{\sqrt{(T-r)(r-t)}}$ . Then we have for all  $t \in [0, T)$  that  $H(t) \leq 2ae^{86b^3 T^2 (T-t)}$ .

*Proof of Corollary 2.4.* Lemma 2.3 (with  $p \curvearrowright \frac{3}{2}$ ,  $q \curvearrowright 3$  in the notation of Lemma 2.3) and the fact that  $\sup_{s \in [0, T]} |H(s)| < \infty$  show for all  $t \in [0, T]$  that

$$\begin{aligned} H(t) &\leq 2^{\frac{3-1}{3}} a \exp \left( \frac{2^{3-1} |b|^3 T^2 \left( B(1 - \frac{3}{4}, 1 - \frac{3}{4}) \right)^2}{3} (T - t) \right) \\ &\leq 2a \exp \left( \frac{4|b|^3 T^2 \delta^2}{3} (T - t) \right) \\ &\leq 2ae^{86b^3 T^2 (T-t)}. \end{aligned} \quad (32)$$

This completes the proof of Corollary 2.4.  $\square$

**2.2. Existence and uniqueness of solutions to SFPEs.** In Lemma 2.6 below we establish existence, uniqueness, and a growth property of solutions to SFPEs.

**Setting 2.5.** Let  $T \in (0, \infty)$ ,  $d \in \mathbb{N}$ ,  $p_\nu, p_z \in (1, \infty)$ ,  $c \in [1, \infty)$ ,  $(L_i)_{i \in [0, d] \cap \mathbb{Z}} \in \mathbb{R}^{d+1}$  satisfy that  $\sum_{i=0}^d L_i \leq c$  and  $\frac{1}{p_\nu} + \frac{1}{p_z} \leq 1$ . Let  $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$  be a norm on  $\mathbb{R}^d$ . Let  $\Lambda = (\Lambda_i)_{i \in [0, d] \cap \mathbb{Z}}: [0, T] \rightarrow \mathbb{R}^{1+d}$  satisfy for all  $t \in [0, T]$  that  $\Lambda(t) = (1, \sqrt{t}, \dots, \sqrt{t})$ . Let  $\text{pr} = (\text{pr}_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  satisfy for all  $w = (w_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}$ ,  $i \in [0, d] \cap \mathbb{Z}$  that  $\text{pr}_i(w) = w_i$ . Let  $f \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1}, \mathbb{R})$ ,  $g \in C(\mathbb{R}^d, \mathbb{R})$ ,  $V \in C([0, T] \times \mathbb{R}^d, [0, \infty))$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. For every random variable  $\mathfrak{X}: \Omega \rightarrow \mathbb{R}$ ,  $s \in [1, \infty)$  let  $\|\mathfrak{X}\|_s \in [0, \infty]$  satisfy that  $\|\mathfrak{X}\|_s = (\mathbb{E}[|\mathfrak{X}|^s])^{\frac{1}{s}}$ . Let  $(X_t^{s,x})_{s \in [0, T], t \in [s, T], x \in \mathbb{R}^d}: \{(\tau, \sigma) \in [0, T]^2: \tau \leq \sigma\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ ,  $(Z_t^{s,x})_{s \in [0, T], t \in [s, T], x \in \mathbb{R}^d}: \{(\tau, \sigma) \in [0, T]^2: \tau < \sigma\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^{d+1}$  be measurable. Assume for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $r \in (t, T]$ ,  $x \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$  that

$$|g(x)| \leq V(T, x), \quad |Tf(t, x, 0)| \leq V(t, x), \quad (33)$$

$$|f(t, x, w_1) - f(t, x, w_2)| \leq \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) |\text{pr}_\nu(w_1 - w_2)|], \quad (34)$$

$$\|V(r, X_r^{t,x})\|_{p_\nu} \leq V(t, x), \quad \|\text{pr}_i(Z_r^{t,x})\|_{p_z} \leq \frac{c}{\Lambda_i(r-t)}. \quad (35)$$

**Lemma 2.6.** Assume Setting 2.5. Then the following items hold.

(i) There exists a unique measurable function  $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  such that for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T], \xi \in \mathbb{R}^d} \left[ \Lambda_\nu(T - \tau) \frac{|\text{pr}_\nu(u(\tau, \xi))|}{V(\tau, \xi)} \right] < \infty, \quad (36)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E}[|g(X_T^{t,x}) \text{pr}_\nu(Z_T^{t,x})|] + \int_t^T \mathbb{E}[|f(r, X_r^{t,x}, u(r, X_r^{t,x})) \text{pr}_\nu(Z_r^{t,x})|] dr \right] < \infty, \quad (37)$$

and

$$u(t, x) = \mathbb{E}[g(X_T^{t,x}) Z_T^{t,x}] + \int_t^T \mathbb{E}[f(r, X_r^{t,x}, u(r, X_r^{t,x})) Z_r^{t,x}] dr. \quad (38)$$

(ii) For all  $t \in [0, T]$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T - t) \frac{|\text{pr}_\nu(u(t, y))|}{V(t, y)} \right] \leq 6ce^{86c^6 T^2 (T-t)}. \quad (39)$$

*Proof of Lemma 2.6.* Denote by  $M$  the set of all measurable functions  $w: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  and by  $B \subseteq M$  the set which satisfies that

$$B = \left\{ w \in M: \sup_{t \in [0, T], x \in \mathbb{R}^d} \max_{\nu \in [0, d] \cap \mathbb{Z}} \frac{|\Lambda_\nu(T - t) \text{pr}_\nu(w(t, x))|}{V(t, x)} < \infty \right\}. \quad (40)$$

For every  $\lambda \in [0, \infty)$  let  $\|\cdot\|_\lambda : M \rightarrow [0, \infty]$  satisfy for all  $w \in M$  that

$$\|w\|_\lambda = \sup_{t \in [0, T], x \in \mathbb{R}^d} \max_{\nu \in [0, d] \cap \mathbb{Z}} \frac{e^{\lambda t} |\Lambda_\nu(T-t) \text{pr}_\nu(w(t, x))|}{V(t, x)}. \quad (41)$$

Then it is easy to show for all  $\lambda \in [0, \infty)$  that  $(B, \|\cdot\|_\lambda \upharpoonright_B)$  is an  $\mathbb{R}$ -Banach space.

Next, Hölder's inequality, (33), and (35) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \|\Lambda_i(T-t)g(X_T^{t,x})\text{pr}_i(Z_T^{t,x})\|_1 &\leq \|\Lambda_i(T-t)V(T, X_T^{t,x})\text{pr}_i(Z_T^{t,x})\|_1 \\ &\leq \Lambda_i(T-t) \|V(T, X_T^{t,x})\|_{p_\nu} \|\text{pr}_i(Z_T^{t,x})\|_{p_z} \\ &\leq \Lambda_i(T-t)V(t, x) \frac{c}{\Lambda_i(T-t)} \\ &\leq cV(t, x). \end{aligned} \quad (42)$$

In addition, Hölder's inequality, the fact that  $\frac{1}{p_\nu} + \frac{1}{p_z} \leq 1$ , (33), (35), and the fact that  $\forall t \in [0, T)$ :  $\int_t^T \frac{dr}{\sqrt{r-t}} = 2\sqrt{r-t}|_{r=t}^T = 2\sqrt{T-t}$  show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \int_t^T \|\Lambda_i(T-t)f(r, X_r^{t,x}, 0)\text{pr}_i(Z_r^{t,x})\|_1 dr &\leq \int_t^T \left\| \Lambda_i(T-t) \frac{1}{T} V(r, X_r^{t,x}) \text{pr}_i(Z_r^{t,x}) \right\|_1 dr \\ &\leq \int_t^T \Lambda_i(T-t) \frac{1}{T} \|V(r, X_r^{t,x})\|_{p_\nu} \|\text{pr}_i(Z_r^{t,x})\|_{p_z} dr \\ &\leq \int_t^T \Lambda_i(T-t) \frac{1}{T} V(t, x) \frac{c}{\Lambda_i(r-t)} dr \\ &\leq \int_t^T \frac{c}{T} V(t, x) \frac{\sqrt{T-t}}{\sqrt{r-t}} dr \\ &\leq 2cV(t, x). \end{aligned} \quad (43)$$

Next, Hölder's inequality, the fact that  $\frac{1}{p_\nu} + \frac{1}{p_z} \leq 1$ , (35), and Lemma 2.1 imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $w \in B$  that

$$\begin{aligned} &\int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) \|\text{pr}_\nu(w(r, X_r^{t,x}))\text{pr}_i(Z_r^{t,x})\|_1] dr \\ &\leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \|\text{pr}_\nu(w(r, X_r^{t,x}))\text{pr}_i(Z_r^{t,x})\|_1 \right] dr \\ &\leq \int_t^T \Lambda_i(T-t) \frac{c\sqrt{T}}{\sqrt{T-r}} \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(w(r, y))|}{V(r, y)} \right] \|V(r, X_r^{t,x})\text{pr}_i(Z_r^{t,x})\|_1 dr \\ &\leq \int_t^T \Lambda_i(T-t) \frac{c\sqrt{T}}{\sqrt{T-r}} \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-\tau) |\text{pr}_\nu(w(\tau, y))|}{V(\tau, y)} \right] \|V(r, X_r^{t,x})\|_{p_\nu} \|\text{pr}_i(Z_r^{t,x})\|_{p_z} dr \\ &\leq \int_t^T \Lambda_i(T-t) \frac{c\sqrt{T}}{\sqrt{T-r}} \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-\tau) |\text{pr}_\nu(w(\tau, y))|}{V(\tau, y)} \right] V(t, x) \frac{c}{\Lambda_i(r-t)} dr \\ &\leq \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-\tau) |\text{pr}_\nu(w(\tau, y))|}{V(\tau, y)} \right] V(t, x) \int_t^T \frac{c^2 \sqrt{T}}{\sqrt{T-r}} \frac{\Lambda_i(T-t)}{\Lambda_i(r-t)} dr \\ &\leq \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-\tau) |\text{pr}_\nu(w(\tau, y))|}{V(\tau, y)} \right] V(t, x) \int_t^T \frac{c^2 \sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t}}{\sqrt{r-t}} dr < \infty. \end{aligned} \quad (44)$$

This, the triangle inequality, (34), and (43) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $w \in B$  that

$$\begin{aligned} & \int_t^T \Lambda_i(T-t) \|f(r, X_r^{t,x}, w(r, X_r^{t,x})) \text{pr}_i(Z_r^{t,x})\|_1 dr \\ & \leq \int_t^T \|\Lambda_i(T-t) f(r, X_r^{t,x}, 0) \text{pr}_i(Z_r^{t,x})\|_1 dr \\ & \quad + \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) \|\text{pr}_\nu(w(r, X_r^{t,x})) \text{pr}_i(Z_r^{t,x})\|_1] dr < \infty. \end{aligned} \quad (45)$$

Let  $\Phi: B \rightarrow (\mathbb{R}^{d+1})^{[0,T) \times \mathbb{R}^d}$  satisfy for all  $w \in B$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$(\Phi(w))(t, x) = \mathbb{E}[g(X_T^{t,x}) Z_T^{t,x}] + \int_t^T \mathbb{E}[f(r, X_r^{t,x}, w(r, X_r^{t,x})) Z_r^{t,x}] dr, \quad (46)$$

where the expectations are well-defined due to (45) and (42). Moreover, Fubini's theorem implies for all  $w \in B$  that  $\Phi(w) \in M$ . Next, (46), the triangle inequality, (42), and (43) show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \Lambda_i(T-t) |\text{pr}_i((\Phi(0))(t, x))| & \leq \Lambda_i(T-t) \|g(X_T^{t,x}) Z_T^{t,x}\|_1 + \int_t^T \Lambda_i(T-t) \|f(r, X_r^{t,x}, 0) Z_r^{t,x}\|_1 dr \\ & \leq cV(t, x) + 2cV(t, x) = 3cV(t, x). \end{aligned} \quad (47)$$

This implies for all  $\lambda \in [0, \infty)$  that

$$\|\Phi(0)\|_\lambda \leq 3ce^{\lambda T} < \infty. \quad (48)$$

Moreover, (46), (34), Hölder's inequality, the fact that  $\frac{1}{p_\nu} + \frac{1}{p_z} \leq 1$ , and (35) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $w, v \in B$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $\lambda \in [0, \infty)$  that

$$\begin{aligned} & |\Lambda_i(T-t) \text{pr}_i((\Phi(w))(t, x) - (\Phi(v))(t, x))| \\ & = \left| \Lambda_i(T-t) \int_t^T \mathbb{E}[(f(r, X_r^{t,x}, w(r, X_r^{t,x})) - f(r, X_r^{t,x}, v(r, X_r^{t,x}))) \text{pr}_i(Z_r^{t,x})] dr \right| \\ & \leq \Lambda_i(T-t) \int_t^T \|\|(f(r, X_r^{t,x}, w(r, X_r^{t,x})) - f(r, X_r^{t,x}, v(r, X_r^{t,x})))\| \text{pr}_i(Z_r^{t,x})\|_1 dr \\ & \leq \Lambda_i(T-t) \int_t^T \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(w(r, X_r^{t,x}) - v(r, X_r^{t,x}))| |\text{pr}_i(Z_r^{t,x})| \right\|_1 dr \\ & \leq \int_t^T \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \|\text{pr}_\nu(w(r, X_r^{t,x}) - v(r, X_r^{t,x}))\|_{p_\nu} \Lambda_i(T-t) \|\text{pr}_i(Z_r^{t,x})\|_{p_z} dr \\ & \leq \int_t^T \sum_{\nu=0}^d L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \|\text{pr}_\nu(w(r, X_r^{t,x}) - v(r, X_r^{t,x}))\|_{p_\nu} \Lambda_i(T-t) \|\text{pr}_i(Z_r^{t,x})\|_{p_z} dr \\ & \leq \int_t^T \sum_{\nu=0}^d L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \left[ \sup_{y \in \mathbb{R}^d} \frac{|\text{pr}_\nu(w(r, y) - v(r, y))|}{V(r, y)} \right] \|V(r, X_r^{t,x})\|_{p_\nu} \frac{c\Lambda_i(T-t)}{\Lambda_i(r-t)} dr \\ & \leq \int_t^T \sum_{\nu=0}^d L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \left[ \sup_{y \in \mathbb{R}^d} \frac{|\text{pr}_\nu(w(r, y) - v(r, y))|}{V(r, y)} \right] V(t, x) \frac{c\sqrt{T-t}}{\sqrt{r-t}} dr \\ & \leq \int_t^T \sum_{\nu=0}^d L_\nu \left[ \sup_{y \in \mathbb{R}^d} \frac{e^{\lambda r} \Lambda_\nu(T-r) |\text{pr}_\nu(w(r, y) - v(r, y))|}{V(r, y)} \right] V(t, x) \frac{cT e^{-\lambda r}}{\sqrt{(T-r)(r-t)}} dr \\ & \leq c^2 \|w - v\|_\lambda V(t, x) \int_t^T \frac{T e^{-\lambda r}}{\sqrt{(T-r)(r-t)}} dr. \end{aligned} \quad (49)$$

Next, Lemma 2.2 (with  $p \curvearrowright \frac{3}{2}$ ,  $q \curvearrowright 3$ ) and the fact that  $\forall t \in [0, T], \lambda \in (0, \infty): \int_t^T e^{-3\lambda r} dr = \frac{e^{-3\lambda t}}{-3\lambda} \Big|_{r=t}^T = \frac{e^{-3\lambda t} - e^{-3\lambda T}}{3\lambda} \leq \frac{e^{-3\lambda t}}{3\lambda}$  establish for all  $t \in [0, T]$  that

$$\begin{aligned} T \int_t^T \frac{e^{-\lambda r} dr}{\sqrt{(T-r)(r-t)}} &\leq T^{\frac{2}{3}} \left( B(1 - \frac{3}{4}, 1 - \frac{3}{4}) \right)^{\frac{2}{3}} \left( \int_t^T e^{-3\lambda r} dr \right)^{\frac{1}{3}} \\ &\leq T^{\frac{2}{3}} \left( B(\frac{1}{4}, \frac{1}{4}) \right)^{\frac{2}{3}} \frac{e^{-\lambda t}}{(3\lambda)^{\frac{1}{3}}}. \end{aligned} \quad (50)$$

This and (49) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $w, v \in B$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\lambda \in (0, \infty)$  that

$$|\Lambda_i(T-t) \text{pr}_i((\Phi(w))(t, x) - (\Phi(v))(t, x))| \leq c^2 \|w - v\|_\lambda V(t, x) T^{\frac{2}{3}} \left( B(\frac{1}{4}, \frac{1}{4}) \right)^{\frac{2}{3}} \frac{e^{-\lambda t}}{(3\lambda)^{\frac{1}{3}}}. \quad (51)$$

Hence, (41) proves for all  $w, v \in B$ ,  $\lambda \in (0, \infty)$  that

$$\|\Phi(w) - \Phi(v)\|_\lambda \leq c^2 \|w - v\|_\lambda \frac{T^{\frac{2}{3}} \left( B(\frac{1}{4}, \frac{1}{4}) \right)^{\frac{2}{3}}}{(3\lambda)^{\frac{1}{3}}}. \quad (52)$$

Therefore, there exists  $\lambda_0 \in (0, \infty)$  such that for all  $w, v \in B$  we have that

$$\|\Phi(w) - \Phi(v)\|_{\lambda_0} \leq \frac{1}{2} \|w - v\|_{\lambda_0}. \quad (53)$$

This, the triangle inequality, and (48) imply for all  $w \in B$  that

$$\|\Phi(w)\|_{\lambda_0} \leq \|\Phi(0)\|_{\lambda_0} + \|\Phi(w) - \Phi(0)\|_{\lambda_0} \leq \|\Phi(0)\|_{\lambda_0} + \frac{1}{2} \|w\|_{\lambda_0} < \infty. \quad (54)$$

Thus,  $\Phi(w) \in B$ . This, (53), and the Banach fixed point theorem show that there exists  $u \in B$  such that  $\Phi(u) = u$ . Therefore, (46) and (40) imply (i).

Next, Hölder's inequality and (35) show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} &\int_t^T \left\| \Lambda_i(T-t) \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(u(r, X_r^{t,x}))| |\text{pr}_i(Z_r^{t,x})| \right\|_1 dr \\ &\leq \int_t^T \sum_{\nu=0}^d \left[ L_\nu \Lambda_\nu(T) \|\text{pr}_\nu(u(r, X_r^{t,x}))\|_{p_\nu} \right] \left[ \Lambda_i(T-t) \|\text{pr}_i(Z_r^{t,x})\|_{p_z} \right] dr \\ &\leq \int_t^T \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \|\text{pr}_\nu(u(r, X_r^{t,x}))\|_{p_\nu} \right] \left[ \Lambda_i(T-t) \|\text{pr}_i(Z_r^{t,x})\|_{p_z} \right] dr \\ &\leq \int_t^T c \frac{\sqrt{T}}{\sqrt{T-r}} \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \Lambda_\nu(T-r) \|\text{pr}_\nu(u(r, X_r^{t,x}))\|_{p_\nu} \right] \right] \left[ \frac{c \Lambda_i(T-t)}{\Lambda_i(r-t)} \right] dr \\ &\leq \int_t^T c \frac{\sqrt{T}}{\sqrt{T-r}} \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \Lambda_\nu(T-r) \|\text{pr}_\nu(u(r, X_r^{t,x}))\|_{p_\nu} \right] \right] \frac{c \sqrt{T-t}}{\sqrt{r-t}} dr \\ &\leq \int_t^T c \frac{\sqrt{T}}{\sqrt{T-r}} \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-r) \frac{|\text{pr}_\nu(u(r, y))|}{V(r, y)} \right] \right] \|V(r, X_r^{t,x})\|_{p_\nu} \frac{c \sqrt{T-t}}{\sqrt{r-t}} dr \\ &\leq \int_t^T c^2 T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-r) \frac{|\text{pr}_\nu(u(r, y))|}{V(r, y)} \right] \right] \frac{V(t, x)}{\sqrt{(T-r)(r-t)}} dr. \end{aligned} \quad (55)$$

In addition, the triangle inequality and (34) imply for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $w: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  that

$$\begin{aligned} |f(t, x, w(t, x))| &\leq |f(t, x, 0)| + |f(t, x, w(t, x)) - f(t, x, 0)| \\ &\leq |f(t, x, 0)| + \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) |\text{pr}_\nu(w)|]. \end{aligned} \quad (56)$$

This, (38), the triangle inequality, (42), (43), and (55) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& |\Lambda_i(T-t)\text{pr}_i(u(t, x))| \\
&= \left| \mathbb{E}[\Lambda_i(T-t)g(X_T^{t,x})\text{pr}_i(Z_T^{t,x})] + \int_t^T \mathbb{E}[\Lambda_i(T-t)f(r, X_r^{t,x}, u(r, X_r^{t,x}))\text{pr}_i(Z_r^{t,x})] dr \right| \\
&\leq \|\Lambda_i(T-t)g(X_T^{t,x})\text{pr}_i(Z_T^{t,x})\|_1 + \int_t^T \|\Lambda_i(T-t)f(r, X_r^{t,x}, u(r, X_r^{t,x}))\text{pr}_i(Z_r^{t,x})\|_1 dr \\
&\leq \|\Lambda_i(T-t)g(X_T^{t,x})\text{pr}_i(Z_T^{t,x})\|_1 + \int_t^T \|\Lambda_i(T-t)f(r, X_r^{t,x}, 0)\text{pr}_i(Z_r^{t,x})\|_1 dr \\
&\quad + \int_t^T \left\| \Lambda_i(T-t) \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(u(r, X_r^{t,x}))| |\text{pr}_i(Z_r^{t,x})| \right\|_1 dr \\
&\leq cV(t, x) + 2cV(t, x) + \int_t^T c^2 T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-r) \frac{|\text{pr}_\nu(u(r, y))|}{V(r, y)} \right] \right] \frac{V(t, x)}{\sqrt{(T-r)(r-t)}} dr.
\end{aligned} \tag{57}$$

Dividing by  $V(t, x)$  then proves for all  $t \in [0, T]$  that

$$\begin{aligned}
& \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-t) \frac{|\text{pr}_\nu(u(t, y))|}{V(t, y)} \right] \\
&\leq 3c + \int_t^T c^2 T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-r) \frac{|\text{pr}_\nu(u(r, y))|}{V(r, y)} \right] \right] \frac{dr}{\sqrt{(T-r)(r-t)}}
\end{aligned} \tag{58}$$

Thus, (36) and Corollary 2.4 imply for all  $t \in [0, T]$  that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-t) \frac{|\text{pr}_\nu(u(t, y))|}{V(t, y)} \right] \leq 6ce^{86c^6 T^2 (T-t)}. \tag{59}$$

This completes the proof of Lemma 2.6.  $\square$

**2.3. Spatial Lipschitz continuity of solutions to SFPEs.** In Lemma 2.8 below we prove spatial Lipschitz continuity of solutions to SFPEs. The key assumptions here are the Lipschitz-type conditions (60)–(63) together with the so-called flow property (64), which is satisfied, e.g., when the corresponding process  $X$  is a solution to an SDE.

**Setting 2.7.** Assume Setting 2.5. Suppose that  $\max\{c, 48e^{86c^6 T^3}\} \leq V$ ,  $\frac{2}{p_x} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$ , and  $\frac{1}{2} + \frac{1}{p_z} \leq 1$ . Furthermore, suppose for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $r \in (t, T]$ ,  $x \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$ ,  $A \in (\mathcal{B}(\mathbb{R}^d))^{\otimes \mathbb{R}^d}$ ,  $B \in (\mathcal{B}(\mathbb{R}^d))^{\otimes ([t, T] \times \mathbb{R}^d)}$  that

$$|f(t, x, w_1) - f(t, y, w_2)| \leq \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) |\text{pr}_\nu(w_1 - w_2)|] + \frac{1}{T} \frac{V(t, x) + V(t, y)}{2} \frac{\|x - y\|}{\sqrt{T}}, \tag{60}$$

$$|g(x) - g(y)| \leq \frac{V(T, x) + V(T, y)}{2} \frac{\|x - y\|}{\sqrt{T}}, \tag{61}$$

$$\|X_r^{t,x} - X_r^{t,y}\|_{p_x} \leq \frac{V(t, x) + V(t, y)}{2} \|x - y\|, \tag{62}$$

$$\|\text{pr}_i(Z_r^{t,x} - Z_r^{t,y})\|_{p_z} \leq \frac{V(t, x) + V(t, y)}{2} \frac{\|x - y\|}{\sqrt{T} \Lambda_i(r-t)}, \tag{63}$$

$$\mathbb{P}(X_r^{t, X_t^{s,x}} = X_r^{s,x}) = 1, \quad \mathbb{P}(X_t^{s,(\cdot)} \in A, X_{(\cdot)}^{t,(\cdot)} \in B) = \mathbb{P}(X_t^{s,(\cdot)} \in A) \mathbb{P}(X_{(\cdot)}^{t,(\cdot)} \in B), \tag{64}$$

and  $\mathbb{P}(X_s^{s,x} = x) = 1$ . Moreover, let  $u: [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  be the unique measurable function (cf. Lemma 2.6) such that for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), \xi \in \mathbb{R}^d} \left[ \Lambda_\nu(T - \tau) \frac{|\text{pr}_\nu(u(\tau, \xi))|}{V(\tau, \xi)} \right] < \infty, \quad (65)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E}[|g(X_T^{t,x}) \text{pr}_\nu(Z_T^{t,x})|] + \int_t^T \mathbb{E}[|f(r, X_r^{t,x}, u(r, X_r^{t,x})) \text{pr}_\nu(Z_r^{t,x})|] dr \right] < \infty, \quad (66)$$

and

$$u(t, x) = \mathbb{E}[g(X_T^{t,x}) Z_T^{t,x}] + \int_t^T \mathbb{E}[f(r, X_r^{t,x}, u(r, X_r^{t,x})) Z_r^{t,x}] dr. \quad (67)$$

**Lemma 2.8** (Lipschitz continuity of the fixed point). *Assume Setting 2.7. Then for all  $x, y \in \mathbb{R}^d$ ,  $t \in [0, T)$  we have that*

$$\max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T - t) |\text{pr}_i(u(t, x) - u(t, y))|] \leq e^{c^2 T} \frac{V^6(t, x) + V^6(t, y)}{2} \frac{\|x - y\|}{\sqrt{T}}. \quad (68)$$

*Proof of Lemma 2.8.* Lemma 2.6 and the assumption of Lemma 2.8 prove that for all  $t \in [0, T)$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T - t) \frac{|\text{pr}_\nu(u(t, y))|}{V(t, y)} \right] \leq 6c e^{86c^6 T^2 (T-t)}. \quad (69)$$

Next, Hölder's inequality, the fact that  $\frac{1}{p_\nu} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$ , (35), (62), the fact that  $\forall x, y, p, q \in [0, \infty)$ :  $\frac{x^p + y^p}{2} \frac{x^q + y^q}{2} \leq \frac{x^{p+q} + y^{p+q}}{2}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $\tilde{x}, \tilde{y} \in \mathbb{R}^d$ ,  $t \in [0, T)$ ,  $r \in (t, T)$  that

$$\begin{aligned} & \Lambda_i(T - t) \left\| \frac{V(r, X_r^{t, \tilde{x}}) + V(r, X_r^{t, \tilde{y}})}{2} \frac{\|X_r^{t, \tilde{x}} - X_r^{t, \tilde{y}}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t, \tilde{x}}) \right\|_1 \\ & \leq \Lambda_i(T - t) \frac{\|V(r, X_r^{t, \tilde{x}})\|_{p_\nu} + \|V(r, X_r^{t, \tilde{y}})\|_{p_\nu}}{2} \frac{\| \|X_r^{t, \tilde{x}} - X_r^{t, \tilde{y}}\| \| \text{pr}_i(Z_r^{t, \tilde{x}}) \|_{p_z}}{\sqrt{T}} \\ & \leq \Lambda_i(T - t) \frac{V(t, \tilde{x}) + V(t, \tilde{y})}{2} \frac{V(t, \tilde{x}) + V(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T}} \frac{c}{\Lambda_i(r - t)} \\ & \leq c \frac{V^2(t, \tilde{x}) + V^2(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}}. \end{aligned} \quad (70)$$

Hence, Hölder's inequality, the fact that  $\frac{2}{p_\nu} + \frac{1}{p_x} \leq 1$ , the fact that  $c \leq V$ , (35), (62), and the fact that  $\forall x, y, p, q \in [0, \infty)$ :  $\frac{x^p + y^p}{2} \frac{x^q + y^q}{2} \leq \frac{x^{p+q} + y^{p+q}}{2}$  show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T)$ ,  $t \in [s, T)$ ,  $r \in (0, T)$  that

$$\begin{aligned} & \Lambda_i(T - t) \left\| \left\| \frac{V(r, X_r^{t, \tilde{x}}) + V(r, X_r^{t, \tilde{y}})}{2} \frac{\|X_r^{t, \tilde{x}} - X_r^{t, \tilde{y}}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t, \tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\ & \leq \left\| c \frac{V^2(t, \tilde{x}) + V^2(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}} \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\ & = \left\| c \frac{V^2(t, X_t^{s,x}) + V^2(t, X_t^{s,y})}{2} \frac{\|X_t^{s,x} - X_t^{s,y}\|}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}} \right\|_2 \\ & \leq c \frac{\|V^2(t, X_t^{s,x})\|_{\frac{p_\nu}{2}} + \|V^2(t, X_t^{s,y})\|_{\frac{p_\nu}{2}}}{2} \frac{\| \|X_t^{s,x} - X_t^{s,y}\| \| \sqrt{T - t} \|_{p_x}}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}} \\ & \leq \frac{V(s, x) + V(s, y)}{2} \frac{V^2(s, x) + V^2(s, y)}{2} \frac{V(s, x) + V(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}} \\ & \leq \frac{V^4(s, x) + V^4(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}}. \end{aligned} \quad (71)$$

This and (61) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that

$$\begin{aligned}
& \Lambda_i(T-t) \left\| \left\| \left( g(X_T^{t,\tilde{x}}) - g(X_T^{t,\tilde{y}}) \right) \text{pr}_i(Z_T^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
& \leq \Lambda_i(T-t) \left\| \left\| \frac{V(T, X_T^{t,\tilde{x}}) + V(T, X_T^{t,\tilde{y}})}{2} \frac{\|X_T^{t,\tilde{x}} - X_T^{t,\tilde{y}}\|}{\sqrt{T}} \text{pr}_i(Z_T^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
& \leq \frac{V^4(s, x) + V^4(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}}.
\end{aligned} \tag{72}$$

Next, Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (35), (63), and the fact that  $\forall x, y, p, q \in [0, \infty)$ :  $\frac{x^p + y^p}{2} \frac{x^q + y^q}{2} \leq \frac{x^{p+q} + y^{p+q}}{2}$  prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $\tilde{x}, \tilde{y} \in \mathbb{R}^d$ ,  $t \in [0, T]$ ,  $r \in (t, T]$  that

$$\begin{aligned}
& \Lambda_i(T-t) \|V(r, X_r^{t,\tilde{y}}) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}})\|_1 \\
& \leq \Lambda_i(T-t) \|V(r, X_r^{t,\tilde{y}})\|_{p_v} \| \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \|_{p_z} \\
& \leq \Lambda_i(T-t) V(t, \tilde{y}) \frac{V(t, \tilde{x}) + V(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T} \Lambda_i(r-t)} \\
& \leq 2 \frac{V(t, \tilde{x}) + V(t, \tilde{y})}{2} \frac{V(t, \tilde{x}) + V(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}} \\
& \leq 2 \frac{V^2(t, \tilde{x}) + V^2(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}}.
\end{aligned} \tag{73}$$

Therefore, (71) and the fact that  $1 \leq c$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $r \in (0, T]$  that

$$\begin{aligned}
& \Lambda_i(T-t) \left\| \left\| V(r, X_r^{t,\tilde{y}}) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
& \leq \left\| \left\| 2 \frac{V^2(t, \tilde{x}) + V^2(t, \tilde{y})}{2} \frac{\|\tilde{x} - \tilde{y}\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}} \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
& \leq 2 \frac{V^4(s, x) + V^4(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}}.
\end{aligned} \tag{74}$$

This and (33) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that

$$\begin{aligned}
& \Lambda_i(T-t) \left\| \left\| g(X_T^{t,\tilde{y}}) \text{pr}_i(Z_T^{t,\tilde{x}} - Z_T^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
& \leq \Lambda_i(T-t) \left\| \left\| V(T, X_T^{t,\tilde{y}}) \text{pr}_i(Z_T^{t,\tilde{x}} - Z_T^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
& \leq 2 \frac{V^4(s, x) + V^4(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}}.
\end{aligned} \tag{75}$$

Next, the triangle inequality, Hölder's inequality, the fact that  $\frac{1}{2} + \frac{1}{p_z} \leq 1$ , and (35) establish for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $\tilde{x}, \tilde{y} \in \mathbb{R}^d$ ,  $t \in [0, T]$ ,  $r \in (t, T]$  that

$$\begin{aligned}
& \Lambda_i(T-t) \left\| \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \left| \text{pr}_\nu(u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{y}}) \right| \right\|_1 \right\|_2 \\
& \leq \Lambda_i(T-t) \sum_{\nu=0}^d \left\| \left\| L_\nu \Lambda_\nu(T) \text{pr}_\nu(u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{y}}) \right\|_1 \right\|_2 \\
& \leq \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \left\| \Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{y}}) \right\|_1 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \right\|_2 \left\| \text{pr}_i(Z_r^{t,\tilde{y}}) \right\|_{p_z} \right] \\
&= \sum_{\nu=0}^d \left[ L_\nu \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \right\|_2 \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_i(T-t) \left\| \text{pr}_i(Z_r^{t,\tilde{y}}) \right\|_{p_z} \right] \\
&\leq \sum_{\nu=0}^d \left[ L_\nu \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \right\|_2 \frac{\sqrt{T}}{\sqrt{T-r}} \frac{c \Lambda_i(T-t)}{\Lambda_i(r-t)} \right] \\
&\leq \sum_{\nu=0}^d \left[ L_\nu \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \right\|_2 \frac{\sqrt{T}}{\sqrt{T-r}} \frac{c \sqrt{T-t}}{\sqrt{r-t}} \right]. \tag{76}
\end{aligned}$$

This, the triangle inequality, the disintegration theorem, (64), and the fact that  $\sum_{\nu=0}^d L_\nu \leq c$  prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $r \in (t, T]$  that

$$\begin{aligned}
&\Lambda_i(T-t) \left\| \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \left| \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{y}}) \right| \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
&\leq \left\| \sum_{\nu=0}^d \left[ L_\nu \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \right\|_2 \frac{\sqrt{T}}{\sqrt{T-r}} \frac{c \sqrt{T-t}}{\sqrt{r-t}} \right] \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
&\leq \sum_{\nu=0}^d \left[ L_\nu \left\| \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \right\|_2 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \frac{cT}{\sqrt{(T-r)(r-t)}} \right] \\
&\leq \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{s,x}) - u(r, X_r^{s,y})) \right\|_2 \right] \frac{c^2 T}{\sqrt{(T-r)(r-t)}}. \tag{77}
\end{aligned}$$

Hence, the triangle inequality, (60), and (71) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $r \in (t, T]$  that

$$\begin{aligned}
&\Lambda_i(T-t) \left\| \left\| (f(r, X_r^{t,\tilde{x}}, u(r, X_r^{t,\tilde{x}})) - f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}}))) \text{pr}_i(Z_r^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
&\leq \Lambda_i(T-t) \left\| \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \left| \text{pr}_\nu (u(r, X_r^{t,\tilde{x}}) - u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{y}}) \right| \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
&\quad + \Lambda_i(T-t) \frac{1}{T} \left\| \left\| \frac{V(r, X_r^{t,\tilde{x}}) + V(r, X_r^{t,\tilde{y}})}{2} \frac{\|X_r^{t,\tilde{x}} - X_r^{t,\tilde{y}}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \\
&\leq \max_{\nu \in [0, d] \cap \mathbb{Z}} \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{s,x}) - u(r, X_r^{s,y})) \right\|_2 \frac{c^2 T}{\sqrt{(T-r)(r-t)}} \\
&\quad + \frac{1}{T} \frac{V^4(s, x) + V^4(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}}. \tag{78}
\end{aligned}$$

This and the fact that  $\forall t \in [0, T]: \int_t^T \frac{dr}{\sqrt{r-t}} = 2\sqrt{r-t} \Big|_{r=t}^T = 2\sqrt{T-t}$  prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that

$$\begin{aligned}
&\int_t^T \max_{i \in [0, d] \cap \mathbb{Z}} \left[ \Lambda_i(T-t) \left\| \left\| (f(r, X_r^{t,\tilde{x}}, u(r, X_r^{t,\tilde{x}})) - f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}}))) \text{pr}_i(Z_r^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \right] dr \\
&\leq \int_t^T \max_{\nu \in [0, d] \cap \mathbb{Z}} \left\| \Lambda_\nu(T-r) \text{pr}_\nu (u(r, X_r^{s,x}) - u(r, X_r^{s,y})) \right\|_2 \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr \\
&\quad + \frac{1}{T} \frac{V^4(s, x) + V^4(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} \int_t^T \frac{\sqrt{T-t}}{\sqrt{r-t}} dr
\end{aligned}$$

$$\begin{aligned}
&\leq \int_t^T \max_{\nu \in [0, d] \cap \mathbb{Z}} \|\Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{s,x}) - u(r, X_r^{s,y}))\|_2 \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr \\
&\quad + 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}}.
\end{aligned} \tag{79}$$

Next, (33), (74), and the fact that  $\forall t \in [0, T]: \int_t^T \frac{dr}{\sqrt{r-t}} = 2\sqrt{r-t}|_{r=t}^T = 2\sqrt{T-t}$  show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T)$ ,  $t \in [s, T)$  that

$$\begin{aligned}
&\int_t^T \Lambda_i(T-t) \left\| \left\| f(r, X_r^{t,\tilde{y}}, 0) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 dr \\
&\leq \int_t^T \Lambda_i(T-t) \frac{1}{T} \left\| \left\| V(r, X_r^{t,\tilde{y}}) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 dr \\
&\leq \int_t^T \frac{1}{T} \cdot 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}} dr \\
&\leq 4 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}}.
\end{aligned} \tag{80}$$

Moreover, (69), the triangle inequality, the fact that  $\sum_{\nu=0}^d L_\nu \leq c$ , (74), Lemma 2.1, the fact that  $48e^{86c^6T^3} \leq V$ , and the fact that  $\forall x, y, p, q \in [0, \infty): \frac{x^p+y^p}{2} \frac{x^q+y^q}{2} \leq \frac{x^{p+q}+y^{p+q}}{2}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T)$ ,  $t \in [s, T)$  that

$$\begin{aligned}
&\int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \left\| \left\| \Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \right] dr \\
&\leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \left\| \left\| \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \right] dr \\
&\leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \left\| \left\| \frac{\sqrt{T}}{\sqrt{T-r}} 6c e^{86c^6T^3} V(r, X_r^{t,\tilde{y}}) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \right] dr \\
&\leq \int_t^T \Lambda_i(T-t) \left\| \left\| V(r, X_r^{t,\tilde{y}}) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \frac{\sqrt{T}}{\sqrt{T-r}} 6c^2 e^{86c^6T^3} dr \\
&\leq \int_t^T 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} \frac{\sqrt{T-t}}{\sqrt{r-t}} \frac{\sqrt{T}}{\sqrt{T-r}} 6c^2 e^{86c^6T^3} dr \\
&\leq 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} \int_t^T \frac{6c^2 T e^{86c^6T^3}}{\sqrt{(T-r)(r-t)}} dr \\
&\leq 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} 6c^2 T e^{86c^6T^3} \cdot 4 \\
&\leq \frac{V^4(s,x) + V^4(s,y)}{2} \frac{V(s,x) + V(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} c^2 T \\
&\leq \frac{V^5(s,x) + V^5(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} c^2 T.
\end{aligned} \tag{81}$$

This, the triangle inequality, (80), and the fact that  $1 + c^2 T \leq e^{c^2 T}$  prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T)$ ,  $t \in [s, T)$  that

$$\begin{aligned}
&\int_t^T \Lambda_i(T-t) \left\| \left\| f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 dr \\
&\leq \int_t^T \Lambda_i(T-t) \left\| \left\| f(r, X_r^{t,\tilde{y}}, 0) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 dr
\end{aligned}$$

$$\begin{aligned}
& + \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \left\| \left\| \Lambda_\nu(T) \text{pr}_\nu(u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2 \right] dr \\
& \leq 4 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} + \frac{V^5(s,x) + V^5(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} c^2 T \\
& \leq 4 \frac{V^5(s,x) + V^5(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} e^{c^2 T}. \tag{82}
\end{aligned}$$

Next, (67) and the triangle inequality show for all  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that

$$\begin{aligned}
& \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \|\text{pr}_i(u(t, X_t^{s,x}) - u(t, X_t^{s,y}))\|_2] \\
& = \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \|\text{pr}_i(u(t, \tilde{x}) - u(t, \tilde{y}))\|_2 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}}] \\
& = \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \left\| \left[ \mathbb{E} \left[ \text{pr}_i \left( g(X_T^{t,\tilde{x}}) Z_T^{t,\tilde{x}} - g(X_T^{t,\tilde{y}}) Z_T^{t,\tilde{y}} \right) \right] \right. \right. \\
& \quad \left. \left. + \int_t^T \mathbb{E} \left[ \text{pr}_i \left( f(r, X_r^{t,\tilde{x}}, u(r, X_r^{t,\tilde{x}})) Z_r^{t,\tilde{x}} - f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}})) Z_r^{t,\tilde{y}} \right) \right] dr \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] \\
& \leq \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \left\| \left\| \text{pr}_i \left( g(X_T^{t,\tilde{x}}) Z_T^{t,\tilde{x}} - g(X_T^{t,\tilde{y}}) Z_T^{t,\tilde{y}} \right) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] \\
& \quad + \max_{i \in [0, d] \cap \mathbb{Z}} \int_t^T [\Lambda_i(T-t) \left\| \left\| \text{pr}_i \left( f(r, X_r^{t,\tilde{x}}, u(r, X_r^{t,\tilde{x}})) Z_r^{t,\tilde{x}} - f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}})) Z_r^{t,\tilde{y}} \right) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] dr \\
& \leq \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \left\| \left\| \left( g(X_T^{t,\tilde{x}}) - g(X_T^{t,\tilde{y}}) \right) \text{pr}_i(Z_T^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] \\
& \quad + \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \left\| \left\| g(X_T^{t,\tilde{y}}) \text{pr}_i(Z_T^{t,\tilde{x}} - Z_T^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] \\
& \quad + \max_{i \in [0, d] \cap \mathbb{Z}} \int_t^T [\Lambda_i(T-t) \left\| \left\| \left( f(r, X_r^{t,\tilde{x}}, u(r, X_r^{t,\tilde{x}})) - f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}})) \right) \text{pr}_i(Z_r^{t,\tilde{x}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] dr \\
& \quad + \max_{i \in [0, d] \cap \mathbb{Z}} \int_t^T [\Lambda_i(T-t) \left\| \left\| f(r, X_r^{t,\tilde{y}}, u(r, X_r^{t,\tilde{y}})) \text{pr}_i(Z_r^{t,\tilde{x}} - Z_r^{t,\tilde{y}}) \right\|_1 \Big|_{\tilde{x}=X_t^{s,x}, \tilde{y}=X_t^{s,y}} \right\|_2] dr. \tag{83}
\end{aligned}$$

This, (72), (75), (79), (82), and the fact that  $1 \leq V$  prove for all  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that

$$\begin{aligned}
& \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \|\text{pr}_i(u(t, X_t^{s,x}) - u(t, X_t^{s,y}))\|_2] \\
& \leq \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} + 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} \\
& \quad + \int_t^T \max_{\nu \in [0, d] \cap \mathbb{Z}} \|\Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{s,x}) - u(r, X_r^{s,y}))\|_2 \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr \\
& \quad + 2 \frac{V^4(s,x) + V^4(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} + 4 \frac{V^5(s,x) + V^5(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} e^{c^2 T} \\
& \leq 9e^{c^2 T} \frac{V^5(s,x) + V^5(s,y)}{2} \frac{\|x-y\|}{\sqrt{T}} \\
& \quad + \int_t^T \max_{\nu \in [0, d] \cap \mathbb{Z}} \|\Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{s,x}) - u(r, X_r^{s,y}))\|_2 \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr. \tag{84}
\end{aligned}$$

Moreover, (69), Jensen's inequality, the fact that  $2 \leq p_\nu$ , and (35) imply for all  $x \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that

$$\max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \|\text{pr}_i(u(t, X_t^{s,x}))\|_2] \leq 6ce^{86c^6 T^3} \|V(t, X_t^{s,x})\|_{p_\nu} \leq 6ce^{86c^6 T^3} V(s, x). \tag{85}$$

Therefore, the Grönwall-type inequality (see Corollary 2.4), (84), the fact that  $48e^{86c^6T^3} \leq V$ , and the fact that  $\forall x, y, p, q \in [0, \infty): \frac{x^p+y^p}{2} \frac{x^q+y^q}{2} \leq \frac{x^{p+q}+y^{p+q}}{2}$  show for all  $x, y \in \mathbb{R}^d$ ,  $s \in [0, T)$ ,  $t \in [s, T)$  that

$$\begin{aligned} & \max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \|\text{pr}_i(u(t, X_t^{s,x}) - u(t, X_t^{s,y}))\|_2] \\ & \leq 18e^{c^2T} \frac{V^5(s, x) + V^5(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} e^{86c^6T^3} \\ & \leq e^{c^2T} \frac{V^5(s, x) + V^5(s, y)}{2} \frac{V(s, x) + V(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}} \\ & \leq e^{c^2T} \frac{V^6(s, x) + V^6(s, y)}{2} \frac{\|x - y\|}{\sqrt{T}}. \end{aligned} \quad (86)$$

This and the fact that  $\forall s \in [0, T), x \in \mathbb{R}^d: \mathbb{P}(X_s^{s,x} = x) = 1$  imply (68). The proof of Lemma 2.8 is thus completed.  $\square$

**2.4. Temporal regularity of the fixed point.** After establishing the spatial Lipschitz continuity we establish the temporal regularity for solutions to SFPEs, see Lemma 2.9 below. Together with the spatial Lipschitz continuity the temporal regularity implies that the fixed point is continuous. Later, continuity of the fixed point is an important property to ensure that the fixed point is the unique viscosity solution to the corresponding semilinear parabolic PDE.

**Lemma 2.9** (Temporal regularity of the fixed point). *Assume Setting 2.7. Suppose that  $\frac{6}{p_v} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$  and  $\frac{1}{2} + \frac{1}{p_z} \leq 1$ . Assume for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $s \in [0, T)$ ,  $t \in [s, T)$ ,  $r \in (t, T]$ ,  $x \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$  that*

$$\| \|X_r^{t,x} - X_r^{t,y}\| \|_{p_x} \leq c\|x - y\|, \quad (87)$$

$$\|\text{pr}_i(Z_r^{t,x} - Z_r^{s,x})\|_{p_z} \leq \frac{V(t, x) + V(s, x)}{2} \frac{\sqrt{t-s}}{\sqrt{r-t}\Lambda_i(r-s)}, \quad (88)$$

$$\| \|X_t^{s,x} - x\| \|_{p_x} \leq V(s, x)\sqrt{t-s}. \quad (89)$$

Moreover, let  $u: [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  be the unique measurable function (cf. Lemma 2.6) such that for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), \xi \in \mathbb{R}^d} \left[ \Lambda_\nu(T-\tau) \frac{|\text{pr}_\nu(u(\tau, \xi))|}{V(\tau, \xi)} \right] < \infty, \quad (90)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E}[|g(X_T^{t,x}) \text{pr}_\nu(Z_T^{t,x})|] + \int_t^T \mathbb{E}[|f(r, X_r^{t,x}, u(r, X_r^{t,x})) \text{pr}_\nu(Z_r^{t,x})|] dr \right] < \infty, \quad (91)$$

and

$$u(t, x) = \mathbb{E}[g(X_T^{t,x}) Z_T^{t,x}] + \int_t^T \mathbb{E}[f(r, X_r^{t,x}, u(r, X_r^{t,x})) Z_r^{t,x}] dr. \quad (92)$$

Then  $u$  is continuous.

*Proof of Lemma 2.9.* First, Lemmas 2.6 and 2.8 and the assumptions of Lemma 2.9 prove that for all  $t \in [0, T)$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-t) \frac{|\text{pr}_\nu(u(t, y))|}{V(t, y)} \right] \leq 6ce^{86c^6T^2(T-t)} \quad (93)$$

and that for all  $x, y \in \mathbb{R}^d$ ,  $t \in [0, T)$  we have that

$$\max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T-t) \|\text{pr}_i(u(t, x) - u(t, y))\|] \leq e^{c^2T} \frac{V^6(t, x) + V^6(t, y)}{2} \frac{\|x - y\|}{\sqrt{T}}. \quad (94)$$

Next, (64), the disintegration theorem, (87), the fact that  $c \leq V$ , (89), and the fact that  $\forall x, y, p, q \in [0, \infty)$ :  $\frac{x^p+y^p}{2} \frac{x^q+y^q}{2} \leq \frac{x^{p+q}+y^{p+q}}{2}$  prove for all  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $r \in (t_2, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\left\| \left\| X_r^{t_1, x} - X_r^{t_2, x} \right\| \right\|_{p_x} &= \left\| \left\| X_r^{t_2, X_{t_2}^{t_1, x}} - X_r^{t_2, x} \right\| \right\|_{p_x} \\
&= \left\| \left\| X_r^{t_2, y} - X_r^{t_2, x} \right\| \right\|_{p_x} \Big|_{y=X_{t_2}^{t_1, x}} \Big\|_{p_x} \\
&\leq \left\| c \|y - x\| \right\|_{y=X_{t_2}^{t_1, x}} \Big\|_{p_x} \\
&= \|c \|X_{t_2}^{t_1, x} - x\|\|_{p_x} \\
&\leq 2 \frac{V(t_1, x) + V(t_2, x)}{2} \frac{V(t_1, x) + V(t_2, x)}{2} \sqrt{t_2 - t_1} \\
&\leq 2 \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \sqrt{t_2 - t_1}. \tag{95}
\end{aligned}$$

Hence, Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$ , (35), the fact that  $c \leq V$ , and the fact that  $\forall x, y, p, q \in [0, \infty)$ :  $\frac{x^p+y^p}{2} \frac{x^q+y^q}{2} \leq \frac{x^{p+q}+y^{p+q}}{2}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $r \in (t_2, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\Lambda_i(T - t_1) &\left\| \frac{V(r, X_r^{t_1, x}) + V(r, X_r^{t_2, x})}{2} \frac{\|X_r^{t_1, x} - X_r^{t_2, x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t_1, x}) \right\|_1 \\
&\leq \Lambda_i(T - t_1) \frac{\|V(r, X_r^{t_1, x})\|_{p_v} + \|V(r, X_r^{t_2, x})\|_{p_v}}{2} \frac{\left\| \left\| X_r^{t_1, x} - X_r^{t_2, x} \right\| \right\|_{p_x}}{\sqrt{T}} \left\| \text{pr}_i(Z_r^{t_1, x}) \right\|_{p_z} \\
&\leq \Lambda_i(T - t_1) \frac{V(t_1, x) + V(t_2, x)}{2} 2 \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} \frac{c}{\Lambda_i(r - t_1)} \\
&\leq 2 \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} \frac{\sqrt{T - t_1}}{\sqrt{r - t_1}}. \tag{96}
\end{aligned}$$

This and (61) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\Lambda_i(T - t_1) &\left\| (g(X_T^{t_1, x}) - g(X_T^{t_2, x})) \text{pr}_i(Z_T^{t_1, x}) \right\|_1 \\
&\leq \Lambda_i(T - t_1) \left\| \frac{V(T, X_T^{t_1, x}) + V(T, X_T^{t_2, x})}{2} \frac{\|X_T^{t_1, x} - X_T^{t_2, x}\|}{\sqrt{T}} \text{pr}_i(Z_T^{t_1, x}) \right\|_1 \\
&\leq 2 \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}}. \tag{97}
\end{aligned}$$

Next, Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (35), and (88) show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $r \in (t_2, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\Lambda_i(T - t_1) &\left\| V(r, X_r^{t_2, x}) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1 \\
&\leq \Lambda_i(T - t_1) \left\| V(r, X_r^{t_2, x}) \right\|_{p_v} \left\| \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_{p_z} \\
&\leq \Lambda_i(T - t_1) 2 \frac{V(t_1, x) + V(t_2, x)}{2} \frac{V(t_1, x) + V(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{r - t_2} \Lambda_i(r - t_1)} \\
&\leq 2 \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{r - t_2}} \frac{\sqrt{T - t_1}}{\sqrt{r - t_1}}. \tag{98}
\end{aligned}$$

This and (33) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\Lambda_i(T - t_1) \left\| g(X_T^{t_2, x}) \text{pr}_i(Z_T^{t_1, x} - Z_T^{t_2, x}) \right\|_1$$

$$\leq \Lambda_i(T - t_1) \|V(T, X_T^{t_2, x}) \text{pr}_i(Z_T^{t_1, x} - Z_T^{t_2, x})\|_1 \leq 2 \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T - t_2}}. \quad (99)$$

Therefore, the triangle inequality and (97) imply that for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|g(X_T^{t_n, x}) \text{pr}_i(Z_T^{t_n, x}) - g(X_T^{t, x}) \text{pr}_i(Z_T^{t, x})\|_1 \\ & \leq \limsup_{n \rightarrow \infty} \|(g(X_T^{t_n, x}) - g(X_T^{t, x})) \text{pr}_i(Z_T^{t_n, x})\|_1 + \limsup_{n \rightarrow \infty} \|g(X_T^{t, x}) (\text{pr}_i(Z_T^{t_n, x}) - \text{pr}_i(Z_T^{t, x}))\|_1 \\ & \leq \limsup_{n \rightarrow \infty} \left[ 2 \frac{V^4(t_n, x) + V^4(t, x)}{2} \frac{\sqrt{|t_n - t|}}{\sqrt{T}} \frac{1}{\Lambda_i(T - \min\{t, t_n\})} \right] \\ & \quad + \limsup_{n \rightarrow \infty} \left[ 2 \frac{V^2(t_n, x) + V^2(t, x)}{2} \frac{\sqrt{|t_n - t|}}{\sqrt{T - \max\{t, t_n\}}} \frac{1}{\Lambda_i(T - \min\{t, t_n\})} \right] \\ & = 0. \end{aligned} \quad (100)$$

This and Jensen's inequality prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $x \in \mathbb{R}^d$  that

$$([0, T) \ni t \mapsto \mathbb{E}[g(X_T^{t, x}) \text{pr}_i(Z_T^{t, x})] \in \mathbb{R}) \in C([0, T), \mathbb{R}). \quad (101)$$

Next, Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (33), (35), the fact that  $c \leq V$ , and the fact that  $\forall t_1 \in [0, T), t_2 \in [t_1, T)$ :  $\int_{t_1}^{t_2} \frac{dr}{\sqrt{r - t_1}} = 2\sqrt{r - t_1}|_{r=t_1}^{t_2} = 2\sqrt{t_2 - t_1}$  show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T), t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_{t_1}^{t_2} \Lambda_i(T - t_1) \|f(r, X_r^{t_1, x}, 0) \text{pr}_i(Z_r^{t_1, x})\|_1 dr \\ & \leq \int_{t_1}^{t_2} \frac{1}{T} \Lambda_i(T - t_1) \|V(r, X_r^{t_1, x}) \text{pr}_i(Z_r^{t_1, x})\|_1 dr \\ & \leq \int_{t_1}^{t_2} \frac{1}{T} \Lambda_i(T - t_1) \|V(r, X_r^{t_1, x})\|_{p_v} \|\text{pr}_i(Z_r^{t_1, x})\|_{p_z} dr \\ & \leq \int_{t_1}^{t_2} \frac{1}{T} \Lambda_i(T - t_1) V(t_1, x) \frac{c}{\Lambda_i(r - t_1)} dr \\ & \leq \frac{V^2(t_1, x)}{T} \int_{t_1}^{t_2} \frac{\sqrt{T - t_1}}{\sqrt{r - t_1}} dr \\ & = \frac{V^2(t_1, x) \sqrt{T - t_1}}{T} 2\sqrt{t_2 - t_1} \\ & \leq 2V^2(t_1, x) \frac{\sqrt{t_2 - t_1}}{\sqrt{T}}. \end{aligned} \quad (102)$$

Furthermore, the triangle inequality, (93), Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_x} \leq 1$ , (35), the fact that  $\max\{c, 6e^{86c^6 T^3}\} \leq V$ , and the fact that  $\forall t_1 \in [0, T), t_2 \in [t_1, T)$ :  $\int_{t_1}^{t_2} \frac{dr}{r - t_1} = 2\sqrt{r - t_1}|_{r=t_1}^{t_2} = 2\sqrt{t_2 - t_1}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T), t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_{t_1}^{t_2} \Lambda_i(T - t_1) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(u(r, X_r^{t_1, x}))| |\text{pr}_i(Z_r^{t_1, x})| \right\|_1 dr \\ & \leq \int_{t_1}^{t_2} \Lambda_i(T - t_1) \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) \|\text{pr}_\nu(u(r, X_r^{t_1, x})) \text{pr}_i(Z_r^{t_1, x})\|_1] dr \\ & \leq \int_{t_1}^{t_2} \Lambda_i(T - t_1) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T - r}} \Lambda_\nu(T - r) \|\text{pr}_\nu(u(r, X_r^{t_1, x})) \text{pr}_i(Z_r^{t_1, x})\|_1 \right] dr \end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2} \Lambda_i(T-t_1) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} 6ce^{86c^6T^3} \|V(r, X_r^{t_1,x})\|_{p_\nu} \|\text{pr}_i(Z_r^{t_1,x})\|_1 \right] dr \\
&\leq \int_{t_1}^{t_2} \Lambda_i(T-t_1) \frac{c\sqrt{T}}{\sqrt{T-r}} 6ce^{86c^6T^3} \|V(r, X_r^{t_1,x})\|_{p_\nu} \|\text{pr}_i(Z_r^{t_1,x})\|_{p_z} dr \\
&\leq \int_{t_1}^{t_2} \Lambda_i(T-t_1) \frac{c\sqrt{T}}{\sqrt{T-r}} 6ce^{86c^6T^3} V(t_1, x) \frac{c}{\Lambda_i(r-t_1)} \\
&\leq c^2 V^3(t_1, x) \int_{t_1}^{t_2} \frac{\sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t_1}}{\sqrt{r-t_1}} dr \\
&\leq c^2 T V^3(t_1, x) \frac{1}{\sqrt{T-t_2}} \int_{t_1}^{t_2} \frac{dr}{\sqrt{r-t_1}} \\
&\leq 2c^2 T V^3(t_1, x) \frac{\sqrt{t_1-t_2}}{\sqrt{T-t_2}}. \tag{103}
\end{aligned}$$

This, the triangle inequality, (60), (102), the fact that  $1 \leq V$ , and the fact that  $1 + c^2T \leq e^{c^2T}$  prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T]$ ,  $t_2 \in [t_1, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\Lambda_i(T-t_1) \int_{t_1}^{t_2} \|f(r, X_r^{t_1,x}, u(r, X_r^{t_1,x}))\|_1 \text{pr}_i(Z_r^{t_1,x}) dr \\
&\leq \int_{t_1}^{t_2} \Lambda_i(T-t_1) \|f(r, X_r^{t_1,x}, 0)\|_1 \text{pr}_i(Z_r^{t_1,x}) dr \\
&\quad + \int_{t_1}^{t_2} \Lambda_i(T-t_1) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(u(r, X_r^{t_1,x}))| |\text{pr}_i(Z_r^{t_1,x})| \right\|_1 dr \\
&\leq 2V^2(t_1, x) \frac{\sqrt{t_2-t_1}}{\sqrt{T}} + 2c^2 T V^3(t_1, x) \frac{\sqrt{t_1-t_2}}{\sqrt{T-t_2}} \\
&\leq 4 \frac{V^3(t_1, x) + V^3(t_2, x)}{2} e^{c^2T} \frac{\sqrt{t_1-t_2}}{\sqrt{T-t_2}}. \tag{104}
\end{aligned}$$

Next, the triangle inequality, (94), the fact that  $\sum_{\nu=0}^d L_i \leq c$ , Hölder's inequality, the fact that  $\frac{6}{p_\nu} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$ , (35), (95), the fact that  $\forall x, y, p, q \in [0, \infty)$ :  $\frac{x^p+y^p}{2} \frac{x^q+y^q}{2} \leq \frac{x^{p+q}+y^{p+q}}{2}$ , and Lemma 2.1 show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T]$ ,  $t_2 \in [t_1, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\int_{t_2}^T \Lambda_i(T-t_1) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(u(r, X_r^{t_1,x}) - u(r, X_r^{t_2,x}))| |\text{pr}_i(Z_r^{t_1,x})| \right\|_1 dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \|\Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{t_1,x}) - u(r, X_r^{t_2,x}))\|_1 \|\text{pr}_i(Z_r^{t_1,x})\|_1 \right] dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \frac{c\sqrt{T}}{\sqrt{T-r}} \left\| e^{c^2T} \frac{V^6(r, X_r^{t_1,x}) + V^6(r, X_r^{t_2,x})}{2} \frac{\|X_r^{t_1,x} - X_r^{t_2,x}\|}{\sqrt{T}} \|\text{pr}_i(Z_r^{t_1,x})\|_1 \right\|_1 dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \frac{c\sqrt{T}}{\sqrt{T-r}} e^{c^2T} \\
&\quad \frac{\|V^6(r, X_r^{t_1,x})\|_{\frac{p_x}{6}} + \|V^6(r, X_r^{t_2,x})\|_{\frac{p_x}{6}}}{2} \frac{\|X_r^{t_1,x} - X_r^{t_2,x}\|_{p_x}}{\sqrt{T}} \|\text{pr}_i(Z_r^{t_1,x})\|_{p_z} dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \frac{c\sqrt{T}}{\sqrt{T-r}} e^{c^2T} \frac{V(t_1, x) + V(t_2, x)}{2} \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2-t_1}}{\sqrt{T}} \frac{c}{\Lambda_i(r-t_1)} dr \\
&\leq \int_{t_2}^T 2c^2 \frac{V^3(t_1, x) + V^3(t_2, x)}{2} \frac{\sqrt{t_2-t_1}}{\sqrt{T}} \frac{\sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t_1}}{\sqrt{r-t_1}} dr
\end{aligned}$$

$$\begin{aligned}
&\leq 2c^2T \frac{V^3(t_1, x) + V^3(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} \int_{t_1}^T \frac{dr}{\sqrt{T-r}\sqrt{r-t_1}} \\
&\leq 8c^2T \frac{V^3(t_1, x) + V^3(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}}.
\end{aligned} \tag{105}$$

Furthermore, (96) and the fact that  $\forall t_1 \in [0, T), t_2 \in [0, T): \int_{t_2}^T \frac{dr}{\sqrt{r-t_1}} = 2\sqrt{r-t_1}|_{r=t_2}^T \leq 2\sqrt{T-t_1}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\int_{t_2}^T \Lambda_i(T-t_1) \left\| \frac{1}{T} \frac{V(r, X_r^{t_1, x}) + V(r, X_r^{t_2, x})}{2} \frac{\|X_r^{t_1, x} - X_r^{t_2, x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t_1, x}) \right\|_1 dr \\
&\leq \frac{2}{T} \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} \int_{t_2}^T \frac{\sqrt{T-t_1}}{\sqrt{r-t_1}} dr \\
&\leq \frac{2}{T} \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} 2(T-t_1) \\
&\leq 4 \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}}.
\end{aligned} \tag{106}$$

This, the triangle inequality, (60), (105), the fact that  $1 \leq V$ , and the fact that  $1 + c^2T \leq e^{c^2T}$  prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\int_{t_2}^T \Lambda_i(T-t_1) \left\| [f(r, X_r^{t_1, x}, u(r, X_r^{t_1, x})) - f(r, X_r^{t_2, x}, u(r, X_r^{t_2, x}))] \text{pr}_i(Z_r^{t_1, x}) \right\|_1 dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |u(r, X_r^{t_1, x}) - u(r, X_r^{t_2, x})| \text{pr}_i(Z_r^{t_1, x}) \right\|_1 dr \\
&\quad + \int_{t_2}^T \Lambda_i(T-t_1) \left\| \frac{1}{T} \frac{V(r, X_r^{t_1, x}) + V(r, X_r^{t_2, x})}{2} \frac{\|X_r^{t_1, x} - X_r^{t_2, x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t_1, x}) \right\|_1 dr \\
&\leq 8c^2T \frac{V^3(t_1, x) + V^3(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} + 4 \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} \\
&\leq 8e^{c^2T} \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}}.
\end{aligned} \tag{107}$$

Next, (33) and (98) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\int_{t_2}^T \Lambda_i(T-t_1) \left\| f(r, X_r^{t_2, x}, 0) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1 dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \left\| \frac{1}{T} V(r, X_r^{t_2, x}) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1 dr \\
&\leq \int_{t_2}^T \frac{2}{T} \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{r-t_2}} \frac{\sqrt{T-t_1}}{\sqrt{r-t_1}} dr.
\end{aligned} \tag{108}$$

In addition, the triangle inequality, (93), Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (35), (88), the fact that  $\max\{c, 6e^{86c^6T^3}\} \leq V$ , and the fact that  $\forall x, y, p, q \in [0, \infty): \frac{x^p + y^p}{2} \frac{x^q + y^q}{2} \leq \frac{x^{p+q} + y^{p+q}}{2}$  show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\int_{t_2}^T \Lambda_i(T-t_1) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \text{pr}_\nu(u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1 dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) \left\| \text{pr}_\nu(u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1] dr
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \left\| \text{pr}_\nu(u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1 \right] dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \left\| 6ce^{86c^6 T^3} V(r, X_r^{t_2, x}) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x}) \right\|_1 \right] dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) c \frac{\sqrt{T}}{\sqrt{T-r}} 6ce^{86c^6 T^3} \|V(r, X_r^{t_2, x})\|_{p_\nu} \|\text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x})\|_{p_z} dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \frac{\sqrt{T}}{\sqrt{T-r}} V^4(t_2, x) \frac{V(t_1, x) + V(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{r - t_2} \Lambda_i(r - t_1)} dr \\
&\leq \int_{t_2}^T 2 \frac{V^5(t_1, x) + V^5(t_2, x)}{2} \frac{\sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t_1} \sqrt{t_2-t_1}}{\sqrt{r-t_1} \sqrt{r-t_2}} dr. \tag{109}
\end{aligned}$$

Thus, the triangle inequality, (60), and (108) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\int_{t_2}^T \Lambda_i(T-t_1) \|f(r, X_r^{t_2, x}, u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x})\|_1 dr \\
&\leq \int_{t_2}^T \Lambda_i(T-t_1) \|f(r, X_r^{t_2, x}, 0) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x})\|_1 dr \\
&\quad + \int_{t_2}^T \Lambda_i(T-t_1) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) |\text{pr}_\nu(u(r, X_r^{t_2, x}))| |\text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x})| \right\|_1 dr \\
&\leq \int_{t_2}^T \frac{2}{T} \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{r - t_2}} \frac{\sqrt{T - t_1}}{\sqrt{r - t_1}} dr \\
&\quad + \int_{t_2}^T 2 \frac{V^5(t_1, x) + V^5(t_2, x)}{2} \frac{\sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t_1} \sqrt{t_2-t_1}}{\sqrt{r-t_1} \sqrt{r-t_2}} dr. \tag{110}
\end{aligned}$$

Therefore, the triangle inequality, (104), and (107) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t_1 \in [0, T)$ ,  $t_2 \in [t_1, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left| \int_{t_1}^T \mathbb{E}[f(r, X_r^{t_1, x}, u(r, X_r^{t_1, x})) \text{pr}_i(Z_r^{t_1, x})] dr - \int_{t_2}^T \mathbb{E}[f(r, X_r^{t_2, x}, u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_2, x})] dr \right| \\
&\leq \int_{t_1}^{t_2} \|f(r, X_r^{t_1, x}, u(r, X_r^{t_1, x})) \text{pr}_i(Z_r^{t_1, x})\|_1 dr \\
&\quad + \int_{t_2}^T \|f(r, X_r^{t_1, x}, u(r, X_r^{t_1, x})) \text{pr}_i(Z_r^{t_1, x}) - f(r, X_r^{t_2, x}, u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_2, x})\|_1 dr \\
&\leq \int_{t_1}^{t_2} \|f(r, X_r^{t_1, x}, u(r, X_r^{t_1, x})) \text{pr}_i(Z_r^{t_1, x})\|_1 dr \\
&\quad + \int_{t_2}^T \|[f(r, X_r^{t_1, x}, u(r, X_r^{t_1, x})) - f(r, X_r^{t_2, x}, u(r, X_r^{t_2, x}))] \text{pr}_i(Z_r^{t_1, x})\|_1 dr \\
&\quad + \int_{t_2}^T \|f(r, X_r^{t_2, x}, u(r, X_r^{t_2, x})) \text{pr}_i(Z_r^{t_1, x} - Z_r^{t_2, x})\|_1 dr \\
&\leq \left[ 4 \frac{V^3(t_1, x) + V^3(t_2, x)}{2} e^{c^2 T} \frac{\sqrt{t_2 - t_1}}{\sqrt{T - t_2}} \right. \\
&\quad \left. + 8e^{c^2 T} \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{T}} \right. \\
&\quad \left. + \int_{t_2}^T \frac{2}{T} \frac{V^2(t_1, x) + V^2(t_2, x)}{2} \frac{\sqrt{t_2 - t_1}}{\sqrt{r - t_2}} \frac{\sqrt{T - t_1}}{\sqrt{r - t_1}} dr \right]
\end{aligned}$$

$$+ \int_{t_2}^T 2 \frac{V^5(t_1, x) + V^5(t_2, x)}{2} \frac{\sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t_1}\sqrt{t_2-t_1}}{\sqrt{r-t_1}\sqrt{r-t_2}} dr \left] \frac{1}{\Lambda_i(T-t_1)}. \quad (111)$$

This shows for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left| \int_{t_n}^T \mathbb{E}[f(r, X_r^{t_n, x}, u(r, X_r^{t_n, x})) \text{pr}_i(Z_r^{t_n, x})] dr - \int_t^T \mathbb{E}[f(r, X_r^{t, x}, u(r, X_r^{t, x})) \text{pr}_i(Z_r^{t, x})] dr \right| \\ & \leq \left[ 4 \frac{V^3(t_n, x) + V^3(t, x)}{2} e^{c^2 T} \frac{\sqrt{|t_n - t|}}{\sqrt{T - \max\{t, t_n\}}} \right. \\ & \quad + 8e^{c^2 T} \frac{V^4(t_1, x) + V^4(t_2, x)}{2} \frac{\sqrt{|t_n - t|}}{\sqrt{T}} \\ & \quad + \int_{\max\{t, t_n\}}^T \frac{2}{T} \frac{V^2(t_n, x) + V^2(t, x)}{2} \frac{\sqrt{|t_n - t|}}{\sqrt{r - t_n}} \frac{\sqrt{T - \min\{t, t_n\}}}{\sqrt{r - t}} dr \\ & \quad \left. + \int_{\max\{t, t_n\}}^T 2 \frac{V^5(t_1, x) + V^5(t_2, x)}{2} \frac{\sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T - \min\{t, t_n\}}\sqrt{|t_n - t|}}{\sqrt{r - t_n}\sqrt{r - t}} dr \right] \frac{1}{\Lambda_i(T - \min\{t, t_n\})}. \end{aligned} \quad (112)$$

Next, note that for all  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \int_0^T \left[ \frac{\sqrt{|t_n - t|}}{\sqrt{r - t_n}\sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) \right]^{\frac{3}{2}} dr &= \sup_{n \in \mathbb{N}} \int_{\max\{t, t_n\}}^T \left[ \frac{\sqrt{|t_n - t|}}{\sqrt{r - \min\{t, t_n\}}\sqrt{r - \max\{t, t_n\}}} \right]^{\frac{3}{2}} dr \\ &\leq \sup_{n \in \mathbb{N}} \int_{\max\{t, t_n\}}^T \left[ \frac{\sqrt{|t_n - t|}}{\sqrt{|t_n - t|}\sqrt{r - \max\{t, t_n\}}} \right]^{\frac{3}{2}} dr \\ &= \sup_{n \in \mathbb{N}} \int_{\max\{t, t_n\}}^T (r - \max\{t, t_n\})^{-\frac{3}{4}} dr \\ &= \sup_{n \in \mathbb{N}} 4(r - \max\{t, t_n\})^{\frac{1}{4}} \Big|_{r=\max\{t, t_n\}}^T \\ &\leq T^{\frac{1}{4}}. \end{aligned} \quad (113)$$

This implies that for all  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that  $([0, T) \ni r \mapsto \frac{\sqrt{|t_n - t|}}{\sqrt{r - t_n}\sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) \in \mathbb{R})_{n \in \mathbb{N}}$  is uniformly integrable. Hence, for all  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\max\{t, t_n\}}^T \frac{\sqrt{|t_n - t|}}{\sqrt{r - t_n}\sqrt{r - t}} dr &= \lim_{n \rightarrow \infty} \int_0^T \frac{\sqrt{|t_n - t|}}{\sqrt{r - t_n}\sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) dr \\ &= \int_0^T \lim_{n \rightarrow \infty} \frac{\sqrt{|t_n - t|}}{\sqrt{r - t_n}\sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) dr = 0. \end{aligned} \quad (114)$$

Next, the substitution  $s = \frac{r-a}{T-a}$ ,  $dr = (T-a)ds$  and the definition of the Beta function prove for all  $a \in [0, T)$  that

$$\int_a^T \frac{dr}{(T-r)^{\frac{3}{4}}(r-a)^{\frac{3}{4}}} = \int_0^1 \frac{(T-a)ds}{[(1-s)(T-a)]^{\frac{3}{4}}[s(T-a)]^{\frac{3}{4}}} = B\left(\frac{1}{4}, \frac{1}{4}\right)(T-a)^{-\frac{1}{2}}. \quad (115)$$

This shows that for all  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that

$$\sup_{n \in \mathbb{N}} \int_0^T \left[ \frac{\sqrt{|t_n - t|}}{\sqrt{T-r}\sqrt{r-t_n}\sqrt{r-t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) \right]^{\frac{3}{2}} dr$$

$$\begin{aligned}
&= \sup_{n \in \mathbb{N}} \int_{\max\{t, t_n\}}^T \left[ \frac{\sqrt{|t_n - t|}}{\sqrt{T - r} \sqrt{r - \min\{t, t_n\}} \sqrt{r - \max\{t, t_n\}}} \right]^{\frac{3}{2}} dr \\
&\leq \sup_{n \in \mathbb{N}} \int_{\max\{t, t_n\}}^T \left[ \frac{\sqrt{|t_n - t|}}{\sqrt{T - r} \sqrt{|t_n - t|} \sqrt{r - \max\{t, t_n\}}} \right]^{\frac{3}{2}} dr \\
&\leq \sup_{n \in \mathbb{N}} \int_{\max\{t, t_n\}}^T \frac{dr}{(T - r)^{\frac{3}{4}} (r - \max\{t, t_n\})^{\frac{3}{4}}} \\
&= \sup_{n \in \mathbb{N}} B\left(\frac{1}{4}, \frac{1}{4}\right) (T - \max\{t, t_n\})^{-\frac{1}{2}} < \infty.
\end{aligned} \tag{116}$$

Therefore, for all  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that  $([0, T) \ni r \mapsto \frac{\sqrt{|t_n - t|}}{\sqrt{T - r} \sqrt{r - t_n} \sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) \in \mathbb{R})_{n \in \mathbb{N}}$  is uniformly integrable. Hence, for all  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$  with  $\lim_{n \rightarrow \infty} t_n = t$  we have that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_{\max\{t, t_n\}}^T \frac{\sqrt{|t_n - t|}}{\sqrt{T - r} \sqrt{r - t_n} \sqrt{r - t}} dr &= \lim_{n \rightarrow \infty} \int_0^T \frac{\sqrt{|t_n - t|}}{\sqrt{T - r} \sqrt{r - t_n} \sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) dr \\
&= \int_0^T \lim_{n \rightarrow \infty} \frac{\sqrt{|t_n - t|}}{\sqrt{T - r} \sqrt{r - t_n} \sqrt{r - t}} \mathbb{1}_{(\max\{t, t_n\}, T)}(r) dr \\
&= 0.
\end{aligned} \tag{117}$$

This, (114), and (112) prove that for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  with  $t = \lim_{n \rightarrow \infty} t_n$  we have that

$$\lim_{n \rightarrow \infty} \left| \int_{t_n}^T \mathbb{E}[f(r, X_r^{t_n, x}, u(r, X_r^{t_n, x})) \text{pr}_i(Z_r^{t_n, x})] dr - \int_t^T \mathbb{E}[f(r, X_r^{t, x}, u(r, X_r^{t, x})) \text{pr}_i(Z_r^{t, x})] dr \right| = 0. \tag{118}$$

Hence, for all  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that  $[0, T) \ni t \mapsto \int_t^T \mathbb{E}[f(r, X_r^{t, x}, u(r, X_r^{t, x})) \text{pr}_i(Z_r^{t, x})] dr \in \mathbb{R}$  is continuous. This, (101), and (92) show for all  $x \in \mathbb{R}^d$  that  $[0, T) \ni t \mapsto u(t, x) \in \mathbb{R}^{d+1}$  is continuous. Therefore, (94) and the fact that  $V$  and  $\Lambda$  are continuous imply that for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $(t_n)_{n \in \mathbb{N}} \subseteq [0, T)$ ,  $t \in [0, T)$ ,  $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$  with  $\lim_{n \rightarrow \infty} (t_n, x_n) = (t, x)$  we have that

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} |\text{pr}_i(u(t_n, x_n) - u(t, x))| \\
&\leq \limsup_{n \in \mathbb{N}} |\text{pr}_i(u(t_n, x_n) - u(t_n, x))| + \limsup_{n \rightarrow \infty} |\text{pr}_i(u(t_n, x) - u(t, x))| \\
&\leq \limsup_{n \rightarrow \infty} \left[ e^{c_2 T} \frac{V^6(t_n, x_n) + V^6(t_n, x)}{2} \frac{\|x_n - x\|}{\sqrt{T}} \frac{1}{\Lambda_i(T - t_n)} \right] = 0.
\end{aligned} \tag{119}$$

Hence,  $u$  is continuous. This completes the proof of Lemma 2.9.  $\square$

### 3. PERTURBATION LEMMA

In Lemma 3.1 below we estimate the difference between two fixed points, roughly speaking, one generated from the solution to an SDE and one from an approximation schema (e.g. Euler-Maruyama schema), see (126) and (127).

**Lemma 3.1** (Perturbation lemma). *Assume Setting 2.7. Let  $\delta \in (0, 1)$ . Suppose that  $\frac{\delta}{p_v} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$  and  $\frac{1}{2} + \frac{1}{p_z} \leq 1$ . Let  $(\mathcal{X}_t^{s, x})_{s \in [0, T], t \in [s, T], x \in \mathbb{R}^d} : \{(\sigma, \tau) \in [0, T]^2 : \sigma \leq \tau\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ ,  $(\mathcal{Z}_t^{s, x})_{s \in [0, T], t \in (s, T], x \in \mathbb{R}^d} : \{(\sigma, \tau) \in [0, T]^2 : \sigma < \tau\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^{d+1}$  be measurable. Assume for all  $t \in [0, T)$ ,  $r \in (t, T)$ ,  $x \in \mathbb{R}^d$  that*

$$\|V(r, \mathcal{X}_r^{t, x})\|_{p_v} \leq V(t, x), \tag{120}$$

$$\|X_r^{t,x} - \mathcal{X}_r^{t,x}\|_{p_x} \leq \delta^{\frac{1}{2}} V(t, x), \quad \|\text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_{p_z} \leq \frac{\delta^{\frac{1}{2}} V(t, x)}{\sqrt{T} \Lambda_i(r-t)}. \quad (121)$$

Then the following items hold.

(i) There exist unique measurable functions  $u, \mathbf{u}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  such that for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T], \xi \in \mathbb{R}^d} \left[ \Lambda_\nu(T - \tau) \frac{|\text{pr}_\nu(u(\tau, \xi))|}{V(\tau, \xi)} \right] < \infty, \quad (122)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T], \xi \in \mathbb{R}^d} \left[ \Lambda_\nu(T - \tau) \frac{|\text{pr}_\nu(\mathbf{u}(\tau, \xi))|}{V(\tau, \xi)} \right] < \infty, \quad (123)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E}[|g(X_T^{t,x}) \text{pr}_\nu(Z_T^{t,x})|] + \int_t^T \mathbb{E}[|f(r, X_r^{t,x}, u(r, X_r^{t,x})) \text{pr}_\nu(Z_r^{t,x})|] dr \right] < \infty, \quad (124)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E}[|g(\mathcal{X}_T^{t,x}) \text{pr}_\nu(\mathcal{Z}_T^{t,x})|] + \int_t^T \mathbb{E}[|f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_\nu(\mathcal{Z}_r^{t,x})|] dr \right] < \infty, \quad (125)$$

$$u(t, x) = \mathbb{E}[g(X_T^{t,x}) Z_T^{t,x}] + \int_t^T \mathbb{E}[f(r, X_r^{t,x}, u(r, X_r^{t,x})) Z_r^{t,x}] dr, \quad (126)$$

and

$$\mathbf{u}(t, x) = \mathbb{E}[g(\mathcal{X}_T^{t,x}) \mathcal{Z}_T^{t,x}] + \int_t^T \mathbb{E}[f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x})) \mathcal{Z}_r^{t,x}] dr. \quad (127)$$

(ii) For all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $y \in \mathbb{R}^d$  we have that

$$\Lambda_\nu(T - t) |\text{pr}_\nu(u(t, y) - \mathbf{u}(t, y))| \leq \frac{\delta^{\frac{1}{2}} e^{c^2 T}}{\sqrt{T}} V^9(t, y). \quad (128)$$

*Proof of Lemma 3.1.* First, Lemma 2.6 and the assumptions of Lemma 3.1 prove (i) and imply that for all  $t \in [0, T]$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T - t) \frac{|\text{pr}_\nu(\mathbf{u}(t, y))|}{V(t, y)} \right] \leq 6c e^{86c^6 T^2 (T-t)}. \quad (129)$$

Furthermore, Lemma 2.8 and the assumptions of Lemma 3.1 show for all  $x, y \in \mathbb{R}^d$ ,  $t \in [0, T]$  that

$$\max_{i \in [0, d] \cap \mathbb{Z}} [\Lambda_i(T - t) |\text{pr}_i(u(t, x) - u(t, y))|] \leq e^{c^2 T} \frac{V^6(t, x) + V^6(t, y)}{2} \frac{\|x - y\|}{\sqrt{T}}. \quad (130)$$

Next, Hölder's inequality, the fact that  $\frac{6}{p_\nu} + \frac{1}{p_x} + \frac{1}{p_z} \leq 1$ , (35), (120), (121), and the fact that  $c \leq V$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $r \in (t, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \Lambda_i(T - t) \left\| \frac{V(r, X_r^{t,x}) + V(r, \mathcal{X}_r^{t,x})}{2} \frac{\|X_r^{t,x} - \mathcal{X}_r^{t,x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t,x}) \right\|_1 \\ & \leq \Lambda_i(T - t) \left\| \frac{V^6(r, X_r^{t,x}) + V^6(r, \mathcal{X}_r^{t,x})}{2} \frac{\|X_r^{t,x} - \mathcal{X}_r^{t,x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t,x}) \right\|_1 \\ & \leq \frac{\|V^6(r, X_r^{t,x})\|_{\frac{p_\nu}{6}} + \|V^6(r, \mathcal{X}_r^{t,x})\|_{\frac{p_\nu}{6}}}{2} \frac{\|X_r^{t,x} - \mathcal{X}_r^{t,x}\|_{p_x}}{\sqrt{T}} \Lambda_i(T - t) \|\text{pr}_i(Z_r^{t,x})\|_{p_z} \\ & \leq V^6(t, x) \frac{\delta^{\frac{1}{2}} V(t, x)}{\sqrt{T}} \frac{c \Lambda_i(T - t)}{\Lambda_i(r - t)} \\ & \leq \frac{c \delta^{\frac{1}{2}} V^7(t, x)}{\sqrt{T}} \frac{\sqrt{T - t}}{\sqrt{r - t}} \end{aligned}$$

$$\leq \frac{\delta^{\frac{1}{2}} V^8(t, x) \sqrt{T-t}}{\sqrt{T} \sqrt{r-t}}. \quad (131)$$

This and (61) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \Lambda_i(T-t) \left\| (g(X_T^{t,x}) - g(\mathcal{X}_T^{t,x})) \text{pr}_i(Z_T^{t,x}) \right\|_1 \\ & \leq \Lambda_i(T-t) \left\| \frac{V(T, X_T^{t,x}) + V(T, \mathcal{X}_T^{t,x})}{2} \frac{\|X_T^{t,x} - \mathcal{X}_T^{t,x}\|}{\sqrt{T}} \text{pr}_i(Z_T^{t,x}) \right\|_1 \leq \frac{\delta^{\frac{1}{2}} V^8(t, x)}{\sqrt{T}}. \end{aligned} \quad (132)$$

Next, Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (35), and (121) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \Lambda_i(T-t) \left\| g(X_T^{t,x}) \text{pr}_i(Z_T^{t,x} - \mathcal{Z}_T^{t,x}) \right\|_1 & \leq \Lambda_i(T-t) \left\| V(T, X_T^{t,y}) \text{pr}_i(Z_T^{t,x} - \mathcal{Z}_T^{t,x}) \right\|_1 \\ & \leq \Lambda_i(T-t) \left\| V(T, X_T^{t,y}) \right\|_{p_v} \left\| \text{pr}_i(Z_T^{t,x} - \mathcal{Z}_T^{t,x}) \right\|_{p_z} \\ & \leq \Lambda_i(T-t) V(t, x) \frac{\delta^{\frac{1}{2}} V(t, x)}{\sqrt{T} \Lambda_i(T-t)} \\ & \leq \frac{\delta^{\frac{1}{2}} V^2(t, x)}{\sqrt{T}}. \end{aligned} \quad (133)$$

In addition, the triangle inequality, the fact that  $\sum_{\nu=0}^d L_\nu \leq c$ , (130), (131) prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $r \in (0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \Lambda_i(T-t) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \left| \text{pr}_\nu(u(r, X_r^{t,x}) - u(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x}) \right| \right\|_1 \\ & \leq \Lambda_i(T-t) \sum_{\nu=0}^d \left\| L_\nu \Lambda_\nu(T) \text{pr}_\nu(u(r, X_r^{t,x}) - u(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x}) \right\|_1 \\ & \leq \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \left\| \Lambda_\nu(T-r) \text{pr}_\nu(u(r, X_r^{t,x}) - u(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x}) \right\|_1 \right] \\ & \leq \frac{c\sqrt{T}}{\sqrt{T-r}} \Lambda_i(T-t) \left\| e^{c^2 T} \frac{V^6(r, X_r^{t,x}) + V^6(r, \mathcal{X}_r^{t,x})}{2} \frac{\|X_r^{t,x} - \mathcal{X}_r^{t,x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t,x}) \right\|_1 \\ & \leq \frac{e^{c^2 T} c\sqrt{T}}{\sqrt{T-r}} \frac{c\delta^{\frac{1}{2}} V^7(t, x) \sqrt{T-t}}{\sqrt{T} \sqrt{r-t}} \\ & \leq \frac{\delta^{\frac{1}{2}} e^{c^2 T} c^2 T V^7(t, x)}{\sqrt{T}} \frac{1}{\sqrt{(T-r)(r-t)}}. \end{aligned} \quad (134)$$

This, (60), the triangle inequality, the fact that  $\forall t \in [0, T): \int_t^T \frac{dr}{\sqrt{r-t}} = 2\sqrt{r-t}|_{r=t}^T = 2\sqrt{T-t}$ , Lemma 2.1, the fact that  $1 \leq V$ , the fact that  $1 + c^2 T \leq e^{c^2 T}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_t^T \Lambda_i(T-t) \left\| (f(r, X_r^{t,x}, u(r, X_r^{t,x})) - f(r, \mathcal{X}_r^{t,x}, u(r, \mathcal{X}_r^{t,x}))) \text{pr}_i(Z_r^{t,x}) \right\|_1 dr \\ & \leq \int_t^T \Lambda_i(T-t) \frac{1}{T} \left\| \frac{V(r, X_r^{t,x}) + V(r, \mathcal{X}_r^{t,x})}{2} \frac{\|X_r^{t,x} - \mathcal{X}_r^{t,x}\|}{\sqrt{T}} \text{pr}_i(Z_r^{t,x}) \right\|_1 dr \\ & \quad + \int_t^T \Lambda_i(T-t) \left\| \sum_{\nu=0}^d L_\nu \Lambda_\nu(T) \left| \text{pr}_\nu(u(r, X_r^{t,x}) - u(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x}) \right| \right\|_1 dr \\ & \leq \frac{1}{T} \int_t^T \frac{\delta^{\frac{1}{2}} V^8(t, x) \sqrt{T-t}}{\sqrt{T} \sqrt{r-t}} dr + \int_t^T \frac{\delta^{\frac{1}{2}} e^{c^2 T} c^2 T V^7(t, x)}{\sqrt{T}} \frac{dr}{\sqrt{(T-r)(r-t)}} \end{aligned}$$

$$\leq 2 \frac{\delta^{\frac{1}{2}} V^8(t, x)}{\sqrt{T}} + \frac{\delta^{\frac{1}{2}} e^{c^2 T} c^2 T V^7(t, x)}{\sqrt{T}} \leq 2 \frac{\delta^{\frac{1}{2}} e^{2c^2 T} V^8(t, x)}{\sqrt{T}}. \quad (135)$$

Next, (60), the triangle inequality, Hölder's inequality, the fact that  $\frac{8}{p_v} + \frac{1}{p_z} \leq 1$ , (120), and (35) show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_t^T \Lambda_i(T-t) \left\| (f(r, \mathcal{X}_r^{t,x}, u(r, \mathcal{X}_r^{t,x})) - f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x}))) \text{pr}_i(Z_r^{t,x}) \right\|_1 dr \\ & \leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) \left\| \text{pr}_\nu(u(r, \mathcal{X}_r^{t,x}) - \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x}) \right\|_1] dr \\ & \leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \Lambda_\nu(T-r) \left\| \text{pr}_\nu(u(r, \mathcal{X}_r^{t,x}) - \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x}) \right\|_1 \right] dr \\ & \leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \left\| V^8(r, \mathcal{X}_r^{t,x}) \text{pr}_i(Z_r^{t,x}) \right\|_1 \right] dr \\ & \leq \int_t^T \Lambda_i(T-t) \frac{c\sqrt{T}}{\sqrt{T-r}} \\ & \quad \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \right] \left\| V^8(r, \mathcal{X}_r^{t,x}) \right\|_{\frac{p_v}{8}} \left\| \text{pr}_i(Z_r^{t,x}) \right\|_{p_z} dr \\ & \leq \int_t^T \Lambda_i(T-t) \frac{c\sqrt{T}}{\sqrt{T-r}} \\ & \quad \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \right] V^8(t, x) \frac{c}{\Lambda_i(r-t)} dr \\ & \leq \int_t^T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \right] V^8(t, x) \frac{c^2 \sqrt{T}}{\sqrt{T-r}} \frac{\Lambda_i(T-t)}{\Lambda_i(r-t)} dr \\ & \leq \int_t^T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \right] V^8(t, x) \frac{c^2 \sqrt{T}}{\sqrt{T-r}} \frac{\sqrt{T-t}}{\sqrt{r-t}} dr \\ & \leq \int_t^T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \right] V^8(t, x) \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr. \quad (136) \end{aligned}$$

Furthermore, (33), Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (120), (121), and the fact that  $\forall t \in [0, T): \int_t^T \frac{dr}{\sqrt{r-t}} = 2\sqrt{r-t}|_{r=t}^T = 2\sqrt{T-t}$  imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \int_t^T \Lambda_i(T-t) \left\| f(r, \mathcal{X}_r^{t,x}, 0) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x}) \right\|_1 dr \\ & \leq \int_t^T \Lambda_i(T-t) \frac{1}{T} \left\| V(r, \mathcal{X}_r^{t,x}) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x}) \right\|_1 dr \\ & \leq \frac{1}{T} \int_t^T \Lambda_i(T-t) \left\| V(r, \mathcal{X}_r^{t,x}) \right\|_{p_v} \left\| \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x}) \right\|_{p_z} dr \\ & \leq \frac{1}{T} \int_t^T \Lambda_i(T-t) V(t, x) \frac{\delta^{\frac{1}{2}} V(t, x)}{\sqrt{T} \Lambda_i(r-t)} dr \\ & \leq \frac{\delta^{\frac{1}{2}} V^2(t, x)}{T \sqrt{T}} \int_t^T \frac{\sqrt{T-t}}{\sqrt{r-t}} dr \\ & \leq 2 \frac{\delta^{\frac{1}{2}} V^2(t, x)}{\sqrt{T}}. \quad (137) \end{aligned}$$

Next, the triangle inequality, (129), the fact that  $\sum_{\nu=0}^d L_i \leq c$ , Hölder's inequality, the fact that  $\frac{1}{p_v} + \frac{1}{p_z} \leq 1$ , (120), (121), and Lemma 2.1 prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d [L_\nu \|\Lambda_\nu(T) \text{pr}_\nu(\mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1] dr \\
& \leq \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d \left[ L_\nu \frac{\sqrt{T}}{\sqrt{T-r}} \|\Lambda_\nu(T-r) \text{pr}_\nu(\mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1 \right] dr \\
& \leq \int_t^T \Lambda_i(T-t) \frac{c\sqrt{T}}{\sqrt{T-r}} \left\| 6ce^{86c^6T^3} V(r, \mathcal{X}_r^{t,x}) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x}) \right\|_1 dr \\
& \leq \int_t^T \Lambda_i(T-t) \frac{6c^2 e^{86c^6T^3} \sqrt{T}}{\sqrt{T-r}} \|V(r, \mathcal{X}_r^{t,x})\|_{p_v} \|\text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_{p_z} \\
& \leq \int_t^T \Lambda_i(T-t) \frac{6c^2 e^{86c^6T^3} \sqrt{T}}{\sqrt{T-r}} V(t, x) \frac{\delta^{\frac{1}{2}} V(t, x)}{\sqrt{T} \Lambda_i(r-t)} \\
& \leq \int_t^T \frac{6\delta^{\frac{1}{2}} c^2 e^{86c^6T^3} V^2(t, x)}{\sqrt{T}} \frac{\sqrt{T} \Lambda_i(T-t)}{\sqrt{T-r} \Lambda_i(r-t)} dr \\
& \leq \int_t^T \frac{6\delta^{\frac{1}{2}} c^2 e^{86c^6T^3} V^2(t, x)}{\sqrt{T}} \frac{\sqrt{T} \sqrt{T-t}}{\sqrt{T-r} \sqrt{r-t}} dr \\
& \leq \frac{24\delta^{\frac{1}{2}} c^2 T e^{86c^6T^3} V^2(t, x)}{\sqrt{T}}. \tag{138}
\end{aligned}$$

This, the triangle inequality, (137), and the fact that  $24e^{86c^6T^3} \leq V$  show for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $r \in (0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \int_t^T \Lambda_i(T-t) \|f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1 dr \\
& \leq \int_t^T \Lambda_i(T-t) \|f(r, \mathcal{X}_r^{t,x}, 0) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1 dr \\
& \quad + \int_t^T \Lambda_i(T-t) \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) \|\text{pr}_\nu(\mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1] dr \\
& \leq 2 \frac{\delta^{\frac{1}{2}} V^2(t, x)}{\sqrt{T}} + \frac{24\delta^{\frac{1}{2}} c^2 T e^{86c^6T^3} V^2(t, x)}{\sqrt{T}} \\
& \leq \frac{24\delta^{\frac{1}{2}} e^{c^2T} e^{86c^6T^3} V^2(t, x)}{\sqrt{T}} \\
& \leq \frac{\delta^{\frac{1}{2}} e^{c^2T} V^3(t, x)}{\sqrt{T}}. \tag{139}
\end{aligned}$$

Next, (126), (127), and the triangle inequality prove for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \Lambda_i(T-t) |\text{pr}_i(\mathbf{u}(t, x) - \mathbf{u}(t, x))| \\
& \leq \Lambda_i(T-t) |\mathbb{E}[g(X_T^{t,x}) \text{pr}_i(Z_T^{t,x}) - g(\mathcal{X}_T^{t,x}) \text{pr}_i(\mathcal{Z}_T^{t,x})]| \\
& \quad + \int_t^T |\Lambda_i(T-t) \mathbb{E}[f(r, X_r^{t,x}, \mathbf{u}(r, X_r^{t,x})) \text{pr}_i(Z_r^{t,x}) - f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(\mathcal{Z}_r^{t,x})]| dr \\
& \leq \Lambda_i(T-t) \|(g(X_T^{t,x}) - g(\mathcal{X}_T^{t,x})) \text{pr}_i(Z_T^{t,x})\|_1 + \Lambda_i(T-t) \|g(\mathcal{X}_T^{t,x}) \text{pr}_i(Z_T^{t,x} - \mathcal{Z}_T^{t,x})\|_1 \\
& \quad + \int_t^T \Lambda_i(T-t) \|(f(r, X_r^{t,x}, \mathbf{u}(r, X_r^{t,x})) - f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x}))) \text{pr}_i(Z_r^{t,x})\|_1 dr
\end{aligned}$$

$$\begin{aligned}
& + \int_t^T \Lambda_i(T-t) \|f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1 dr \\
& \leq \Lambda_i(T-t) \|(g(X_T^{t,x}) - g(\mathcal{X}_T^{t,x})) \text{pr}_i(Z_T^{t,x})\|_1 + \Lambda_i(T-t) \|g(\mathcal{X}_T^{t,y}) \text{pr}_i(Z_T^{t,x} - \mathcal{Z}_T^{t,x})\|_1 \\
& + \int_t^T \Lambda_i(T-t) \|(f(r, X_r^{t,x}, u(r, X_r^{t,x})) - f(r, \mathcal{X}_r^{t,x}, u(r, \mathcal{X}_r^{t,x}))) \text{pr}_i(Z_r^{t,x})\|_1 dr \\
& + \int_t^T \Lambda_i(T-t) \|(f(r, \mathcal{X}_r^{t,x}, u(r, \mathcal{X}_r^{t,x})) - f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x}))) \text{pr}_i(Z_r^{t,x})\|_1 dr \\
& + \int_t^T \Lambda_i(T-t) \|f(r, \mathcal{X}_r^{t,x}, \mathbf{u}(r, \mathcal{X}_r^{t,x})) \text{pr}_i(Z_r^{t,x} - \mathcal{Z}_r^{t,x})\|_1 dr. \tag{140}
\end{aligned}$$

Thus, (132), (133), (135), (136), and (139) imply for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \Lambda_i(T-t) |\text{pr}_i(u(t, x) - \mathbf{u}(t, x))| \\
& \leq \frac{\delta^{\frac{1}{2}} V^8(t, x)}{\sqrt{T}} + \frac{\delta^{\frac{1}{2}} V^2(t, x)}{\sqrt{T}} + 2 \frac{\delta^{\frac{1}{2}} e^{c^2 T} V^8(t, x)}{\sqrt{T}} \\
& + \int_t^T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) \text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))}{V^8(r, y)} \right] V^8(t, x) \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr \\
& + \frac{\delta^{\frac{1}{2}} e^{c^2 T} V^3(t, x)}{\sqrt{T}} \\
& \leq 5 \frac{\delta^{\frac{1}{2}} e^{c^2 T} V^8(t, x)}{\sqrt{T}} + \int_t^T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) \text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))}{V^8(r, y)} \right] V^8(t, x) \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr. \tag{141}
\end{aligned}$$

This shows for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$  that

$$\begin{aligned}
& \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-t) |\text{pr}_\nu(u(t, y) - \mathbf{u}(t, y))|}{V^8(t, y)} \\
& \leq 5 \frac{\delta^{\frac{1}{2}} e^{c^2 T}}{\sqrt{T}} + \int_t^T \left[ \max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(r, y) - \mathbf{u}(r, y))|}{V^8(r, y)} \right] \frac{c^2 T}{\sqrt{(T-r)(r-t)}} dr. \tag{142}
\end{aligned}$$

Therefore, (122), (123), and the Grönwall-type inequality in Corollary 2.4 imply for all  $t \in [0, T)$  that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) |\text{pr}_\nu(u(t, y) - \mathbf{u}(t, y))|}{V^8(t, y)} \leq 10 \frac{\delta^{\frac{1}{2}} e^{c^2 T}}{\sqrt{T}} e^{86c^6 T^3}. \tag{143}$$

Hence, the fact that  $10e^{86c^6 T^3} \leq V$  proves for all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $y \in \mathbb{R}^d$  that

$$\Lambda_\nu(T-t) |\text{pr}_\nu(u(t, y) - \mathbf{u}(t, y))| \leq 10 \frac{\delta^{\frac{1}{2}} e^{c^2 T}}{\sqrt{T}} e^{86c^6 T^3} V^8(t, y) \leq \frac{\delta^{\frac{1}{2}} e^{c^2 T}}{\sqrt{T}} V^9(t, y). \tag{144}$$

This shows (ii) and completes the proof of Lemma 3.1.  $\square$

#### 4. MLP APPROXIMATIONS

In Setting 4.1 below we introduce MLP approximations for solutions to SFPEs (see (153)).

**Setting 4.1.** Let  $d \in \mathbb{N}$ ,  $\Theta = \cup_{n \in \mathbb{N}} \mathbb{Z}^n$ ,  $T \in (0, \infty)$ ,  $p_\nu, p_z, p_x \in (1, \infty)$ ,  $c \in [1, \infty)$ ,  $(L_i)_{i \in [0, d] \cap \mathbb{Z}} \in \mathbb{R}^{d+1}$  satisfy that  $\sum_{i=0}^d L_i \leq c$ . Let  $\|\cdot\|: \mathbb{R}^d \rightarrow [0, \infty)$  be a norm on  $\mathbb{R}^d$ . Let  $\Lambda = (\Lambda_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}: [0, T] \rightarrow \mathbb{R}^{1+d}$  satisfy for all  $t \in [0, T]$  that  $\Lambda(t) = (1, \sqrt{t}, \dots, \sqrt{t})$ . Let  $\text{pr} = (\text{pr}_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  satisfy for all  $w = (w_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}$ ,  $i \in [0, d] \cap \mathbb{Z}$  that  $\text{pr}_i(w) = w_i$ . Let  $f \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1}, \mathbb{R})$ ,

$g \in C(\mathbb{R}^d, \mathbb{R})$ ,  $V \in C([0, T] \times \mathbb{R}^d, [0, \infty))$  satisfy that  $\max\{c, 48e^{86c^6T^3}\} \leq V$ . To shorten the notation we write for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $w: [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  that

$$(F(w))(t, x) = f(t, x, w(t, x)). \quad (145)$$

Let  $\varrho: \{(\tau, \sigma) \in [0, T]^2: \tau < \sigma\} \rightarrow \mathbb{R}$  satisfy for all  $t \in [0, T)$ ,  $s \in (t, T)$  that

$$\varrho(t, s) = \frac{1}{\mathbb{B}(\frac{1}{2}, \frac{1}{2})} \frac{1}{\sqrt{(T-s)(s-t)}}. \quad (146)$$

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. For every random variable  $\mathfrak{X}: \Omega \rightarrow \mathbb{R}$ ,  $s \in [1, \infty)$  let  $\|\mathfrak{X}\|_s \in [0, \infty]$  satisfy that  $\|\mathfrak{X}\|_s = (\mathbb{E}[|\mathfrak{X}|^s])^{\frac{1}{s}}$ . Let  $\mathfrak{r}^\theta: \Omega \rightarrow (0, 1)$ ,  $\theta \in \Theta$ , be independent and identically distributed random variables and satisfy for all  $b \in (0, 1)$  that

$$\mathbb{P}(\mathfrak{r}^\theta \leq b) = \frac{1}{\mathbb{B}(\frac{1}{2}, \frac{1}{2})} \int_0^b \frac{dr}{\sqrt{r(1-r)}}. \quad (147)$$

Let  $\mathcal{X}^\theta = (\mathcal{X}_t^{\theta, s, x})_{s \in [0, T], t \in [s, T], x \in \mathbb{R}^d}: \{(\sigma, \tau) \in [0, T]^2: \sigma \leq \tau\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ ,  $\theta \in \Theta$ , be measurable. Let  $\mathcal{Z}^\theta = (\mathcal{Z}_t^{\theta, s, x})_{s \in [0, T], t \in (s, T], x \in \mathbb{R}^d}: \{(\sigma, \tau) \in [0, T]^2: \sigma < \tau\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^{d+1}$ ,  $\theta \in \Theta$ , be measurable. Assume that  $(\mathcal{X}^\theta, \mathcal{Z}^\theta)$ ,  $\theta \in \Theta$ , are independent and identically distributed. Assume that  $(\mathcal{X}^\theta, \mathcal{Z}^\theta)_{\theta \in \Theta}$  and  $(\mathfrak{r}^\theta)_{\theta \in \Theta}$  are independent. Assume for all  $i \in [0, d] \cap \mathbb{Z}$ ,  $s \in [0, T)$ ,  $t \in [s, T)$ ,  $r \in (t, T]$ ,  $x \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$  that

$$|g(x)| \leq V(T, x), \quad |Tf(t, x, 0)| \leq V(t, x), \quad (148)$$

$$|f(t, x, w_1) - f(t, y, w_2)| \leq \sum_{\nu=0}^d [L_\nu \Lambda_\nu(T) |\text{pr}_\nu(w_1 - w_2)|] + \frac{1}{T} \frac{V(t, x) + V(t, y)}{2} \frac{\|x - y\|}{\sqrt{T}}, \quad (149)$$

$$\|V(r, \mathcal{X}_r^{0, t, x})\|_{p_\nu} \leq V(t, x), \quad \|\|\mathcal{X}_t^{0, s, x} - x\|\|_{p_x} \leq V(s, x) \sqrt{t - s}, \quad \|\text{pr}_i(\mathcal{Z}_r^{0, t, x})\|_{p_z} \leq \frac{c}{\Lambda_i(r - t)}, \quad (150)$$

$$V(T, x) \leq V(t, x), \quad |g(x) - g(y)| \leq \frac{V(T, x) + V(T, y)}{2} \frac{\|x - y\|}{\sqrt{T}}, \quad (151)$$

$$\mathbb{P}(\text{pr}_0(\mathcal{Z}_t^{0, s, x}) = 1) = 1, \quad \mathbb{E}[\mathcal{Z}_t^{0, s, x}] = (1, 0, \dots, 0). \quad (152)$$

Let  $U_{n, m}^\theta: [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ ,  $n, m \in \mathbb{Z}$ ,  $\theta \in \Theta$ , satisfy for all  $n, m \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that  $U_{-1, m}^\theta(t, x) = U_{0, m}^\theta(t, x) = 0$  and

$$\begin{aligned} U_{n, m}^\theta(t, x) &= (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta, 0, -i), t, x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta, 0, -i), t, x} \\ &+ \sum_{\ell=0}^{n-1} \sum_{i=1}^{m^{n-\ell}} \frac{\left(F(U_{\ell, m}^{(\theta, \ell, i)}) - \mathbb{1}_{\mathbb{N}}(\ell) F(U_{\ell-1, m}^{(\theta, \ell, -i)})\right) \left(t + (T-t)\mathfrak{r}^{(\theta, \ell, i)}, \mathcal{X}_{t+(T-t)\mathfrak{r}^{(\theta, \ell, i)}}^{(\theta, \ell, i), t, x}\right)}{m^{n-\ell} \varrho(t, t + (T-t)\mathfrak{r}^{(\theta, \ell, i)})} \mathcal{Z}_{t+(T-t)\mathfrak{r}^{(\theta, \ell, i)}}^{(\theta, \ell, i), t, x}. \end{aligned} \quad (153)$$

In Lemma 4.2 below we first study independence and distributional properties of MLP approximations.

**Lemma 4.2** (Independence and distributional properties). *Assume Setting 4.1. Then the following items hold.*

- (i) We have for all  $n \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ ,  $\theta \in \Theta$  that  $U_{n, m}^\theta$  is measurable,
- (ii) We have for all  $n \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ ,  $\theta \in \Theta$  that

$$\begin{aligned} &\sigma(\{U_{n, m}^\theta(t, x): t \in [0, T), x \in \mathbb{R}^d\}) \\ &\subseteq \sigma(\{\mathfrak{r}^{(\theta, \nu)}, \mathcal{X}_t^{(\theta, \nu), s, x}, \mathcal{Z}_t^{(\theta, \nu), s, x}: \nu \in \Theta, s \in [0, T), t \in (s, T], x \in \mathbb{R}^d\}) \end{aligned} \quad (154)$$

(iii) We have for all  $\theta \in \Theta$ ,  $m \in \mathbb{N}$  that  $(U_{\ell,m}^{\theta,\ell,i})_{t \in [0,T], x \in \mathbb{R}^d}$ ,  $(U_{\ell-1,m}^{\theta,\ell,-i})_{t \in [0,T], x \in \mathbb{R}^d}$ ,  $((\mathcal{X}_t^{(\theta,\ell,i),s,x})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d}, (\mathcal{Z}_t^{(\theta,\ell,i),s,x})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d})$ ,  $\mathbf{r}^{(\theta,\ell,i)}$ ,  $i \in \mathbb{N}$ ,  $\ell \in \mathbb{N}_0$ , are independent,

(iv) We have for all  $n \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$  that  $(U_{n,m}^\theta(t, x))_{t \in [0,T], x \in \mathbb{R}^d}$ ,  $\theta \in \Theta$ , are identically distributed.

(v) We have for all  $\theta \in \Theta$ ,  $\ell \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\frac{\left(F(U_{\ell,m}^{\theta,\ell,i}) - \mathbb{1}_{\mathbb{N}}(\ell)F(U_{\ell-1,m}^{\theta,\ell,-i})\right)\left(t + (T-t)\mathbf{r}^{(\theta,\ell,i)}, \mathcal{X}_{t+(T-t)\mathbf{r}^{(\theta,\ell,i)}}^{\theta,\ell,i,t,x}\right) \mathcal{Z}_{t+(T-t)\mathbf{r}^{(\theta,\ell,i)}}^{\theta,\ell,i,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^{(\theta,\ell,i)})}, \quad i \in \mathbb{N}, \quad (155)$$

are independent and identically distributed and have the same distribution as

$$\frac{\left(F(U_{\ell,m}^0) - \mathbb{1}_{\mathbb{N}}(\ell)F(U_{\ell-1,m}^1)\right)\left(t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}\right) \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^0)}, \quad i \in \mathbb{N}. \quad (156)$$

*Proof of Lemma 4.2.* The assumptions on measurability and distributions, basic properties of measurable functions, and induction prove (i) and (ii). In addition, (ii) and the assumptions on independence prove (iii). Furthermore, (iii), the fact that  $\forall \theta \in \Theta, m \in \mathbb{N}: U_{0,m}^\theta = 0$ , (153), the disintegration theorem, the assumptions on distributions, and induction establish (iv) and (v).  $\square$

In Proposition 4.3 below we establish an upper bounds for the  $L^2$ -distances between the exact solutions of the considered stochastic fixed point equations and the proposed MLP approximations. Our main idea here is the use of a family of semi-norms which allows us to get the recursive inequality (161).

**Proposition 4.3** (Error analysis by semi-norms). *Assume Setting 4.1. Let  $q_1 \in [3, \infty)$ . Assume that  $\frac{1}{p_v} + \frac{1}{p_x} + \frac{1}{p_z} \leq \frac{1}{2}$ . For every random field  $H: [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^{d+1}$  let  $\|H\|_s$ ,  $s \in [0, T]$ , satisfy for all  $s \in [0, T]$  that*

$$\|H\|_s = \max_{\nu \in [0,d] \cap \mathbb{Z}} \sup_{r \in [s,T], x \in \mathbb{R}^d} \frac{\Lambda_\nu(T-r) \|\text{pr}_\nu(H(r, x))\|_2}{V^{q_1}(r, x)}. \quad (157)$$

Then the following items hold.

(i) There exists a unique measurable function  $\mathbf{u}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  such that for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0,d] \cap \mathbb{Z}} \sup_{\tau \in [0,T], \xi \in \mathbb{R}^d} \left[ \Lambda_\nu(T-\tau) \frac{|\text{pr}_\nu(\mathbf{u}(\tau, \xi))|}{V(\tau, \xi)} \right] < \infty, \quad (158)$$

$$\max_{\nu \in [0,d] \cap \mathbb{Z}} \left[ \mathbb{E}[|g(\mathcal{X}_T^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_T^{0,t,x})|] + \int_t^T \mathbb{E}[|f(r, \mathcal{X}_r^{0,t,x}, \mathbf{u}(r, \mathcal{X}_r^{0,t,x})) \text{pr}_\nu(\mathcal{Z}_r^{0,t,x})|] dr \right] < \infty, \quad (159)$$

and

$$\mathbf{u}(t, x) = \mathbb{E}[g(\mathcal{X}_T^{0,t,x}) \mathcal{Z}_T^{0,t,x}] + \int_t^T \mathbb{E}[f(r, \mathcal{X}_r^{0,t,x}, \mathbf{u}(r, \mathcal{X}_r^{0,t,x})) \mathcal{Z}_r^{0,t,x}] dr. \quad (160)$$

(ii) For all  $n, m \in \mathbb{N}$ ,  $t \in [0, T]$  we have that

$$\|U_{n,m}^0 - \mathbf{u}\|_t \leq \frac{4}{\sqrt{m^n}} + \sum_{\ell=0}^{n-1} \left[ \frac{8c^2 T^{\frac{5}{6}}}{\sqrt{m^{n-\ell-1}}} \left[ \int_t^T \|U_{\ell,m}^0 - \mathbf{u}\|_s^6 \right]^{\frac{1}{6}} \right]. \quad (161)$$

(iii) For all  $n, m \in \mathbb{N}$ ,  $t \in [0, T]$  we have that

$$\|U_{n,m}^0 - \mathbf{u}\|_t \leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2 T}. \quad (162)$$

(iv) For all  $n, m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  we have that

$$\Lambda_\nu(T-t) \|\text{pr}_\nu(U_{n,m}^0(t, x) - \mathbf{u}(t, x))\|_2 \leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2 T} V^{q_1}(t, x). \quad (163)$$

*Proof of Proposition 4.3.* First, Lemma 2.6 and the assumption of Proposition 4.3 show (i) and imply that for all  $t \in [0, T)$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{y \in \mathbb{R}^d} \left[ \Lambda_\nu(T-t) \frac{|\text{pr}_\nu(\mathbf{u}(t, y))|}{V(t, y)} \right] \leq 6ce^{86c^6 T^2(T-t)}. \quad (164)$$

Thus, the fact that  $\max\{c, 6e^{86c^6 T^3}\} \leq V$  implies for all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $y \in \mathbb{R}^d$  that  $\Lambda_\nu(T-t)|\text{pr}_\nu(\mathbf{u}(t, y))| \leq V^3(t, y)$ . This, (157), and the fact that  $q_1 \geq 3$  prove for all  $t \in [0, T)$  that

$$\|\mathbf{u}\|_t \leq 1. \quad (165)$$

Next, (148), the fact that  $p_\nu \geq 2$ , Jensen's inequality, and (150) show for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\|g(\mathcal{X}_T^{0,t,x})\|_2 \leq \|V(T, \mathcal{X}_T^{0,t,x})\|_2 \leq \|V(T, \mathcal{X}_T^{0,t,x})\|_{p_\nu} \leq V(t, x). \quad (166)$$

Furthermore, the definition of  $\Lambda$ , (151), Hölder's inequality, the fact that  $\frac{1}{p_\nu} + \frac{1}{p_x} + \frac{1}{p_z} \leq \frac{1}{2}$ , and (150) prove for all  $\nu \in [1, d] \cap \mathbb{Z}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \|\Lambda_\nu(T-t)(g(\mathcal{X}_T^{0,t,x}) - g(x))\text{pr}_\nu(\mathcal{Z}_T^{0,t,x})\|_2 \\ & \leq \left\| \sqrt{T-t} \frac{V(T, \mathcal{X}_T^{0,t,x}) + V(T, x)}{2} \frac{\|\mathcal{X}_T^{0,t,x} - x\|}{\sqrt{T}} \text{pr}_\nu(\mathcal{Z}_T^{0,t,x}) \right\|_2 \\ & \leq \sqrt{T-t} \frac{\|V(T, \mathcal{X}_T^{0,t,x})\|_{p_\nu} + V(t, x)}{2} \frac{\|\|\mathcal{X}_T^{0,t,x} - x\|\|_{p_x}}{\sqrt{T}} \|\text{pr}_\nu(\mathcal{Z}_T^{0,t,x})\|_{p_z} \\ & \leq \sqrt{T-t} V(t, x) \frac{V(t, x) \sqrt{T-t}}{\sqrt{T}} \frac{c}{\sqrt{T-t}} \\ & \leq V^3(t, x). \end{aligned} \quad (167)$$

In addition, (152), the triangle inequality, (166), the independence and distributional properties (cf. Lemma 4.2), and a standard property of the variance imply for all  $m, n \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left\| \Lambda_0(T-t)\text{pr}_0 \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta,0,-i),t,x} \right) \right\|_2 \\ & = \left\| \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x})}{m^n} \right\|_2 \leq \|g(\mathcal{X}_T^{0,t,x})\|_2 \leq V(t, x) \end{aligned} \quad (168)$$

and

$$\begin{aligned} & \left( \text{var} \left[ \Lambda_0(T-t)\text{pr}_0 \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta,0,-i),t,x} \right) \right] \right)^{\frac{1}{2}} \\ & = \left( \text{var} \left[ \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x})}{m^n} \right] \right)^{\frac{1}{2}} = \frac{(\text{var}[g(\mathcal{X}_T^{0,t,x})])^{\frac{1}{2}}}{\sqrt{m^n}} \leq \frac{\|g(\mathcal{X}_T^{0,t,x})\|_2}{\sqrt{m^n}} \leq \frac{V(t, x)}{\sqrt{m^n}}. \end{aligned} \quad (169)$$

Next, (152), the triangle inequality, (167), the independence and distributional properties (cf. Lemma 4.2), and a standard property of the variance show for all  $m, n \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $\nu \in [1, d] \cap \mathbb{Z}$  that

$$\begin{aligned} & \left\| \Lambda_\nu(T-t)\text{pr}_\nu \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta,0,-i),t,x} \right) \right\|_2 \\ & = \left\| \Lambda_\nu(T-t) \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \text{pr}_\nu(\mathcal{Z}_T^{(\theta,0,-i),t,x}) \right\|_2 \end{aligned}$$

$$\leq \left\| \Lambda_\nu(T-t)(g(\mathcal{X}_T^{0,t,x}) - g(x)) \text{pr}_\nu(\mathcal{Z}_T^{0,t,x}) \right\|_2 \leq V^3(t, x) \quad (170)$$

and

$$\begin{aligned} & \left( \text{var} \left[ \Lambda_\nu(T-t) \text{pr}_\nu \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta,0,-i),t,x} \right) \right] \right)^{\frac{1}{2}} \\ &= \left( \text{var} \left[ \Lambda_\nu(T-t) \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \text{pr}_\nu(\mathcal{Z}_T^{(\theta,0,-i),t,x}) \right] \right)^{\frac{1}{2}} \\ &= \left( \frac{\text{var} [\Lambda_\nu(T-t)(g(\mathcal{X}_T^{0,t,x}) - g(x)) \text{pr}_\nu(\mathcal{Z}_T^{0,t,x})]}{m^n} \right)^{\frac{1}{2}} \\ &\leq \frac{\left\| \Lambda_\nu(T-t)(g(\mathcal{X}_T^{0,t,x}) - g(x)) \text{pr}_\nu(\mathcal{Z}_T^{0,t,x}) \right\|_2}{\sqrt{m^n}} \leq \frac{V^3(t, x)}{\sqrt{m^n}}. \end{aligned} \quad (171)$$

This, (168), (169), and the fact that  $1 \leq V$  imply for all  $m, n \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \Lambda_\nu(T-t) \text{pr}_\nu \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta,0,-i),t,x} \right) \right\|_2 \leq V^3(t, x) \quad (172)$$

and

$$\left( \text{var} \left[ \Lambda_\nu(T-t) \text{pr}_\nu \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(\theta,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(\theta,0,-i),t,x} \right) \right] \right)^{\frac{1}{2}} \leq \frac{V^3(t, x)}{\sqrt{m^n}}. \quad (173)$$

Furthermore, Hölder's inequality, the fact that  $\frac{1}{p_\nu} + \frac{1}{p_z} \leq \frac{1}{2}$ , (150), and the fact that  $c \leq V$  show for all  $t \in [0, T)$ ,  $s \in (t, T]$ ,  $x \in \mathbb{R}^d$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  that

$$\begin{aligned} \left\| V(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x}) \right\|_2 &\leq \left\| V(s, \mathcal{X}_s^{0,t,x}) \right\|_{p_\nu} \left\| \text{pr}_\nu(\mathcal{Z}_s^{0,t,x}) \right\|_{p_z} \\ &\leq V(t, x) \frac{c}{\Lambda_\nu(s-t)} \leq \frac{V^2(t, x)}{\Lambda_\nu(s-t)}. \end{aligned} \quad (174)$$

Next, the substitution  $s = t + (T-t)r$ ,  $ds = (T-t)dr$ ,  $r = 0 \Rightarrow s = t$ ,  $r = \frac{b-t}{T-t} \Rightarrow s = b$ ,  $r = \frac{s-t}{T-t}$ ,  $1-r = 1 - \frac{s-t}{T-t} = \frac{T-s}{T-t}$  and (146) prove for all  $t \in [0, T)$ ,  $b \in (t, T)$  that

$$\begin{aligned} \mathbb{P}(t + (T-t)\mathfrak{r}^0 \leq b) &= \mathbb{P}(\mathfrak{r}^0 \leq \frac{b-t}{T-t}) = \frac{1}{\text{B}(\frac{1}{2}, \frac{1}{2})} \int_0^{\frac{b-t}{T-t}} \frac{dr}{\sqrt{r(1-r)}} \\ &= \frac{1}{\text{B}(\frac{1}{2}, \frac{1}{2})} \int_t^b \frac{\frac{ds}{T-t}}{\sqrt{\frac{s-t}{T-t} \frac{T-s}{T-t}}} = \frac{1}{\text{B}(\frac{1}{2}, \frac{1}{2})} \int_t^b \frac{ds}{\sqrt{(T-s)(s-t)}} = \int_t^b \varrho(t, s) ds. \end{aligned} \quad (175)$$

This shows for all  $t \in [0, T)$  and all measurable functions  $h: (t, T) \rightarrow [0, \infty)$  that

$$\mathbb{E}[h(t + (T-t)\mathfrak{r}^0)] = \int_t^T h(s) \varrho(t, s) ds. \quad (176)$$

Hence, the independence and distributional properties (cf. Lemma 4.2), the disintegration theorem, (148), (174), (146), the fact that  $\forall t \in [0, T): \int_t^T \frac{dr}{\sqrt{r-t}} = 2\sqrt{r-t}|_{r=t}^T = 2\sqrt{T-t}$ , the fact that  $\sqrt{2\text{B}(\frac{1}{2}, \frac{1}{2})} \leq 3$  imply for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \frac{\Lambda_\nu(T-t)(F(0)) \left( t + (T-t)\mathfrak{r}^0, \mathcal{X}_{t+(T-t)\mathfrak{r}^0}^{0,t,x} \right) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathfrak{r}^0}^{0,t,x} \right)}{\varrho(t, t + (T-t)\mathfrak{r}^0)} \right\|_2$$

$$\begin{aligned}
&= \left[ \int_t^T \left\| \frac{\Lambda_\nu(T-t)(F(0))(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})}{\varrho(t,s)} \right\|_2^2 \varrho(t,s) ds \right]^{\frac{1}{2}} \\
&= \left[ \int_t^T \frac{\|\Lambda_\nu(T-t)(F(0))(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})\|_2^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\|\Lambda_\nu(T-t)\frac{1}{T}V(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})\|_2^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\Lambda_\nu^2(T-t)\frac{1}{T^2}\frac{V^4(t,x)}{\Lambda_\nu^2(s-t)}}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\frac{1}{T^2}V^4(t,x)\frac{T-t}{s-t}}{\frac{1}{B(\frac{1}{2},\frac{1}{2})}\sqrt{(T-s)(s-t)}} ds \right]^{\frac{1}{2}} \\
&= \frac{V^2(t,x)}{T} \sqrt{B(\frac{1}{2},\frac{1}{2})} \left[ (T-t)^{\frac{3}{2}} \int_t^T \frac{ds}{\sqrt{s-t}} \right]^{\frac{1}{2}} \\
&= \frac{V^2(t,x)}{T} \sqrt{B(\frac{1}{2},\frac{1}{2})} \left[ (T-t)^{\frac{3}{2}} 2\sqrt{T-t} \right]^{\frac{1}{2}} \\
&\leq 3V^2(t,x). \tag{177}
\end{aligned}$$

Next, (176), the independence and distributional properties (cf. Lemma 4.2), the disintegration theorem, (149), the triangle inequality, (157), (174), (146) prove for all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $j \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| \frac{\Lambda_\nu(T-t)(F(U_{j,m}^0) - F(\mathbf{u}))\left(t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}\right) \text{pr}_\nu\left(\mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}\right)}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right\|_2 \\
&= \left[ \int_t^T \left\| \frac{\Lambda_\nu(T-t)(F(U_{j,m}^0) - F(\mathbf{u}))(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})}{\varrho(t,s)} \right\|_2^2 \varrho(t,s) ds \right]^{\frac{1}{2}} \\
&= \left[ \int_t^T \frac{\|\Lambda_\nu(T-t)(F(U_{j,m}^0) - F(\mathbf{u}))(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})\|_2^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\|\Lambda_\nu(T-t) \sum_{i=0}^d L_i \Lambda_i(T) \|(U_{j,m}^0 - \mathbf{u})(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})\|_2\|_2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\left[ \Lambda_\nu(T-t) \sum_{i=0}^d L_i \Lambda_i(T) \|(U_{j,m}^0 - \mathbf{u})(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})\|_2 \right]^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\left[ \Lambda_\nu(T-t) \sum_{i=0}^d L_i \frac{\sqrt{T}}{\sqrt{T-s}} \|\Lambda_i(T-s)(U_{j,m}^0 - \mathbf{u})(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x})\|_2 \right]^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \left[ \int_t^T \frac{\left[ \Lambda_\nu(T-t) \sum_{i=0}^d L_i \frac{\sqrt{T}}{\sqrt{T-s}} \left\| \Lambda_i(T-s)(U_{j,m}^0 - \mathbf{u})(s, \tilde{x}) \text{pr}_\nu(\tilde{z}) \right\|_2 \Big|_{\tilde{x}=\mathcal{X}_s^{0,t,x}, \tilde{z}=\mathcal{Z}_s^{0,t,x}} \right]_2^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\left[ \Lambda_\nu(T-t) c \frac{\sqrt{T}}{\sqrt{T-s}} \left\| U_{j,m}^0 - \mathbf{u} \right\|_s \left\| V^{q_1}(s, \mathcal{X}_s^{0,t,x}) \text{pr}_\nu(\mathcal{Z}_s^{0,t,x}) \right\|_2 \right]^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\left[ \Lambda_\nu(T-t) c \frac{\sqrt{T}}{\sqrt{T-s}} \left\| U_{j,m}^0 - \mathbf{u} \right\|_s V^{q_1}(t,x) \frac{c}{\Lambda_\nu(s-t)} \right]^2}{\varrho(t,s)} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{\left[ c^2 \left\| U_{j,m}^0 - \mathbf{u} \right\|_s V^{q_1}(t,x) \frac{\sqrt{T}}{\sqrt{T-s}} \frac{\sqrt{T-t}}{\sqrt{s-t}} \right]^2}{\frac{1}{B(\frac{1}{2}, \frac{1}{2})} \frac{1}{\sqrt{(T-s)(s-t)}}}} ds \right]^{\frac{1}{2}} \\
&\leq \left[ \int_t^T \frac{c^4 B(\frac{1}{2}, \frac{1}{2}) \left[ \left\| U_{j,m}^0 - \mathbf{u} \right\|_s V^{q_1}(t,x) \sqrt{T} \sqrt{T-t} \right]^2}{\sqrt{(T-s)(s-t)}} ds \right]^{\frac{1}{2}} \\
&= c^2 \sqrt{T} \sqrt{T-t} \sqrt{B(\frac{1}{2}, \frac{1}{2})} V^{q_1}(t,x) \left[ \int_t^T \frac{\left\| U_{j,m}^0 - \mathbf{u} \right\|_s^2}{\sqrt{(T-s)(s-t)}} ds \right]^{\frac{1}{2}}. \tag{178}
\end{aligned}$$

Furthermore, the substitution  $s = t + (T-t)r$ ,  $ds = (T-t)dr$ ,  $s = t \Rightarrow r = 0$ ,  $s = T \Rightarrow r = 1$ ,  $T-s = T-t-(T-t)r = (T-t)(1-r)$ ,  $s-t = (T-t)r$  and the definition of the beta functions imply for all  $t \in [0, T)$  that

$$\int_t^T \frac{ds}{(T-s)^{\frac{3}{4}}(s-t)^{\frac{3}{4}}} = \int_0^1 \frac{(T-t)dr}{[(T-t)(1-r)]^{\frac{3}{4}}[(T-t)r]^{\frac{3}{4}}} = (T-t)^{-\frac{1}{2}} B(\frac{1}{4}, \frac{1}{4}). \tag{179}$$

Therefore, (178), Hölder's inequality, the fact that  $\frac{2}{3} + \frac{1}{3} = 1$ , and the fact that  $(B(\frac{1}{4}, \frac{1}{4}))^{\frac{1}{3}} \leq 2$  show for all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $j \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| \frac{\Lambda_\nu(T-t) (F(U_{j,m}^0) - F(\mathbf{u})) \left( t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x} \right) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x} \right)}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right\|_2 \\
&\leq c^2 \sqrt{T} \sqrt{T-t} \sqrt{B(\frac{1}{2}, \frac{1}{2})} V^{q_1}(t,x) \left[ \int_t^T \frac{ds}{(T-s)^{\frac{1}{2}}(s-t)^{\frac{3}{2}}} \right]^{\frac{2}{3}} \left[ \int_t^T \left\| U_{j,m}^0 - \mathbf{u} \right\|_s^{2 \cdot 3} \right]^{\frac{1}{3}} \\
&\leq c^2 \sqrt{T} \sqrt{T-t} \sqrt{B(\frac{1}{2}, \frac{1}{2})} V^{q_1}(t,x) \left[ \int_t^T \frac{ds}{(T-s)^{\frac{3}{4}}(s-t)^{\frac{3}{4}}} \right]^{\frac{1}{3}} \left[ \int_t^T \left\| U_{j,m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \\
&\leq 2c^2 \sqrt{T} \sqrt{T-t} V^{q_1}(t,x) \left[ (T-t)^{-\frac{1}{2}} B(\frac{1}{4}, \frac{1}{4}) \right]^{\frac{1}{3}} \left[ \int_t^T \left\| U_{j,m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \\
&\leq 4c^2 T^{\frac{5}{6}} V^{q_1}(t,x) \left[ \int_t^T \left\| U_{j,m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}}. \tag{180}
\end{aligned}$$

This, the triangle inequality, and the distributional and independence properties (cf. Lemma 4.2) prove for all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $\ell, m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left\| \frac{\Lambda_\nu(T-t) (F(U_{\ell,m}^0) - F(U_{\ell-1,m}^1)) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \text{Pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x} \right)}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right\|_2 \\ & \leq \sum_{j=\ell-1}^{\ell} \left\| \frac{\Lambda_\nu(T-t) (F(U_{j,m}^0) - F(\mathbf{u})) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \text{Pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x} \right)}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right\|_2 \\ & \leq 4c^2 T^{\frac{5}{8}} V^{q_1}(t, x) \sum_{j=\ell-1}^{\ell} \left[ \int_t^T \|U_{j,m}^0 - \mathbf{u}\|_s^6 \right]^{\frac{1}{6}}. \end{aligned} \quad (181)$$

Hence, (153), the fact that  $\forall \theta \in \Theta, m \in \mathbb{N}: U_{0,m}^\theta = 0$ , (172), (177), (165), the independence and distributional properties (cf. Lemma 4.2), and an induction argument prove for all  $n, m \in \mathbb{N}$ ,  $\ell \in [0, n-1] \cap \mathbb{Z}$ ,  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\|U_{n,m}^0\|_t + \left\| \frac{\Lambda_\nu(T-t) (F(U_{\ell,m}^0)) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \text{Pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x} \right)}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right\|_2 < \infty. \quad (182)$$

This, linearity of the expectation, and the independence and distributional properties (cf. Lemma 4.2) imply for all  $n, m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \mathbb{E}[U_{n,m}^0(t, x)] \\ & = (g(x), 0) + \sum_{i=1}^{m^n} \mathbb{E} \left[ \frac{g(\mathcal{X}_T^{(0,0,-i),t,x}) - g(x)}{m^n} \mathcal{Z}_T^{(0,0,-i),t,x} \right] \\ & + \sum_{\ell=0}^{n-1} \mathbb{E} \left[ \sum_{i=1}^{m^{n-\ell}} \frac{(F(U_{\ell,m}^{(0,\ell,i)}) - \mathbb{1}_{\mathbb{N}}(\ell) F(U_{\ell-1,m}^{(0,\ell,-i)})) (t + (T-t)\mathbf{r}^{(0,\ell,i)}, \mathcal{X}_{t+(T-t)\mathbf{r}^{(0,\ell,i)}}^{(0,\ell,i),t,x}) \mathcal{Z}_{t+(T-t)\mathbf{r}^{(0,\ell,i)}}^{(0,\ell,i),t,x}}{m^{n-\ell} \varrho(t, t + (T-t)\mathbf{r}^{(0,\ell,i)})} \right] \\ & = (g(x), 0) + \mathbb{E}[(g(\mathcal{X}_T^{0,t,x}) - g(x)) \mathcal{Z}_T^{0,t,x}] \\ & + \sum_{\ell=0}^{n-1} \left( \mathbb{E} \left[ \frac{F(U_{\ell,m}^0) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right] \right. \\ & \quad \left. - \mathbb{1}_{\mathbb{N}}(\ell) \mathbb{E} \left[ \frac{F(U_{\ell-1,m}^0) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right] \right) \\ & = \mathbb{E}[(g(\mathcal{X}_T^{0,t,x}) \mathcal{Z}_T^{0,t,x})] + \mathbb{E} \left[ \frac{F(U_{n-1,m}^0) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right]. \end{aligned} \quad (183)$$

Next, (160), (176), the disintegration theorem, the independence and distributional properties (cf. Lemma 4.2) prove for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \mathbf{u}(t, x) & = \mathbb{E}[g(\mathcal{X}_T^{0,t,x}) \mathcal{Z}_T^{0,t,x}] + \int_t^T \mathbb{E} \left[ \frac{f(r, \mathcal{X}_r^{0,t,x}, \mathbf{u}(r, \mathcal{X}_r^{0,t,x})) \mathcal{Z}_r^{0,t,x}}{\varrho(t, r)} \right] \varrho(t, r) dr \\ & = \mathbb{E}[(g(\mathcal{X}_T^{0,t,x}) \mathcal{Z}_T^{0,t,x})] + \mathbb{E} \left[ \frac{F(\mathbf{u})(t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right]. \end{aligned} \quad (184)$$

This, (183), Jensen's inequality, and (180) imply for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  that

$$\begin{aligned} & \Lambda_\nu(T-t) |\text{Pr}_\nu(\mathbb{E}[U_{n,m}^0(t, x)] - \mathbf{u}(t, x))| \\ & \leq \left\| \Lambda_\nu(T-t) \frac{(F(U_{n-1,m}^0) - F(\mathbf{u})) (t + (T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0,t,x}) \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0,t,x}}{\varrho(t, t + (T-t)\mathbf{r}^0)} \right\|_2 \end{aligned}$$

$$\leq 4c^2 T^{\frac{5}{6}} V^{q_1}(t, x) \left[ \int_t^T \left\| U_{n-1, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}}. \quad (185)$$

Furthermore, (153), the triangle inequality, the independence and distributional properties (cf. Lemma 4.2), (171), (177), (178), and the fact that  $1 \leq V$  prove for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  that

$$\begin{aligned} & \Lambda_\nu(T-t) \left( \text{var} \left[ \text{pr}_\nu(U_{n, m}^0(t, x)) \right] \right)^{\frac{1}{2}} \\ &= \Lambda_\nu(T-t) \left( \text{var} \left[ \text{pr}_\nu \left( (g(x), 0) + \sum_{i=1}^{m^n} \frac{g(\mathcal{X}_T^{(0,0,-i), t, x}) - g(x)}{m^n} \mathcal{Z}_T^{(0,0,-i), t, x} \right) \right] \right)^{\frac{1}{2}} \\ &+ \sum_{\ell=0}^{n-1} \left( \text{var} \left[ \sum_{i=1}^{m^{n-\ell}} \frac{\Lambda_\nu(T-t) (F(U_{\ell, m}^{(0, \ell, i)}) - \mathbb{1}_N(\ell) F(U_{\ell-1, m}^{(0, \ell, -i)})) (t+(T-t)\mathbf{r}^{(0, \ell, i)}, \mathcal{X}_{t+(T-t)\mathbf{r}^{(0, \ell, i)}}^{(0, \ell, i), t, x}) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^{(0, \ell, i)}}^{(0, \ell, i), t, x} \right)}{m^{n-\ell} \varrho(t, t+(T-t)\mathbf{r}^{(0, \ell, i)})} \right] \right)^{\frac{1}{2}} \\ &\leq \frac{V^3(t, x)}{\sqrt{m^n}} + \sum_{\ell=0}^{n-1} \frac{1}{\sqrt{m^{n-\ell}}} \left( \text{var} \left[ \frac{\Lambda_\nu(T-t) (F(U_{\ell, m}^0) - \mathbb{1}_N(\ell) F(U_{\ell-1, m}^1)) (t+(T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0, t, x}) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0, t, x} \right)}{\varrho(t, t+(T-t)\mathbf{r}^0)} \right] \right)^{\frac{1}{2}} \\ &\leq \frac{V^3(t, x)}{\sqrt{m^n}} + \sum_{\ell=0}^{n-1} \left[ \frac{1}{\sqrt{m^{n-\ell}}} \left\| \frac{\Lambda_\nu(T-t) (F(U_{\ell, m}^0) - \mathbb{1}_N(\ell) F(U_{\ell-1, m}^1)) (t+(T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0, t, x}) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0, t, x} \right)}{\varrho(t, t+(T-t)\mathbf{r}^0)} \right\|_2 \right] \\ &\leq \frac{V^3(t, x)}{\sqrt{m^n}} + \frac{1}{\sqrt{m^n}} \left\| \frac{\Lambda_\nu(T-t) (F(0)) (t+(T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0, t, x}) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0, t, x} \right)}{\varrho(t, t+(T-t)\mathbf{r}^0)} \right\|_2 \\ &+ \sum_{\ell=1}^{n-1} \left[ \frac{1}{\sqrt{m^{n-\ell}}} \left\| \frac{\Lambda_\nu(T-t) (F(U_{\ell, m}^0) - F(U_{\ell-1, m}^1)) (t+(T-t)\mathbf{r}^0, \mathcal{X}_{t+(T-t)\mathbf{r}^0}^{0, t, x}) \text{pr}_\nu \left( \mathcal{Z}_{t+(T-t)\mathbf{r}^0}^{0, t, x} \right)}{\varrho(t, t+(T-t)\mathbf{r}^0)} \right\|_2 \right] \\ &\leq \frac{V^3(t, x)}{\sqrt{m^n}} + \frac{3V^2(t, x)}{\sqrt{m^n}} + \sum_{\ell=1}^{n-1} \left[ \frac{1}{\sqrt{m^{n-\ell}}} \left[ 4c^2 T^{\frac{5}{6}} V^{q_1}(t, x) \sum_{j=\ell-1}^{\ell} \left[ \int_t^T \left\| U_{j, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \right] \right] \\ &\leq \frac{4V^3(t, x)}{\sqrt{m^n}} + \sum_{\ell=1}^{n-1} \left[ \sum_{j=\ell-1}^{\ell} \frac{4c^2 T^{\frac{5}{6}} V^{q_1}(t, x)}{\sqrt{m^{n-j-1}}} \left[ \int_t^T \left\| U_{j, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \right] \\ &= \frac{4V^3(t, x)}{\sqrt{m^n}} + \sum_{j=0}^{n-1} \left[ \sum_{\ell \in [1, n-1] \cap \{j, j+1\}} \frac{4c^2 T^{\frac{5}{6}} V^{q_1}(t, x)}{\sqrt{m^{n-j-1}}} \left[ \int_t^T \left\| U_{j, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \right] \\ &= \frac{4V^3(t, x)}{\sqrt{m^n}} + \sum_{j=0}^{n-1} \left[ (2 - \mathbb{1}_{n-1}(j)) \frac{4c^2 T^{\frac{5}{6}} V^{q_1}(t, x)}{\sqrt{m^{n-j-1}}} \left[ \int_t^T \left\| U_{j, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \right]. \quad (186) \end{aligned}$$

Thus, the triangle inequality, (185), and the fact that  $V^3 \leq V^{q_1}$  prove for all  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $n, m \in \mathbb{N}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left\| \Lambda_\nu(T-t) \text{pr}_\nu(U_{n, m}^0(t, x) - \mathbf{u}(t, x)) \right\|_2 \\ &\leq \Lambda_\nu(T-t) \left( \text{var} \left[ \text{pr}_\nu(U_{n, m}^0(t, x)) \right] \right)^{\frac{1}{2}} + \Lambda_\nu(T-t) \left| \text{pr}_\nu(\mathbb{E}[U_{n, m}^0(t, x)] - \mathbf{u}(t, x)) \right| \\ &\leq \frac{4V^{q_1}(t, x)}{\sqrt{m^n}} + \sum_{j=0}^{n-1} \left[ \frac{8c^2 T^{\frac{5}{6}} V^{q_1}(t, x)}{\sqrt{m^{n-j-1}}} \left[ \int_t^T \left\| U_{j, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \right]. \quad (187) \end{aligned}$$

Therefore, (157) implies for all  $n, m \in \mathbb{N}$ ,  $t \in [0, T)$  that

$$\left\| U_{n, m}^0 - \mathbf{u} \right\|_t \leq \frac{4}{\sqrt{m^n}} + \sum_{\ell=0}^{n-1} \left[ \frac{8c^2 T^{\frac{5}{6}}}{\sqrt{m^{n-\ell-1}}} \left[ \int_t^T \left\| U_{\ell, m}^0 - \mathbf{u} \right\|_s^6 \right]^{\frac{1}{6}} \right]. \quad (188)$$

This shows (ii).

Next, (188), [33, Lemma 3.11], (165), and the fact that  $1 + c^2T \leq e^{c^2T}$  prove for all  $n, m \in \mathbb{N}$ ,  $t \in [0, T)$  that

$$\begin{aligned} \left\| U_{n,m}^0 - \mathbf{u} \right\|_t &\leq (4 + 8c^2T^{\frac{5}{6}} \cdot T^{\frac{1}{6}} \cdot 1) e^{\frac{m^3}{6}} m^{-\frac{n}{2}} \left[ 1 + 8c^2T^{\frac{5}{6}} T^{\frac{1}{6}} \right]^{n-1} \\ &\leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} (4 + 8c^2T)^n \\ &\leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n (1 + c^2T)^n \\ &\leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2T}. \end{aligned} \quad (189)$$

This shows (iii).

Next, (189) and (157) prove for all  $n, m \in \mathbb{N}$ ,  $t \in [0, T)$ ,  $\nu \in [0, d] \cap \mathbb{Z}$  that

$$\Lambda_\nu(T-t) \left\| \text{pr}_\nu(U_{n,m}^0(t, x) - \mathbf{u}(t, x)) \right\|_2 \leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2T} V^{q_1}(t, x). \quad (190)$$

This implies (iv) and completes the proof of Proposition 4.3.  $\square$

## 5. EULER-MARUYAMA APPROXIMATIONS REVISITED

In this section we provide some results of moment estimates, stability, continuity, and discretization errors for solutions to SDEs with explicit constants independent of the dimension  $d \in \mathbb{N}$ .

**Setting 5.1.** Let  $\|\cdot\| : \cup_{k, \ell \in \mathbb{N}} \mathbb{R}^{k \times \ell} \rightarrow [0, \infty)$  satisfy for all  $k, \ell \in \mathbb{N}$ ,  $s = (s_{ij})_{i \in [1, k] \cap \mathbb{N}, j \in [1, \ell] \cap \mathbb{N}} \in \mathbb{R}^{k \times \ell}$  that  $\|s\|^2 = \sum_{i=1}^k \sum_{j=1}^\ell |s_{ij}|^2$ . Let  $d \in \mathbb{N}$ ,  $T \in (0, \infty)$ ,  $c, \bar{c}, b \in [1, \infty)$ ,  $p \in [8, \infty)$ ,  $\mu \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma \in C^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$ ,  $\varphi \in C^2(\mathbb{R}^d, [1, \infty))$  satisfy for all  $x, y, h \in \mathbb{R}^d$  that  $\sigma(x)$  is invertible,

$$\|\mu(0)\| + \|\sigma(0)\| + c\|x\| \leq (\varphi(x))^{\frac{1}{p}}, \quad (191)$$

$$|((D\varphi)(x))(y)| \leq \bar{c}(\varphi(x))^{\frac{p-1}{p}} \|y\|, \quad |((D^2\varphi)(x))(y, y)| \leq \bar{c}(\varphi(x))^{\frac{p-2}{p}} \|y\|^2, \quad (192)$$

$$\max \{ \|((D\mu)(x))(h)\|, \|((D\sigma)(x))(h)\|, \|\sigma^{-1}(x)h\| \} \leq c\|h\|, \quad (193)$$

and

$$\begin{aligned} &\max \{ \|((D\mu)(x) - (D\mu)(y))(h)\|, \|((D\sigma)(x) - (D\sigma)(y))(h)\|, \|[(\sigma(x))^{-1} - (\sigma(y))^{-1}]h\| \} \\ &\leq b\|x - y\|\|h\|. \end{aligned} \quad (194)$$

Let  $\iota : [0, T] \rightarrow [0, T]$  satisfy for all  $t \in [0, T]$  that  $\iota(t) = t$ . Let  $\mathbb{S}$  satisfy that

$$\mathbb{S} = \left\{ \delta : [0, T] \rightarrow [0, T] : \begin{array}{l} \exists n \in \mathbb{N}, t_0, t_1, \dots, t_n \in [0, T] : 0 = t_0 < t_1 < \dots < t_n = T, \\ \delta([t_0, t_1]) = \{t_0\}, \delta((t_1, t_2]) = \{t_1\}, \dots, \delta((t_{n-1}, t_n]) = \{t_{n-1}\} \end{array} \right\}. \quad (195)$$

Let  $\tilde{\mathbb{S}} = \mathbb{S} \cup \{\iota\}$ . Let  $|\cdot| : \tilde{\mathbb{S}} \rightarrow [0, T]$  satisfy for all  $\delta \in \mathbb{S}$  that  $|\iota| = 0$  and

$$|\delta| = \max \left\{ |s - t| : s, t \in \delta([0, T]), s < t, (s, t) \cap \delta([0, T]) = \emptyset \right\}. \quad (196)$$

For every  $k \in [1, d] \cap \mathbb{Z}$  let  $e_k \in \mathbb{R}^d$  denote the  $d$ -dimensional vector with a 1 in the  $k$ -th coordinate and 0's elsewhere. Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$  be a filtered probability space which satisfies the usual conditions. For every  $s \in [1, \infty)$ ,  $k, \ell \in \mathbb{N}$  and every random variable  $\mathfrak{X} : \Omega \rightarrow \mathbb{R}^{k \times \ell}$  let  $\|\mathfrak{X}\|_s \in [0, \infty)$  satisfy that  $\|\mathfrak{X}\|_s^s = \mathbb{E}[\|\mathfrak{X}\|_s^s]$ . Let  $W = (W_t)_{t \in [0, T]} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$  be a standard  $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motion with continuous sample paths. For every  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  let  $(\mathcal{X}_t^{\delta, s, x})_{t \in [s, T]} : [s, T] \times \Omega \rightarrow \mathbb{R}^d$  be an  $(\mathbb{F}_t)_{t \in [s, T]}$ -adapted stochastic process with continuous sample paths such that for all  $t \in [s, T]$  we have  $\mathbb{P}$ -a.s. that

$$\mathcal{X}_t^{\delta, s, x} = x + \int_s^t \mu(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) dr + \int_s^t \sigma(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) dW_r. \quad (197)$$

For every  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  let  $(\mathcal{D}_t^{\delta, s, x, k})_{t \in [s, T]}: [s, T] \times \Omega \rightarrow \mathbb{R}^d$  be an  $(\mathbb{F}_t)_{t \in [s, T]}$ -adapted stochastic process with continuous sample paths such that for all  $t \in [s, T]$  we have  $\mathbb{P}$ -a.s. that

$$\begin{aligned} \mathcal{D}_t^{\delta, s, x, k} &= e_k + \int_s^t \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dr \\ &\quad + \int_s^t \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dW_r. \end{aligned} \quad (198)$$

For every  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  let  $\mathcal{D}_t^{\delta, s, x} = (\mathcal{D}_t^{\delta, s, x, k})_{k \in [1, d] \cap \mathbb{Z}}$ . For every  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$  let  $\mathcal{V}_t^{\delta, s, x} = (\mathcal{V}_t^{\delta, s, x, k})_{k \in [1, d] \cap \mathbb{Z}}: \Omega \rightarrow \mathbb{R}^d$  satisfy that

$$\mathcal{V}_t^{\delta, s, x} = \frac{1}{t-s} \int_s^t \left( \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x} \right)^\top dW_r. \quad (199)$$

**Remark 5.2.** If  $\delta = \iota$  in (197) and (198) we have for all  $k \in [0, d] \cap \mathbb{Z}$ ,  $x \in \mathbb{R}^d$ ,  $s \in [0, T]$ ,  $t \in [s, T]$  that  $\mathbb{P}$ -a.s.

$$\mathcal{X}_t^{\iota, s, x} = x + \int_s^t \mu(\mathcal{X}_r^{\iota, s, x}) dr + \int_s^t \sigma(\mathcal{X}_r^{\iota, s, x}) dW_r \quad (200)$$

and

$$\mathcal{D}_t^{\iota, s, x, k} = e_k + \int_s^t \left( (D\mu)(\mathcal{X}_r^{\iota, s, x}) \right) (\mathcal{D}_r^{\iota, s, x, k}) dr + \int_s^t \left( (D\sigma)(\mathcal{X}_r^{\iota, s, x}) \right) (\mathcal{D}_r^{\iota, s, x, k}) dW_r. \quad (201)$$

In this case  $\mathcal{D}_t^{\iota, s, x, k} = \frac{\partial}{\partial x_k} \mathcal{X}_t^{\iota, s, x}$  where  $\frac{\partial}{\partial x_k} \mathcal{X}_t^{\iota, s, x}$  is the derivative process (cf. [46, Theorem 3.4]) defined through the following SDE:

$$\frac{\partial}{\partial x_k} \mathcal{X}_t^{\iota, s, x} = e_k + \int_s^t \left( (D\mu)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \frac{\partial}{\partial x_k} \mathcal{X}_r^{\iota, s, x} \right) dr + \int_s^t \left( (D\sigma)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \frac{\partial}{\partial x_k} \mathcal{X}_r^{\iota, s, x} \right) dW_r. \quad (202)$$

**Lemma 5.3.** Assume Setting 5.1. Then the following items hold.

(i) For all  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that  $\mathbb{E} \left[ \varphi(\mathcal{X}_t^{\delta, s, x}) \right] \leq e^{1.5c|t-s|} \varphi(x)$ .

(ii) For all  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \mathcal{X}_t^{\delta, s, x} - \mathcal{X}_t^{\iota, s, x} \right\|_p \leq \sqrt{2}c \left[ \sqrt{T} + p \right]^2 e^{c^2[\sqrt{T}+p]^2 T} \left( e^{1.5cT} \varphi(x) \right)^{\frac{1}{p}} |t-s|^{\frac{1}{2}} |\delta|^{\frac{1}{2}}. \quad (203)$$

(iii) For all  $\delta \in \tilde{\mathbb{S}}$ ,  $s, \tilde{s} \in [0, T]$ ,  $t \in [s, T]$ ,  $\tilde{t} \in [\tilde{s}, T]$ ,  $x, \tilde{x} \in \mathbb{R}^d$  we have that

$$\begin{aligned} &\left\| \mathcal{X}_t^{\delta, s, x} - \mathcal{X}_{\tilde{t}}^{\delta, \tilde{s}, \tilde{x}} \right\|_p \\ &\leq \sqrt{2} \|x - \tilde{x}\| e^{c^2[\sqrt{T}+p]^2 T} + 5e^{c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5cT}{p}} \frac{\varphi^{\frac{1}{p}}(x) + \varphi^{\frac{1}{p}}(\tilde{x})}{2} \left[ |s - \tilde{s}|^{\frac{1}{2}} + |t - \tilde{t}|^{\frac{1}{2}} \right]. \end{aligned} \quad (204)$$

(iv) For all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \mathcal{V}_t^{\delta, s, x, k} \right\|_p \leq \frac{\sqrt{2}pce^{c^2[p+\sqrt{T}]^2 T}}{\sqrt{t-s}}. \quad (205)$$

(v) For all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  we have that

$$\left\| \mathcal{V}_t^{\delta, s, x, k} - \mathcal{V}_t^{\delta, s, y, k} \right\|_{\frac{p}{4}} \leq \frac{2bc \left[ \sqrt{T} + p \right]^3 e^{3c^2[\sqrt{T}+p]^2 T} \|x - y\|}{\sqrt{t-s}}. \quad (206)$$

(vi) For all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \mathcal{V}_t^{\delta, s, x, k} - \mathcal{V}_t^{\iota, s, x, k} \right\|_{\frac{p}{2}} \leq \frac{15c(bc + c^2) \left[ \sqrt{T} + p \right]^6 e^{3c^2[\sqrt{T}+p]^2 T} \left( e^{1.5cT} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}}{\sqrt{t-s}}. \quad (207)$$

(vii) For all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \mathcal{V}_t^{t, \tilde{s}, x, k} - \mathcal{V}_t^{t, s, x, k} \right\|_{\frac{p}{2}} \leq \frac{13(b+c)ce^{3c^2[\sqrt{T}+p]^2 T} [\sqrt{T}+p]^4 e^{\frac{1.5\epsilon T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}}{\sqrt{t-s}\sqrt{t-\tilde{s}}}. \quad (208)$$

*Proof of Lemma 5.3.* First, [39, Theorem 3.2] (with  $b \curvearrowright \infty$ ,  $V \curvearrowright \varphi$  in the notation of [39, Theorem 3.2]) shows (i)–(iii).

Next, Hölder's inequality and (193) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  and all stopping times  $\tau: \Omega \rightarrow [0, T]$  that

$$\begin{aligned} & \left\| \int_s^{\max\{s, \delta(\min\{t, \tau\})\}} \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dr \right\|_p \\ &= \left\| \int_s^t \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \mathbb{1}_{r \leq \max\{s, \delta(\min\{t, \tau\})\}} dr \right\|_p \\ &= \left\| \int_s^t \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right) \mathbb{1}_{r \leq \max\{s, \delta(\min\{t, \tau\})\}} dr \right\|_p \\ &\leq \int_s^t \left\| \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right) \mathbb{1}_{r \leq \max\{s, \delta(\min\{t, \tau\})\}} \right\|_p dr \\ &\leq \sqrt{T} \left[ \int_s^t \left\| \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right) \right\|_p^2 dr \right]^{\frac{1}{2}} \\ &\leq c\sqrt{T} \left[ \int_s^t \left\| \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}}. \end{aligned} \quad (209)$$

In addition, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]) and (193) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  and all stopping times  $\tau: \Omega \rightarrow [0, T]$  that

$$\begin{aligned} & \left\| \int_s^{\max\{s, \delta(\min\{t, \tau\})\}} \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dW_r \right\|_p \\ &= \left\| \int_s^t \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \mathbb{1}_{r \leq \max\{s, \delta(\min\{t, \tau\})\}} dW_r \right\|_p \\ &= \left\| \int_s^t \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right) \mathbb{1}_{r \leq \max\{s, \delta(\min\{t, \tau\})\}} dW_r \right\|_p \\ &\leq p \left[ \int_s^t \left\| \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right) \right\|_p^2 dr \right]^{\frac{1}{2}} \\ &\leq pc \left[ \int_s^t \left\| \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}}. \end{aligned} \quad (210)$$

Thus, the triangle inequality, (198), and (209) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \left\| \mathcal{D}_{\max\{s, \delta(\min\{t, \tau\})\}}^{\delta, s, x, k} \right\|_p &\leq 1 + \left\| \int_s^{\max\{s, \delta(\min\{t, \tau\})\}} \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dr \right\|_p \\ &\quad + \left\| \int_s^{\max\{s, \delta(\min\{t, \tau\})\}} \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dW_r \right\|_p \\ &\leq 1 + c [\sqrt{T} + p] \left[ \int_s^t \left\| \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}}. \end{aligned} \quad (211)$$

This and the fact that  $\forall x, y \in \mathbb{R}: (x+y)^2 \leq 2x^2 + 2y^2$  show for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \mathcal{D}_{\max\{s, \delta(\min\{t, \tau\})\}}^{\delta, s, x, k} \right\|_p^2 \leq 2 + 2c^2 \left[ \sqrt{T} + p \right]^2 \int_s^t \left\| \mathcal{D}_{\max\{s, \delta(\min\{r, \tau\})\}}^{\delta, s, x, k} \right\|_p^2 dr. \quad (212)$$

For every  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  let  $\tau_n^{\delta, s, x, k} = \min \left\{ \inf \left\{ t \in [0, T]: \left\| \mathcal{D}_{\max\{s, \delta(t)\}}^{\delta, s, x, k} \right\| \geq n \right\}, T \right\}$ . Then for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$ ,  $n \in \mathbb{N}$  we have that  $\left\| \mathcal{D}_{\max\{s, \delta(\min\{t, \tau_n^{\delta, s, x, k}\})\}}^{\delta, s, x, k} \right\| \leq n$ . Then (212) and Grönwall's lemma imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$ ,  $n \in \mathbb{N}$  that

$$\left\| \mathcal{D}_{\max\{s, \delta(\min\{t, \tau_n^{\delta, s, x, k}\})\}}^{\delta, s, x, k} \right\|_p^2 \leq 2e^{2c^2[\sqrt{T}+p]^2(t-s)}. \quad (213)$$

This and Fatou's lemma prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \mathcal{D}_{\max\{s, \delta(t)\}}^{\delta, s, x, k} \right\|_p^2 \leq 2e^{2c^2[\sqrt{T}+p]^2(t-s)}. \quad (214)$$

Next, the triangle inequality, (198), Hölder's inequality, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), and (193) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \left\| \mathcal{D}_t^{\delta, s, x, k} \right\|_p &\leq 1 + \left\| \int_s^t \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dr \right\|_p \\ &\quad + \left\| \int_s^t \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dW_r \right\|_p \\ &\leq 1 + \sqrt{T} \left[ \int_s^t \left\| \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \right\|_p^2 dr \right]^{\frac{1}{2}} \\ &\quad + p \left[ \int_s^t \left\| \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \right\|_p^2 dr \right]^{\frac{1}{2}} \\ &\leq 1 + c \left[ \sqrt{T} + p \right] \left[ \int_s^t \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}}. \end{aligned} \quad (215)$$

This and the fact that  $\forall x, y \in \mathbb{R}: (x+y)^2 \leq 2x^2 + 2y^2$  show for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \left\| \mathcal{D}_t^{\delta, s, x, k} \right\|_p^2 &\leq 2 + 2c^2 \left[ \sqrt{T} + p \right]^2 \int_s^t \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p^2 dr \\ &\leq 2 + 2c^2 \left[ \sqrt{T} + p \right]^2 \int_s^t 2e^{2c^2[\sqrt{T}+p]^2(r-s)} dr \\ &= 2 + \left[ 2e^{2c^2[\sqrt{T}+p]^2(r-s)} \Big|_{r=s}^t \right] \\ &= 2e^{2c^2[\sqrt{T}+p]^2(t-s)} \end{aligned} \quad (216)$$

and hence

$$\left\| \mathcal{D}_t^{\delta, s, x, k} \right\|_p \leq \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T}. \quad (217)$$

Therefore, (199), the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), and (193) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \mathcal{V}_t^{\delta, s, x, k} \right\|_p = \frac{1}{t-s} \left\| \int_s^t \left( \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right)^\top dW_r \right\|_p$$

$$\begin{aligned}
&\leq \frac{1}{t-s} p \left[ \int_s^t \left\| \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
&\leq \frac{1}{t-s} pc \left[ \int_s^t \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
&\leq \frac{1}{t-s} pc \sqrt{t-s} \sqrt{2} e^{c^2 [p+\sqrt{T}]^2 T} \\
&= \frac{\sqrt{2} p c e^{c^2 [p+\sqrt{T}]^2 T}}{\sqrt{t-s}}.
\end{aligned} \tag{218}$$

This shows (iv).

Next, the triangle inequality, Hölder's inequality, (194), (193), (iii), and (217) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) - \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right\|_{\frac{p}{2}} \\
&\leq \left\| \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) - (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) \right\|_{\frac{p}{2}} \\
&\quad + \left\| \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right\|_{\frac{p}{2}} \\
&\leq b \left\| \mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x} - \mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y} \right\|_p \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right\|_p + c \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{2}} \\
&\leq b\sqrt{2} \|x - y\| e^{c^2 [\sqrt{T}+p]^2 T} \sqrt{2} e^{c^2 [p+\sqrt{T}]^2 T} + c \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{2}} \\
&= 2b \|x - y\| e^{2c^2 [p+\sqrt{T}]^2 T} + c \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{2}}.
\end{aligned} \tag{219}$$

Similarly, we have for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| \left( (D\sigma)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) - \left( (D\sigma)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right\|_{\frac{p}{2}} \\
&\leq 2b \|x - y\| e^{2c^2 [p+\sqrt{T}]^2 T} + c \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{2}}.
\end{aligned} \tag{220}$$

This, (198), the triangle inequality, Hölder's inequality, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), and (219) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| \mathcal{D}_t^{\delta,s,x,k} - \mathcal{D}_t^{\delta,s,y,k} \right\|_{\frac{p}{2}} \\
&\leq \left\| \int_s^t \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) - \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) dr \right\|_{\frac{p}{2}} \\
&\quad + \left\| \int_s^t \left( (D\sigma)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) - \left( (D\sigma)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) dW_r \right\|_{\frac{p}{2}} \\
&\leq \sqrt{T} \left[ \int_s^t \left\| \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) - \left( (D\mu)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
&\quad + p \left[ \int_s^t \left\| \left( (D\sigma)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right) - \left( (D\sigma)(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
&\leq \left[ \sqrt{T} + p \right] \left[ \sqrt{t-s} \cdot 2b \|x - y\| e^{2c^2 [p+\sqrt{T}]^2 T} + c \left[ \int_s^t \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{2}}^2 \right]^{\frac{1}{2}} \right]
\end{aligned}$$

$$= 2b \left[ \sqrt{T} + p \right] e^{2c^2[p+\sqrt{T}]^2 T} \sqrt{t-s} \|x - y\| + c \left[ \sqrt{T} + p \right] \left[ \int_s^t \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{2}}^2 \right]^{\frac{1}{2}}. \quad (221)$$

Therefore, (217), a Grönwall-type inequality (cf. [39, Corollary 2.2]), and the fact that  $2\sqrt{2} \leq 3$  imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \mathbb{S}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} \left\| \mathcal{D}_t^{\delta,s,x,k} - \mathcal{D}_t^{\delta,s,y,k} \right\|_{\frac{p}{2}} &\leq 2\sqrt{2}b \left[ \sqrt{T} + p \right] e^{2c^2[\sqrt{T}+p]^2 T} \sqrt{t-s} \|x - y\| e^{c^2[\sqrt{T}+p]^2 T} \\ &\leq 3b \left[ \sqrt{T} + p \right]^2 e^{3c^2[\sqrt{T}+p]^2 T} \|x - y\|. \end{aligned} \quad (222)$$

Furthermore, Hölder's inequality, (194), (iii), and (217) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} &\left\| \left( \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) - \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right\|_{\frac{p}{4}} \\ &\leq b \left\| \mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x} - \mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y} \right\|_{\frac{p}{2}} \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right\|_{\frac{p}{2}} \\ &\leq b\sqrt{2} \|x - y\| e^{c^2[\sqrt{T}+p]^2 T} \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T} \\ &= 2be^{2c^2[\sqrt{T}+p]^2 T} \|x - y\|. \end{aligned} \quad (223)$$

Next, (193) and (222) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} &\left\| \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right\|_{\frac{p}{4}} \\ &\leq c \left\| \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right\|_{\frac{p}{4}} \\ &\leq 3bc \left[ \sqrt{T} + p \right]^2 e^{3c^2[\sqrt{T}+p]^2 T} \|x - y\|. \end{aligned} \quad (224)$$

Hence, (199), the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), the triangle inequality, and (223) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} &\left\| \mathcal{V}_t^{\delta,s,x,k} - \mathcal{V}_t^{\delta,s,y,k} \right\|_{\frac{p}{4}} \\ &= \frac{1}{t-s} \left\| \int_s^t \left( \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right)^\top dW_r \right\|_{\frac{p}{4}} \\ &\leq \frac{\frac{p}{4}}{t-s} \left[ \int_s^t \left\| \left( \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right)^\top \right\|_{\frac{p}{4}}^2 dr \right]^{\frac{1}{2}} \\ &\leq \frac{\frac{p}{4}}{t-s} \left[ \int_s^t \left\| \left( \left( \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,x}) - \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \right) \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} \right)^\top \right\|_{\frac{p}{4}}^2 dr \right]^{\frac{1}{2}} \\ &\quad + \frac{\frac{p}{4}}{t-s} \left[ \int_s^t \left\| \left( \sigma^{-1}(\mathcal{X}_{\max\{s,\delta(r)\}}^{\delta,s,y}) \left( \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,x,k} - \mathcal{D}_{\max\{s,\delta(r)\}}^{\delta,s,y,k} \right) \right)^\top \right\|_{\frac{p}{4}}^2 dr \right]^{\frac{1}{2}} \\ &\leq \frac{\frac{p}{4}}{\sqrt{t-s}} \left[ 2be^{2c^2[\sqrt{T}+p]^2 T} \|x - y\| + 3bc \left[ \sqrt{T} + p \right]^2 e^{3c^2[\sqrt{T}+p]^2 T} \|x - y\| \right] \\ &\leq \frac{5bc\frac{p}{4} \left[ \sqrt{T} + p \right]^2 e^{3c^2[\sqrt{T}+p]^2 T} \|x - y\|}{\sqrt{t-s}} \\ &\leq \frac{2bc \left[ \sqrt{T} + p \right]^3 e^{3c^2[\sqrt{T}+p]^2 T} \|x - y\|}{\sqrt{t-s}}. \end{aligned} \quad (225)$$

This proves (v).

Next, (198), the triangle inequality, Hölder's inequality, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), (193), and (217) show for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $\tilde{t} \in [t, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \mathcal{D}_{\tilde{t}}^{\delta, s, x, k} - \mathcal{D}_t^{\delta, s, x, k} \right\|_p \\
& \leq \left\| \int_t^{\tilde{t}} \left( (\mathrm{D}\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dr \right\|_p + \left\| \int_t^{\tilde{t}} \left( (\mathrm{D}\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) dW_r \right\|_p \\
& \leq \sqrt{T} \left[ \int_t^{\tilde{t}} \left\| \left( (\mathrm{D}\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \right\|_p^2 dr \right]^{\frac{1}{2}} \\
& \quad + p \left[ \int_t^{\tilde{t}} \left\| \left( (\mathrm{D}\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \right\|_p^2 dr \right]^{\frac{1}{2}} \\
& \leq \left[ \sqrt{T} + p \right] c \left[ \int_t^{\tilde{t}} \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
& \leq \left[ \sqrt{T} + p \right] c \sqrt{\tilde{t} - t} \sqrt{2} e^{c^2 [\sqrt{T} + p]^2 T} \\
& = \sqrt{2} c \left[ \sqrt{T} + p \right] e^{c^2 [\sqrt{T} + p]^2 T} \sqrt{\tilde{t} - t}.
\end{aligned} \tag{226}$$

In addition, (194), Hölder's inequality, the triangle inequality, (iii), (ii), (217), the fact that  $(5 + \sqrt{2})\sqrt{2} \leq 10$ , and the fact that  $c \geq 1$  imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \left( (\mathrm{D}\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) - (\mathrm{D}\mu)(\mathcal{X}_r^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq b \left\| \mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x} - \mathcal{X}_r^{\delta, s, x} \right\|_p \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p \\
& \leq b \left[ \left\| \mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x} - \mathcal{X}_r^{\delta, s, x} \right\|_p + \left\| \mathcal{X}_r^{\delta, s, x} - \mathcal{X}_r^{\delta, s, x} \right\|_p \right] \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p \\
& \leq b \left[ 5e^{c^2 [\sqrt{T} + p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |\delta|^{\frac{1}{2}} + \sqrt{2} c \left[ \sqrt{T} + p \right]^3 e^{c^2 [\sqrt{T} + p]^2 T} \left( e^{1.5\tilde{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \right] \\
& \quad \sqrt{2} e^{c^2 [\sqrt{T} + p]^2 T} \\
& \leq 10bc \left[ \sqrt{T} + p \right]^3 e^{2c^2 [\sqrt{T} + p]^2 T} \left( e^{1.5\tilde{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}.
\end{aligned} \tag{227}$$

Furthermore, (193), the triangle inequality, and (226) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| (\mathrm{D}\mu)(\mathcal{X}_r^{\delta, s, x}) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq c \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right\|_{\frac{p}{2}} \\
& \leq c \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right\|_{\frac{p}{2}} + c \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right\|_{\frac{p}{2}} \\
& \leq c\sqrt{2} c \left[ \sqrt{T} + p \right] e^{c^2 [\sqrt{T} + p]^2 T} |\delta|^{\frac{1}{2}} + c \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right\|_{\frac{p}{2}} \\
& = \sqrt{2} c^2 \left[ \sqrt{T} + p \right] e^{c^2 [\sqrt{T} + p]^2 T} |\delta|^{\frac{1}{2}} + c \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right\|_{\frac{p}{2}}.
\end{aligned} \tag{228}$$

This, the triangle inequality, and (227) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) - \left( (D\mu)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq \left\| \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) - (D\mu)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) \right\|_{\frac{p}{2}} + \left\| (D\mu)(\mathcal{X}_r^{\iota, s, x}) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq 10bc \left[ \sqrt{T} + p \right]^3 e^{2c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \\
& \quad + \sqrt{2}c^2 \left[ \sqrt{T} + p \right] e^{c^2[\sqrt{T}+p]^2 T} |\delta|^{\frac{1}{2}} + c \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}} \\
& \leq 12(bc + c^2) \left[ \sqrt{T} + p \right]^3 e^{2c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} + c \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}}. \tag{229}
\end{aligned}$$

Similarly, we have for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) - \left( (D\sigma)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq 12(bc + c^2) \left[ \sqrt{T} + p \right]^3 e^{2c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} + c \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}}. \tag{230}
\end{aligned}$$

Thus, (198), the triangle inequality, Hölder's inequality, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), and (229) show for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \mathcal{D}_t^{\delta, s, x, k} - \mathcal{D}_t^{\iota, s, x, k} \right\|_{\frac{p}{2}} \\
& \leq \left\| \int_s^t \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) - \left( (D\mu)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_r^{\iota, s, x, k} \right) dr \right\|_{\frac{p}{2}} \\
& \quad + \left\| \int_s^t \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) - \left( (D\sigma)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_r^{\iota, s, x, k} \right) dW_r \right\|_{\frac{p}{2}} \\
& \leq \sqrt{T} \left[ \int_s^t \left\| \left( (D\mu)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) - \left( (D\mu)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
& \quad + p \left[ \int_s^t \left\| \left( (D\sigma)(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \right) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right) - \left( (D\sigma)(\mathcal{X}_r^{\iota, s, x}) \right) \left( \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
& \leq \left[ \sqrt{T} + p \right] \sqrt{t-s} \cdot 12(bc + c^2) \left[ \sqrt{T} + p \right]^3 e^{2c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \\
& \quad + \left[ \sqrt{T} + p \right] c \left[ \int_s^t \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
& \leq 12(bc + c^2) \left[ \sqrt{T} + p \right]^5 e^{2c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \\
& \quad + \left[ \sqrt{T} + p \right] c \left[ \int_s^t \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}}. \tag{231}
\end{aligned}$$

This, a Grönwall-type inequality (cf. [39, Corollary 2.2]) and the fact that  $12\sqrt{2} \leq 18$  prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \tilde{\mathbb{S}}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \mathcal{D}_t^{\delta, s, x, k} - \mathcal{D}_t^{\iota, s, x, k} \right\|_{\frac{p}{2}} \leq \sqrt{2} \cdot 12(bc + c^2) \left[ \sqrt{T} + p \right]^5 e^{2c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \cdot e^{c^2[\sqrt{T}+p]^2 T} \\
& \leq 18(bc + c^2) \left[ \sqrt{T} + p \right]^5 e^{3c^2[\sqrt{T}+p]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}. \tag{232}
\end{aligned}$$

Next, (194), Hölder's inequality, the triangle inequality, (ii), (iii), (217), the fact that  $(5 + \sqrt{2})\sqrt{2} \leq 10$  imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \mathbb{S}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \left( \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) - \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \right) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_{\frac{p}{2}} \\
& \leq b \left\| \mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x} - \mathcal{X}_r^{\iota, s, x} \right\|_p \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p \\
& \leq b \left[ \left\| \mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x} - \mathcal{X}_r^{\delta, s, x} \right\|_p + \left\| \mathcal{X}_r^{\delta, s, x} - \mathcal{X}_r^{\iota, s, x} \right\|_p \right] \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_p \\
& \leq b \left[ 5e^{c^2[\sqrt{T+p}]^2 T} \left[ \sqrt{T+p} \right] e^{\frac{1.5\bar{c}T}{p}} \varphi^{\frac{1}{p}}(x) |\delta|^{\frac{1}{2}} + \sqrt{2}c \left[ \sqrt{T+p} \right]^3 e^{c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \right] \\
& \quad \sqrt{2}e^{c^2[\sqrt{T+p}]^2 T} \\
& \leq 10bc \left[ \sqrt{T+p} \right]^3 e^{2c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}. \tag{233}
\end{aligned}$$

Moreover, (193), the triangle inequality, (226), and (232) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \mathbb{S}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq c \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_p \\
& \leq c \left[ \left\| \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\delta, s, x, k} \right\|_p + \left\| \mathcal{D}_r^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right\|_p \right] \\
& \leq c \left[ \sqrt{2}c \left[ \sqrt{T+p} \right] e^{c^2[\sqrt{T+p}]^2 T} |\delta|^{\frac{1}{2}} + 18(bc + c^2) \left[ \sqrt{T+p} \right]^5 e^{3c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \right] \\
& \leq 20c(bc + c^2) \left[ \sqrt{T+p} \right]^5 e^{3c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}. \tag{234}
\end{aligned}$$

This, the triangle inequality, (233), and the fact that  $b, c \geq 1$  show for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \mathbb{S}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}} \\
& \leq \left\| \left( \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) - \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \right) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} \right\|_{\frac{p}{2}} + \left\| \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \left( \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \mathcal{D}_r^{\iota, s, x, k} \right) \right\|_{\frac{p}{2}} \\
& \leq 10bc \left[ \sqrt{T+p} \right]^3 e^{2c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \\
& \quad + 20c(bc + c^2) \left[ \sqrt{T+p} \right]^5 e^{3c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}} \\
& \leq 30c(bc + c^2) \left[ \sqrt{T+p} \right]^5 e^{3c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}. \tag{235}
\end{aligned}$$

Therefore, (199) and the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $\delta \in \mathbb{S}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\left\| \mathcal{V}_t^{\delta, s, x, k} - \mathcal{V}_t^{\iota, s, x, k} \right\|_{\frac{p}{2}} &= \left\| \frac{1}{t-s} \int_s^t \left( \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \mathcal{D}_r^{\iota, s, x, k} \right)^\top dW_r \right\|_{\frac{p}{2}} \\
&\leq \frac{\frac{p}{2}}{t-s} \left[ \int_s^t \left\| \sigma^{-1}(\mathcal{X}_{\max\{s, \delta(r)\}}^{\delta, s, x}) \mathcal{D}_{\max\{s, \delta(r)\}}^{\delta, s, x, k} - \sigma^{-1}(\mathcal{X}_r^{\iota, s, x}) \mathcal{D}_r^{\iota, s, x, k} \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
&\leq \frac{\frac{p}{2}}{\sqrt{t-s}} 30c(bc + c^2) \left[ \sqrt{T+p} \right]^5 e^{3c^2[\sqrt{T+p}]^2 T} \left( e^{1.5\bar{c}T} \varphi(x) \right)^{\frac{1}{p}} |\delta|^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{15c(bc + c^2) [\sqrt{T} + p]^6 e^{3c^2[\sqrt{T}+p]^2 T} (e^{1.5cT} \varphi(x))^{\frac{1}{p}} |\delta|^{\frac{1}{2}}}{\sqrt{t-s}}. \quad (236)$$

This shows (vi).

Next, Hölder's inequality, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), (193), and (217) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left\| \int_s^{\tilde{s}} ((D\mu)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) dr \right\|_p + \left\| \int_s^{\tilde{s}} ((D\sigma)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) dW_r \right\|_p \\ & \leq \sqrt{T} \left[ \int_s^{\tilde{s}} \|((D\mu)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k})\|_p^2 dr \right]^{\frac{1}{2}} + p \left[ \int_s^{\tilde{s}} \|((D\sigma)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k})\|_p^2 dr \right]^{\frac{1}{2}} \\ & \leq c \left[ \sqrt{T} + p \right] \left[ \int_s^{\tilde{s}} \|\mathcal{D}_r^{\iota, s, x, k}\|_p^2 dr \right]^{\frac{1}{2}} \\ & \leq c \left[ \sqrt{T} + p \right] \sqrt{\tilde{s} - s} \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T} = \sqrt{2} c \left[ \sqrt{T} + p \right] e^{c^2[\sqrt{T}+p]^2 T} \sqrt{\tilde{s} - s}. \end{aligned} \quad (237)$$

Furthermore, (194), Hölder's inequality, (iii), (217), and the fact that  $5\sqrt{2} \leq 8$  imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $r \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \|((D\mu)(\mathcal{X}_r^{\iota, s, x}) - (D\mu)(\mathcal{X}_r^{\iota, \tilde{s}, x})) \mathcal{D}_r^{\iota, s, x, k}\|_{\frac{p}{2}} \\ & \leq b \|\mathcal{X}_r^{\iota, s, x} - \mathcal{X}_r^{\iota, \tilde{s}, x}\|_p \|\mathcal{D}_r^{\iota, s, x, k}\|_p \\ & \leq b \cdot 5e^{c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5cT}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} \cdot \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T} \\ & \leq 8be^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5cT}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}. \end{aligned} \quad (238)$$

This, the triangle inequality, and (193) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \|((D\mu)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) - ((D\mu)(\mathcal{X}_r^{\iota, \tilde{s}, x})) (\mathcal{D}_r^{\iota, \tilde{s}, x, k})\|_{\frac{p}{2}} \\ & \leq \|((D\mu)(\mathcal{X}_r^{\iota, s, x}) - (D\mu)(\mathcal{X}_r^{\iota, \tilde{s}, x})) \mathcal{D}_r^{\iota, s, x, k}\|_{\frac{p}{2}} + \|(D\mu)(\mathcal{X}_r^{\iota, \tilde{s}, x}) (\mathcal{D}_r^{\iota, s, x, k} - \mathcal{D}_r^{\iota, \tilde{s}, x, k})\|_{\frac{p}{2}} \\ & \leq 8be^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5cT}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} + c \|\mathcal{D}_r^{\iota, s, x, k} - \mathcal{D}_r^{\iota, \tilde{s}, x, k}\|_{\frac{p}{2}}. \end{aligned} \quad (239)$$

Similarly, for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\begin{aligned} & \|((D\mu)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) - ((D\mu)(\mathcal{X}_r^{\iota, \tilde{s}, x})) (\mathcal{D}_r^{\iota, \tilde{s}, x, k})\|_{\frac{p}{2}} \\ & \leq 8be^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5cT}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} + c \|\mathcal{D}_r^{\iota, s, x, k} - \mathcal{D}_r^{\iota, \tilde{s}, x, k}\|_{\frac{p}{2}}. \end{aligned} \quad (240)$$

Next, (198) shows for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} \mathcal{D}_t^{\iota, s, x, k} - \mathcal{D}_t^{\iota, \tilde{s}, x, k} &= \int_s^{\tilde{s}} ((D\mu)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) dr + \int_s^{\tilde{s}} ((D\sigma)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) dW_r \\ & \quad + \int_{\tilde{s}}^t ((D\mu)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) - ((D\mu)(\mathcal{X}_r^{\iota, \tilde{s}, x})) (\mathcal{D}_r^{\iota, \tilde{s}, x, k}) dr \\ & \quad + \int_{\tilde{s}}^t ((D\sigma)(\mathcal{X}_r^{\iota, s, x})) (\mathcal{D}_r^{\iota, s, x, k}) - ((D\sigma)(\mathcal{X}_r^{\iota, \tilde{s}, x})) (\mathcal{D}_r^{\iota, \tilde{s}, x, k}) dW_r. \end{aligned} \quad (241)$$

Thus, the triangle inequality, (237), Hölder's inequality, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), (239), and (240) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \mathcal{D}_t^{\iota, s, x, k} - \mathcal{D}_t^{\iota, \tilde{s}, x, k} \right\|_{\frac{p}{2}}$$

$$\begin{aligned}
&\leq \sqrt{2}c \left[ \sqrt{T} + p \right] e^{c^2[\sqrt{T}+p]^2 T} \sqrt{\tilde{s} - s} \\
&\quad + \sqrt{T} \left[ \int_{\tilde{s}}^t \left\| ((D\mu)(\mathcal{X}_r^{\iota,s,x})) (\mathcal{D}_r^{\iota,s,x,k}) - ((D\mu)(\mathcal{X}_r^{\iota,\tilde{s},x})) (\mathcal{D}_r^{\iota,\tilde{s},x,k}) \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
&\quad + p \left[ \int_{\tilde{s}}^t \left\| ((D\sigma)(\mathcal{X}_r^{\iota,s,x})) (\mathcal{D}_r^{\iota,s,x,k}) - ((D\sigma)(\mathcal{X}_r^{\iota,\tilde{s},x})) (\mathcal{D}_r^{\iota,\tilde{s},x,k}) \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
&\leq \sqrt{2}c \left[ \sqrt{T} + p \right] e^{c^2[\sqrt{T}+p]^2 T} \sqrt{\tilde{s} - s} \\
&\quad + \left[ \sqrt{T} + p \right] \left[ \sqrt{T} \cdot 8be^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} + c \left[ \int_{\tilde{s}}^t \left\| \mathcal{D}_r^{\iota,s,x,k} - \mathcal{D}_r^{\iota,\tilde{s},x,k} \right\|_{\frac{p}{2}}^2 \right]^{\frac{1}{2}} \right] \\
&\leq 8(b+c)e^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} + c \left[ \sqrt{T} + p \right] \left[ \int_{\tilde{s}}^t \left\| \mathcal{D}_r^{\iota,s,x,k} - \mathcal{D}_r^{\iota,\tilde{s},x,k} \right\|_{\frac{p}{2}}^2 \right]^{\frac{1}{2}}. \tag{242}
\end{aligned}$$

This, (217), and a Grönwall-type inequality (cf. [39, Corollary 2.2]) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\left\| \mathcal{D}_t^{\iota,s,x,k} - \mathcal{D}_t^{\iota,\tilde{s},x,k} \right\|_{\frac{p}{2}} &\leq \sqrt{2} \cdot 8(b+c)e^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} \cdot e^{c^2[\sqrt{T}+p]^2 T} \\
&\leq 12(b+c)e^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}. \tag{243}
\end{aligned}$$

Next, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), (193), and (217) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\left| \frac{1}{t - \tilde{s}} - \frac{1}{t - s} \right| \left\| \int_{\tilde{s}}^t (\sigma^{-1}(\mathcal{X}_r^{\iota,\tilde{s},x}) \mathcal{D}_r^{\iota,\tilde{s},x,k})^\top dW_r \right\|_p &\leq \frac{|\tilde{s} - s|}{(t - s)(t - \tilde{s})} p \left[ \int_{\tilde{s}}^t \left\| \sigma^{-1}(\mathcal{X}_r^{\iota,\tilde{s},x}) \mathcal{D}_r^{\iota,\tilde{s},x,k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
&\leq \frac{|\tilde{s} - s|}{(t - s)(t - \tilde{s})} pc \left[ \int_{\tilde{s}}^t \left\| \mathcal{D}_r^{\iota,\tilde{s},x,k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
&\leq \frac{|\tilde{s} - s|}{(t - s)(t - \tilde{s})} pc \sqrt{t - \tilde{s}} \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T} \\
&\leq \frac{\sqrt{2} p c e^{c^2[\sqrt{T}+p]^2 T} \sqrt{\tilde{s} - s}}{\sqrt{t - s} \sqrt{t - \tilde{s}}}. \tag{244}
\end{aligned}$$

In addition, (194), (iii), (217), and the fact that  $5\sqrt{2} \leq 8$  imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| (\sigma^{-1}(\mathcal{X}_r^{\iota,\tilde{s},x}) - \sigma^{-1}(\mathcal{X}_r^{\iota,s,x})) \mathcal{D}_r^{\iota,\tilde{s},x,k} \right\|_{\frac{p}{2}} \\
&\leq b \left\| \mathcal{X}_r^{\iota,\tilde{s},x} - \mathcal{X}_r^{\iota,s,x} \right\|_p \left\| \mathcal{D}_r^{\iota,\tilde{s},x,k} \right\|_p \\
&\leq b \cdot 5e^{c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} \cdot \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T} \\
&\leq 8be^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}. \tag{245}
\end{aligned}$$

Next, (193) and (243) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
&\left\| \sigma^{-1}(\mathcal{X}_r^{\iota,s,x}) (\mathcal{D}_r^{\iota,\tilde{s},x,k} - \mathcal{D}_r^{\iota,s,x,k}) \right\|_{\frac{p}{2}} \\
&\leq c \left\| \mathcal{D}_r^{\iota,\tilde{s},x,k} - \mathcal{D}_r^{\iota,s,x,k} \right\|_{\frac{p}{2}} \\
&\leq c \cdot 12(b+c)e^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}. \tag{246}
\end{aligned}$$

Thus, the triangle inequality and (245) prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k} - \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k} \right\|_{\frac{p}{2}} \\
& \leq \left\| (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) - \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x})) \mathcal{D}_r^{\tilde{s}, x, k} \right\|_{\frac{p}{2}} + \left\| \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) (\mathcal{D}_r^{\tilde{s}, x, k} - \mathcal{D}_r^{\tilde{s}, x, k}) \right\|_{\frac{p}{2}} \\
& \leq 8be^{2c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right] e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} \\
& \quad + 12(b+c)ce^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} \\
& \leq 20(b+c)ce^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}. \tag{247}
\end{aligned}$$

This and the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]) show for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
& \left\| \frac{1}{t-s} \int_{\tilde{s}}^t (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k} - \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r \right\|_{\frac{p}{2}} \\
& \leq \frac{\frac{p}{2}}{t-s} \left[ \int_{\tilde{s}}^t \left\| (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k} - \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top \right\|_{\frac{p}{2}}^2 dr \right]^{\frac{1}{2}} \\
& \leq \frac{\frac{p}{2}}{t-s} \sqrt{t-\tilde{s}} \cdot 20(b+c)ce^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}} \\
& \leq \frac{10(b+c)ce^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^3 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}}{\sqrt{t-s}} \\
& \leq \frac{10(b+c)ce^{3c^2[\sqrt{T}+p]^2 T} \left[ \sqrt{T} + p \right]^4 e^{\frac{1.5\tilde{c}T}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}}{\sqrt{t-s}\sqrt{t-\tilde{s}}}. \tag{248}
\end{aligned}$$

Next, the Burkholder-Davis-Gundy inequality (cf. [12, Lemma 7.7]), (193), and (217) imply for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\frac{1}{t-s} \left\| \int_s^{\tilde{s}} (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r \right\|_p & \leq \frac{1}{t-s} p \left[ \int_s^{\tilde{s}} \left\| \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
& \leq \frac{1}{t-s} pc \left[ \int_s^{\tilde{s}} \left\| \mathcal{D}_r^{\tilde{s}, x, k} \right\|_p^2 dr \right]^{\frac{1}{2}} \\
& \leq \frac{1}{t-s} pc \sqrt{\tilde{s}-s} \sqrt{2} e^{c^2[\sqrt{T}+p]^2 T} \\
& \leq \frac{\sqrt{2} p c e^{c^2[\sqrt{T}+p]^2 T} \sqrt{\tilde{s}-s}}{\sqrt{t-s}\sqrt{t-\tilde{s}}}. \tag{249}
\end{aligned}$$

Moreover, (199) shows for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\mathcal{V}_t^{\tilde{s}, x, k} - \mathcal{V}_t^{\tilde{s}, x, k} & = \frac{1}{t-\tilde{s}} \int_{\tilde{s}}^t (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r - \frac{1}{t-s} \int_s^t (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r \\
& = \left( \frac{1}{t-\tilde{s}} - \frac{1}{t-s} \right) \int_{\tilde{s}}^t (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r \\
& \quad + \frac{1}{t-s} \int_{\tilde{s}}^t (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k} - \sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r \\
& \quad - \frac{1}{t-s} \int_s^{\tilde{s}} (\sigma^{-1}(\mathcal{X}_r^{\tilde{s}, x}) \mathcal{D}_r^{\tilde{s}, x, k})^\top dW_r. \tag{250}
\end{aligned}$$

Hence, the triangle inequality, (244), (248), (249), and the fact that  $2\sqrt{2} \leq 3$  prove for all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $t \in (\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned}
\left\| \mathcal{V}_t^{i, \tilde{s}, x, k} - \mathcal{V}_t^{i, s, x, k} \right\|_{\frac{p}{2}} &\leq \frac{\sqrt{2} p c e^{c^2 [\sqrt{T+p}]^2 T} \sqrt{\tilde{s} - s}}{\sqrt{t-s} \sqrt{t-\tilde{s}}} \\
&+ \frac{10(b+c) c e^{3c^2 [\sqrt{T+p}]^2 T} [\sqrt{T+p}]^4 e^{\frac{1.5cT}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}}{\sqrt{t-s} \sqrt{t-\tilde{s}}} \\
&+ \frac{\sqrt{2} p c e^{c^2 [\sqrt{T+p}]^2 T} \sqrt{\tilde{s} - s}}{\sqrt{t-s} \sqrt{t-\tilde{s}}} \\
&\leq \frac{13(b+c) c e^{3c^2 [\sqrt{T+p}]^2 T} [\sqrt{T+p}]^4 e^{\frac{1.5cT}{p}} \varphi^{\frac{1}{p}}(x) |s - \tilde{s}|^{\frac{1}{2}}}{\sqrt{t-s} \sqrt{t-\tilde{s}}} \tag{251}
\end{aligned}$$

This shows (vii). This completes the proof of Lemma 5.3.  $\square$

## 6. COMPLEXITY ANALYSIS

In this section we study the MLP approximations which have been introduced in (153) in the case when  $(\mathcal{X}, \mathcal{Z})$  is replaced by the Euler-Maruyama approximations (see (261)–(264)). We will prove that when the coefficients satisfy (253)–(255) then the corresponding processes  $\mathcal{X}_{(\cdot)}^{d, \theta, K, (\cdot), (\cdot)}$ ,  $\mathcal{Z}_{(\cdot)}^{d, \theta, K, (\cdot), (\cdot)}$  defined in Theorem 6.1 below fulfill (150) as well as (62)–(121). This allows us to combine Proposition 4.3 with Lemma 3.1 and Lemma 5.3 (see the proof of Theorem 6.1 below).

**Theorem 6.1.** *Let  $\Theta = \cup_{n \in \mathbb{N}} \mathbb{Z}^n$ ,  $T \in (0, \infty)$ ,  $\mathbf{k} \in [0, \infty)$ ,  $\beta, c \in [1, \infty)$ . Let  $\|\cdot\|: \cup_{k, \ell \in \mathbb{N}} \mathbb{R}^{k \times \ell} \rightarrow [0, \infty)$  satisfy for all  $k, \ell \in \mathbb{N}$ ,  $s = (s_{ij})_{i \in [1, k] \cap \mathbb{N}, j \in [1, \ell] \cap \mathbb{N}} \in \mathbb{R}^{k \times \ell}$  that  $\|s\|^2 = \sum_{i=1}^k \sum_{j=1}^{\ell} |s_{ij}|^2$ . For every  $d \in \mathbb{N}$  let  $(L_i^d)_{i \in [0, d] \cap \mathbb{Z}} \in \mathbb{R}^{d+1}$  satisfy that  $\sum_{i=0}^d L_i^d \leq c$ . For every  $K \in \mathbb{N}$  let  $[\cdot]_K: \mathbb{R} \rightarrow \mathbb{R}$  satisfy for all  $t \in \mathbb{R}$  that  $[t]_K = \max(\{0, \frac{t}{K}, \frac{2t}{K}, \dots, T\} \cap ((-\infty, t) \cup \{0\}))$ . For every  $d \in \mathbb{N}$  let  $\Lambda^d = (\Lambda_\nu^d)_{\nu \in [0, d] \cap \mathbb{Z}}: [0, T] \rightarrow \mathbb{R}^{1+d}$  satisfy for all  $t \in [0, T]$  that  $\Lambda^d(t) = (1, \sqrt{t}, \dots, \sqrt{t})$ . For every  $d \in \mathbb{N}$  let  $\text{pr}^d = (\text{pr}_\nu^d)_{\nu \in [0, d] \cap \mathbb{Z}}: \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  satisfy for all  $w = (w_\nu)_{\nu \in [0, d] \cap \mathbb{Z}}$ ,  $i \in [0, d] \cap \mathbb{Z}$  that  $\text{pr}_i^d(w) = w_i$ . For every  $d \in \mathbb{N}$ ,  $k \in [1, d] \cap \mathbb{Z}$  let  $e_k^d \in \mathbb{R}^d$  denote the  $d$ -dimensional vector with a 1 in the  $k$ -th coordinate and 0's elsewhere. For every  $d \in \mathbb{N}$  let  $f_d \in C([0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1}, \mathbb{R})$ ,  $g_d \in C(\mathbb{R}^d, \mathbb{R})$ ,  $\mu_d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ ,  $\sigma_d \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$ . To shorten the notation we write for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $w: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  that*

$$(F_d(w))(t, x) = f_d(t, x, w(t, x)). \tag{252}$$

Assume for all  $d \in \mathbb{N}$ ,  $i \in [0, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $r \in (t, T]$ ,  $x, y, h \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$  that  $\sigma_d$  is invertible,

$$\|\mu_d(0)\| + \|\sigma_d(0)\| \leq cd^c, \tag{253}$$

$$\max \{ \|((D\mu_d)(x))(h)\|, \|((D\sigma_d)(x))(h)\|, \|\sigma_d^{-1}(x)h\| \} \leq c\|h\|, \tag{254}$$

$$\begin{aligned}
&\max \{ \|((D\mu_d)(x) - (D\mu_d)(y))(h)\|, \|((D\sigma_d)(x) - (D\sigma_d)(y))(h)\|, \|[(\sigma_d(x))^{-1} - (\sigma_d(y))^{-1}]h\| \} \\
&\leq cd^c \|x - y\| \|h\|, \tag{255}
\end{aligned}$$

$$|g_d(x)| + |T f_d(t, x, 0)| \leq [(cd^c)^2 + c^2 \|x\|^2]^\beta, \tag{256}$$

$$\begin{aligned}
&|f_d(t, x, w_1) - f_d(t, y, w_2)| \\
&\leq \sum_{\nu=0}^d [L_\nu^d \Lambda_\nu^d(T) |\text{pr}_\nu^d(w_1 - w_2)|] + \frac{1}{T} \frac{((cd^c)^2 + c^2 \|x\|^2)^\beta + ((cd^c)^2 + c^2 \|y\|^2)^\beta}{2} \frac{\|x - y\|}{\sqrt{T}}, \tag{257}
\end{aligned}$$

and

$$|g_d(x) - g_d(y)| \leq \frac{((cd^c)^2 + c^2\|x\|^2)^\beta + ((cd^c)^2 + c^2\|y\|^2)^\beta}{2} \frac{\|x - y\|}{\sqrt{T}}. \quad (258)$$

Let  $\varrho: \{(\tau, \sigma) \in [0, T]^2: \tau < \sigma\} \rightarrow \mathbb{R}$  satisfy for all  $t \in [0, T)$ ,  $s \in (t, T)$  that

$$\varrho(t, s) = \frac{1}{B(\frac{1}{2}, \frac{1}{2})} \frac{1}{\sqrt{(T-s)(s-t)}}. \quad (259)$$

Let  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$  be a filtered probability space which satisfies the usual conditions. Let  $\mathbf{r}^\theta: \Omega \rightarrow (0, 1)$ ,  $\theta \in \Theta$ , be independent and identically distributed random variables and satisfy for all  $b \in (0, 1)$  that

$$\mathbb{P}(\mathbf{r}^\theta \leq b) = \frac{1}{B(\frac{1}{2}, \frac{1}{2})} \int_0^b \frac{dr}{\sqrt{r(1-r)}}. \quad (260)$$

For every  $d \in \mathbb{N}$  let  $W^{d, \theta}: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ ,  $\theta \in \Theta$ , be independent  $(\mathbb{F}_t)_{t \in [0, T]}$ -Brownian motions with continuous sample paths. Assume for every  $d \in \mathbb{N}$  that  $(W^{d, \theta})_{\theta \in \Theta}$  and  $(\mathbf{r}^\theta)_{\theta \in \Theta}$  are independent. For every  $\theta \in \Theta$ ,  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $k \in [1, d] \cap \mathbb{Z}$  let  $(\mathcal{X}_t^{d, \theta, K, s, x})_{t \in [s, T]}$ ,  $(\mathcal{D}_t^{d, \theta, K, s, x, k})_{t \in [s, T]}: [s, T] \times \Omega \rightarrow \mathbb{R}^d$  be  $(\mathbb{F}_t)_{t \in [s, T]}$ -adapted stochastic processes with continuous sample paths which satisfy for all  $t \in [s, T]$  that  $\mathbb{P}$ -a.s. we have that

$$\mathcal{X}_t^{d, \theta, K, s, x} = x + \int_s^t \mu_d(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) dr + \int_s^t \sigma_d(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) dW_r^{d, \theta} \quad (261)$$

and

$$\begin{aligned} \mathcal{D}_t^{d, \theta, K, s, x, k} &= e_k^d + \int_s^t \left( (D\mu_d)(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) \right) \left( \mathcal{D}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x, k} \right) dr \\ &\quad + \int_s^t \left( (D\sigma_d)(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) \right) \left( \mathcal{D}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x, k} \right) dW_r^{d, \theta}. \end{aligned} \quad (262)$$

For every  $\theta \in \Theta$ ,  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$  let  $\mathcal{V}_t^{d, \theta, K, s, x} = (\mathcal{V}_t^{d, \theta, K, s, x, k})_{k \in [1, d] \cap \mathbb{Z}}: \Omega \rightarrow \mathbb{R}^d$ ,  $\mathcal{Z}_t^{d, \theta, K, s, x} = (\mathcal{Z}_t^{d, \theta, K, s, x, k})_{k \in [0, d] \cap \mathbb{Z}}: \Omega \rightarrow \mathbb{R}^{d+1}$  satisfy that

$$\mathcal{V}_t^{d, \theta, K, s, x} = \frac{1}{t-s} \int_s^t \left( \sigma_d^{-1}(\mathcal{X}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x}) \mathcal{D}_{\max\{s, [r]_K\}}^{d, \theta, K, s, x} \right)^\top dW_r^{d, \theta} \quad (263)$$

and  $\mathcal{Z}_t^{d, \theta, K, s, x} = (1, \mathcal{V}_t^{d, \theta, K, s, x})$ . Let  $U_{n, m, K}^{d, \theta}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ ,  $d \in \mathbb{N}$ ,  $d, n, m, K \in \mathbb{Z}$ ,  $\theta \in \Theta$ , satisfy for all  $d, n, m, K \in \mathbb{N}$ ,  $\theta \in \Theta$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  that  $U_{-1, m, K}^{d, \theta}(t, x) = U_{0, m, K}^{d, \theta}(t, x) = 0$  and

$$\begin{aligned} U_{n, m, K}^{d, \theta}(t, x) &= (g_d(x), 0) + \sum_{i=1}^{m^n} \frac{g_d(\mathcal{X}_T^{d, (\theta, 0, -i), K, t, x}) - g_d(x)}{m^n} \mathcal{Z}_T^{d, (\theta, 0, -i), K, t, x} \\ &\quad + \sum_{\ell=0}^{n-1} \sum_{i=1}^{m^{n-\ell}} \frac{\left( F_d(U_{\ell, m, K}^{d, (\theta, \ell, i)}) - \mathbb{1}_{\mathbb{N}}(\ell) F_d(U_{\ell-1, m, K}^{d, (\theta, \ell, -i)}) \right) \left( t + (T-t)\mathbf{r}^{(\theta, \ell, i)}, \mathcal{X}_{t+(T-t)\mathbf{r}^{(\theta, \ell, i)}}^{d, (\theta, \ell, i), K, t, x} \right)}{m^{n-\ell} \varrho(t, t + (T-t)\mathbf{r}^{(\theta, \ell, i)})} \mathcal{Z}_{t+(T-t)\mathbf{r}^{(\theta, \ell, i)}}^{d, (\theta, \ell, i), K, t, x}. \end{aligned} \quad (264)$$

For every  $d \in \mathbb{N}$  let  $\mathbf{e}_d, \mathbf{f}_d, \mathbf{g}_d \in [0, \infty)$  satisfy that

$$\mathbf{e}_d + \mathbf{f}_d + \mathbf{g}_d \leq cd^c. \quad (265)$$

Let  $\mathfrak{C}_{n, m, K}^d \in [0, \infty)$ ,  $n, m \in \mathbb{Z}$ ,  $d, K \in \mathbb{N}$ , satisfy for all  $n \in \mathbb{Z}$ ,  $d, m, K \in \mathbb{N}$  that

$$\mathfrak{C}_{n, m, K}^d \leq m^n (K\mathbf{e}_d + \mathbf{g}_d) \mathbb{1}_{\mathbb{N}}(n) + \sum_{\ell=0}^{n-1} \left[ m^{n-\ell} (K\mathbf{e}_d + \mathbf{f}_d + \mathfrak{C}_{\ell, m, K}^d + \mathfrak{C}_{\ell-1, m, K}^d) \right]. \quad (266)$$

Then the following items hold.

(i) For all  $d \in \mathbb{N}$  there exists an up to indistinguishability continuous random field  $(X_t^{d,s,x})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d} : \{(\sigma, \tau) \in [0, T] : \sigma \leq \tau\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$  such that for all  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that  $(X_t^{d,s,x})_{t \in [s,T]}$  is  $(\mathbb{F}_t)_{t \in [s,T]}$ -adapted and such that for all  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have  $\mathbb{P}$ -a.s. that

$$X_t^{d,s,x} = x + \int_s^t \mu_d(X_r^{d,s,x}) dr + \int_s^t \sigma_d(X_r^{d,s,x}) dW_r^{d,0}. \quad (267)$$

(ii) For all  $d \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  there exists an  $(\mathbb{F}_t)_{t \in [s,T]}$ -adapted stochastic process  $(D_t^{d,s,x})_{t \in [s,T]} = (D_t^{d,s,x,k})_{k \in [1,d] \cap \mathbb{Z}} : [s, T] \times \Omega \rightarrow \mathbb{R}^{d \times d}$  which satisfies for all  $t \in [s, T]$ ,  $k \in [1, d] \cap \mathbb{Z}$  that  $\mathbb{P}$ -a.s. we have that

$$D_t^{d,s,x,k} = e_k^d + \int_s^t ((D\mu_d)(X_r^{d,s,x}))(D_r^{d,s,x,k}) dr + \int_s^t ((D\sigma_d)(X_r^{d,s,x}))(D_r^{d,s,x,k}) dW_r^{d,0}. \quad (268)$$

(iii) For all  $d \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  there exist  $(\mathbb{F}_t)_{t \in (s,T]}$ -adapted stochastic processes  $(V_t^{d,s,x})_{t \in (s,T]} = (V_t^{d,s,x,k})_{t \in (s,T], k \in [1,d] \cap \mathbb{Z}} : (s, T] \times \Omega \rightarrow \mathbb{R}^d$  and  $(Z_t^{d,s,x})_{t \in (s,T]} : (s, T] \times \Omega \rightarrow \mathbb{R}^{d+1}$  which satisfy for all  $t \in (s, T]$  that  $\mathbb{P}$ -a.s. we have that

$$V_t^{d,s,x} = \frac{1}{t-s} \int_s^t (\sigma_d^{-1}(X_r^{d,s,x}) D_r^{d,s,x})^\top dW_r^{d,0} \quad (269)$$

and  $Z_t^{d,s,x} = (1, V_t^{d,s,x})$ .

(iv) For all  $d \in \mathbb{N}$  there exists a unique continuous function  $u_d : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  such that for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0,d] \cap \mathbb{Z}} \sup_{\tau \in [0,T], \xi \in \mathbb{R}^d} \left[ \Lambda_\nu^d(T-\tau) \frac{|\text{pr}_\nu^d(u_d(\tau, \xi))|}{[(cd^c)^2 + c^2 \|x\|^2]^\beta} \right] < \infty, \quad (270)$$

$$\max_{\nu \in [0,d] \cap \mathbb{Z}} \left[ \mathbb{E} \left[ \left| g_d(X_T^{d,t,x}) \text{pr}_\nu^d(Z_T^{d,t,x}) \right| \right] + \int_t^T \mathbb{E} [f_d(r, X_r^{d,t,x}, u_d(r, X_r^{d,t,x})) \text{pr}_\nu^d(Z_r^{d,t,x})] dr \right] < \infty, \quad (271)$$

and

$$u_d(t, x) = \mathbb{E} [g_d(X_T^{d,t,x}) Z_T^{d,t,x}] + \int_t^T \mathbb{E} [f_d(r, X_r^{d,t,x}, u_d(r, X_r^{d,t,x})) Z_r^{d,t,x}] dr. \quad (272)$$

(v) For all  $d \in \mathbb{N}$  we have that

$$\limsup_{n \rightarrow \infty} \sup_{\nu \in [0,d] \cap \mathbb{Z}, t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, n^{\frac{1}{3}}, n^{\frac{2}{3}}}^{d,0}(t, x) - u_d(t, x) \right) \right\|_2 \right] = 0. \quad (273)$$

(vi) There exist  $(C_\delta)_{\delta \in (0,1)} \subseteq (0, \infty)$ ,  $\eta \in (0, \infty)$ ,  $(N_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1)} \subseteq \mathbb{N}$  such that for all  $d \in \mathbb{N}$ ,  $\delta, \varepsilon \in (0, 1)$  we have that

$$\sup_{\nu \in [0,d] \cap \mathbb{Z}, t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{2}{3}}}^{d,0}(t, x) - u_d(t, x) \right) \right\|_2 \right] < \varepsilon \quad (274)$$

and

$$\mathfrak{C}_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{2}{3}}}^d \leq C_\delta \varepsilon^{-(4+\delta)} \eta d^m. \quad (275)$$

**Proof of Theorem 6.1.** First, the fundamental theorem of calculus and (254) imply for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} \|\mu_d(x) - \mu_d(y)\| &= \left\| \int_0^1 \frac{d}{ds} [\mu_d(y + s(x-y))] ds \right\| = \left\| \int_0^1 ((D\mu_d)(y + s(x-y)))(x-y) ds \right\| \\ &\leq c \|x - y\| \end{aligned} \quad (276)$$

and similarly,

$$\|\sigma_d(x) - \sigma_d(y)\| \leq c\|x - y\|. \quad (277)$$

This and a standard result on SDEs with Lipschitz continuous coefficients (see, e.g., [45, Theorem 4.5.1]) prove (i). Next, a standard result on the existence and uniqueness of the derivative process (see, e.g., [46, Theorem 3.4]) and the regularity assumptions of  $\mu_d, \sigma_d, d \in \mathbb{N}$ , imply (ii)–(iii).

Throughout the rest of this proof let  $q, \mathbf{c}, \bar{c} \in \mathbb{R}$  satisfy that

$$q = 40\beta, \quad \bar{c} = 16q^2c^2, \quad \mathbf{c} = 2\sqrt{2}qce^{c^2[2q+\sqrt{T}]^2T} \quad (278)$$

and for every  $d \in \mathbb{N}$  let  $\varphi_d: \mathbb{R}^d \rightarrow [1, \infty)$ ,  $\mathbf{V}_d: [0, T] \times \mathbb{R}^d \rightarrow [1, \infty)$  satisfy for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$\varphi_d(x) = \left( \mathbf{c} + 48e^{86c^6T^3} + e^{c^2T} + 4(cd^c)^2 + 4c^2\|x\|^2 \right)^q \quad (279)$$

and

$$\mathbf{V}_d(t, x) = \left[ 15c((cd^c)c + c^2) \left[ \sqrt{T} + 2q \right]^{10} e^{3c^2[\sqrt{T}+2q]^2T} e^{\frac{1.5\bar{c}T}{2q}} \right] e^{\frac{1.5\bar{c}(T-t)}{40}} \varphi_d^{\frac{1}{40}}(x). \quad (280)$$

Then (256) shows for all  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  that

$$|g_d(x)| + |Tf_d(t, x, 0)| \leq ((cd^c)^2 + c^2\|x\|^2)^\beta \leq \varphi_d^{\frac{\beta}{q}}(x) = \varphi_d^{\frac{1}{40}}(x) \leq \mathbf{V}_d(t, x). \quad (281)$$

Next, (258), (279), the fact that  $q = 40\beta$ , and (280) prove for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} |g_d(x) - g_d(y)| &\leq \frac{((cd^c)^2 + c^2\|x\|^2)^\beta + ((cd^c)^2 + c^2\|y\|^2)^\beta}{2} \frac{\|x - y\|}{\sqrt{T}} \\ &\leq \frac{\varphi_d^{\frac{\beta}{q}}(x) + \varphi_d^{\frac{\beta}{q}}(y)}{2} \frac{\|x - y\|}{\sqrt{T}} \\ &= \frac{\varphi_d^{\frac{1}{40}}(x) + \varphi_d^{\frac{1}{40}}(y)}{2} \frac{\|x - y\|}{\sqrt{T}} \\ &\leq \frac{\mathbf{V}_d(T, x) + \mathbf{V}_d(T, y)}{2} \frac{\|x - y\|}{\sqrt{T}}. \end{aligned} \quad (282)$$

Similarly, we have for all  $d \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^d$ ,  $w_1, w_2 \in \mathbb{R}$  that

$$\begin{aligned} &|f_d(t, x, w_1) - f_d(t, y, w_2)| \\ &\leq \sum_{\nu=0}^d [L_\nu^d \Lambda_\nu^d(T) |\text{pr}_\nu^d(w_1 - w_2)|] + \frac{1}{T} \frac{\mathbf{V}_d(t, x) + \mathbf{V}_d(t, y)}{2} \frac{\|x - y\|}{\sqrt{T}}. \end{aligned} \quad (283)$$

Next, (253), the fact that  $\forall x, y \in \mathbb{R}: x + y \leq 2\sqrt{x^2 + y^2}$ , and (279) imply for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that

$$\|\mu_d(0)\| + \|\sigma_d(0)\| + c\|x\| \leq cd^c + c\|x\| \leq 2((cd^c)^2 + c^2\|x\|^2)^{\frac{1}{2}} = \varphi_d^{\frac{1}{2q}}(x). \quad (284)$$

Moreover, (279), [39, Lemma 3.1] (applied for every  $d \in \mathbb{N}$  with  $p \curvearrowright q$ ,  $a \curvearrowright \mathbf{c} + 48e^{86c^6T^3} + e^{c^2T} + 4(cd^c)^2$ ,  $c \curvearrowright 2c$ ,  $V \curvearrowright \varphi_d$  in the notation of [39, Lemma 3.1]), and the fact that  $\bar{c} = 16q^2c^2$  prove for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  that

$$|(D\varphi_d(x))(y)| \leq 4qc\varphi_d^{\frac{2q-1}{2q}}(x)\|y\| \leq \bar{c}\varphi_d^{\frac{2q-1}{2q}}(x)\|y\| \quad (285)$$

and

$$|(D^2\varphi_d(x))(y, y)| \leq 16q^2c^2\varphi_d^{\frac{2q-2}{2q}}(x)\|y\|^2 = \bar{c}\varphi_d^{\frac{2q-2}{2q}}(x)\|y\|^2 \quad (286)$$

where  $D, D^2$  denote the first and second derivative operators. This, (284), (254), (255), and Lemma 5.3 (applied for every  $d, K \in \mathbb{N}$  with  $b \curvearrowright cd^c$ ,  $p \curvearrowright 2q$ ,  $\mu \curvearrowright \mu_d$ ,  $\sigma \curvearrowright \sigma_d$ ,  $\delta \curvearrowright [\cdot]_K$  in the notation of Lemma 5.3) prove that the following items hold.

- For all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\mathbb{E} \left[ \varphi_d(\mathcal{X}_t^{d,0,K,s,x}) \right] \leq e^{1.5\bar{c}|t-s|} \varphi_d(x). \quad (287)$$

- For all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \left\| \mathcal{X}_t^{d,0,K,s,x} - X_t^{d,s,x} \right\|_{2q} \right\| \leq \sqrt{2}c \left[ \sqrt{T} + 2q \right]^2 e^{c^2[\sqrt{T}+2q]^2 T} \left( e^{1.5\bar{c}T} \varphi_d(x) \right)^{\frac{1}{2q}} |t-s|^{\frac{1}{2}} \frac{\sqrt{T}}{\sqrt{K}}. \quad (288)$$

- For all  $d, K \in \mathbb{N}$ ,  $s, \tilde{s} \in [0, T]$ ,  $t \in [s, T]$ ,  $\tilde{t} \in [\tilde{s}, T]$ ,  $x, \tilde{x} \in \mathbb{R}^d$  we have that

$$\begin{aligned} & \left\| \left\| \mathcal{X}_t^{d,0,K,s,x} - \mathcal{X}_{\tilde{t}}^{d,0,K,\tilde{s},\tilde{x}} \right\|_{2q} \right\| \\ & \leq \sqrt{2} \|x - \tilde{x}\| e^{c^2[\sqrt{T}+2q]^2 T} \\ & \quad + 5e^{c^2[\sqrt{T}+2q]^2 T} \left[ \sqrt{T} + 2q \right] e^{\frac{1.5\bar{c}T}{2q}} \frac{\varphi_d^{\frac{1}{2q}}(x) + \varphi_d^{\frac{1}{2q}}(\tilde{x})}{2} \left[ |s - \tilde{s}|^{\frac{1}{2}} + |t - \tilde{t}|^{\frac{1}{2}} \right]. \end{aligned} \quad (289)$$

- For all  $d, K \in \mathbb{N}$ ,  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \left\| \mathcal{V}_t^{d,0,K,s,x,k} \right\|_{2q} \right\| \leq \frac{2\sqrt{2}qce^{c^2[2q+\sqrt{T}]^2 T}}{\sqrt{t-s}}. \quad (290)$$

- For all  $d, K \in \mathbb{N}$ ,  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  we have that

$$\left\| \left\| \mathcal{V}_t^{d,0,K,s,x,k} - \mathcal{V}_t^{d,0,K,s,y,k} \right\|_{\frac{q}{2}} \right\| \leq \frac{2cd^c c [\sqrt{T} + 2q]^3 e^{3c^2[\sqrt{T}+2q]^2 T} \|x - y\|}{\sqrt{t-s}}. \quad (291)$$

- For all  $d, K \in \mathbb{N}$ ,  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \left\| \mathcal{V}_t^{d,0,K,s,x,k} - V_t^{d,s,x,k} \right\|_q \right\| \leq \frac{15c(cd^c c + c^2) [\sqrt{T} + 2q]^6 e^{3c^2[\sqrt{T}+2q]^2 T} \left( e^{1.5\bar{c}T} \varphi_d(x) \right)^{\frac{1}{2q}} \frac{\sqrt{T}}{\sqrt{K}}}{\sqrt{t-s}}. \quad (292)$$

- For all  $k \in [1, d] \cap \mathbb{Z}$ ,  $s \in [0, T]$ ,  $\tilde{s} \in [s, T]$ ,  $r \in [\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$  we have that

$$\left\| \left\| V_t^{d,\tilde{s},x,k} - V_t^{d,s,x,k} \right\|_q \right\| \leq \frac{13(cd^c + c)ce^{3c^2[\sqrt{T}+2q]^2 T} [\sqrt{T} + 2q]^4 e^{\frac{1.5\bar{c}T}{2q}} \varphi_d^{\frac{1}{2q}}(x) |s - \tilde{s}|^{\frac{1}{2}}}{\sqrt{t-s}\sqrt{t-\tilde{s}}}. \quad (293)$$

Now, (287) proves for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \left\| \left\| e^{\frac{1.5\bar{c}(T-t)}{20}} \varphi_d^{\frac{1}{20}}(\mathcal{X}_t^{d,0,K,s,x}) \right\|_{20} \right\| \leq \left( \mathbb{E} \left[ e^{1.5\bar{c}(T-t)} \varphi_d(\mathcal{X}_t^{d,0,K,s,x}) \right] \right)^{\frac{1}{20}} \\ & \leq \left( e^{1.5\bar{c}(T-t)} e^{1.5\bar{c}(t-s)} \varphi_d(x) \right)^{\frac{1}{20}} = e^{1.5\bar{c}(T-s)} \varphi_d^{\frac{1}{20}}(x). \end{aligned} \quad (294)$$

This and (280) imply for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| \mathbf{V}_d(t, \mathcal{X}_t^{d,0,K,s,x}) \right\|_{20} \right\| \leq \mathbf{V}_d(s, x). \quad (295)$$

Next, (288), (280), the fact that  $q \geq 20$ , and Jensen's inequality show for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| \mathcal{X}_t^{d,0,K,s,x} - X_t^{d,s,x} \right\|_{20} \right\| \leq \frac{\sqrt{T} \mathbf{V}_d(t, x)}{\sqrt{K}}. \quad (296)$$

This proves for all  $d \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that  $X_t^{d,s,x} = \lim_{K \rightarrow \infty} \mathcal{X}_t^{d,0,K,s,x}$  in probability. This, (295), the fact that for all  $d \in \mathbb{N}$ ,  $\mathbf{V}_d$  is continuous, and Fatou's lemma imply for all  $d \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| \mathbf{V}_d(t, X_t^{d,s,x}) \right\|_{20} \right\| \leq \mathbf{V}_d(s, x). \quad (297)$$

Next, (292), (280), Jensen's inequality, and the fact that  $q \geq 40$  prove for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $k \in [1, d] \cap \mathbb{Z}$  that

$$\left\| V_t^{d,0,K,s,x,k} - V_t^{d,s,x,k} \right\|_{20} \leq \frac{\frac{\sqrt{T}}{\sqrt{K}} \mathbf{V}_d(t, x)}{\sqrt{T} \sqrt{t-s}} \quad (298)$$

Therefore, the fact that  $\forall d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $x \in \mathbb{R}^d$ :  $\mathbb{P}(\mathcal{Z}^{d,0,K,s,x} = (1, \mathcal{V}^{d,0,K,s,x})) = 1$ , the fact that  $\forall d \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $x \in \mathbb{R}^d$ :  $\mathbb{P}(Z^{d,s,x} = (1, V^{d,s,x})) = 1$ , and the definition of  $\Lambda^d$ ,  $d \in \mathbb{N}$ , prove for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \text{pr}_i^d(\mathcal{Z}_t^{d,0,K,s,x} - Z_t^{d,s,x}) \right\|_{20} \leq \frac{\frac{\sqrt{T}}{\sqrt{K}} \mathbf{V}_d(t, x)}{\sqrt{T} \Lambda_i^d(t-s)}. \quad (299)$$

In addition, (290), (278), Jensen's inequality, and the fact that  $q \geq 40$  show for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $k \in [1, d] \cap \mathbb{Z}$  that

$$\left\| \mathcal{V}_t^{d,0,K,s,x,k} \right\|_{20} \leq \frac{\mathbf{c}}{\sqrt{t-s}}, \quad (300)$$

Thus, the fact that  $\forall d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $x \in \mathbb{R}^d$ :  $\mathbb{P}(\mathcal{Z}^{d,0,K,s,x} = (1, \mathcal{V}^{d,0,K,s,x})) = 1$ , the fact that  $\forall s \in [0, T)$ ,  $x \in \mathbb{R}^d$ :  $\mathbb{P}(Z^{d,s,x} = (1, V^{d,s,x})) = 1$ , and the definition of  $\Lambda^d$ ,  $d \in \mathbb{N}$ , imply for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \text{pr}_i^d(\mathcal{Z}_t^{d,0,K,s,x}) \right\|_{20} \leq \frac{\mathbf{c}}{\Lambda_i^d(t-s)}. \quad (301)$$

This and (299) prove for all  $d \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \text{pr}_i^d(Z_t^{d,s,x}) \right\|_{20} \leq \frac{\mathbf{c}}{\Lambda_i^d(t-s)}. \quad (302)$$

Next, (289), Jensen's inequality, the fact that  $q \geq 40$ , and (278) show for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x, y \in \mathbb{R}^d$  that

$$\left\| \left\| \mathcal{X}_t^{d,0,K,s,x} - \mathcal{X}_t^{d,0,K,s,y} \right\|_{20} \right\| \leq \mathbf{c} \|x - y\|. \quad (303)$$

Moreover, (288), Jensen's inequality, the fact that  $q \geq 40$ , (279), and (280) imply for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| \mathcal{X}_t^{d,0,K,s,x} - X_t^{d,s,x} \right\|_{20} \right\| \leq \frac{\sqrt{T}}{\sqrt{K}} \mathbf{V}_d(t, x). \quad (304)$$

This, (303), and the fact that  $\forall d \in \mathbb{N}$ :  $\mathbf{c} \leq \mathbf{V}_d$  (see (279)–(280)) imply for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| X_t^{d,s,x} - X_t^{d,s,y} \right\|_{20} \right\| \leq \mathbf{c} \|x - y\| \leq \frac{\mathbf{V}_d(t, x) + \mathbf{V}_d(t, y)}{2} \|x - y\|. \quad (305)$$

Next, (291), Jensen's inequality, the fact that  $q \geq 40$ , (278), and (279) show for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $k \in [1, d] \cap \mathbb{Z}$  that

$$\left\| \mathcal{V}_t^{d,0,K,s,x,k} - \mathcal{V}_t^{d,0,K,s,y,k} \right\|_{20} \leq \frac{\mathbf{V}_d(t, x) + \mathbf{V}_d(t, y)}{2} \frac{\|x - y\|}{\sqrt{T} \sqrt{t-s}}. \quad (306)$$

Hence, the fact that  $\forall d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $x \in \mathbb{R}^d$ :  $\mathbb{P}(\mathcal{Z}^{d,0,K,s,x} = (1, \mathcal{V}^{d,0,K,s,x})) = 1$  and the definition of  $\Lambda^d$ ,  $d \in \mathbb{N}$ , prove for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \text{pr}_i^d(\mathcal{Z}_t^{d,0,K,s,x} - \mathcal{Z}_t^{d,0,K,s,y}) \right\|_{20} \leq \frac{\mathbf{V}_d(t, x) + \mathbf{V}_d(t, y)}{2} \frac{\|x - y\|}{\sqrt{T} \Lambda_i^d(t-s)}. \quad (307)$$

This and (299) show for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in (s, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \text{pr}_i^d(Z_t^{d,s,x} - Z_t^{d,s,y}) \right\|_{20} \leq \frac{\mathbf{V}_d(t, x) + \mathbf{V}_d(t, y)}{2} \frac{\|x - y\|}{\sqrt{T} \Lambda_i^d(t-s)}. \quad (308)$$

Next, (293), Jensen's inequality, the fact that  $q \geq 40$ , (279), and (280) imply for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $\tilde{s} \in [s, T)$ ,  $t \in (\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| V_t^{d, \tilde{s}, x, k} - V_t^{d, s, x, k} \right\|_q \leq \frac{\mathbf{V}_d(\tilde{s}, x) + \mathbf{V}_d(s, x)}{2} \frac{\sqrt{\tilde{s} - s}}{\sqrt{t - \tilde{s}} \sqrt{t - s}}. \quad (309)$$

Thus, the fact that  $\forall d \in \mathbb{N}, s \in [0, T), x \in \mathbb{R}^d: \mathbb{P}(Z^{d, s, x} = (1, V^{d, s, x})) = 1$  and the definition of  $\Lambda^d$ ,  $d \in \mathbb{N}$ , show for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $\tilde{s} \in [s, T)$ ,  $t \in (\tilde{s}, T]$ ,  $x \in \mathbb{R}^d$ ,  $i \in [0, d] \cap \mathbb{Z}$  that

$$\left\| \text{pr}_i^d(Z_t^{d, s, x} - Z_t^{d, \tilde{s}, x}) \right\|_{p_z} \leq \frac{\mathbf{V}_d(\tilde{s}, x) + \mathbf{V}_d(s, x)}{2} \frac{\sqrt{\tilde{s} - s}}{\sqrt{t - \tilde{s}} \Lambda_i^d(t - s)}. \quad (310)$$

Moreover, (289), Jensen's inequality, the fact that  $q \geq 40$ , the fact that  $\forall d, K \in \mathbb{N}, s \in [0, T), x \in \mathbb{R}^d: \mathbb{P}(\mathcal{X}_s^{d, 0, K, s, x} = s) = 1$ , (279), and (280) imply for all  $d, K \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| \mathcal{X}_t^{d, 0, K, s, x} - x \right\| \right\|_{20} \leq \mathbf{V}_d(s, x) \sqrt{t - s}. \quad (311)$$

This and (304) prove for all  $d \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  that

$$\left\| \left\| X_t^{d, s, x} - x \right\| \right\|_{20} \leq \mathbf{V}_d(s, x) \sqrt{t - s}. \quad (312)$$

Now, Lemma 3.1 (applied for every  $d, K \in \mathbb{N}$  with  $\delta \curvearrowright \frac{\sqrt{T}}{\sqrt{K}}$ ,  $p_v \curvearrowright 20$ ,  $p_x \curvearrowright 20$ ,  $p_z \curvearrowright 20$ ,  $c \curvearrowright c$ ,  $V \curvearrowright \mathbf{V}_d$ ,  $(L_i)_{i \in [1, d] \cap \mathbb{Z}} \curvearrowright (L_i^d)_{i \in [1, d] \cap \mathbb{Z}}$ ,  $\Lambda \curvearrowright \Lambda^d$ ,  $\text{pr} \curvearrowright \text{pr}^d$ ,  $f \curvearrowright f_d$ ,  $g \curvearrowright g_d$ ,  $\mathcal{X} \curvearrowright \mathcal{X}_{(\cdot)}^{d, 0, K, (\cdot), (\cdot)}$ ,  $\mathcal{Z} \curvearrowright \mathcal{Z}_{(\cdot)}^{d, 0, K, (\cdot), (\cdot)}$ ,  $X \curvearrowright \mathcal{X}_{(\cdot)}^{d, (\cdot), (\cdot)}$ ,  $Z \curvearrowright \mathcal{Z}_{(\cdot)}^{d, (\cdot), (\cdot)}$  in the notation of Lemma 3.1), (281)–(283), (295), (297), (302), (305), (308), the flow property of  $X$  as solution to the SDE in (i), (304), and (299) imply that

- there exist unique measurable functions  $u_d, \mathbf{u}_{d, K}: [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ ,  $d, K \in \mathbb{N}$ , such that for all  $d, K \in \mathbb{N}$ ,  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), \xi \in \mathbb{R}^d} \left[ \Lambda_\nu^d(T - \tau) \frac{|\text{pr}_\nu^d(u_d(\tau, \xi))|}{\mathbf{V}_d(\tau, \xi)} \right] < \infty, \quad (313)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), \xi \in \mathbb{R}^d} \left[ \Lambda_\nu^d(T - \tau) \frac{|\text{pr}_\nu^d(\mathbf{u}_{d, K}(\tau, \xi))|}{\mathbf{V}_d(\tau, \xi)} \right] < \infty, \quad (314)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E} \left[ \left| g_d(X_T^{d, t, x}) \text{pr}_\nu^d(Z_T^{d, t, x}) \right| \right] + \int_t^T \mathbb{E} \left[ \left| f_d(r, X_r^{d, t, x}, u_d(r, X_r^{d, t, x})) \text{pr}_\nu^d(Z_r^{d, t, x}) \right| \right] dr \right] < \infty, \quad (315)$$

$$\begin{aligned} & \max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E} \left[ \left| g_d(\mathcal{X}_T^{d, 0, K, t, x}) \text{pr}_\nu^d(\mathcal{Z}_T^{d, 0, K, t, x}) \right| \right] \right. \\ & \left. + \int_t^T \mathbb{E} \left[ \left| f_d(r, \mathcal{X}_r^{d, 0, K, t, x}, \mathbf{u}_{d, K}(r, \mathcal{X}_r^{d, 0, K, t, x})) \text{pr}_\nu^d(\mathcal{Z}_r^{d, 0, K, t, x}) \right| \right] dr \right] < \infty, \end{aligned} \quad (316)$$

$$u_d(t, x) = \mathbb{E} \left[ g_d(X_T^{d, t, x}) Z_T^{d, t, x} \right] + \int_t^T \mathbb{E} \left[ f_d(r, X_r^{d, t, x}, u_d(r, X_r^{d, t, x})) Z_r^{d, t, x} \right] dr, \quad (317)$$

and

$$\begin{aligned} \mathbf{u}_{d, K}(t, x) &= \mathbb{E} \left[ g_d(\mathcal{X}_T^{d, 0, K, t, x}) \mathcal{Z}_T^{d, 0, K, t, x} \right] \\ &+ \int_t^T \mathbb{E} \left[ f_d(r, \mathcal{X}_r^{d, 0, K, t, x}, \mathbf{u}_{d, K}(r, \mathcal{X}_r^{d, 0, K, t, x})) \mathcal{Z}_r^{d, 0, K, t, x} \right] dr \end{aligned} \quad (318)$$

and

- for all  $d \in \mathbb{N}$ ,  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $y \in \mathbb{R}^d$  we have that

$$\Lambda_\nu^d(T-t) |\text{pr}_\nu^d(u_d(t, y) - \mathbf{u}_{d,K}(t, y))| \leq \frac{\sqrt{T} e^{c^2 T}}{\sqrt{K}} \mathbf{V}_d^9(t, y). \quad (319)$$

This and the fact that  $\forall d \in \mathbb{N}: e^{c^2 T} \leq \mathbf{V}_d$  (see (279)–(280)) imply for all  $d \in \mathbb{N}$ ,  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $t \in [0, T]$ ,  $y \in \mathbb{R}^d$  that

$$\Lambda_\nu^d(T-t) |\text{pr}_\nu^d(u_d(t, y) - \mathbf{u}_{d,K}(t, y))| \leq \frac{\mathbf{V}_d^{10}(t, y)}{\sqrt{K}}. \quad (320)$$

Furthermore, Lemma 2.9 (applied for every  $d \in \mathbb{N}$  with  $p_\nu \curvearrowright 20$ ,  $p_x \curvearrowright 20$ ,  $p_z \curvearrowright 20$ ,  $c \curvearrowright \mathbf{c}$ ,  $(L_i)_{i \in [0, d] \cap \mathbb{Z}} \curvearrowright (L_i^d)_{i \in [0, d] \cap \mathbb{Z}}$ ,  $\Lambda \curvearrowright \Lambda^d$ ,  $\text{pr} \curvearrowright \text{pr}^d$ ,  $f \curvearrowright f_d$ ,  $g \curvearrowright g_d$ ,  $V \curvearrowright \mathbf{V}_d$ ,  $X \curvearrowright X_{(\cdot)}^{d, (\cdot), (\cdot)}$ ,  $Z \curvearrowright Z_{(\cdot)}^{d, (\cdot), (\cdot)}$  in the notation of Lemma 2.9), (281)–(283), (297), (302), (305), (308), and (312) prove for all  $d \in \mathbb{N}$  that  $u_d$  is continuous. This, (313), (315), and (317) show (iv).

Next, Proposition 4.3 (applied for every  $d, K \in \mathbb{N}$  with  $p_\nu \curvearrowright 20$ ,  $p_x \curvearrowright 20$ ,  $p_z \curvearrowright 20$ ,  $c \curvearrowright \mathbf{c}$ ,  $(L_i)_{i \in [0, d] \cap \mathbb{Z}} \curvearrowright (L_i^d)_{i \in [0, d] \cap \mathbb{Z}}$ ,  $\Lambda \curvearrowright \Lambda^d$ ,  $\text{pr} \curvearrowright \text{pr}^d$ ,  $f \curvearrowright f_d$ ,  $g \curvearrowright g_d$ ,  $V \curvearrowright \mathbf{V}_d$ ,  $\mathcal{X} \curvearrowright \mathcal{X}_{(\cdot)}^{d, (\cdot), K, (\cdot), (\cdot)}$ ,  $\mathcal{Z} \curvearrowright \mathcal{Z}_{(\cdot)}^{d, (\cdot), K, (\cdot), (\cdot)}$ ,  $U \curvearrowright U_{(\cdot), (\cdot), K}^{d, (\cdot)}$ ,  $q_1 \curvearrowright 3$  in the notation of Proposition 4.3), (281)–(283), (295), (311), (301), (280), the fact that  $\forall d, K \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$ :  $\mathbb{P}(\mathcal{Z}^{d, 0, K, s, x} = (1, \mathcal{V}^{d, 0, K, s, x}))$ , (263), and (264) prove for all  $d, K, n, m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $x \in \mathbb{R}^d$  that

$$\Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, m, K}^{d, 0}(t, x) - \mathbf{u}_{d, K}(t, x) \right) \right\|_2 \leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2 T} \mathbf{V}_d^3(t, x). \quad (321)$$

Hence, the triangle inequality and (320) show for all  $d, K, n, m \in \mathbb{N}$ ,  $t \in [0, T]$ ,  $\nu \in [0, d] \cap \mathbb{Z}$ ,  $x \in \mathbb{R}^d$  that

$$\begin{aligned} & \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, m, K}^{d, 0}(t, x) - u_d(t, x) \right) \right\|_2 \\ & \leq \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, m, K}^{d, 0}(t, x) - \mathbf{u}_{d, K}(t, x) \right) \right\|_2 + \Lambda_\nu^d(T-t) |\text{pr}_\nu^d(u_d(t, x) - \mathbf{u}_{d, K}(t, x))| \\ & \leq e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2 T} \mathbf{V}_d^3(t, x) + \frac{\mathbf{V}_d^{10}(t, x)}{\sqrt{K}} \\ & \leq \left[ e^{\frac{m^3}{6}} m^{-\frac{n}{2}} 8^n e^{nc^2 T} + \frac{1}{\sqrt{K}} \right] \mathbf{V}_d^{10}(t, x). \end{aligned} \quad (322)$$

This implies for all  $d, n \in \mathbb{N}$  that

$$\begin{aligned} & \sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T], x \in \mathbb{R}^d} \frac{\Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, n^{\frac{1}{3}}, n^{\frac{n}{3}}}^{d, 0}(t, x) - u_d(t, x) \right) \right\|_2}{\mathbf{V}_d^{10}(t, x)} \\ & = \left[ e^{\frac{n}{6}} n^{-\frac{n}{6}} 8^n e^{nc^2 T} + n^{-\frac{n}{6}} \right] \leq e^{\frac{n}{6}} n^{-\frac{n}{6}} 9^n e^{nc^2 T}. \end{aligned} \quad (323)$$

Therefore, (279) and (280) show for all  $d \in \mathbb{N}$  that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n, n^{\frac{1}{3}}, n^{\frac{n}{3}}}^{d, 0}(t, x) - u_d(t, x) \right) \right\|_2 \right] \\ & \leq \limsup_{n \rightarrow \infty} \sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ e^{\frac{n}{6}} n^{-\frac{n}{6}} 9^n e^{nc^2 T} \mathbf{V}_d^{10}(t, x) \right] = 0. \end{aligned} \quad (324)$$

This proves (v).

For the next step let  $(N_{d, \varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0, 1)}$ ,  $(C_\delta)_{\delta \in (0, 1)} \subseteq [0, \infty]$  satisfy for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  that

$$N_{d, \varepsilon} = \inf \left\{ n \in \mathbb{N} : \sup_{k \in [0, \infty) \cap \mathbb{Z}, \nu \in [0, d] \cap \mathbb{Z}, t \in [0, T], x \in [-\mathbf{k}, \mathbf{k}]^d} \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{k, k^{\frac{1}{3}}, k^{\frac{k}{3}}}^{d, 0}(t, x) - u_d(t, x) \right) \right\|_2 \leq \varepsilon \right\} \quad (325)$$

and

$$C_\delta = \sup_{n \in \mathbb{N}} \frac{\left( e^{\frac{n}{6}} 9^n e^{nc^2 T} \right)^{4+\delta} 6^{n+1} n^{\frac{2}{3}}}{n^{\frac{\delta n}{6}}}. \quad (326)$$

Then (324) shows for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  that  $N_{d,\varepsilon} \in \mathbb{N}$ . Furthermore, for all  $\delta \in (0, 1)$  we have that  $C_\delta < \infty$ . Next, [5, Lemma 3.14], (266), and (265) imply for all  $d, K, m, n \in \mathbb{N}$  that

$$\mathfrak{C}_{n,m,K}^d \leq \frac{K\mathfrak{e} + \mathfrak{g}_d + K\mathfrak{e}_d + \mathfrak{f}_d}{2} (3m)^n \leq K(\mathfrak{e}_d + \mathfrak{g}_d + \mathfrak{f}_d) (3m)^n \leq Kcd^c (3m)^n. \quad (327)$$

This proves for all  $d, K, m, n \in \mathbb{N}$  that

$$\begin{aligned} \mathfrak{C}_{n+1,(n+1)^{\frac{1}{3}},(n+1)^{\frac{n+1}{3}}}^d &\leq (n+1)^{\frac{n+1}{3}} cd^c (3(n+1)^{\frac{1}{3}})^{n+1} \\ &= cd^c 3^{n+1} (n+1)^{\frac{2(n+1)}{3}} \\ &\leq cd^c 3^{n+1} (2n)^{\frac{2(n+1)}{3}} \\ &\leq cd^c 6^{n+1} n^{\frac{2(n+1)}{3}} \\ &= cd^c 6^{n+1} n^{\frac{2}{3}} n^{\frac{2n}{3}}. \end{aligned} \quad (328)$$

Hence, (323) shows for all  $d, n \in \mathbb{N}$ ,  $\delta \in (0, 1)$  that

$$\begin{aligned} &\left[ \sup_{\nu \in [0,d] \cap \mathbb{Z}, t \in [0,T], x \in \mathbb{R}^d} \frac{\Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n,n^{\frac{1}{3}},n^{\frac{n}{3}}}^{d,0}(t,x) - u_d(t,x) \right) \right\|_2}{\mathbf{V}_d^{10}(t,x)} \right]^{4+\delta} \mathfrak{C}_{n+1,(n+1)^{\frac{1}{3}},(n+1)^{\frac{n+1}{3}}}^d \\ &\leq \left( e^{\frac{n}{6}} n^{-\frac{n}{6}} 9^n e^{nc^2 T} \right)^{4+\delta} cd^c 6^{n+1} n^{\frac{2}{3}} n^{\frac{2n}{3}} \\ &\leq \frac{\left( e^{\frac{n}{6}} 9^n e^{nc^2 T} \right)^{4+\delta} 6^{n+1} n^{\frac{2}{3}}}{n^{\frac{\delta n}{6}}} cd^c \\ &\leq C_\delta cd^c. \end{aligned} \quad (329)$$

Next, (279) and (280) imply that there exists  $\eta \in (0, \infty)$  such that for all  $d \in \mathbb{N}$  we have that  $\sup_{t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \mathbf{V}_d(t,x) \leq \eta d^n$ . This and (329) imply that there exists  $\eta \in (0, \infty)$  such that for all  $d \in \mathbb{N}$ ,  $\delta \in (0, 1)$  we have that

$$\begin{aligned} &\sup_{\nu \in [0,d] \cap \mathbb{Z}, t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{n,n^{\frac{1}{3}},n^{\frac{n}{3}}}^{d,0}(t,x) - u_d(t,x) \right) \right\|_2 \right]^{4+\delta} \mathfrak{C}_{n+1,(n+1)^{\frac{1}{3}},(n+1)^{\frac{n+1}{3}}}^d \\ &\leq C_\delta cd^c \left[ \sup_{t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \mathbf{V}_d^{10}(t,x) \right]^{4+\delta} \leq C_\delta cd^c \left[ \sup_{t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \mathbf{V}_d^{10}(t,x) \right]^5 \leq C_\delta \eta d^n. \end{aligned} \quad (330)$$

In addition, (327) shows for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  that if  $N_{d,\varepsilon} = 1$  then

$$\varepsilon^{4+\delta} \mathfrak{C}_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{N_{d,\varepsilon}}{3}}} \leq \mathfrak{C}_{1,1,1} \leq 3cd^c. \quad (331)$$

Moreover, (325) and (330) implies that there exists  $\eta \in (0, \infty)$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  we have that if  $N_{d,\varepsilon} \geq 2$  then

$$\begin{aligned} &\varepsilon^{4+\delta} \mathfrak{C}_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{N_{d,\varepsilon}}{3}}} \\ &\leq \sup_{\nu \in [0,d] \cap \mathbb{Z}, t \in [0,T], x \in [-\mathbf{k}, \mathbf{k}]^d} \left[ \Lambda_\nu^d(T-t) \left\| \text{pr}_\nu^d \left( U_{N_{d,\varepsilon}-1, (N_{d,\varepsilon}-1)^{\frac{1}{3}}, (N_{d,\varepsilon}-1)^{\frac{N_{d,\varepsilon}-1}{3}}}(t,x) - u_d(t,x) \right) \right\|_2 \right]^{4+\delta} \\ &\mathfrak{C}_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{N_{d,\varepsilon}}{3}}}^d \end{aligned}$$

$$\leq C_\delta \eta d^n \quad (332)$$

This and (331) prove that there exists  $\eta \in (0, \infty)$  such that for all  $d \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  we have that

$$\varepsilon^{4+\delta} \mathfrak{C}^d \Big|_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{N_{d,\varepsilon}}{3}}} \leq C_\delta \eta d^n. \quad (333)$$

Thus, the fact that  $\forall d \in \mathbb{N}, \varepsilon \in (0, 1): N_{d,\varepsilon} < \infty$ , (325), and the fact that  $\forall \delta \in (0, 1): C_\delta < \infty$  establish (vi). The proof of Theorem 6.1 is thus completed.  $\square$

Finally, we provide the proof of Theorem 1.1.

*Proof of Theorem 1.1.* First, the fundamental theorem of calculus and (4) prove for all  $d \in \mathbb{N}$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} \|\sigma_d(x) - \sigma_d(y)\| &= \left\| \int_0^1 \frac{d}{ds} [\sigma_d(y + s(x-y))] ds \right\| = \left\| \int_0^1 ((D\sigma_d)(y + s(x-y)))(x-y) ds \right\| \\ &\leq c \|x - y\| \end{aligned} \quad (334)$$

and similarly

$$\|\mu_d(x) - \mu_d(y)\| \leq c \|x - y\|. \quad (335)$$

Next, (5) shows for all  $d \in \mathbb{N}$ ,  $x, h \in \mathbb{R}^d$  that

$$\|h\|^2 = h^\top \sigma_d^{-1}(x) \sigma_d(x) (\sigma_d(x))^\top ((\sigma_d(x))^{-1})^\top h \geq \frac{1}{c} \|((\sigma_d(x))^{-1})^\top h\|^2. \quad (336)$$

This implies for all  $d \in \mathbb{N}$ ,  $x, h \in \mathbb{R}^d$  that

$$\|((\sigma_d(x))^{-1})^\top h\|^2 \leq c \|h\|^2. \quad (337)$$

Therefore, the fact that for all  $d \in \mathbb{N}$ ,  $\Sigma \in \mathbb{R}^{d \times d}$  we have that the operator norm of  $\Sigma$  is equal to the operator norm of  $\Sigma^\top$  proves for all  $d \in \mathbb{N}$ ,  $x, h \in \mathbb{R}^d$  that

$$\|((\sigma_d(x))^{-1})h\|^2 \leq c \|h\|^2. \quad (338)$$

Next, the Cauchy-Schwarz inequality implies for all  $d \in \mathbb{N}$ ,  $s = (s_{ij})_{i,j \in [1,d] \cap \mathbb{Z}} \in \mathbb{R}^{d \times d}$ ,  $h = (h_j)_{j \in [1,d] \cap \mathbb{Z}} \in \mathbb{R}^d$  that

$$\|sh\|^2 = \sum_{i=1}^d \left| \sum_{j=1}^d s_{ij} h_j \right|^2 \leq \sum_{i=1}^d \left[ \left( \sum_{j=1}^d |s_{ij}|^2 \right) \left( \sum_{j=1}^d |h_j|^2 \right) \right] = \|s\|^2 \|h\|^2. \quad (339)$$

This, (338), (334), and the fact that  $c \geq 1$  show for all  $d \in \mathbb{N}$ ,  $x, y, h \in \mathbb{R}^d$  that

$$\begin{aligned} \|(\sigma_d^{-1}(y) - \sigma_d^{-1}(x))h\| &= \|\sigma_d^{-1}(y)(\sigma_d(x) - \sigma_d(y))\sigma_d^{-1}(x)h\| \\ &\leq c \|(\sigma_d(x) - \sigma_d(y))\sigma_d^{-1}(x)h\| \\ &\leq c \|\sigma_d(x) - \sigma_d(y)\| \|\sigma_d^{-1}(x)h\| \\ &\leq c^3 \|x - y\| \|h\|. \end{aligned} \quad (340)$$

Hence, Theorem 6.1, (338), and the assumption of Theorem 1.1 prove that the following items hold.

- For all  $d \in \mathbb{N}$  there exists an up to indistinguishability continuous random field  $(X_t^{d,s,x})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d}: \{(\sigma, \tau) \in [0, T]: \sigma \leq \tau\} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$  such that for all  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  we have that  $(X_t^{d,s,x})_{t \in [s,T]}$  is  $(\mathbb{F}_t)_{t \in [s,T]}$ -adapted and such that for all  $s \in [0, T]$ ,  $t \in [s, T]$ ,  $x \in \mathbb{R}^d$  we have  $\mathbb{P}$ -a.s. that

$$X_t^{d,s,x} = x + \int_s^t \mu_d(X_r^{d,s,x}) dr + \int_s^t \sigma_d(X_r^{d,s,x}) dW_r^{d,0}. \quad (341)$$

- For all  $d \in \mathbb{N}$ ,  $s \in [0, T]$ ,  $x \in \mathbb{R}^d$  there exists an  $(\mathbb{F}_t)_{t \in [s, T]}$ -adapted stochastic process  $(D_t^{d,s,x})_{t \in [s, T]} = (D_t^{d,s,x,k})_{k \in [1, d] \cap \mathbb{Z}} : [s, T] \times \Omega \rightarrow \mathbb{R}^{d \times d}$  which satisfies for all  $t \in [s, T]$ ,  $k \in [1, d] \cap \mathbb{Z}$  that  $\mathbb{P}$ -a.s. we have that

$$D_t^{d,s,x,k} = \int_s^t ((D\mu_d)(X_r^{d,s,x})) (D_r^{d,s,x,k}) dr + \int_s^t ((D\sigma_d)(X_r^{d,s,x})) (D_r^{d,s,x,k}) dW_r^{d,0}. \quad (342)$$

- For all  $d \in \mathbb{N}$ ,  $s \in [0, T)$ ,  $x \in \mathbb{R}^d$  there exist  $(\mathbb{F}_t)_{t \in (s, T]}$ -adapted stochastic processes  $(V_t^{d,s,x})_{t \in (s, T]} = (V_t^{d,s,x,k})_{k \in [1, d] \cap \mathbb{Z}} : (s, T] \times \Omega \rightarrow \mathbb{R}^d$  and  $(Z_t^{d,s,x})_{t \in (s, T]} : (s, T] \times \Omega \rightarrow \mathbb{R}^{d+1}$  which satisfy for all  $t \in (s, T]$  that  $\mathbb{P}$ -a.s. we have that

$$V_t^{d,s,x} = \frac{1}{t-s} \int_s^t (\sigma_d^{-1}(X_r^{d,s,x}) D_r^{d,s,x})^\top dW_r^{d,0} \quad (343)$$

and  $Z_t^{d,s,x} = (1, V_t^{d,s,x})$ .

- For all  $d \in \mathbb{N}$  there exists a unique continuous function  $u_d : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$  such that for all  $t \in [0, T)$ ,  $x \in \mathbb{R}^d$  we have that

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \sup_{\tau \in [0, T), \xi \in \mathbb{R}^d} \left[ \Lambda_\nu^d(T - \tau) \frac{|\text{pr}_\nu^d(u_d(\tau, \xi))|}{[(cd^c)^2 + c^2 \|x\|^2]} \right] < \infty, \quad (344)$$

$$\max_{\nu \in [0, d] \cap \mathbb{Z}} \left[ \mathbb{E} \left[ \left| g_d(X_T^{d,t,x}) \text{pr}_\nu^d(Z_T^{d,t,x}) \right| \right] + \int_t^T \mathbb{E} [f_d(r, X_r^{d,t,x}, u_d(r, X_r^{d,t,x})) \text{pr}_\nu^d(Z_r^{d,t,x})] dr \right] < \infty, \quad (345)$$

and

$$u_d(t, x) = \mathbb{E} \left[ g_d(X_T^{d,t,x}) Z_T^{d,t,x} \right] + \int_t^T \mathbb{E} [f_d(r, X_r^{d,t,x}, u_d(r, X_r^{d,t,x})) Z_r^{d,t,x}] dr. \quad (346)$$

- For all  $d \in \mathbb{N}$  we have that

$$\limsup_{n \rightarrow \infty} \sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T), x \in [-k, k]^d} \left[ \Lambda_\nu^d(T - t) \left\| \text{pr}_\nu^d \left( U_{n, n^{\frac{1}{3}}, n^{\frac{2}{3}}}^{d,0}(t, x) - u_d(t, x) \right) \right\|_2 \right] = 0. \quad (347)$$

- There exist  $(C_\delta)_{\delta \in (0, 1)} \subseteq (0, \infty)$ ,  $\eta \in (0, \infty)$ ,  $(N_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0, 1)} \subseteq \mathbb{N}$  such that for all  $d \in \mathbb{N}$ ,  $\delta, \varepsilon \in (0, 1)$  we have that

$$\sup_{\nu \in [0, d] \cap \mathbb{Z}, t \in [0, T), x \in [-k, k]^d} \left[ \Lambda_\nu^d(T - t) \left\| \text{pr}_\nu^d \left( U_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{2}{3}}}^{d,0}(t, x) - u_d(t, x) \right) \right\|_2 \right] < \varepsilon \quad (348)$$

and

$$\mathfrak{C}_{N_{d,\varepsilon}, |N_{d,\varepsilon}|^{\frac{1}{3}}, |N_{d,\varepsilon}|^{\frac{2}{3}}}^d \leq C_\delta \varepsilon^{-(4+\delta)} \eta d^\eta. \quad (349)$$

Next, (8), the Cauchy-Schwarz inequality, and the fact that  $\forall d \in \mathbb{N} : \sum_{\nu=0}^d L_\nu^d \leq c$  imply for all  $d \in \mathbb{N}$ ,  $t \in [0, T)$ ,  $w_1, w_2 \in \mathbb{R}^{d+1}$  that

$$\begin{aligned} & |f_d(t, x, w_1) - f_d(t, y, w_2)| \\ & \leq \sum_{\nu=0}^d [L_\nu^d \Lambda_\nu^d(T) |\text{pr}_\nu^d(w_1 - w_2)|] + \frac{1}{T} c \frac{\|x - y\|}{\sqrt{T}} \\ & \leq L_0^d |\text{pr}_0^d(w_1 - w_2)| + \left( \sum_{\nu=1}^d |L_\nu^d|^2 \right)^{\frac{1}{2}} \left( \sum_{\nu=1}^d T |\text{pr}_\nu^d(w_1 - w_2)|^2 \right)^{\frac{1}{2}} + \frac{c}{T\sqrt{T}} \|x - y\| \\ & \leq c |\text{pr}_0^d(w_1 - w_2)| + c \left( \sum_{\nu=1}^d T |\text{pr}_\nu^d(w_1 - w_2)|^2 \right)^{\frac{1}{2}} + \frac{c}{T\sqrt{T}} \|x - y\|. \end{aligned} \quad (350)$$

In addition, (334) and (335) imply for all  $d \in \mathbb{N}$ ,  $x \in \mathbb{R}^d$  that

$$\|\mu_d(x)\| \leq \|\mu_d(0)\| + c\|x\| \leq cd^c + c\|x\| = c(d^c + \|x\|) \leq cd^c(1 + \|x\|) \quad (351)$$

and similarly

$$\|\sigma_d(x)\| \leq cd^c(1 + \|x\|). \quad (352)$$

Furthermore, (6) shows for all  $d \in \mathbb{N}$ ,  $k \in [1, d] \cap \mathbb{Z}$ ,  $x, y \in \mathbb{R}^d$  that

$$\begin{aligned} & \max \{ \|(\mathbb{D}_k \mu_d)(x) - (\mathbb{D}_k \mu_d)(y)\|, \|((\mathbb{D}_k \sigma_d)(x) - (\mathbb{D}_k \sigma_d)(y))(h)\| \} \\ & = \max \{ \|((\mathbb{D} \mu_d)(x) - (\mathbb{D} \mu_d)(y))(e_k)\|, \|((\mathbb{D} \sigma_d)(x) - (\mathbb{D} \sigma_d)(y))(e_k)\| \} \leq c\|x - y\|, \end{aligned} \quad (353)$$

where  $\mathbb{D}_k$ ,  $k \in [1, d] \cap \mathbb{Z}$ , denote the partial derivatives. Thus, an existence and uniqueness result on viscosity solution (see, e.g., [49, Proposition 5.1] and [49, Proposition 5.2]), (350), (9), (334), (335), (351), (352), (7), (5), (341)–(346), and the regularity assumptions of  $\mu_d$ ,  $\sigma_d$ ,  $d \in \mathbb{N}$ , show for all  $d \in \mathbb{N}$  that  $v_d := \text{pr}_0^d(u_d)$  is the unique viscosity solution to the following semilinear PDE of parabolic type:

$$\begin{aligned} \frac{\partial v_d}{\partial t}(t, x) + \langle (\nabla_x v_d)(t, x), \mu_d(x) \rangle + \frac{1}{2} \text{trace}(\sigma_d(x)[\sigma_d(x)]^\top \text{Hess}_x v_d(t, x)) \\ + f_d(t, x, v_d(t, x), (\nabla_x v_d)(t, x)) = 0 \quad \forall t \in (0, T), x \in \mathbb{R}^d, \end{aligned} \quad (354)$$

$$v_d(T, x) = g_d(x) \quad \forall x \in \mathbb{R}^d \quad (355)$$

and  $\nabla_x v_d = (\text{pr}_1^d(u_d), \text{pr}_2^d(u_d), \dots, \text{pr}_d^d(u_d))$ . Combining this with (347)–(349) establishes (i)–(iii). The proof of Theorem 1.1 is thus completed.  $\square$

## REFERENCES

- [1] AL-ARADI, A., CORREIA, A., JARDIM, G., DE FREITAS NAIFF, D., AND SAPORITO, Y. Extensions of the deep Galerkin method. *Applied Mathematics and Computation* 430 (2022), 127287.
- [2] BECK, C., BECKER, S., CHERIDITO, P., JENTZEN, A., AND NEUFELD, A. Deep learning based numerical approximation algorithms for stochastic partial differential equations and high-dimensional nonlinear filtering problems. *arXiv preprint arXiv:2012.01194* (2020).
- [3] BECK, C., BECKER, S., CHERIDITO, P., JENTZEN, A., AND NEUFELD, A. Deep splitting method for parabolic PDEs. *SIAM Journal on Scientific Computing* 43, 5 (2021), A3135–A3154.
- [4] BECK, C., E, W., AND JENTZEN, A. Machine learning approximation algorithms for high-dimensional fully nonlinear partial differential equations and second-order backward stochastic differential equations. *Journal of Nonlinear Science* 29 (2019), 1563–1619.
- [5] BECK, C., GONON, L., AND JENTZEN, A. Overcoming the curse of dimensionality in the numerical approximation of high-dimensional semilinear elliptic partial differential equations. *arXiv preprint arXiv:2003.00596* (2020).
- [6] BECK, C., HORNING, F., HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. Overcoming the curse of dimensionality in the numerical approximation of Allen–Cahn partial differential equations via truncated full-history recursive multilevel Picard approximations. *Journal of Numerical Mathematics* 28, 4 (2020), 197–222.
- [7] BECK, C., HUTZENTHALER, M., JENTZEN, A., AND KUCKUCK, B. An overview on deep learning-based approximation methods for partial differential equations. *arXiv preprint arXiv:2012.12348* (2020).
- [8] BECKER, S., BRAUNWARTH, R., HUTZENTHALER, M., JENTZEN, A., AND VON WURSTEMBERGER, P. Numerical simulations for full history recursive multilevel Picard approximations for systems of high-dimensional partial differential equations. *arXiv preprint arXiv:2005.10206* (2020).
- [9] BERNER, J., DABLANDER, M., AND GROHS, P. Numerically solving parametric families of high-dimensional Kolmogorov partial differential equations via deep learning. *Advances in Neural Information Processing Systems* 33 (2020), 16615–16627.
- [10] CASTRO, J. Deep learning schemes for parabolic nonlocal integro-differential equations. *Partial Differential Equations and Applications* 3, 6 (2022), 77.
- [11] CIOICA-LICHT, P. A., HUTZENTHALER, M., AND WERNER, P. T. Deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear partial differential equations. *arXiv preprint arXiv:2205.14398* (2022).
- [12] DA PRATO, G., AND ZABCZYK, J. *Stochastic equations in infinite dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [13] E, W., HAN, J., AND JENTZEN, A. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics* 5, 4 (2017), 349–380.

- [14] E, W., HAN, J., AND JENTZEN, A. Algorithms for solving high dimensional PDEs: from nonlinear Monte Carlo to machine learning. *Nonlinearity* 35, 1 (2021), 278.
- [15] E, W., HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. *Journal of Scientific Computing* 79, 3 (2019), 1534–1571.
- [16] E, W., HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. *Partial Differential Equations and Applications* 2, 6 (2021), 1–31.
- [17] E, W., AND YU, B. The deep Ritz method: A deep learning-based numerical algorithm for solving variational problems. *Commun Math Stat* 6, 1 (2018), 1–12.
- [18] FREY, R., AND KÖCK, V. Convergence analysis of the deep splitting scheme: the case of partial integro-differential equations and the associated FBSDEs with jumps. *arXiv preprint arXiv:2206.01597* (2022).
- [19] FREY, R., AND KÖCK, V. Deep neural network algorithms for parabolic PIDEs and applications in insurance mathematics. In *Methods and Applications in Fluorescence*. Springer, 2022, pp. 272–277.
- [20] GERMAIN, M., PHAM, H., AND WARIN, X. Approximation error analysis of some deep backward schemes for nonlinear PDEs. *SIAM Journal on Scientific Computing* 44, 1 (2022), A28–A56.
- [21] GILES, M. B., JENTZEN, A., AND WELTI, T. Generalised multilevel Picard approximations. *arXiv preprint arXiv:1911.03188* (2019).
- [22] GNOATTO, A., PATACCA, M., AND PICARELLI, A. A deep solver for BSDEs with jumps. *arXiv preprint arXiv:2211.04349* (2022).
- [23] GONON, L. Random feature neural networks learn Black-Scholes type PDEs without curse of dimensionality. *Journal of Machine Learning Research* 24, 189 (2023), 1–51.
- [24] GONON, L., AND SCHWAB, C. Deep ReLU network expression rates for option prices in high-dimensional, exponential Lévy models. *Finance and Stochastics* 25, 4 (2021), 615–657.
- [25] GONON, L., AND SCHWAB, C. Deep ReLU neural networks overcome the curse of dimensionality for partial integrodifferential equations. *Analysis and Applications* 21, 01 (2023), 1–47.
- [26] GROHS, P., HORNUNG, F., JENTZEN, A., AND VON WURSTEMBERGER, P. A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black–Scholes partial differential equations. *Mem. Am. Math. Soc.* 284 (2023).
- [27] HAN, J., JENTZEN, A., AND E, W. Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences* 115, 34 (2018), 8505–8510.
- [28] HAN, J., AND LONG, J. Convergence of the deep BSDE method for coupled FBSDEs. *Probability, Uncertainty and Quantitative Risk* 5 (2020), 1–33.
- [29] HAN, J., ZHANG, L., AND E, W. Solving many-electron Schrödinger equation using deep neural networks. *Journal of Computational Physics* 399 (2019), 108929.
- [30] HURÉ, C., PHAM, H., AND WARIN, X. Deep backward schemes for high-dimensional nonlinear PDEs. *Mathematics of Computation* 89, 324 (2020), 1547–1579.
- [31] HUTZENTHALER, M., JENTZEN, A., AND KRUSE, T. Overcoming the curse of dimensionality in the numerical approximation of parabolic partial differential equations with gradient-dependent nonlinearities. *Foundations of Computational Mathematics* 22, 4 (2022), 905–966.
- [32] HUTZENTHALER, M., JENTZEN, A., KRUSE, T., NGUYEN, T., AND VON WURSTEMBERGER, P. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *Proceedings of the Royal Society A* 476, 2244 (2020), 20190630.
- [33] HUTZENTHALER, M., JENTZEN, A., KRUSE, T., AND NGUYEN, T. A. Multilevel Picard approximations for high-dimensional semilinear second-order PDEs with Lipschitz nonlinearities. *arXiv:2009.02484* (2020).
- [34] HUTZENTHALER, M., JENTZEN, A., KRUSE, T., AND NGUYEN, T. A. A proof that rectified deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear heat equations. *SN partial differential equations and applications* 1 (2020), 1–34.
- [35] HUTZENTHALER, M., JENTZEN, A., AND VON WURSTEMBERGER, P. Overcoming the curse of dimensionality in the approximative pricing of financial derivatives with default risks. *Electron. J. Probab.* 25, 101 (2020), 1–73.
- [36] HUTZENTHALER, M., AND KRUSE, T. Multilevel Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. *SIAM Journal on Numerical Analysis* 58, 2 (2020), 929–961.
- [37] HUTZENTHALER, M., KRUSE, T., AND NGUYEN, T. A. Multilevel Picard approximations for McKean-Vlasov stochastic differential equations. *Journal of Mathematical Analysis and Applications* 507, 1 (2022), 125761.
- [38] HUTZENTHALER, M., AND NGUYEN, T. A. Multilevel Picard approximations of high-dimensional semilinear partial differential equations with locally monotone coefficient functions. *Applied Numerical Mathematics* 181 (2022), 151–175.
- [39] HUTZENTHALER, M., AND NGUYEN, T. A. Strong convergence rate of Euler-Maruyama approximations in temporal-spatial Hölder-norms. *Journal of Computational and Applied Mathematics* 412 (2022), 114391.

- [40] HUTZENTHALER, M., AND POHL, K. On nonlinear Feynman-Kac formulas for viscosity solutions of semilinear parabolic partial differential equations with gradient-dependent nonlinearities. *arXiv preprint arXiv:2310.18197* (2023).
- [41] ITO, K., REISINGER, C., AND ZHANG, Y. A neural network-based policy iteration algorithm with global  $H^2$ -superlinear convergence for stochastic games on domains. *Foundations of Computational Mathematics* 21, 2 (2021), 331–374.
- [42] JACQUIER, A., AND OUMGARI, M. Deep curve-dependent PDEs for affine rough volatility. *SIAM Journal on Financial Mathematics* 14, 2 (2023), 353–382.
- [43] JACQUIER, A., AND ZURIC, Z. Random neural networks for rough volatility. *arXiv preprint arXiv:2305.01035* (2023).
- [44] JENTZEN, A., SALIMOVA, D., AND WELTI, T. A proof that deep artificial neural networks overcome the curse of dimensionality in the numerical approximation of Kolmogorov partial differential equations with constant diffusion and nonlinear drift coefficients. *arXiv preprint arXiv:1809.07321* (2018).
- [45] KUNITA, H. *Stochastic flows and stochastic differential equations*, vol. 24. Cambridge university press, 1990.
- [46] KUNITA, H. *Real and Stochastic Analysis. New Perspectives*. Birkhäuser, 2004, ch. Stochastic Differential Equations Based on Lévy Processes and Stochastic Flows of Diffeomorphisms, pp. 304–373.
- [47] LU, L., MENG, X., MAO, Z., AND KARNIADAKIS, G. E. DeepXDE: A deep learning library for solving differential equations. *SIAM review* 63, 1 (2021), 208–228.
- [48] NEUFELD, A., NGUYEN, T. A., AND WU, S. Deep ReLU neural networks overcome the curse of dimensionality in the numerical approximation of semilinear partial integro-differential equations. *Analysis and Applications, accepted for publication. arXiv preprint arXiv:2310.15581* (2023).
- [49] NEUFELD, A., AND WU, S. Multilevel Picard algorithm for general semilinear parabolic PDEs with gradient-dependent nonlinearities. *Journal of Numerical Mathematics, accepted for publication. arXiv preprint arXiv:2310.12545* (2023).
- [50] NEUFELD, A., AND WU, S. Multilevel Picard approximation algorithm for semilinear partial integro-differential equations and its complexity analysis. *Stochastics and Partial Differential Equations: Analysis and Computations* (2025), 1–59.
- [51] NGUWI, J. Y., PENENT, G., AND PRIVAULT, N. A deep branching solver for fully nonlinear partial differential equations. *arXiv preprint arXiv:2203.03234* (2022).
- [52] NGUWI, J. Y., PENENT, G., AND PRIVAULT, N. Numerical solution of the incompressible Navier-Stokes equation by a deep branching algorithm. *arXiv preprint arXiv:2212.13010* (2022).
- [53] NGUWI, J. Y., AND PRIVAULT, N. A deep learning approach to the probabilistic numerical solution of path-dependent partial differential equations. *Partial Differential Equations and Applications* 4, 4 (2023), 37.
- [54] POHL, K., AND HUTZENTHALER, M. On existence and uniqueness properties for solutions of stochastic fixed point equations with gradient-dependent nonlinearities. *arXiv preprint arXiv:2310.17182* (2023).
- [55] RAISSI, M., PERDIKARIS, P., AND KARNIADAKIS, G. E. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational physics* 378 (2019), 686–707.
- [56] REISINGER, C., AND ZHANG, Y. Rectified deep neural networks overcome the curse of dimensionality for nonsmooth value functions in zero-sum games of nonlinear stiff systems. *Analysis and Applications* 18, 06 (2020), 951–999.
- [57] SIRIGNANO, J., AND SPILIOPOULOS, K. DGM: A deep learning algorithm for solving partial differential equations. *Journal of computational physics* 375 (2018), 1339–1364.
- [58] ZHANG, D., GUO, L., AND KARNIADAKIS, G. E. Learning in modal space: Solving time-dependent stochastic PDEs using physics-informed neural networks. *SIAM Journal on Scientific Computing* 42, 2 (2020), A639–A665.

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