Martingale Exercise

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables satisfying

$$\mathbb{P}[X_1 = 1] = \frac{1}{2}, \qquad \mathbb{P}[X_1 = -1] = \frac{1}{2}.$$

For a fixed $S_0 \in \mathbb{R}$, define

$$S_n := S_0 + \sum_{i=1}^{n} X_i, \qquad n \in \mathbb{N}_0,$$

where the empty sum is understood to be zero. Let $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \in \mathbb{N}$.

(a) For any nonzero constant $c \in \mathbb{R}$, define the stochastic process $(M_n)_{n\in\mathbb{N}_0}$ by

$$M_n := c^n \left(2^{S_n} + 2^{-S_n}\right), \qquad n \in \mathbb{N}_0.$$

Determine all constants $c \neq 0$ such that $(M_n)_{n\in\mathbb{N}_0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. Justify your answer.

Solution

We verify adaptedness, integrability, and the martingale condition. For the conditional expectation calculation, we use the numbering in the notes *Properties of the Conditional Expectation*: property 6) is linearity, property 4) allows us to take out an $\mathcal{F}_n$-measurable factor, and property 5) applies to random variables independent of $\mathcal{F}_n$.

Adaptedness. For every $n \in \mathbb{N}_0$, the random variable S_n is $\mathcal{F}_n$-measurable. Consequently, $M_n = c^n(2^{S_n} + 2^{-S_n})$ is also $\mathcal{F}_n$-measurable, so $(M_n)_{n\in\mathbb{N}_0}$ is adapted.

Integrability. Since $-n \leq S_n - S_0 \leq n$, we have

$$|M_n| = |c|^n \left(2^{S_n} + 2^{-S_n}\right) \leq (2|c|)^n \left(2^{S_0} + 2^{-S_0}\right).$$

The right-hand side is a finite constant. Hence $\mathbb{E}[|M_n|] < \infty$ for every $n \in \mathbb{N}_0$.

Conditional expectation

Fix $n \in \mathbb{N}_0$. We calculate $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n]$ step by step.

Step 1. Expand the definition. Using $S_{n+1} = S_n + X_{n+1}$, we obtain

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\Big[c^{n+1}\left(2^{S_n}2^{X_{n+1}} + 2^{-S_n}2^{-X_{n+1}}\right)\Big|\,\mathcal{F}_n\Big].$$

Step 2. Apply property 6). By linearity, we can take out the constant c^{n+1} and split the sum:

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = c^{n+1}\left(\mathbb{E}[2^{S_n}2^{X_{n+1}} \mid \mathcal{F}_n] + \mathbb{E}[2^{-S_n}2^{-X_{n+1}} \mid \mathcal{F}_n]\right).$$

Step 3. Apply property 4). The factors 2^{S_n} and 2^{-S_n} are $\mathcal{F}_n$-measurable. Therefore,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = c^{n+1}\left(2^{S_n}\mathbb{E}[2^{X_{n+1}} \mid \mathcal{F}_n] + 2^{-S_n}\mathbb{E}[2^{-X_{n+1}} \mid \mathcal{F}_n]\right).$$

Step 4. Apply property 5). Since X_{n+1} is independent of $\mathcal{F}_n$, so are $2^{X_{n+1}}$ and $2^{-X_{n+1}}$. Thus

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = c^{n+1}\left(2^{S_n}\mathbb{E}[2^{X_{n+1}}] + 2^{-S_n}\mathbb{E}[2^{-X_{n+1}}]\right).$$

Step 5. Evaluate the expectations. Since $\mathbb{P}[X_{n+1} = 1] = \mathbb{P}[X_{n+1} = -1] = \frac{1}{2}$,

$$\mathbb{E}[2^{X_{n+1}}] = \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{5}{4},$$
$$\mathbb{E}[2^{-X_{n+1}}] = \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot 2 = \tfrac{5}{4}.$$

Consequently,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \frac{5}{4}\,c^{n+1}\left(2^{S_n} + 2^{-S_n}\right) = \frac{5c}{4}M_n.$$

Determine the constant. Because $c \neq 0$ and $2^{S_n} + 2^{-S_n} > 0$, we have $M_n \neq 0$. Thus $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$ for every $n \in \mathbb{N}_0$ if and only if $5c/4 = 1$. Together with adaptedness and integrability, this proves that

$$\boxed{(M_n)_{n\in\mathbb{N}_0} \text{ is a martingale if and only if } c = \frac{4}{5}.}$$

Property numbering: Ariel Neufeld, *Properties of the Conditional Expectation.*