Monotone Convergence Theorem &
Lebesgue's Dominated Convergence Theorem

• Let Ω be any space, let $\mathbb{P}$ be a probability measure on Ω, and let $\mathbb{E}_{\mathbb{P}}[\cdot]$ denote the expectation.

• Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables which converge to a random variable X, i.e. $\lim_{n\to\infty} X_n = X$.

Question: When can we interchange limit with expectation, i.e. when does it hold:

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[X_n] \stackrel{?}{=} \mathbb{E}_{\mathbb{P}}\big[\lim_{n\to\infty} X_n\big] = \mathbb{E}_{\mathbb{P}}[X]$$

The following theorems are *sufficient* conditions to interchange limit with expectations.

Theorem 1 (Monotone Convergence Theorem)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables which are nonnegative and increasingly converge to a random variable X, meaning that $0 \le X_1 \le X_2 \le \cdots \le X$ with $\lim_{n\to\infty} X_n = X$. Then we have*

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[X_n] = \mathbb{E}_{\mathbb{P}}\big[\lim_{n\to\infty} X_n\big] = \mathbb{E}_{\mathbb{P}}[X].$$

Theorem 2 (Lebesgue's Dominated Convergence Theorem)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables which converge to a random variable X, meaning that $\lim_{n\to\infty} X_n = X$. Moreover, assume that there exists a random variable Y with $\mathbb{E}_{\mathbb{P}}[|Y|] < \infty$ such that $|X_n| \le Y$ for every $n \in \mathbb{N}$. Then we have*

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[X_n] = \mathbb{E}_{\mathbb{P}}\big[\lim_{n\to\infty} X_n\big] = \mathbb{E}_{\mathbb{P}}[X].$$