### UNIVERSAL APPROXIMATION PROPERTY OF RANDOM NEURAL NETWORKS

ARIEL NEUFELD AND PHILIPP SCHMOCKER

ABSTRACT. In this paper, we study *random neural networks* which are single-hidden-layer feedforward neural networks whose weights and biases are randomly initialized. After this random initialization, only the linear readout needs to be trained, which can be performed efficiently, e.g., by the least squares method. By viewing random neural networks as Banach space-valued random variables, we prove a universal approximation theorem within a large class of Bochner spaces. Hereby, the corresponding Banach space can be significantly more general than the space of continuous functions over a compact subset of a Euclidean space, namely, e.g., an  $L^p$ -space or a Sobolev space, where the latter includes the approximation of the derivatives. Moreover, we derive approximation rates and an explicit algorithm to learn a deterministic function by a random neural network. In addition, we provide a full error analysis and study when random neural networks overcome the curse of dimensionality in the sense that the training costs scale at most polynomially in the input and output dimension. Furthermore, we show in two numerical examples the empirical advantages of random neural networks compared to fully trained deterministic neural networks.

### 1. INTRODUCTION

Inspired by the functionality of human brains, (artificial) neural networks have been discovered in [McCulloch and Pitts, 1943] and provide as a machine learning technique an algorithmic approach for the quest of artificial intelligence (see [Turing, 1950] and [Mitchell, 1997]). Fundamentally, a neural network consists of nodes arranged in hierarchical layers with connections between adjacent layers, which can be mathematically expressed as the concatenation of affine and non-linear functions.

However, the theoretical approximation properties of neural networks were only proven later by, e.g., [Cybenko, 1989], [Hornik et al., 1989], [Hornik, 1991], [Leshno et al., 1993], [Chen and Chen, 1995], and [Pinkus, 1999]. In mathematical terms, this property is usually shown in universal approximation theorems, which establish density of the set of neural networks within a given function space. For example, neural networks are able to approximate any continuous function arbitrarily well on a given compact subset of a Euclidean space. Subsequently, different works have established approximation rates, which describe the relation between the approximation error and the number of network parameters; see e.g. [Barron, 1992], [Barron, 1993], [Darken et al., 1993], [Mhaskar and Micchelli, 1995], [Darken et al., 1997], [Kůrková, 2012], [Bölcskei et al., 2019], and [Siegel and Xu, 2020].

Despite the theoretical progress in the 1990s, neural networks have only attracted wider attention after the turn of the millennium by showing promising applications in the fields of image classification (see e.g. [Krizhevsky et al., 2012]), speech recognition (see e.g. [Hinton et al., 2012]) and computer games (see e.g. [Silver et al., 2016]). This was due to the drastic improvements in computational power and new optimization techniques such as stochastic gradient descent algorithms like, e.g., the Adam algorithm (see [Kingma and Ba, 2015]). However, the training of a neural network remains a challenging task. First of all, the learning procedure is a non-convex optimization problem, i.e. the algorithm locates one of many local minimas, but possibly not the optimal solution (see [Goodfellow et al., 2016, p. 282]). Moreover, the iterative backpropagation improving the solution at each training step is slow, in particular for deep neural networks (see [Montavon et al., 2012, p. 13]). In addition, one would like to overcome the *curse of dimensionality*, i.e. that the training costs scale at most polynomially in the input and output dimension, which is still an open problem for neural networks (see [Goodfellow et al., 2016, p. 155]).

In order to tackle these training limitations of deterministic neural networks, we suggest to use *random* neural networks instead. Inspired by the works on extreme learning machines (see [Huang et al., 2006]), random feature models (see [Rahimi and Recht, 2007] and [Rudi and Rosasco, 2017]), as well as reservoir computing (see [Maass et al., 2002], [Jaeger and Haas, 2004], [Grigoryeva and Ortega, 2018], and

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[Gonon and Ortega, 2021]), random neural networks are single-hidden-layer feed-forward neural networks whose weights and biases are randomly initialized, and only the linear readout is trained (see [Gonon et al., 2023a] with ReLU activation function). In this form, we retrieve a convex optimization problem without iterative backpropagation, which can be solved efficiently on any average computer.

Our first contribution consists of a universal approximation theorem for random neural networks. To this end, we view random neural networks as random functions that return for every random initialization the corresponding network as a function in a suitable Banach space. This allows us to apply the strong law of large numbers for Banach space-valued random variables (see [Hytönen et al., 2016, Theorem 3.1.10]) to lift the universal approximation property of deterministic neural networks to random neural networks, where the approximation error is quantified in a Bochner norm. This allows us to significantly generalize the first universal approximation results in [Gonon et al., 2023a] from deterministic functions to random functions, from the ReLU activation function to more general non-polynomial activation functions, as well as from  $L^2$ -spaces to more general function spaces including, e.g., the derivatives.

In order to obtain this universal approximation result of random neural networks, we first generalize the universal approximation theorem for deterministic neural networks from the classical formulation on compacta to function spaces over unbounded Euclidean domains, e.g.  $L^p$ -spaces and Sobolev spaces (see also [Hornik et al., 1990] and [Hornik, 1991]). More precisely, for  $k \in \mathbb{N}_0$ , we consider Banach spaces that are obtained as completions of the space of bounded and k-times differentiable functions with bounded derivatives with respect to a weighted norm related to the polynomial growth of the activation function. In order to establish density of the set of deterministic neural networks in those function spaces, we apply the classical Hahn-Banach separation argument (as in [Cybenko, 1989, Theorem 1]) and use a Riesz representation theorem (similar to [Dörsek and Teichmann, 2010, Theorem 2.4]) to express any continuous linear functional on the dense subspace with the help of finite signed Radon measures. Hence, by assuming that the activation function is non-polynomial, we can use Korevaar's distributional extension (see [Korevaar, 1965]) of Wiener's Tauberian theorem (see [Wiener, 1932]) to obtain global universal approximation results beyond compact subsets of a Euclidean space (see also [Cuchiero et al., 2023]).

Our second contribution consists of approximation rates for learning a deterministic function by a random neural network, which relates the required size of the random neural network to the pre-given approximation error. To this end, we assume that the function to be approximated has a Fourier transform that is sufficiently regular and integrable, whereas the approximation error is measured with respect to a weighted Sobolev norm. In particular, we use the Ridgelet transform introduced by [Candès, 1998] and its distributional extension in [Sonoda and Murata, 2017] to represent the function to be approximated as expectation of a particular random neural network. Then, we follow the derivations for the approximation rates of deterministic neural networks and use a symmetrization argument with Rademacher averages. This generalizes the approximation rates in [Gonon et al., 2023a, Section 4.2] (see also [Gonon et al., 2023b] for an infinite dimensional version) for random neural networks with ReLU activation function to more general activation functions and the inclusion of the (weak) derivatives into the approximation.

Moreover, by using the least squares method, we provide an algorithm to learn a deterministic function, where we show in a full error analysis that random neural networks can overcome the curse of dimensionality, i.e. that the training costs scale at most polynomially in the input and output dimension. Therefore, random neural networks are suited as non-parametric regression method to learn high-dimensional functions (see [Györfi et al., 2002], [Rahimi and Recht, 2007], [Rudi and Rosasco, 2017], [Carratino et al., 2018], [Chen et al., 2020], [Mannelli et al., 2020], [Mei and Montanari, 2022], and [Heiss et al., 2023]).

The theoretical foundations of random neural networks are relevant in scientific computing. For example, in mathematical physics, random neural networks have been successfully applied to solve partial differential equations (PDEs) (see [Yang et al., 2018], [Dwivedi and Srinivasan, 2020], [Dong and Li, 2021], and [Wang and Dong, 2023]), for photonic systems (see [Lupo et al., 2021]), and for quantum reservoirs (see [Gonon and Jacquier, 2023]). Moreover, random neural networks have been applied in mathematical finance, e.g., for learning option prices in the Black-Scholes model (see [Gonon, 2023]), for optimal stopping (see [Herrera et al., 2021]), for learning the hedging strategy via Wiener-Ito chaos expansion (see [Neufeld and Schmocker, 2022] and [Neufeld and Schmocker, 2023]), for solving path-dependent PDEs in the context of rough volatility (see [Jacquier and Žurič, 2023]), for pricing American options (see [Yang et al., 2023]), and for random deep splitting methods (see [Neufeld et al., 2023]).

We complement these numerical examples by learning the solution of the heat equation and the price of a Basket option, showing the empirical advantages of random neural networks over deterministic ones.

1.1. Outline. In Section 2, we recall deterministic neural networks and generalize their universal approximation property. In Section 3, we define random neural networks as Banach space-valued random variables and show their universal approximation property. In Section 4, we prove some approximation rates and develop an explicit algorithm to learn a deterministic function by a random neural network including a full error analysis. In Section 5, we show in numerical examples how to apply random neural networks and demonstrate their numerical advantages. Finally, all proofs are contained in Section 6-10.

1.2. Notation. In the following, we introduce the notation of some standard function spaces and the Fourier transform for distributions. Readers who are familiar with these concepts may skip this section.

As usual,  $\mathbb{N} := \{1, 2, 3, ...\}$  and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  denote the sets of natural numbers, whereas  $\mathbb{Z}$  represents the set of integers. Moreover,  $\mathbb{R}$  and  $\mathbb{C}$  denote the sets of real and complex numbers, respectively, where  $i := \sqrt{-1} \in \mathbb{C}$  represents the imaginary unit. In addition, for any  $r \in \mathbb{R}$ , we define |r| := $\max\{k \in \mathbb{Z} : k \leq r\}$  and  $[r] := \min\{k \in \mathbb{Z} : k \geq r\}$ . Furthermore, for any  $z \in \mathbb{C}$ , we denote its real and imaginary part as  $\operatorname{Re}(z)$  and  $\operatorname{Im}(z)$ , respectively, whereas its complex conjugate is  $\overline{z} := \operatorname{Re}(z) - \operatorname{Im}(z)i$ .

Moreover, for any  $m \in \mathbb{N}$ , we denote by  $\mathbb{R}^m$  (and  $\mathbb{C}^m$ ) the *m*-dimensional (complex) Euclidean space, which is equipped with the Euclidean norm  $||u|| = \sqrt{\sum_{i=1}^{m} |u_i|^2}$ . In addition, for any  $m, n \in \mathbb{N}$ , we denote by  $\mathbb{R}^{m \times n}$  the vector space of matrices  $A = (a_{i,j})_{i=1,\dots,m}^{j=1,\dots,n} \in \mathbb{R}^{m \times n}$ , which is equipped with the matrix 2-norm  $||A|| = \sup_{x \in \mathbb{R}^n, ||x|| \leq 1} ||Ax||$ , where  $I_m \in \mathbb{R}^{m \times m}$  is the identity matrix.

Furthermore, for  $U \subseteq \mathbb{R}^m$ , we denote by  $\mathcal{B}(U)$  the  $\sigma$ -algebra of Borel-measurable subsets of U. Moreover, for  $U \in \mathcal{B}(\mathbb{R}^m)$ , we denote by  $\mathcal{L}(U)$  the  $\sigma$ -algebra of Lebesgue-measurable subsets of U, while  $du: \mathcal{L}(U) \to [0,\infty]$  denotes the Lebesgue measure on U. Then, a property is said to hold true almost everywhere (shortly a.e.) if it holds everywhere true except on a set of Lebesgue measure zero. Moreover, for every fixed  $m, d \in \mathbb{N}$  and  $U \subseteq \mathbb{R}^m$ , we introduce the following function spaces:

- (i)  $C^0(U; \mathbb{R}^d)$  denotes the vector space of continuous functions  $f: U \to \mathbb{R}^d$ .
- (ii)  $C^k(U; \mathbb{R}^d)$ , with  $k \in \mathbb{N}$  and  $U \subseteq \mathbb{R}^m$  open, denotes the vector space of k-times continuously differentiable functions  $f: U \to \mathbb{R}^d$  such that for every multi-index  $\alpha \in \mathbb{N}_{0,k}^m :=$  $\{\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}_0^m : |\alpha| := \alpha_1 + ... + \alpha_m \leq k\} \text{ the partial derivative } U \ni u \mapsto \partial_\alpha f(u) := \frac{\partial^{|\alpha|} f}{\partial u_1^{\alpha_1} \cdots \partial u_m^{\alpha_m}}(u) \in \mathbb{R}^d \text{ is continuous. If } m = 1, \text{ we write } f^{(j)} := \frac{\partial^j f}{\partial u_j} : U \to \mathbb{R}^d, j = 0, ..., k.$ (iii)  $C_b^k(U; \mathbb{R}^d)$ , with  $k \in \mathbb{N}_0$  and  $U \subseteq \mathbb{R}^m$  (open, if  $k \geq 1$ ), denotes the vector space of bounded
- functions  $f \in C^k(U; \mathbb{R}^d)$  such that  $\partial_{\alpha} f : U \to \mathbb{R}^d$  is bounded for all  $\alpha \in \mathbb{N}^m_{0,k}$ . Then, the norm

$$\|f\|_{C_b^k(U;\mathbb{R}^d)} := \max_{\alpha \in \mathbb{N}_{0,k}^m} \sup_{u \in U} \|\partial_\alpha f(u)\|$$

turns  $(C_b^k(U; \mathbb{R}^d), \|\cdot\|_{C_c^k(U; \mathbb{R}^d)})$  into a Banach space. Note that for k = 0 and  $U \subset \mathbb{R}^m$  being compact, we obtain the usual Banach space  $(C^0(U; \mathbb{R}^d), \|\cdot\|_{C^0(U; \mathbb{R}^d)})$  of continuous functions, which is equipped with the supremum norm  $||f||_{C^0(U;\mathbb{R}^d)} := ||f||_{C^0_{\mathfrak{b}}(U;\mathbb{R}^d)} = \sup_{u \in U} ||f(u)||.$ 

(iv)  $C_{pol,\gamma}^k(U;\mathbb{R}^d)$ , with  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in [0,\infty)$ , denotes the vector space of functions  $f \in C^k(U;\mathbb{R})$  of  $\gamma$ -polynomial growth such that

$$\|f\|_{C^k_{pol,\gamma}(U;\mathbb{R}^d)} := \max_{\alpha \in \mathbb{N}^m_{0,k}} \sup_{u \in U} \frac{\|\partial_\alpha f(u)\|}{(1+\|u\|)^{\gamma}} < \infty.$$

- (v)  $\overline{C_b^k(U;\mathbb{R}^d)}^{\gamma}$ , with  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ , is defined as the closure of  $C_b^k(U; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{C_{pol,\gamma}^k(U; \mathbb{R}^d)}$ . Then,  $(\overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}, \|\cdot\|_{C_{pol,\gamma}^k(U; \mathbb{R}^d)})$  is by definition a Banach space. If  $U \subseteq \mathbb{R}^m$  is bounded, then  $\overline{C_b^k(U;\mathbb{R}^d)}^{\gamma} = C_b^k(U;\mathbb{R}^d)$ . Otherwise,  $f \in$  $\overline{C_b^k(U;\mathbb{R}^d)}^{\gamma} \text{ if and only if } f \in C^k(U;\mathbb{R}^d) \text{ and } \lim_{r \to \infty} \max_{\alpha \in \mathbb{N}_{0,k}^m} \sup_{u \in U, \|u\| \ge r} \frac{\|\partial_{\alpha} f(u)\|}{(1+\|u\|)^{\gamma}} = 0$ (see Lemma 6.1). For example, if  $f \in C^k_{pol,\gamma_0}(U; \mathbb{R}^d)$ , with  $\gamma_0 \in [0, \gamma)$ , then  $f \in \overline{C^k_b(U; \mathbb{R}^d)}^{\gamma}$ .
- (vi)  $C_c^{\infty}(U; \mathbb{R}^d)$ , with  $U \subseteq \mathbb{R}^m$  open, denotes the vector space of smooth functions  $f: U \to \mathbb{R}^d$  such that  $\operatorname{supp}(f) \subseteq U$ , where  $\operatorname{supp}(f)$  is defined as the closure of  $\{u \in U : f(u) \neq 0\}$  in  $\mathbb{R}^m$ .
- (vii)  $L^1_{loc}(U; \mathbb{R}^d)$ , with  $U \subseteq \mathbb{R}^m$ , denotes the space of Lebesgue measurable functions  $f: U \to \mathbb{R}^d$  such that for every compact subset  $K \subset \mathbb{R}^m$  with  $K \subset U$  it holds that  $\int_K \|f(u)\| du < \infty$ .
- (viii)  $\mathcal{S}(\mathbb{R}^m;\mathbb{C})$  denotes the Schwartz space consisting of smooth functions  $f:\mathbb{R}^m\to\mathbb{C}$  such that the seminorms  $\max_{\alpha \in \mathbb{N}_{0,n}^m} \sup_{u \in \mathbb{R}^m} (1 + ||u||^2)^n |\partial_{\alpha} f(u)|$  are finite, for all  $n \in \mathbb{N}_0$ . Then, we equip

 $\mathcal{S}(\mathbb{R}^m;\mathbb{C})$  with the locally convex topology induced by these seminorms (see [Folland, 1992, p. 330]). Moreover, its dual space  $\mathcal{S}'(\mathbb{R}^m;\mathbb{C})$  consists of continuous linear functionals  $T : \mathcal{S}(\mathbb{R}^m;\mathbb{C}) \to \mathbb{C}$  called tempered distributions (see [Folland, 1992, p. 332]). Hereby, we say that  $f \in L^1_{loc}(\mathbb{R}^m;\mathbb{C})$  induces  $T_f \in \mathcal{S}'(\mathbb{R}^m;\mathbb{C})$  if  $\mathcal{S}(\mathbb{R}^m;\mathbb{C}) \ni g \mapsto T_f(g) := \int_{\mathbb{R}^m} f(u)g(u)du \in \mathbb{C}$  is continuous. For example, if there exists some  $n \in \mathbb{N}$  such that  $\int_{\mathbb{R}^m} (1 + ||u||^2)^{-n} |f(u)| du < \infty$ , then the function  $f \in L^1_{loc}(\mathbb{R}^m;\mathbb{C})$  induces the tempered distribution  $T_f \in \mathcal{S}'(\mathbb{R}^m;\mathbb{C})$  (see [Folland, 1992, Equation 9.26]). Conversely, for an open subset  $U \subseteq \mathbb{R}^m$ , a tempered distribution  $T \in \mathcal{S}'(\mathbb{R}^m;\mathbb{C})$  is said to *coincide on* U with  $f_T \in L^1_{loc}(U;\mathbb{C})$  if  $T(g) = T_{f_T}(g)$  for all  $g \in C^\infty_c(U;\mathbb{C})$ . In addition, the support of any tempered distribution  $T \in \mathcal{S}'(\mathbb{R}^m;\mathbb{C})$  is defined as the complement of the largest open set  $U \subseteq \mathbb{R}^m$  on which  $T \in \mathcal{S}'(\mathbb{R}^m;\mathbb{C})$  vanishes, i.e. T(g) = 0 for all  $g \in C^\infty_c(U;\mathbb{C})$ .

- (ix)  $S_0(\mathbb{R}; \mathbb{C}) \subseteq S(\mathbb{R}; \mathbb{C})$  denotes the vector subspace of functions  $f \in S(\mathbb{R}; \mathbb{C})$  with  $\int_{\mathbb{R}} u^j f(u) du = 0$  for all  $j \in \mathbb{N}_0$  (see [Grafakos, 2014, Definition 1.1.1]). Using the Fourier transform (see (1) below) and [Folland, 1992, Theorem 7.5 (c)], this is equivalent to  $\hat{f}^{(j)}(0) = 0$  for all  $j \in \mathbb{N}_0$ .
- (x)  $L^p(U, \Sigma, \mu; \mathbb{R}^d)$ , with  $p \in [1, \infty)$ ,  $U \subseteq \mathbb{R}^m$ , and (possibly non-finite) measure space  $(U, \Sigma, \mu)$ , denotes the vector space of (equivalence classes of)  $\Sigma/\mathcal{B}(\mathbb{R}^d)$ -measurable functions  $f: U \to \mathbb{R}^d$  such that

$$\|f\|_{L^{p}(U,\Sigma,\mu;\mathbb{R}^{d})} := \left(\int_{U} \|f(u)\|^{p} \mu(du)\right)^{\frac{1}{p}} < \infty$$

Then,  $(L^p(U, \Sigma, \mu; \mathbb{R}^d), \|\cdot\|_{L^p(U, \Sigma, \mu; \mathbb{R}^d)})$  is a Banach space (see [Rudin, 1987, p. 96]).

(xi)  $W^{k,p}(U, \mathcal{L}(U), du; \mathbb{R}^d)$ , with  $k \in \mathbb{N}$ ,  $p \in [1, \infty)$ , and  $U \subseteq \mathbb{R}^m$  open, denotes the Sobolev space of (equivalence classes of) k-times weakly differentiable functions  $f : U \to \mathbb{R}^d$  such that  $\partial_{\alpha} f \in L^p(U, \mathcal{L}(U), du; \mathbb{R}^d)$  for all  $\alpha \in \mathbb{N}_{0,k}^m$  (see [Adams, 1975, Chapter 3]). Then, the norm

$$\|f\|_{W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^d)} := \left(\sum_{\alpha\in\mathbb{N}_{0,k}^m}\int_U \|\partial_\alpha f(u)\|^p du\right)^{\frac{1}{p}}$$

turns  $W^{k,p}(U, \mathcal{L}(U), du; \mathbb{R}^d)$  into a Banach space (see [Adams, 1975, Theorem 3.2]).

(xii)  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ , with  $k \in \mathbb{N}$ ,  $p \in [1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  open, and  $\mathcal{L}(U)/\mathcal{B}(\mathbb{R})$ -measurable  $w: U \to [0, \infty)$ , denotes the weighted Sobolev space of (equivalence classes of) k-times weakly differentiable functions  $f: U \to \mathbb{R}^d$  such that  $\partial_{\alpha} f \in L^p(U, \mathcal{L}(U), w(u)du; \mathbb{R}^d)$  for all  $\alpha \in \mathbb{N}_{0,k}^m$ . Moreover,  $w: U \to [0, \infty)$  is called a *weight* if w is a.e. strictly positive. In this case, the norm

$$\|f\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} := \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \|\partial_\alpha f(u)\|^p w(u) du\right)^{\frac{1}{p}}$$

turns  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  into a Banach space (see [Kufner, 1980, p. 5]).

(xiii)  $W^{0,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ , with  $p \in [1, \infty)$  and  $U \in \mathcal{B}(\mathbb{R}^m)$ , is defined as  $L^p(U, \mathcal{L}(U), w(u)du; \mathbb{R}^d)$ . In addition, if the functions are real-valued, we abbreviate  $C^k(U) := C^k(U; \mathbb{R}), L^p(U, \Sigma, \mu) := L^p(U, \Sigma, \mu; \mathbb{R})$ , etc. Moreover, we define the complex-valued function spaces  $C^k(U; \mathbb{C}^d) \cong C^k(U; \mathbb{R}^{2d})$ ,  $L^p(U, \Sigma, \mu; \mathbb{C}^d) \cong L^p(U, \Sigma, \mu; \mathbb{R}^{2d})$ , etc. as in (i)-(xii) (except (viii)+(ix)) by identifying  $\mathbb{C}^d \cong \mathbb{R}^{2d}$ .

Furthermore, we say that an open subset  $U \subseteq \mathbb{R}^m$  admits the *segment property* if for every  $u \in \partial U := \overline{U} \setminus U$  there exists an open neighborhood  $V \subseteq \mathbb{R}^m$  around  $u \in \partial U$  and a vector  $y \in \mathbb{R}^m \setminus \{0\}$  such that for every  $z \in \overline{U} \cap V$  and  $t \in (0, 1)$  it holds that  $z + ty \in U$  (see [Adams, 1975, p. 54]).

Moreover, we define the (multi-dimensional) Fourier transform of any  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{C}^d)$  as

$$\mathbb{R}^m \ni \xi \quad \mapsto \quad \widehat{f}(\xi) := \int_{\mathbb{R}^m} e^{-i\xi^\top u} f(u) du \in \mathbb{C}^d, \tag{1}$$

see [Folland, 1992, p. 247]. Then, by using [Hytönen et al., 2016, Proposition 1.2.2], it follows that

$$\sup_{\xi \in \mathbb{R}^m} \left\| \widehat{f}(\xi) \right\| = \sup_{\xi \in \mathbb{R}^m} \left\| \int_{\mathbb{R}^m} e^{-i\xi^\top u} f(u) du \right\| \leq \int_{\mathbb{R}^m} \|f(u)\| du = \|f\|_{L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)}.$$
 (2)

In addition, the Fourier transform of any tempered distribution  $T \in \mathcal{S}'(\mathbb{R}^m; \mathbb{C})$  is defined by  $T(g) := T(\hat{g})$ , for  $g \in \mathcal{S}(\mathbb{R}^m; \mathbb{C})$  (see [Folland, 1992, Equation 9.28]).

### 2. DETERMINISTIC NEURAL NETWORKS

Before we introduce random neural networks in Section 3, we first consider classical (deterministic) neural networks. Inspired by the functionality of a human brain (see [McCulloch and Pitts, 1943]), deterministic neural networks can be described by a composition of affine and non-affine functions.

**Definition 2.1.** For  $\rho \in C^0(\mathbb{R})$ , a function  $\varphi : \mathbb{R}^m \to \mathbb{R}^d$  is called a deterministic (single-hidden-layer feed-forward) neural network *if it is of the form* 

$$\mathbb{R}^m \ni u \quad \mapsto \quad \varphi(u) = \sum_{n=1}^N y_n \rho\left(a_n^\top u - b_n\right) \in \mathbb{R}^d, \tag{3}$$

for some  $N \in \mathbb{N}$  denoting the number of neurons, where  $a_1, ..., a_N \in \mathbb{R}^m$ ,  $b_1, ..., b_N \in \mathbb{R}$ , and  $y_1, ..., y_N \in \mathbb{R}^d$  represent the weight vectors, biases, and linear readouts, respectively.

In this paper, we only consider single-hidden-layer feed-forward neural networks of the form (3) and simply refer to them as (deterministic) neural networks.

**Definition 2.2.** For  $U \subseteq \mathbb{R}^m$  and  $\rho \in C^0(\mathbb{R})$ , we denote by  $\mathcal{NN}_{U,d}^{\rho}$  the set of all neural networks of the form (3) restricted to U with corresponding activation function  $\rho \in C^0(\mathbb{R})$ .



**Figure 1.** A neural network  $\varphi : \mathbb{R}^m \to \mathbb{R}^d$  with m = 3, d = 2, and N = 5.

Deterministic neural networks admit the so-called universal approximation property, which establishes the density of the set of deterministic neural networks within a given function space. For example, every continuous function can be approximated arbitrarily well on a compact subset of a Euclidean space (see e.g. [Cybenko, 1989], [Hornik, 1991], [Pinkus, 1999], and the references therein).

In order to generalize the approximation properties of deterministic neural networks beyond continuous functions on compact subsets, we now consider the following type of function spaces. For this purpose, we fix the input dimension  $m \in \mathbb{N}$  and the output dimension  $d \in \mathbb{N}$  throughout the rest of this paper.

**Definition 2.3.** For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , we call a Banach space  $(X, \|\cdot\|_X)$  a  $(k, U, \gamma)$ -approximable function space if X consists of functions  $f : U \to \mathbb{R}^d$  and the restriction map

$$(C_b^k(\mathbb{R}^m;\mathbb{R}^d), \|\cdot\|_{C_{pol,\gamma}^k(\mathbb{R}^m;\mathbb{R}^d)}) \ni f \quad \mapsto \quad f|_U \in (X, \|\cdot\|_X)$$

$$\tag{4}$$

is a continuous dense embedding.

**Remark 2.4.** The restriction map in (4) is a continuous dense embedding if it is continuous and its image is dense in X with respect to  $\|\cdot\|_X$ . By definition of  $\overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}$  in Notation (v), this is equivalent to  $(\overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}, \|\cdot\|_{C_{pol,\gamma}^k(\mathbb{R}^m;\mathbb{R}^d)}) \ni f \mapsto f|_U \in (X, \|\cdot\|_X)$  being a continuous dense embedding.

The continuous dense embedding in (4) has two important consequences. First, any  $(k, U, \gamma)$ approximable function space  $(X, \|\cdot\|_X)$  is a separable Banach space which is needed for the notion of
Bochner spaces in Section 3. Moreover, for any activation function  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ , the set of deterministic
neural networks  $\mathcal{NN}_{U,d}^{\rho} \subseteq X$  is well-defined in the function space  $(X, \|\cdot\|_X)$ .

**Lemma 2.5.** For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ -approximable function space. Then, the following holds true:

- (i) The Banach space  $(X, \|\cdot\|_X)$  is separable.
- (ii) For every  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  it holds that  $\mathcal{NN}_{\mathbb{R}^m,d}^{\rho} \subseteq \overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}$  and  $\mathcal{NN}_{U,d}^{\rho} \subseteq X$ .

In order to derive the following universal approximation result, we now assume that the activation function  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is non-polynomial. This is equivalent to the condition that the Fourier transform  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  of the tempered distribution<sup>1</sup>  $(g \mapsto T_{\rho}(g) := \int_{\mathbb{R}} \rho(s)g(s)ds) \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  is supported at a non-zero point (see e.g. [Rudin, 1991, Examples 7.16]). The proof can be found in Section 7.2.

**Theorem 2.6** (Universal Approximation). For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ -approximable function space and assume that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is non-polynomial. Then,  $\mathcal{NN}_{U,d}^{\rho}$  is dense in X with respect to  $\|\cdot\|_X$ .

Theorem 2.6 yields a *global* universal approximation result beyond compact subsets of a Euclidean space and is therefore interesting in its own right. Let us compare Theorem 2.6 to the existing literature.

Remark 2.7. Theorem 2.6 unifies the following universal approximation theorems (UATs):

- (i) Theorem 2.6 extends the UATs in [Cybenko, 1989, Theorem 1], [Hornik et al., 1989, Theorem 2.4], [Hornik, 1991, Theorem 2], [Leshno et al., 1993, Theorem 1], [Chen and Chen, 1995, Theorem 3], and [Pinkus, 1999, Theorem 3.1] for continuous functions on compact subsets of a Euclidean space to more general functions on unbounded Euclidean domains. The latter three results are for non-polynomial activation functions.
- (ii) *Theorem 2.6 extends the UAT in* [Hornik, 1991, Theorem 3+4] *and* [Hornik et al., 1990, Corollary 3.6+3.8] *on (weighted) Sobolev spaces to more general function spaces.*
- (iii) Theorem 2.6 extends the UAT in [Cuchiero et al., 2023, Theorem 4.13] for functions defined on weighted (infinite dimensional) domains to differentiable functions.

Towards the end of this section, let us give some examples of  $(k, U, \gamma)$ -approximable function spaces which include some of the standard function spaces introduced in Section 1.2.

Example 2.8.	For any $k \in \mathbb{N}_0$ , $U \subseteq \mathbb{R}^m$ (ope	en, if $k \ge 1$ ), and $\gamma \in (0,\infty)$ , the following Banach spaces
$(X, \ \cdot\ _X)$ are	$e \; (k, U, \gamma)$ -approximable functi	on spaces:
Eunoti	$\log \operatorname{space}(\mathbf{V} \parallel \parallel_{})$	Notation additional assumptions

	Function space $(X, \ \cdot\ _X)$	Notation	additional assumptions
(i)	$(C_b^k(U; \mathbb{R}^d), \ \cdot\ _{C_b^k(U; \mathbb{R}^d)})$ $k \in \mathbb{N}_0 \text{ and } U \subseteq \mathbb{R}^m \text{ (open, if } k \ge 1)$	(iii)	if $U \subset \mathbb{R}^m$ is bounded
(ii)	$(\overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}, \ \cdot\ _{C_{pol,\gamma}^k(U; \mathbb{R}^d)})$ $k \in \mathbb{N}_0, U \subseteq \mathbb{R}^m \ (open, \ if \ k \ge 1), \ and \ \gamma \in (0,$	(v) (∞	none
(iii)	$(L^{p}(U, \Sigma, \mu; \mathbb{R}^{d}), \ \cdot\ _{L^{p}(U, \Sigma, \mu; \mathbb{R}^{d})})$ $k = 0, p \in [1, \infty), U \subseteq \mathbb{R}^{m},$ and measure space $(U, \Sigma, \mu)$	(x)	$\begin{split} & \text{if } \Sigma = \mathcal{B}(U), \\ & \text{if } \mu : \mathcal{B}(U) \to [0, \infty] \text{ is a Borel-measure,} \\ & \text{and if } \int_{U} (1 + \ u\ )^{\gamma p} \mu(du) < \infty \end{split}$
(iv)	$(W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^d), \ \cdot\ _{W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^d)})$ $k \in \mathbb{N}, p \in [1,\infty), and U \subseteq \mathbb{R}^m open$	$^{(d)})$ (xi)	if $U \subset \mathbb{R}^m$ has the segment property and if $U \subset \mathbb{R}^m$ is bounded
(v)	$(W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d), \ \cdot\ _{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}$ $k \in \mathbb{N}, p \in [1,\infty), U \subseteq \mathbb{R}^m \text{ open,}$ and weight $w: U \to [0,\infty)$	) (xii)	if $U \subseteq \mathbb{R}^m$ has the segment property, if $w: U \to [0, \infty)$ is bounded, if $\inf_{u \in B} w(u) > 0$ for all bounded $B \subseteq U$ , and if $\int_U (1 +   u  )^{\gamma p} w(u) du < \infty$

Moreover, let us give some examples of non-polynomial activation functions  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ , which includes the standard activation functions such as, e.g., the sigmoid function and the ReLU function.

**Example 2.9.** For  $k \in \mathbb{N}_0$  and  $\gamma \in (0, \infty)$ , the following functions  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  are non-polynomial, where its Fourier transform  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  coincides on  $\mathbb{R} \setminus \{0\}$  with the function  $f_{\widehat{T_{\rho}}} \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C})$ :

		$\rho\in\overline{C_b^k(\mathbb{R})}^\gamma$	$k \in \mathbb{N}_0$	$\gamma \in (0,\infty)$	$f_{\widehat{T}_{\rho}} \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C})$
(i)	Sigmoid function	$\rho(s) := \frac{1}{1 + e^{-s}}$	$k \in \mathbb{N}_0$	$\gamma > 0$	$f_{\widehat{T_{\rho}}}(\xi) = \frac{-i\pi}{\sinh(\pi\xi)}$
(ii)	Tangens hyperbolicus	$\rho(s) := \tanh(s)$	$k \in \mathbb{N}_0$	$\gamma > 0$	$f_{\widehat{T}_{\rho}}(\xi) = \frac{-i\pi}{\sinh(\pi\xi/2)}$
(iii)	Softplus function	$\rho(s) := \ln\left(1 + e^s\right)$	$k \in \mathbb{N}_0$	$\gamma > 1$	$f_{\widehat{T_{\rho}}}(\xi) = \frac{-\pi}{\xi \sinh(\pi\xi)}$
(iv)	ReLU function	$\rho(s) := \max(s, 0)$	k = 0	$\gamma > 1$	$f_{\widehat{T_{\rho}}}(\xi) = -\frac{1}{\xi^2}$

<sup>1</sup>Note that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  induces the tempered distribution  $(g \mapsto T_{\rho}(g) := \int_{\mathbb{R}} \rho(s)g(s)ds) \in \mathcal{S}'(\mathbb{R};\mathbb{C})$  as  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is of polynomial growth (see [Folland, 1992, Equation 9.26]).

### 3. RANDOM NEURAL NETWORKS

In order to induce randomness in a neural network, we assume throughout this paper the existence of a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  which supports an independent and identically distributed (i.i.d.) sequence of  $\mathbb{R}^m$ -valued random variables  $(A_n)_{n \in \mathbb{N}} : \Omega \to \mathbb{R}^m$  and an i.i.d. sequence of  $\mathbb{R}$ -valued random variables  $(B_n)_{n \in \mathbb{N}} : \Omega \to \mathbb{R}$ . Then, we define the  $\sigma$ -algebra generated by  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$  as

$$\mathcal{F}_{A,B} := \sigma\left(\{A_n, B_n : n \in \mathbb{N}\}\right).$$
(5)

Moreover, we impose the following condition on the distribution of  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$ .

**Assumption 3.1.** The random vector  $(A_1, B_1) : \Omega \to \mathbb{R}^m \times \mathbb{R}$  satisfies for every  $a \in \mathbb{R}^m$ ,  $b \in \mathbb{R}$ , and r > 0 that  $\mathbb{P}\left[\{\omega \in \Omega : \|(A_1(\omega), B_1(\omega)) - (a, b)\| < r\}\right] > 0$ .

Moreover, for  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ -approximable function space (see Definition 2.3). Then, any  $\mathcal{F}_{A,B}$ -strongly measurable random variable  $F : \Omega \to X$  can be seen as random function  $(U \ni u \mapsto F(\omega)(u) \in \mathbb{R}^d) \in X$  for  $\omega \in \Omega$  (see Section 8.1).

Now, we introduce random neural networks as random functions, where the weight vectors and biases inside the activation function are obtained from  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$ , respectively. Hereby, the linear readout is also a random variable, but which is observable with respect to  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$ .

**Definition 3.2.** Let  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$ ,  $\gamma \in (0, \infty)$ , and  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ . Then, we call a map  $\Phi : \Omega \to \overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}$  a random neural network if it is of the form<sup>2</sup>

$$\Omega \ni \omega \quad \mapsto \quad \Phi(\omega) = \sum_{n=1}^{N} W_n(\omega) \rho \left( A_n(\omega)^\top \cdot -B_n(\omega) \right) \in \overline{C_b^k(U; \mathbb{R}^d)}^{\gamma} \tag{6}$$

for some  $N \in \mathbb{N}$  denoting the number of neurons. Moreover,  $A_1, ..., A_N : \Omega \to \mathbb{R}^m$  and  $B_1, ..., B_N : \Omega \to \mathbb{R}$  are the random weight vectors and random bias, respectively, while the  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable random variables  $W_1, ..., W_N : \Omega \to \mathbb{R}^d$  represent the linear readouts.

**Definition 3.3.** We denote by  $\mathcal{RN}_{U,d}^{\rho}$  the set of all random neural networks of the form (6) with corresponding activation function  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ , for some  $k \in \mathbb{N}_0$  and  $\gamma \in (0, \infty)$ .

Let us briefly explain how a random neural network can be trained on a computer (see also Section 4.2). **Remark 3.4.** For the random initialization of  $(A_n, B_n)_{n=1,...,N}$ , we draw some  $\omega \in \Omega$  and fix the values of  $A_1(\omega), ..., A_N(\omega) \in \mathbb{R}^m$  and  $B_1(\omega), ..., B_N(\omega) \in \mathbb{R}$ . Thus, by using that  $W_1, ..., W_N : \Omega \to \mathbb{R}^d$ are  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable, the training of some  $\Phi \in \mathcal{RN}_{U,d}^{\rho}$  consists of finding the optimal vectors  $W_1(\omega), ..., W_n(\omega) \in \mathbb{R}^d$  given  $A_1(\omega), ..., A_N(\omega) \in \mathbb{R}^m$  and  $B_1(\omega), ..., B_N(\omega) \in \mathbb{R}$  (see Algorithm 1).

In the following, we now lift the universal approximation property of deterministic neural networks in Theorem 2.6 to this setting involving randomness. The proof can be found in Section 8.3.

**Theorem 3.5** (Universal Approximation). For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ -approximable function space and assume that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is non-polynomial. Moreover, let  $(A_1, B_1)$  satisfy Assumption 3.1 and let  $F \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  for some  $r \in [1, \infty)$ . Then, the following holds true:

(i) For every  $\varepsilon > 0$  there exists some  $\Phi \in \mathcal{RN}^{\rho}_{U,d} \cap L^{r}(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  such that

$$\|F - \Phi\|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)} := \mathbb{E}\left[\|F - \Phi\|_X^r\right]^{\frac{1}{r}} < \varepsilon.$$

(ii) For every  $\delta, \varepsilon > 0$  there exists some  $\Phi \in \mathcal{RN}^{\rho}_{U,d} \cap L^{r}(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  such that  $\mathbb{P}\left[\{\omega \in \Omega : \|F(\omega) - \Phi(\omega)\|_{X} \ge \varepsilon\}\right] \le \delta.$ 

**Remark 3.6.** Note that every deterministic function  $f \in X$  is a constant random function  $(\omega \mapsto f) \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  and can therefore be approximated by a random neural network  $\Phi \in \mathcal{RN}_{U,d}^{\rho}$ .

Theorem 3.5 generalizes the universal approximation results in [Rahimi and Recht, 2008, Theorem 3.1], [Hart et al., 2020, Theorem 2.4.3], [Gonon et al., 2023a, Corollary 3], and [Gonon, 2023, Corollary 6] from deterministic functions to random functions, from particular activation functions (such as the ReLU function) to more general non-polynomial activation functions, as well as from particular Banach spaces (e.g.  $L^2$ -spaces) to more general  $(k, U, \gamma)$ -approximable function spaces.

<sup>2</sup>The notation  $W_n(\omega)\rho\left(A_n(\omega)^\top \cdot -B_n(\omega)\right)$  refers to the function  $U \ni u \mapsto W_n(\omega)\rho\left(A_n(\omega)^\top u - B_n(\omega)\right) \in \mathbb{R}^d$ .

In this section, we provide approximation rates for learning a deterministic function by a random neural network. For this purpose, we assume the following for the random initialization  $(A_1, B_1) : \Omega \to \mathbb{R}^m \times \mathbb{R}$ .

**Assumption 4.1.**  $A_1: \Omega \to \mathbb{R}^m$  and  $B_1: \Omega \to \mathbb{R}$  are independent. Moreover,  $A_1$  admits a probability density function  $\theta_A: \mathbb{R}^m \to (0, \infty)$  which is strictly positive. In addition,  $B_1: \Omega \to \mathbb{R}$  follows a Student's t-distribution<sup>3</sup>, i.e.  $B_1 \sim t_1(\nu)$  for some  $\nu \in (0, \infty)$ . In this case, we write  $(A_1, B_1) \sim \theta_A \otimes t_1(\nu)$ .

To derive the approximation rates, we apply the reconstruction formula in [Sonoda and Murata, 2017, Theorem 5.6] to obtain an integral representation of the function to be approximated (see also Section 9.2). To this end, we consider the following pairs  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C_{pol,\gamma}^k(\mathbb{R})$  consisting of a ridgelet function  $\psi \in S_0(\mathbb{R}; \mathbb{C})$  (see Notation (ix)) and an activation function  $\rho \in C_{pol,\gamma}^k(\mathbb{R})$  (see Notation (iv)), which is a special case of [Sonoda and Murata, 2017, Definition 5.1] (see also [Candès, 1998]).

**Definition 4.2.** For  $k \in \mathbb{N}_0$ ,  $\gamma \in [0, \infty)$ , and  $m \in \mathbb{N}$ , a pair  $(\psi, \rho) \in \mathcal{S}_0(\mathbb{R}; \mathbb{C}) \times C^k_{pol,\gamma}(\mathbb{R})$  is called *m*-admissible if  $\widehat{T_\rho} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  coincides on  $\mathbb{R} \setminus \{0\}$  with a function  $f_{\widehat{T_\rho}} \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C})$  such that

$$C_m^{(\psi,\rho)} := (2\pi)^{m-1} \int_{\mathbb{R}\setminus\{0\}} \overline{\frac{\widehat{\psi}(\xi)}{f_{\widehat{T}_\rho}(\xi)}} d\xi \in \mathbb{C}\setminus\{0\}.$$
(7)

**Remark 4.3.** If  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C^k_{pol,\gamma}(\mathbb{R})$  is *m*-admissible, then  $\rho \in C^k_{pol,\gamma}(\mathbb{R})$  has to be nonpolynomial. Indeed, otherwise the support of  $\widehat{T_{\rho}} \in S'(\mathbb{R}; \mathbb{C})$  is contained in  $\{0\} \subset \mathbb{R}$  (see e.g. [Rudin, 1991, Examples 7.16]), which implies that (7) vanishes for any choice of  $\psi \in S_0(\mathbb{R}; \mathbb{C})$ .

4.1. **Approximation Rates.** In this section, we now provide the approximation rates for learning a deterministic function by a random neural network. The proof can be found in Section 9.3.

**Theorem 4.4** (Approximation Rates). For  $k \in \mathbb{N}_0$ ,  $p, r \in (1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in [0, \infty)$ , let  $w : U \to [0, \infty)$  be a weight such that

$$C_{U,w}^{(\gamma,p)} := \left( \int_{U} (1 + \|u\|)^{\gamma p} w(u) du \right)^{\frac{1}{p}} < \infty.$$
(8)

Moreover, for  $\nu \in (0, \infty)$ , let  $(A_1, B_1) \sim \theta_A \otimes t_1(\nu)$  satisfy Assumption 4.1, and let  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C^k_{pol,\gamma}(\mathbb{R})$  be *m*-admissible. Then, there exists  $C_1 > 0$  (independent of  $m, d \in \mathbb{N}$ ) such that for every  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform satisfying

$$C_{f} := \max_{\substack{j \in \mathbb{N}_{0} \cap [0, [\gamma] + [\nu] + 1], \\ \beta \in \mathbb{N}_{0}^{m}, [\gamma] + [\nu] + 1}} \int_{\mathbb{R}} \frac{\left| \widehat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{r}}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \widehat{f}(\xi) \|^{r} \frac{(1 + \|\xi/\zeta\|)^{(k+2[\gamma] + [\nu] + 2)r}}{\theta_{A}(\xi/\zeta)^{r-1}} d\xi \right)^{\frac{1}{r}} d\zeta < \infty, \quad (9)$$

the following holds true:

such that

- (i) It holds that  $f \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ .
- (ii) For every  $N \in \mathbb{N}$  there exists some  $\Phi_N \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d))$ having N neurons such that

$$\mathbb{E}\left[\|f - \Phi_N\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r\right]^{\frac{1}{r}} \leq C_1 \frac{C_{U,w}^{(\gamma,p)} C_f}{\left|C_m^{(\psi,\rho)}\right|} \frac{m^{\frac{k}{p} + \lceil\gamma\rceil + \lceil\nu\rceil + 1}}{N^{1 - \frac{1}{\min(2,p,r)}}},\tag{10}$$

(iii) For every  $\delta, \varepsilon > 0$  there exists some  $\Phi_N \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d))$ having  $N \in \mathbb{N}$  neurons, with

$$N \ge \left( C_1 \frac{C_{U,w}^{(\gamma,p)} C_f}{\left| C_m^{(\psi,\rho)} \right|} \frac{m^{\frac{k}{p} + \lceil \gamma \rceil + \lceil \nu \rceil + 1}}{\delta^{1/r} \varepsilon} \right)^{\frac{\min(2,p,r) - 1}{\min(2,p,r) - 1}},$$

$$\mathbb{P} \left[ \left\{ \omega \in \Omega : \| f - \Phi_N(\omega) \|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} \ge \varepsilon \right\} \right] \le \delta.$$
(11)

 ${}^{3}B_{1} \sim t_{1}(\nu)$  has probability density function  $\mathbb{R} \ni b \mapsto \theta_{B}(b) = \frac{\Gamma((1+\nu)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}} (1 + b^{2}/\nu)^{-(1+\nu)/2} \in (0, \infty)$ , where  $\Gamma$  denotes the Gamma function (see [Abramowitz and Stegun, 1970, Section 6.1])

Theorem 4.4 relates the number of neurons  $N \in \mathbb{N}$  needed for a random neural network to approximate a deterministic function with pre-given error tolerance  $\varepsilon > 0$  (and probability threshold  $\delta > 0$  for (iii)).

**Remark 4.5.** Theorem 4.4 is related to the following approximation rates in the literature:

- (i) Theorem 4.4 extends the approximation rates for random neural networks in [Gonon et al., 2023a, Section 4.2] and [Gonon, 2023, Theorem 1] from the ReLU activation function and Bochner space L<sup>2</sup>(Ω, F, P; X), with X := L<sup>2</sup>(ℝ<sup>m</sup>, B(ℝ<sup>m</sup>), µ; ℝ<sup>d</sup>) for some probability measure µ : B(ℝ<sup>m</sup>) → [0, 1] or X := C<sup>0</sup>([−M, M]) for some M > 0, to more general activation functions and more general L<sup>r</sup>-Bochner spaces with weighted Sobolev space X := W<sup>k,p</sup>(U, L(U), w; ℝ<sup>d</sup>).
- (ii) *Theorem 4.4 provides analogous approximation rates as the ones for deterministic neural net-works in* [Barron, 1993], [Darken et al., 1993], [Siegel and Xu, 2020], and the references therein.

Moreover, for r = 2, we use the equivalent characterization of Sobolev spaces via Fourier transform (see [Grubb, 2009, Lemma 6.8]) to provide sufficient conditions for  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  to have a weakly differentiable Fourier transform such that the constant  $C_f \ge 0$  defined in (9) is finite.

**Example 4.6.** Let  $k \in \mathbb{N}_0$ , r = 2,  $\gamma \in [0, \infty)$ , and  $\nu \in (0, \infty)$ . Moreover, let  $(A_1, B_1) \sim t_m(\nu) \otimes t_1(\nu)$ and let  $\psi \in S_0(\mathbb{R}; \mathbb{C})$  such that  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$  for some  $0 < \xi_1 < \xi_2 < \infty$  (see Example 4.7 (b) below). If  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + ||u||)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} du; \mathbb{R}^d) \cap W^{m+2k+4\lceil \gamma \rceil + 3\lceil \nu \rceil + 4, 2}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + || \cdot ||)^{2(\lceil \gamma \rceil + \lceil \nu \rceil + 1)}; \mathbb{R}^d)$ , then the Fourier transform  $\hat{f} : \mathbb{R}^m \to \mathbb{R}^d$  is  $(\lceil \gamma \rceil + \lceil \nu \rceil + 1)$ -times weakly differentiable and the constant  $C_f \ge 0$  defined in (9) is finite.

Finally, we estimate the constants  $C_{U,w}^{(\gamma,p)}$ ,  $|C_m^{(\psi,\rho)}|$ , and  $C_f$  to give some sufficient conditions when learning a deterministic function by a random neural network overcomes the curse of dimensionality.

**Example 4.7.** Let  $k \in \mathbb{N}_0$ ,  $p \in (1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in [0, \infty)$ . Then, the following holds:

- (a)  $\begin{array}{l} For \ C_{U,w}^{(\gamma,p)} \colon Let \ w : U \to [0,\infty) \ be \ a \ weight \ of \ separable \ form \ w(u) := w_0(u_1) \cdots w_0(u_m) \ for \\ \hline all \ u := (u_1, ..., u_m)^\top \in U, \ where \ w_0 : \mathbb{R} \to [0,\infty) \ is \ another \ weight \ satisfying \ \int_{\mathbb{R}} w_0(s) ds = 1, \\ and \ C_{\mathbb{R},w_0}^{(\gamma,p)} := \left( \int_{\mathbb{R}} (1+|s|)^{\gamma p} w_0(s) ds \right)^{1/p} < \infty. \ Then, \ it \ holds \ that \ C_{U,w}^{(\gamma,p)} \leqslant C_{\mathbb{R},w_0}^{(\gamma,p)} m^{\gamma+1/p}. \end{array}$
- (b) <u>For  $C_m^{(\psi,\rho)}$ </u>: Let  $\psi \in S_0(\mathbb{R}; \mathbb{C})$  be such that  $\hat{\psi} \in C_c^{\infty}(\mathbb{R})$  is non-negative with  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$ for some  $0 < \xi_1 < \xi_2 < \infty$ . Then, for every standard activation function  $\rho \in C_{pol,\gamma}^k(\mathbb{R})$  in Example 2.9 and every  $m \in \mathbb{N}$  the pair  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C_{pol,\gamma}^k(\mathbb{R})$  is m-admissible.

		$\rho \in C^{\kappa}_{pol,\gamma}(\mathbb{R})$	$k \in \mathbb{N}_0$	$\gamma \in [0,\infty)$	$f_{\widehat{T_{\rho}}} \in L^{1}_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C})$
(i)	Sigmoid function	$\rho(s) := \frac{1}{1 + e^{-s}}$	$k \in \mathbb{N}_0$	$\gamma = 0$	$f_{\widehat{T_{\rho}}}(\xi) = \frac{-i\pi}{\sinh(\pi\xi)}$
(ii)	Tangens hyperb.	$\rho(s) := \tanh(s)$	$k \in \mathbb{N}_0$	$\gamma = 0$	$f_{\widehat{T_{\rho}}}(\xi) = \frac{-i\pi}{\sinh(\pi\xi/2)}$
(iii)	Softplus function	$\rho(s) := \ln\left(1 + e^s\right)$	$k \in \mathbb{N}_0$	$\gamma = 1$	$f_{\widehat{T_{\rho}}}(\xi) = \frac{-\pi}{\xi \sinh(\pi\xi)}$
(iv)	ReLU function	$\rho(s) := \max(s, 0)$	k = 0	$\gamma = 1$	$f_{\widehat{T}_{\rho}}(\xi) = -\frac{1}{\xi^2}$

Moreover, there exists  $C_{\psi,\rho} > 0$  (independent of  $m, d \in \mathbb{N}$ ) such that  $\left|C_m^{(\psi,\rho)}\right| \ge C_{\psi,\rho}(2\pi/\xi_2)^m$ .

**Remark 4.8.** Assume the setting of Example 4.6+4.7, where we choose without loss of generality  $\xi_1 \in (0, \nu^{-1/2}]$ . Then, there exists some  $C_2 > 0$  (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ) such that

$$\frac{C_{U,w}^{(\gamma,p)}C_f}{\left|C_m^{(\psi,\rho)}\right|} \leqslant C_2 \frac{m^{\gamma+1/p} \xi_2^m \pi^{\frac{m}{4}}}{(2\pi\xi_1)^m \Gamma\left(\frac{m+\nu}{2}\right)^{\frac{1}{2}}} \max_{\beta \in \mathbb{N}_{0,\lceil\gamma\rceil+\lceil\nu\rceil+1}^m} \left(\int_{\mathbb{R}^m} \|\partial_\beta \hat{f}(\xi)\|^2 \left(1 + \|\xi\|^2\right)^{\frac{m+2k+4\lceil\gamma\rceil+3\lceil\nu\rceil+4}{2}} d\xi\right)^{\frac{1}{2}}.$$
(12)

Hence, if the right-hand side of (12) grows at most polynomially in  $m, d \in \mathbb{N}$ , we conclude from (10)+(11) that  $f : \mathbb{R}^m \to \mathbb{R}^d$  can be learned by a random neural network without the curse of dimensionality.

Note that the integral on the right-hand side of (12) with  $\beta = 0 \in \mathbb{N}_{0,[\gamma]+[\nu]+1}^{m}$  also appears as Barron norm in the approximation rates of deterministic neural networks (see [Barron, 1993, Equation 3], [Klusowski and Barron, 2016, Theorem 6], [Siegel and Xu, 2020, Equation 5], and [E et al., 2022, Section 2.1]). However, in our case of random neural networks, we also have to include the weak derivatives.

$${}^{4}A_{1} \sim t_{m}(\nu)$$
 has probability density function  $\mathbb{R}^{m} \ni a \mapsto \theta_{A}(a) = \frac{\Gamma((m+\nu)/2)}{\Gamma(\nu/2)(\pi\nu)^{m/2}} \left(1 + \|a\|^{2}/\nu\right)^{-(m+\nu)/2} \in (0,\infty).$ 

4.2. Algorithm and Complexity. In this section, we provide an explicit algorithm to learn a deterministic function by a random neural network. For some fixed  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and a weight  $w: U \to [0,\infty)$ , which is normalized, i.e.  $\int_U w(u) du = 1$ , let the training data  $(V_j)_{j \in \mathbb{N}} \sim w$  be an i.i.d. sequence of U-valued random variables, which is independent of  $(A_n, B_n)_{n \in \mathbb{N}}$ . Moreover, we define

$$\mathcal{F}_{A,B} \subseteq \mathcal{F}_{A,B,V} := \sigma(\{A_n, B_n, V_n : n \in \mathbb{N}\}) \subseteq \mathcal{F}.$$

Next, we fix some  $N \in \mathbb{N}$  and define  $\mathcal{W}_N$  as the vector space of all  $\mathbb{R}^{d \times N}$ -valued random variables  $W := (W_{i,n})_{i=1,\dots,d}^{n=1,\dots,N}$ , which are  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^{d\times N})$ -measurable. Then, for an activation function  $\rho \in C^k_{pol,\gamma}(\mathbb{R})$  and every  $W \in \mathcal{W}_N$ , we define the corresponding random neural network as

$$\Omega \ni \omega \mapsto \Phi_N^W(\omega) := \left(\sum_{n=1}^N W_{i,n}(\omega)\rho\left(A_n(\omega)^\top \cdot -B_n(\omega)\right)\right)_{i=1,\dots,d}^\top \in C^k_{pol,\gamma}(U;\mathbb{R}^d).$$
(13)

Note that (13) slightly differs from Definition 3.2, as the linear readout  $W \in W_N$  is now measurable with respect to  $\mathcal{F}_{A,B,V}$  (instead of  $\mathcal{F}_{A,B}$ ) and can therefore only be trained after the training data has been drawn. Moreover, we denote by  $\mathcal{RN}_{U,d}^{\rho,V}$  the set of all random neural networks of the form (13).

Now, we fix some  $J \in \mathbb{N}$  and approximate a given deterministic function  $f: U \to \mathbb{R}^d$ . To this end, we use the least squares method to find the linear readout  $W^{(J)} := \left(W_{i,n}^{(J)}\right)_{i=1,\dots,d}^{n=1,\dots,N} \in \mathcal{W}_N$  of the random neural network  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  which minimizes the empirical weighted mean squared error (MSE), i.e.

$$W^{(J)} = \arg\min_{\widetilde{W}\in\mathcal{W}_N} \left( \frac{1}{J} \sum_{j=1}^J \sum_{\alpha\in\mathbb{N}_{0,k}^m} c_\alpha^2 \left\| \partial_\alpha f(V_j) - \partial_\alpha \Phi_N^{\widetilde{W}}(\cdot)(V_j) \right\|^2 \right).$$
(14)

Hereby, the constants  $(c_{\alpha})_{\alpha \in \mathbb{N}_{0,k}^m} \subset (0,\infty)$  control the contribution of the derivatives, e.g.,  $c_{\alpha} := m^{-|\alpha|}$ ,  $\alpha \in \mathbb{N}_{0,k}^{m}, \text{ means equal contribution of each order. Moreover, we define } \kappa \left( (c_{\alpha})_{\alpha \in \mathbb{N}_{0,k}^{m}} \right) := \frac{\max_{\alpha \in \mathbb{N}_{0,k}^{m}} c_{\alpha}}{\min_{\alpha \in \mathbb{N}_{0,k}^{m}} c_{\alpha}}.$ 

## Algorithm 1: Learning a random neural network

**Input:**  $k \in \mathbb{N}_0, U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in [0, \infty)$ , normalized weight  $w : U \to [0, \infty)$ ,  $\rho \in C^k_{pol,\gamma}(\mathbb{R}), (c_{\alpha})_{\alpha \in \mathbb{N}^m_{0,k}} \subset (0,\infty), (A_1, B_1) \sim \theta_A \otimes t_1(\nu) \text{ satisfying Assumption 4.1,}$ and k-times weakly differentiable function  $f = (f_1, ..., f_d)^\top : U \to \mathbb{R}^d$ . **Output:**  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  with linear readout  $W^{(J)} := \left(W_{i,n}^{(J)}\right)_{i=1,...,d}^{n=1,...,N} \in \mathcal{W}_N$  solving (14).

- 1 Generate i.i.d. random variables  $(A_n, B_n)_{n=1,...,N} \sim \theta_A \otimes t_1(\nu)$  (see Assumption 4.1).
- 2 Generate i.i.d. random variables  $(N_n) D_n j_{n=1,...,N} \to Q \in \Gamma(V)$  (see the sum  $p \in V$ ) 3 Generate i.i.d. random variables  $(V_j)_{j=1,...,J} \sim w$ , which are independent of  $(A_n, B_n)_{n=1,...,N}$ . 3 Define the  $\mathbb{R}^{(J \cdot |\mathbb{N}_{0,k}^m|) \times N}$ -valued random variable  $R = (R_{(j,\alpha),n})_{(j,\alpha) \in \{1,...,J\} \times \mathbb{N}_{0,k}^m}$  with components

$$R_{(j,\alpha),n} := c_{\alpha} \rho^{(|\alpha|)} \left( A_n^{\top} V_j - B_n \right) A_n^{\alpha}, \text{ for } (j,\alpha) \in \{1, ..., J\} \times \mathbb{N}_{0,k}^m \text{ and } n = 1, ..., N.$$

- for i = 1, ..., d do 4
- Define the  $\mathbb{R}^{J \cdot |\mathbb{N}_{0,k}^{m}|}$ -valued random variable  $Y_i := (c_\alpha \partial_\alpha f_i(V_j))_{(j,\alpha) \in \{1,\dots,J\} \times \mathbb{N}_{0,k}^{m}}$ . 5
- Solve the least squares problem  $R^{\top}RW_i^{(J)} = R^{\top}Y_i$  for  $W_i^{(J)}$  via Cholesky decomposition and forward/backward substitution (see [Biörck, 1996, Section 2.2.2]), where  $W^{(J)} := (W^{(J)})^{\top}$ 6

forward/backward substitution (see [Bjorck, 1996, Section 2.2.2]), where 
$$W_i^{(r)} := (W_{i,n}^{(r)})_{n=1,\dots,N}$$

7 Return 
$$\Omega \ni \omega \mapsto \Phi_N^{W^{(J)}}(\omega) := \left(\sum_{n=1}^N W_{i,n}^{(J)}(\omega) \rho\left(A_n(\omega)^\top \cdot -B_n(\omega)\right)\right)_{i=1,\dots,d}^\top \in C^k_{pol,\gamma}(U;\mathbb{R}^d).$$

To analyze the complexity of Algorithm 1, we count every elementary operation, function evaluation, and generation of one-dimensional random variable as one unit and define  $\mathscr{C}_{m,d,k}(J,N)$  as this number.

**Proposition 4.9.** For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in [0, \infty)$ , let  $w : U \to [0, \infty)$  be a normalized weight. Moreover, let  $\rho \in C^k_{pol,\gamma}(\mathbb{R})$ ,  $(c_{\alpha})_{\alpha \in \mathbb{N}^m_{0,k}} \subset (0,\infty)$ , let  $(A_1, B_1) \sim \theta_A \otimes t_1(\nu)$  satisfy Assumption 4.1, and let  $f: U \to \mathbb{R}^d$  be k-times weakly differentiable. Then, the following holds true:

- (i) Algorithm 1 terminates and is correct, i.e. returns  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  with  $W^{(J)}$  solving (14).
- (ii) The complexity of Algorithm 1 is of order<sup>5</sup>  $\mathscr{C}_{m,d,k}(J,N) = \mathcal{O}((k+1)dJm^{k+1}N^2 + dN^3).$

<sup>5</sup>We use the Landau notation, i.e.  $a_{J,N} = \mathcal{O}(b_{J,N})$  (as  $J, N \to \infty$ ) if  $\limsup_{J,N\to\infty} \frac{|a_{J,N}|}{|b_{J,N}|} < \infty$ .

For fixed  $k \in \mathbb{N}$ , this shows that the computational costs for learning a deterministic function by a random neural network including the derivatives up to order k scales polynomially in  $J, N, m, d \in \mathbb{N}$ .

4.3. **Generalization Error.** In this section, we bound the generalization error for learning a deterministic function  $f : \mathbb{R}^m \to \mathbb{R}^d$  by the random neural network  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  obtained from Algorithm 1. Since the linear readout  $W^{(J)}$  minimizes the empirical MSE in (14),  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  is the best choice on the training data  $(V_j)_{j=1,...,J}$ . In the following, we bound the error beyond  $(V_j)_{j=1,...,J}$ .

To this end, we combine the approximation rate in Theorem 4.4 (ii) with a result on non-parametric function regression (see [Györfi et al., 2002, Theorem 11.3]). Moreover, we define for every L > 0 the truncation of a vector by  $\mathbb{R}^d \ni y := (y_1, ..., y_d)^\top \mapsto T_L(y) := (\max(\min(y_i, L), -L))_{i=1,...,d}^\top \in \mathbb{R}^d$ . The proof of the following result can be found in Section 9.4.

**Theorem 4.10** (Generalization Error). For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in [0, \infty)$ , let  $w : U \to [0, \infty)$  be a normalized weight such that the constant  $C_{U,w}^{(\gamma,2)} > 0$  defined in (8) is finite. Moreover, for  $\nu \in (0, \infty)$ , let  $(A_1, B_1) \sim \theta_A \otimes t_1(\nu)$  satisfy Assumption 4.1, and let  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C_{pol,\gamma}^k(\mathbb{R})$  be m-admissible. Then, there exists a constant  $C_3 > 0$  (independent of  $m, d \in \mathbb{N}$ ) such that for every  $J, N \in \mathbb{N}, L > 0$ , and  $f := (f_1, ..., f_d)^\top \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $|\partial_{\alpha} f_i(u)| \leq L$  for all  $\alpha \in \mathbb{N}_{0,k}^m$ , i = 1, ..., d, and  $u \in U$ , and with  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform satisfying

$$C_{f} := \max_{\substack{j \in \mathbb{N}_{0} \cap [0, \lceil \gamma \rceil + \lceil \nu \rceil + 1], \\ \beta \in \mathbb{N}_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}^{m}} \int_{\mathbb{R}} \frac{\left| \widehat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{2}}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \widehat{f}(\xi) \|^{2} \frac{(1 + \|\xi/\zeta\|)^{2(k+2\lceil \gamma \rceil + \lceil \nu \rceil + 2)}}{\theta_{A}(\xi/\zeta)} d\xi \right)^{\frac{1}{2}} d\zeta < \infty$$
(15)

we obtain from Algorithm 1 some  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  with N neurons, which is an  $\mathcal{F}_{A,B,V}$ -strongly measurable map  $\Phi_N^{W^{(J)}} : \Omega \to W^{k,2}(U,\mathcal{L}(U),w;\mathbb{R}^d)$  such that

$$\mathbb{E}\left[\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\int_{U}\left\|\partial_{\alpha}f(u)-T_{L}\left(\partial_{\alpha}\Phi_{N}^{W^{(J)}}(\cdot)(u)\right)\right\|^{2}w(u)du\right]^{\frac{1}{2}}$$
$$\leqslant C_{3}L\frac{\sqrt{m^{k}d}\sqrt{\ln(J)+1}\sqrt{N}}{\sqrt{J}}+C_{3}\kappa\left((c_{\alpha})_{\alpha\in\mathbb{N}_{0,k}^{m}}\right)\frac{C_{U,w}^{(\gamma,2)}C_{f}}{\left|C_{m}^{(\psi,\rho)}\right|}\frac{m^{\frac{k}{2}+2[\gamma]+[\nu]+1}}{\sqrt{N}}$$

Theorem 4.10 shows together with Proposition 4.9 that learning a deterministic function by a random neural network overcomes the curse of dimensionality under some conditions (see Remark 4.8).

**Remark 4.11.** Theorem 4.10 is related to the following results in the existing literature:

- (i) Theorem 4.10 extends the generalization error in [Gonon, 2023, Theorem 4.1] for random neural networks with ReLU activation function to more general activation functions and by including the approximation of the weak derivatives.
- (ii) The approximation rate in Theorem 4.10 coincides up to constants with the approximation rate for deterministic neural networks obtained in the seminal work of [Barron, 1994]. There, the neural network parameters (including the weight vectors and biases inside the activation function) are estimated via empirical risk minimization over a constrained parameter set.

4.4. **Conclusion.** Finally, we summarize the advantages of using random neural networks to learn a deterministic function instead of fully trained deterministic neural networks:

- (i) <u>Convexity</u>: The least squares method in (14) forms a convex optimization problem and thus has a minimizer. This is not the case for deterministic neural networks, where the optimization problem is non-convex due to the training of the parameters inside the non-linear activation function.
- (ii) Efficiency: Algorithm 1 with the least squares method is more efficient than the training of deterministic neural network as it does not require the iterative backpropagation procedure and the amount of trainable parameters (i.e. the parameter space) is much smaller.
- (iii) No optimization error: Since the least squares method in (14) directly returns a minimizer, we do not have to consider an additional *optimization error*. This is not the case for deterministic neural networks, which are trained, e.g., via stochastic gradient descent. For example, this additional optimization error has not been addressed in [Barron, 1994] (see Remark 4.11 (ii)).

### 5. NUMERICAL EXAMPLES

In this section, we illustrate in two numerical examples<sup>6</sup> how random neural networks can be applied in empirical tasks and how they numerically outperform fully trained deterministic neural networks.

5.1. **Mathematical Physics: Learning the solution of the heat equation.** In the first example, we follow [Evans, 2010, Section 2.3] and consider the heat equation, which describes the evolution of a given quantity throughout time. More precisely, we consider the partial differential equation (PDE)

$$\frac{\partial f}{\partial t}(t,u) - \lambda \sum_{l=1}^{m} \frac{\partial^2 f}{\partial u_l^2}(t,u) = 0, \qquad (t,u) \in (0,\infty) \times \mathbb{R}^m, \tag{16}$$

where we assume that the quantity is initially described by a function  $g : \mathbb{R}^m \to \mathbb{R}$ , i.e. we impose the initial condition  $f(0, u) := \lim_{t \to 0} f(t, u) = g(u)$  for a.e.  $u \in \mathbb{R}^m$ .

The first part of the following result is a slight generalization of [Evans, 2010, Theorem 2.3.1] to a.e. continuous functions, where we define  $\overline{\mathbb{B}_r(0)} := \{u \in \mathbb{R}^m : ||u|| \leq r\}, r \geq 0$ , for the last part.

**Lemma 5.1.** Let  $\lambda, \nu \in (0, \infty)$ ,  $\gamma \in [0, \infty)$ , and  $g : \mathbb{R}^m \to \mathbb{R}$  be a.e. bounded and a.e. continuous. Then: (i) The function

$$(0,\infty) \times \mathbb{R}^m \ni (t,u) \quad \mapsto \quad f(t,u) = \frac{1}{(4\pi\lambda t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\frac{\|u-v\|^2}{4\lambda t}} g(v) dv \in \mathbb{R}$$
(17)

is the unique solution of the PDE (16) with initial condition  $g : \mathbb{R}^m \to \mathbb{R}$ . (ii) Let  $p \in (1, \infty)$  and assume also that  $g \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + ||u||)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} du)$  with

 $C_g := \|g\|_{L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1+\|u\|)^{\lceil \gamma\rceil + \lceil \nu\rceil + 1} du)} < \infty.$ 

Moreover, let  $(A_1, B_1) \sim t_m(\nu) \otimes t_1(\nu)$  and  $w : \mathbb{R}^m \to [0, \infty)$  be a weight of separable form as in Example 4.7 (a). In addition, let  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C^0_{pol,\gamma}(\mathbb{R})$  be *m*-adimissible as in Example 4.7 (b) with  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$  for some  $0 < \xi_1 < \xi_2 < \infty$ . Then, there exist some constants  $C_4, C_5 > 0$  (independent of  $m \in \mathbb{N}$ ) such that for every  $N \in \mathbb{N}$  there exists some  $\Phi_N \in \mathcal{RN}^\rho_{\mathbb{R}^m,1} \cap L^2(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; L^p(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), w(u)du))$  with N neurons satisfying

$$\mathbb{E}\left[\|f(t,\cdot) - \Phi_N\|_{L^p(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),w(u)du)}^2\right]^{\frac{1}{2}} \leqslant \frac{C_4 m^{C_5} \left(1 + \left(\frac{3([\gamma] + [\nu] + 2)\xi_2^2}{2\pi\sqrt{\lambda t}\xi_1^2}\right)^{\frac{m}{2}}\right)C_g}{N^{1 - \frac{1}{\min(2,p)}}}.$$
 (18)

(iii) For R > 0 and  $\kappa \in [0, 1/2)$ , the function  $g := \mathbb{1}_{\overline{\mathbb{B}_m \kappa_R(0)}} \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1+||u||)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} du)$ is a.e. bounded, a.e. continuous, and the right-hand side of (18) grows polynomially in  $m \in \mathbb{N}$ .

Now, we learn the solution  $f(1, \cdot)$  of the heat equation (16) by deterministic neural networks and random neural networks, where we choose  $\lambda = 4$  and the initial condition  $g(u) := \mathbb{1}_{\mathbb{B}_{m^{\kappa}R}(0)}(u)$ , with R = 4 and  $\kappa = 0.4$ . Moreover, we generate  $J = 2 \cdot 10^5$  i.i.d. data  $(V_j)_{j=1,\dots,J} \sim \mathcal{N}_m(0, I_m)$  which are split up into 80% for training and 20% for testing. Then, we minimize the empirical  $L^2$ -error

$$\left(\frac{1}{J}\sum_{j=1}^{J}|f(1,V_j) - \mathfrak{N}_N(V_j)|^2\right)^{\frac{1}{2}} \text{ with } \mathfrak{N}_N(V_j) = \begin{cases} \varphi_N(V_j), & \varphi_N \in \mathcal{NN}_{\mathbb{R}^m,1}^{\rho} \text{ having } N \text{ neurons,} \\ \Phi_N(\cdot)(V_j), & \Phi_N \in \mathcal{RN}_{\mathbb{R}^m,1}^{\rho} \text{ having } N \text{ neurons,} \end{cases}$$
(19)

over the training data, where  $\mathbb{R} \ni s \mapsto \rho(s) = \tanh(s) \in \mathbb{R}$ . Hereby, we use the Adam algorithm (see [Kingma and Ba, 2015]) for the deterministic neural networks (over 3000 epochs with learning rate  $\gamma = 10^{-5}$  and batchsize 500), whereas for the random neural networks, we let  $(A_1, B_1) \sim t_m(\nu) \otimes t_1(\nu)$ , with  $\nu = 20$ , and apply a batch normalization before the activation function.

Figure 2 shows that random neural networks are indeed able to learn the solution of the heat equation (16). Note that in order to achieve a similar approximation quality as for deterministic neural networks, the number of neurons in the hidden layer of the random neural networks should be about three times larger than in the hidden layer of the deterministic neural networks. However, in terms of computational efficiency, random neural networks outperform deterministic neural networks by far (see also Section 4.4).

<sup>&</sup>lt;sup>6</sup>The numerical experiments have been implemented in Python on an average laptop (Lenovo ThinkPad X13 Gen2a with Processor AMD Ryzen 7 PRO 5850U and Radeon Graphics, 1901 Mhz, 8 Cores, 16 Logical Processors). The code can be found under the following link: https://github.com/psc25/RandomNeuralNetworks



(b) Approximation of  $\mathbb{R} \ni u_1 \mapsto f(1, (u_1, 0.5, ..., 0.5)) \in \mathbb{R}$ 

	=	= 10	m = 20		m = 30	
	$\mathcal{NN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{RN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{NN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{RN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{NN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{RN}^{ ho}_{\mathbb{R}^m,1}$
N7 10	985.93	0.08	994.53	0.11	1056.62	0.12
IV = 10	$6.96 \cdot 10^{10}$	$1.65 \cdot 10^7$	$1.32\cdot 10^{11}$	$2.13\cdot 10^7$	$1.94 \cdot 10^{11}$	$2.61\cdot 10^7$
M = 50	1028.83	0.47	1078.05	0.54	1045.09	0.58
N = 50	$2.81\cdot 10^{11}$	$2.21 \cdot 10^8$	$5.35\cdot10^{11}$	$2.26\cdot 10^8$	$7.90 \cdot 10^{11}$	$2.31\cdot 10^8$
N 100	1105.79	1.07	1110.50	1.13	1179.85	1.12
IV = 100	$5.45 \cdot 10^{11}$	$8.37\cdot 10^8$	$1.04 \cdot 10^{12}$	$8.42\cdot 10^8$	$1.53\cdot 10^{12}$	$8.47\cdot 10^8$
N = 200	1164.95	2.63	1194.23	2.79	1268.09	2.87
N = 200	$1.07 \cdot 10^{12}$	$3.27\cdot 10^9$	$2.05\cdot 10^{12}$	$3.28\cdot 10^9$	$3.02\cdot10^{12}$	$3.28\cdot 10^9$
N = 300		4.93		4.91		5.03
		$7.31 \cdot 10^9$		$7.31\cdot 10^9$		$7.32\cdot 10^9$
N = 400		7.58		7.57		7.59
		$1.29\cdot 10^{10}$		$1.29\cdot 10^{10}$		$1.30\cdot 10^{10}$

(c) Computational time (in seconds, italic font) and complexity  $\mathscr{C}_{m,1,0}(J,N)$  (in scientific format)

**Figure 2.** Learning the solution of the heat equation (16) with deterministic neural networks (label  $\mathcal{NN}_{\mathbb{R}^m,1}^{\rho}$ ) and random neural networks (label  $\mathcal{RN}_{\mathbb{R}^m,1}^{\rho}$ ). In (a), the learning performance is displayed in terms of the empirical  $L^2$ -error (19) on the test set. In (b), the learned networks (with N = 200 for  $\mathcal{NN}_{\mathbb{R}^m,1}^{\rho}$  and N = 400 for  $\mathcal{RN}_{\mathbb{R}^m,1}^{\rho}$ ) are compared to the true solution  $u_1 \mapsto f(1, (u_1, 0.5, ..., 0.5))$ . In (c), the computational time and the complexity  $\mathscr{C}_{m,1,0}(J, N)$  (see also Proposition 4.9 (ii)) are shown.

5.2. Mathematical Finance: Basket option pricing in Black-Scholes model. In the second example, we consider the problem of pricing a financial derivative written on multiple assets in the multidimensional Black scholes model. More precisely, for T > 0 and  $m \in \mathbb{N}$ , we assume that the stock prices processes  $(X_t)_{t \in [0,T]} := (X_t^1, \dots, X_t^d)_{t \in [0,T]}^\top$  are for every  $l = 1, \dots, m$  and  $t \in [0,T]$  given by

$$X_t^l = X_0^l \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^l\right).$$
(20)

where  $X_0^l \in (0,\infty)^m$  is the initial price, where r > 0 is the interest rate, and where  $(W_t)_{t \in [0,T]} :=$  $(W_t^1, ..., W_t^m)_{t \in [0,T]}^{\top}$  is an *m*-dimensional Brownian motion. Then, for a given strike price K > 0, we learn the pricing function of the geometric Basket call option  $g(X_T)$  whose payoff function is given by

$$(0,\infty)^m \ni x := (x_1, \dots, x_m)^\top \quad \mapsto \quad g(x) := \max\left(\left(\prod_{l=1}^m x_l\right)^{\frac{1}{m}} - K, 0\right) \in \mathbb{R}.$$
 (21)

Hence, by using the Feynman-Kac formula, the pricing function is given by the conditional expectation  $v(t,x) := e^{-r(T-t)} \mathbb{E}\left[g(X_T) | X_t = x\right]$  which is the unique viscosity solution of the PDE

$$\frac{\partial v}{\partial t}(t,x) + \frac{\sigma^2}{2} \sum_{l_1, l_2=1}^m x_{l_1} x_{l_2} \frac{\partial^2 v}{\partial x_{l_1} \partial x_{l_2}}(t,u) + r \sum_{l=1}^m x_l \frac{\partial v}{\partial x_l}(t,x) - rv(t,x) = 0, \quad (t,x) \in (0,T) \times (0,\infty)^m,$$

with terminal condition v(T, x) = g(x) for all  $x \in (0, \infty)^m$  (see [Grohs et al., 2023, Proposition 2.23]). Thus, by using log-prices  $u_l := \ln(x_l), l = 1, ..., m$ , the new pricing function

$$f(t,u) := e^{-r(T-t)} \mathbb{E}\left[g(X_T) \middle| X_t = (\exp(u_l))_{l=1,\dots,m}^\top\right]$$
(22)

is the unique viscosity solution of the transformed partial differential equation

$$\frac{\partial f}{\partial t}(t,u) + \frac{\sigma^2}{2} \sum_{l_1,l_2=1}^m \frac{\partial^2 f}{\partial u_{l_1} \partial u_{l_2}}(t,u) + r \sum_{l=1}^m \frac{\partial f}{\partial u_l}(t,u) - rf(t,u) = 0, \quad (t,u) \in (0,T) \times \mathbb{R}^m,$$

with terminal condition f(T, u) = g(u) for all  $u \in \mathbb{R}^m$  (see [Grohs et al., 2023, Corollary 2.24]). Moreover, by using that  $\left(\prod_{l=1}^m X_T^l\right)^{1/m} = \left(\prod_{l=1}^m X_0^l\right)^{1/m} \exp\left(\left(r - \sigma^2/2\right)T + \frac{\sigma}{m}\sum_{l=1}^m W_T^l\right)$ , where the sum of Brownian motions satisfies  $\frac{\sigma}{m} \sum_{l=1}^{m} W_T^l \sim \mathcal{N}(0, \frac{\sigma^2 T}{m})$ , it follows that

$$\left(\prod_{l=1}^{m} X_T^l\right)^{\frac{1}{m}} \sim \left(\prod_{l=1}^{m} X_0^l\right)^{\frac{1}{m}} \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma\sqrt{T}}{\sqrt{m}}Z\right)$$

is log-normally distributed, with  $Z \sim \mathcal{N}(0,1)$ . Hence, the option price v(t,x) can be computed via the classical Black-Scholes formula in [Black and Scholes, 1973] for a one-dimensional put option, i.e.

$$v(t,u) = e^{-r(T-t)} \mathbb{E}\left[g(X_T) \middle| X_t = x\right] = K e^{-r(T-t)} \Theta_{\mathcal{N}(0,1)}(d_2) - \left(\prod_{l=1}^m x_l\right)^{\frac{1}{m}} \Theta_{\mathcal{N}(0,1)}(d_1)$$

where  $d_1 := \frac{\sqrt{m}}{\sigma\sqrt{T}} \left( \frac{1}{\ln(K)m} \sum_{l=1}^m \ln(x_l) + \left(r + \sigma^2/2\right) T \right)$  and  $d_2 := d_1 - \frac{\sqrt{T}}{\sigma\sqrt{m}}$ , and where  $\Theta_{\mathcal{N}(0,1)} := d_1 - \frac{\sqrt{T}}{\sigma\sqrt{m}}$  $\mathbb{R} \to [0,1]$  denotes the cumulative distribution function of the standard normal distribution  $\mathcal{N}(0,1)$ .

Now, we learn  $f(0, \cdot)$  by deterministic and random neural networks, with  $r = 0.01, \sigma = 0.5, K = 82$ , and T = 1. Moreover, we generate  $J = 2 \cdot 10^5$  i.i.d. uniformly distributed data  $(V_j)_{j=1,\dots,J} \sim \mathcal{U}([4,5]^m)$ split up into 80% for training and 20% for testing. Then, we minimize the empirical  $L^2$ -error

$$\left(\frac{1}{J}\sum_{j=1}^{J}|f(0,V_{j})-\mathfrak{N}_{N}(V_{j})|^{2}\right)^{\frac{1}{2}} \text{ with } \mathfrak{N}_{N}(V_{j}) = \begin{cases} \varphi_{N}(V_{j}), & \varphi_{N} \in \mathcal{NN}_{\mathbb{R}^{m},1}^{\rho} \text{ having } N \text{ neurons,} \\ \Phi_{N}(\cdot)(V_{j}), & \Phi_{N} \in \mathcal{RN}_{\mathbb{R}^{m},1}^{\rho} \text{ having } N \text{ neurons,} \end{cases}$$

$$(23)$$

over the training data, where we use the same setting as in Section 5.1 except the learning rate  $\gamma = 0.001$ . Figure 3 shows that the pricing function (22) can indeed be learned by random neural networks.



(**b**) Approximation of  $\mathbb{R} \ni u_1 \mapsto f(0, (u_1, \ln(82), ..., \ln(82))) \in \mathbb{R}$ 

	=	= 10	m = 20		m = 30	
	$\mathcal{NN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{RN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{NN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{RN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{NN}^{ ho}_{\mathbb{R}^m,1}$	$\mathcal{RN}^{ ho}_{\mathbb{R}^m,1}$
N7 10	1055.14	0.08	1048.98	0.10	998.91	0.09
IV = 10	$6.96\cdot10^{10}$	$1.65 \cdot 10^7$	$1.32\cdot 10^{11}$	$2.13 \cdot 10^7$	$1.94\cdot 10^{11}$	$2.61 \cdot 10^7$
N - 50	1001.17	0.44	1020.82	0.51	1037.85	0.51
N = 50	$2.81 \cdot 10^{11}$	$2.21 \cdot 10^8$	$5.35 \cdot 10^{11}$	$2.26 \cdot 10^8$	$7.90\cdot10^{11}$	$2.31 \cdot 10^8$
N 100	1059.16	1.00	1104.42	1.05	1181.24	1.07
IV = 100	$5.45 \cdot 10^{11}$	$8.37\cdot 10^8$	$1.04\cdot10^{12}$	$8.42\cdot 10^8$	$1.53\cdot 10^{12}$	$8.47 \cdot 10^8$
N = 200	1174.53	2.88	1242.82	2.65	1262.51	2.61
	$1.07\cdot 10^{12}$	$3.27\cdot 10^9$	$2.05\cdot 10^{12}$	$3.28\cdot 10^9$	$3.02\cdot10^{12}$	$3.28\cdot 10^9$
N = 300		5.82		4.65		4.63
		$7.31 \cdot 10^9$		$7.31 \cdot 10^9$		$7.32\cdot 10^9$
N = 400		11.03		7.70		7.46
		$1.29\cdot 10^{10}$		$1.29\cdot 10^{10}$		$1.30\cdot10^{10}$

(c) Computational time (in seconds, italic font) and complexity  $\mathscr{C}_{m,1,0}(J,N)$  (in scientific format)

**Figure 3.** Learning the pricing function (22) of the Basket option (21) written on the geometric Brownian motions (20) with deterministic neural networks (label  $\mathcal{NN}_{\mathbb{R}^{m,1}}^{\rho}$ ) and random neural networks (label  $\mathcal{RN}_{\mathbb{R}^{m,1}}^{\rho}$ ). In (a), the learning performance is displayed in terms of the empirical  $L^2$ -error (23) on the test set. In (b), the learned networks (with N = 200 for  $\mathcal{NN}_{\mathbb{R}^{m,1}}^{\rho}$  and N = 400 for  $\mathcal{RN}_{\mathbb{R}^{m,1}}^{\rho}$ ) are compared to the true solution  $u_1 \mapsto f(0, (u_1, \ln(82), ..., \ln(82)))$ . In (c), the computational time and the complexity  $\mathscr{C}_{m,1,0}(J, N)$  (see also Proposition 4.9 (ii)) are shown.

In this section, we show an equivalent characterization for functions in the Banach space  $(\overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}, \|\cdot\|_{C_{pol,\gamma}^k(U; \mathbb{R}^d)})$  introduced in Notation (v), where  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ . This generalizes the results in [Dörsek and Teichmann, 2010, Theorem 2.7] and [Cuchiero et al., 2023, Lemma 2.7] to differentiable functions defined on an open subset of an Euclidean space.

In the following, we denote the factorial of a multi-index  $\alpha := (\alpha_1, ..., \alpha_m) \in \mathbb{N}_0^m$  by  $\alpha! := \prod_{l=1}^m \alpha_l!$ . Moreover, we denote by  $\mathbb{B}_r(u_0) := \{u \in \mathbb{R}^m : ||u - u_0|| < r\}$  and  $\overline{\mathbb{B}_r(u_0)} := \{u \in \mathbb{R}^m : ||u - u_0|| \leq r\}$  the open and closed ball with radius  $r \ge 0$  around  $u_0 \in \mathbb{R}^m$ , respectively.

**Lemma 6.1.** Let  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ . Then, the following holds true:

- (i) If  $U \subseteq \mathbb{R}^m$  is bounded, then  $\overline{C_b^k(U; \mathbb{R}^d)}^{\gamma} = C_b^k(U; \mathbb{R}^d)$ .
- (ii) If  $U \subseteq \mathbb{R}^m$  is unbounded, then  $f \in \overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}$  if and only if  $f \in C^k(U; \mathbb{R}^d)$  and

$$\lim_{r \to \infty} \max_{\alpha \in \mathbb{N}_{0,k}^m} \sup_{u \in U \setminus \overline{\mathbb{B}_r(0)}} \frac{\|\partial_\alpha f(u)\|}{(1+\|u\|)^{\gamma}} = 0.$$
(24)

*Proof.* The conclusion in (i) follows from the definition of  $(\overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}, \|\cdot\|_{C_{pol,\gamma}^k(U; \mathbb{R}^d)})$ . Now, for sufficiency in (ii), fix some  $f \in \overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}$ . Then, by definition of  $\overline{C_b^k(U; \mathbb{R}^d)}^{\gamma}$ , there exists a sequence  $(g_n)_{n \in \mathbb{N}} \subseteq C_b^k(U; \mathbb{R}^d)$  with  $\lim_{n \to \infty} \|f - g_n\|_{C_{pol,\gamma}^k(U; \mathbb{R}^d)} = 0$ , which implies for every fixed r > 0 that

$$\lim_{n \to \infty} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \cap \overline{\mathbb{B}_{r}(0)}} \left\| \partial_{\alpha} f(u) - \partial_{\alpha} g_{n}(u) \right\| \leq (1+r)^{\gamma} \lim_{n \to \infty} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \cap \overline{\mathbb{B}_{r}(0)}} \frac{\left\| \partial_{\alpha} f(u) - \partial_{\alpha} g_{n}(u) \right\|}{(1+\left\| u \right\|)^{\gamma}} \\ = (1+r)^{\gamma} \lim_{n \to \infty} \left\| f - g_{n} \right\|_{C_{pol,\gamma}^{k}(U; \mathbb{R}^{d})} = 0.$$

This together with the Fundamental Theorem of Calculus shows that  $f|_{U \cap \mathbb{B}_r(0)} : U \cap \mathbb{B}_r(0) \to \mathbb{R}^d$  is *k*-times differentiable since for every fixed  $\alpha \in \mathbb{N}_{0,k}^m$  the partial derivative  $\partial_{\alpha} f|_{U \cap \overline{\mathbb{B}_r(0)}} : U \cap \overline{\mathbb{B}_r(0)} \to \mathbb{R}^d$ is continuous as uniform limit of continuous functions. Hence, by using that U is locally compact, it follows from [Munkres, 2014, Lemma 46.3+46.4] that  $\partial_{\alpha} f : U \to \mathbb{R}^d$  is continuous everywhere on U. Since this holds true for every  $\alpha \in \mathbb{N}_{0,k}^m$ , we apply again the Fundamental Theorem of Calculus to conclude that  $f \in C^k(U; \mathbb{R}^d)$ . Moreover, in order to show (24), we fix some  $\varepsilon > 0$  and choose some  $n \in \mathbb{N}$  large enough such that  $||f - g_n||_{C_{pol,\gamma}^k(U; \mathbb{R}^d)} < \varepsilon/2$ . Moreover, we choose r > 0 sufficiently large such that  $(1 + r)^{\gamma} > 2\varepsilon^{-1}||g_n||_{C_b^k(U; \mathbb{R}^d)}$  holds true, which implies that

$$\max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha}f(u)\|}{(1+\|u\|)^{\gamma}} \leq \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U} \frac{\|\partial_{\alpha}f(u) - \partial_{\alpha}g(u)\|}{(1+\|u\|)^{\gamma}} + \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha}g(u)\|}{(1+\|u\|)^{\gamma}} \\
< \frac{\varepsilon}{2} + \frac{\|g\|_{C_{b}^{k}(U;\mathbb{R}^{d})}}{(1+r)^{\gamma}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since  $\varepsilon > 0$  was chosen arbitrarily, we obtain (24).

For necessity in (ii), let  $f \in C^{k}(U; \mathbb{R}^{d})$  such that (24) holds true and fix some  $\varepsilon > 0$ . Moreover, we choose some  $h \in C_{c}^{\infty}(\mathbb{R}^{m})$  such that h(u) = 1 for all  $u \in \overline{\mathbb{B}_{1}(0)}$ , h(u) = 0 for all  $u \in \mathbb{R}^{m} \setminus \mathbb{B}_{2}(0)$ , and that there exists a constant  $C_{h} > 0$  such that for every  $\alpha \in \mathbb{N}_{0,k}^{m}$  and  $u \in \mathbb{R}^{m}$  it holds that  $|\partial_{\alpha}h(u)| \leq C_{h}$ . In addition, by using (24), there exists some r > 1 such that

$$\max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha} f(u)\|}{(1+\|u\|)^{\gamma}} < \frac{\varepsilon}{1+2^{k}C_{h}}.$$
(25)

From this, we define the functions  $\mathbb{R}^m \ni u \mapsto h_r(u) := h(u/r) \in \mathbb{R}$  and  $U \ni u \mapsto g(u) := h_r(u)f(u) \in \mathbb{R}^d$ , which both have bounded support. Furthermore, we use the binomial theorem to conclude for every  $\alpha \in \mathbb{N}_0^m$  that

$$\sum_{\substack{\beta_1,\beta_2\in\mathbb{N}_0^m\\\beta_1+\beta_2=\alpha}}\frac{\alpha!}{\beta_1!\beta_2!} = \sum_{\substack{\beta\in\mathbb{N}_0^m\\\forall l:\beta_l\leqslant\alpha_l}}\prod_{l=1}^m\frac{\alpha_l!}{\beta_l!(\alpha_l-\beta_l)!} \leqslant \prod_{l=1}^m\sum_{\beta_l=0}^{\alpha_l}\frac{\alpha_l!}{\beta_l!(\alpha_l-\beta_l)!} = \prod_{l=1}^m 2^{\alpha_l}\leqslant 2^{|\alpha|}.$$
 (26)

Then, by using the Leibniz product rule together with the triangle inequality, the inequality (26), that  $|\partial_{\alpha}h_r(u)| = |\partial_{\alpha}h(u/r)| r^{-|\alpha|} \leq C_h$  for any  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in \mathbb{R}^m$ , and the inequality (26), it follows for every  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in U$  that

$$\|\partial_{\alpha}g(u)\| \leq \sum_{\substack{\beta_1,\beta_2 \in \mathbb{N}_0^m\\\beta_1+\beta_2=\alpha}} \frac{\alpha!}{\beta_1!\beta_2!} \left|\partial_{\beta_1}h_r(u)\right| \left\|\partial_{\beta_2}f(u)\right\| \leq 2^k C_h \max_{\beta_2 \in \mathbb{N}_{0,k}^m} \left\|\partial_{\beta_2}f(u)\right\|.$$
(27)

Hence, by using that  $\partial_{\alpha}g(u) = \partial_{\alpha}(h_r(u)f(u)) = \partial_{\alpha}f(u)$  for any  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in U \cap \mathbb{B}_r(0)$  (as  $h_r(u) = 1$  for any  $u \in \mathbb{B}_r(0)$ ), and the inequalities (27) and (25), the function  $g \in C_b^k(U; \mathbb{R}^d)$  satisfies

$$\begin{split} \|f - g\|_{C_{pol,\gamma}^{k}(U;\mathbb{R}^{d})} &= \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U} \frac{\|\partial_{\alpha} f(u) - \partial_{\alpha} g(u)\|}{(1 + \|u\|)^{\gamma}} \\ &\leq \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \cap \mathbb{B}_{r}(0)} \frac{\|\partial_{\alpha} f(u) - \partial_{\alpha} g(u)\|}{(1 + \|u\|)^{\gamma}} + \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha} f(u)\|}{(1 + \|u\|)^{\gamma}} \\ &+ \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha} g(u)\|}{(1 + \|u\|)^{\gamma}} \\ &\leq \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha} f(u)\|}{(1 + \|u\|)^{\gamma}} + 2^{k} C_{h} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|\partial_{\alpha} f(u)\|}{(1 + \|u\|)^{\gamma}} \\ &< \frac{\varepsilon}{1 + 2^{k} C_{h}} + 2^{k} C_{h} \frac{\varepsilon}{1 + 2^{k} C_{h}} = \varepsilon. \end{split}$$

Since  $\varepsilon > 0$  was chosen arbitrarily, it follows that  $f \in \overline{C^k(U; \mathbb{R}^d)}^{\gamma}$ .

## 7. PROOFS OF RESULTS IN SECTION 2.

In this section, we provide the proofs of the results Section 2. In Section 7.1, we prove Lemma 2.5, whereas the main result of Section 2, i.e. the universal approximation property of deterministic neural networks in Theorem 2.6, is proven in Section 7.2. Finally, we verify Example 2.8+2.9 in Section 7.3.

# 7.1. Proof of Lemma 2.5.

*Proof of Lemma 2.5.* For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ -approximable function space. Then, in order to show (i), we define the subset

$$\mathcal{A} := \left\{ U \ni u \mapsto \sum_{n=1}^{N} \left( y_{n,1} \cos \left( a_{n,1}^{\top} u \right) + y_{n,2} \sin \left( a_{n,2}^{\top} u \right) \right) \in \mathbb{R} : \begin{array}{c} N \in \mathbb{N} \\ y_{n,1}, y_{n,2} \in \mathbb{Q} \\ a_{n,1}, a_{n,2} \in \mathbb{Q}^m \end{array} \right\} \subseteq C_b^k(\mathbb{R}^m)$$
(28)

and the vector subspace  $\mathcal{W} := \{ U \ni u \mapsto (a_1(u), ..., a_d(u))^\top \in \mathbb{R}^d : a_1, ..., a_d \in \mathcal{A} \} \subseteq C_b^k(\mathbb{R}^m; \mathbb{R}^d).$ In addition, we define the weight  $\mathbb{R}^m \ni u \mapsto \psi(u) := (1 + ||u||)^c \in (0, \infty)$ , which is admissible in the sense of [Cuchiero et al., 2023, Definition 2.1] and [Schmocker, 2022, Definition 3.1].

First, if k = 0, we observe that  $\mathcal{A}$  is a point separating subalgebra of  $C_b^0(\mathbb{R}^m) \subseteq \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$ , which vanishes nowhere and consists only of bounded functions, thus by [Cuchiero et al., 2023, Remark 3.5] point separating of  $\psi$ -moderate growth in the sense of [Cuchiero et al., 2023, Definition 3.4]. Hence, by applying the weighted Stone-Weierstrass theorem in [Cuchiero et al., 2023, Theorem 3.6] componentwise, it follows that  $\mathcal{W}$  is dense in  $\overline{C_b^0(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}$  with respect to  $\|\cdot\|_{C_{pol,\gamma}^0(\mathbb{R}^m;\mathbb{R}^d)}$ .

On the other hand, if  $k \ge 1$ , we observe that  $\mathcal{A}$  is a point separating subalgebra of  $C_b^k(\mathbb{R}^m) \subseteq \overline{C_b^k(\mathbb{R}^m)}^{\gamma}$  which vanishes nowhere and for every  $v \in \mathbb{R}^m \setminus \{0\}$  there exists some  $a \in \mathcal{A}$  such that  $v^{\top}(\partial_{e_1}a(u), ..., \partial_{e_m}a(u)) \ne 0$ . Moreover, since  $\mathcal{A}$  consists only of bounded functions and the function

$$\mathbb{R}^m \ni u \quad \mapsto \quad \left(\cos\left(e_l^\top u\right), \sin\left(e_l^\top u\right), \cos\left(\pi e_l^\top u\right), \sin\left(\pi e_l^\top u\right)\right)_{l=1,\dots,m}^\top \in \mathbb{R}^{4m}$$

with  $e_l \in \mathbb{R}^m$  being the *l*-th unit vector of  $\mathbb{R}^m$ , is a continuous embedding with components from  $\mathcal{A}$ , we conclude that  $\mathcal{A}$  is locally point separating of order *k* in the sense of [Schmocker, 2022, Remark 3.22]. Hence, by applying the weighted Nachbin theorem in [Schmocker, 2022, Theorem 3.40] componentwise, it follows that  $\mathcal{W}$  is dense in  $\overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  with respect to  $\|\cdot\|_{C_{nol,\gamma}^k(\mathbb{R}^m; \mathbb{R}^d)}$ .

Finally, in both cases k = 0 and  $k \ge 1$ , we use that  $(X, \|\cdot\|_X)$  is  $(k, U, \gamma)$ -approximable function space, i.e. that the restriction map  $(\overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}, \|\cdot\|_{C_{pol,\gamma}^k(\mathbb{R}^m; \mathbb{R}^d)}) \ni f \mapsto f|_U \in (X, \|\cdot\|_X)$  is by Remark 2.4 a continuous dense embedding, to conclude that  $\mathcal{W}$  is dense in X with respect to  $\|\cdot\|_X$ . Therefore, since the set  $\mathcal{W}$  is by definition countable, it follows that  $(X, \|\cdot\|_X)$  is separable.

For (ii), we fix some  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ ,  $y \in \mathbb{R}^d$ ,  $a \in \mathbb{R}^m$ , and  $b \in \mathbb{R}$ , and define the constant  $C_{y,a,b} := 1 + \max_{\alpha \in \mathbb{N}_{0,k}^m} \|ya^{\alpha}\| (1 + \|a\| + |b|)^{\gamma} > 0$ , where  $a^{\alpha} := \prod_{l=1}^m a_l^{\alpha_l}$  for  $a := (a_1, ..., a_m)^{\top} \in \mathbb{R}^m$  and  $\alpha := (\alpha_1, ..., \alpha_m) \in \mathbb{N}_{0,k}^m$ . Then, by definition of  $\overline{C_b^k(\mathbb{R})}^{\gamma}$ , there exists some  $\tilde{\rho} \in C_b^k(\mathbb{R})$  such that

$$\|\rho - \widetilde{\rho}\|_{C^k_{pol,\gamma}(\mathbb{R})} := \max_{j=0,\dots,k} \sup_{s \in \mathbb{R}} \frac{\left|\rho^{(j)}(s) - \widetilde{\rho}^{(j)}(s)\right|}{(1+|s|)^{\gamma}} < \frac{\varepsilon}{C_{y,a,b}}$$

Hence, by using the inequality  $1 + |a^{\top}u - b| \leq 1 + ||a|| ||u|| + |b| \leq (1 + ||a|| + |b|)(1 + ||u||)$ , it follows for the function  $y\widetilde{\rho}(a^{\top}\cdot -b) := (u \mapsto y\widetilde{\rho}(a^{\top}u - b)) \in C_b^k(\mathbb{R}^m;\mathbb{R}^d)$  that

$$\begin{split} \|y\rho\left(a^{\top}\cdot-b\right)-y\widetilde{\rho}\left(a^{\top}\cdot-b\right)\|_{C^{k}_{pol,\gamma}(\mathbb{R}^{m};\mathbb{R}^{d})} &= \max_{\alpha\in\mathbb{N}^{m}_{0,k}}\sup_{u\in\mathbb{R}^{m}}\frac{\|y\rho^{(|\alpha|)}\left(a^{\top}u-b\right)a^{\alpha}-y\widetilde{\rho}^{(|\alpha|)}\left(a^{\top}u-b\right)a^{\alpha}}{(1+\|u\|)^{\gamma}}\\ &\leqslant \left(\max_{\alpha\in\mathbb{N}^{m}_{0,k}}\|ya^{\alpha}\|\left(1+\|a\|+|b|\right)\right)\max_{\alpha\in\mathbb{N}^{m}_{0,k}}\sup_{u\in\mathbb{R}^{m}}\frac{\|y\rho\left(a^{\top}u-b\right)-y\widetilde{\rho}\left(a^{\top}u-b\right)\|}{(1+|a^{\top}u-b|)^{\gamma}}\\ &\leqslant C_{y,a,b}\max_{j=0,\dots,k}\sup_{s\in\mathbb{R}}\frac{\left|\rho^{(j)}(s)-\widetilde{\rho}^{(j)}(s)\right|}{(1+|s|)^{\gamma}}\\ &< C_{y,a,b}\frac{\varepsilon}{C_{y,a,b}}=\varepsilon. \end{split}$$

Since  $\varepsilon > 0$  was chosen arbitrarily and  $y \tilde{\rho} (a^{\top} \cdot -b) \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)$ , it follows that  $y \rho (a^{\top} \cdot -b) \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)^{\gamma}$ . Thus, by using that  $\mathcal{NN}_{\mathbb{R}^m,d}^{\rho}$  is defined as vector space consisting of functions of the form  $\mathbb{R}^m \ni u \mapsto y\rho (a^{\top}u - b) \in \mathbb{R}^d$ , with  $y \in \mathbb{R}^d$ ,  $a \in \mathbb{R}^m$ , and  $b \in \mathbb{R}$ , the triangle inequality implies that  $\mathcal{NN}_{\mathbb{R}^m,d}^{\rho} \subseteq \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$ . Finally, by using that  $(X, \|\cdot\|_X)$  is  $(k, U, \gamma)$ -approximable function space, i.e. that the restriction map in (4) is a continuous embedding, it follows that  $\mathcal{NN}_{U,d}^{\rho} \subseteq X$ 

7.2. **Proof of Theorem 2.6.** In this section, we provide the proof of Theorem 2.6, i.e. the universal approximation property of deterministic neural networks  $\mathcal{NN}_{U,d}^{\rho}$  in any  $(k, U, \gamma)$ -approximable function space  $(X, \|\cdot\|_X)$ , where  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , and  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ .

The idea of the proof is the following. By contradiction, we assume that  $\mathcal{NN}_{U,d}^{\rho} \subseteq X$  is not dense in X with respect to  $\|\cdot\|_X$ . Then, by applying the classical Hahn-Banach separation argument (as in [Cybenko, 1989, Theorem 1]), we obtain a non-zero continuous linear functional  $l: X \to \mathbb{R}$  which vanishes on the vector subspace  $\mathcal{NN}_{U,d}^{\rho} \subseteq X$ . Moreover, by using the continuous embedding in (4), we can express  $l: X \to \mathbb{R}$  on the dense subspace  $\overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}$  with finite signed Radon measures. This relies on the Riesz representation theorem in [Dörsek and Teichmann, 2010, Theorem 2.4].

Subsequently, we use the distributional extension of Wiener's Tauberian theorem in [Korevaar, 1965], which generalizes the classical Wiener Tauberian theorem, i.e. that span  $\{\mathbb{R} \ni s \mapsto \rho(s+b) \in \mathbb{R} : b \in \mathbb{R}\}$  is dense in  $L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), du)$  if and only if the Fourier transform  $\hat{\rho}$  (in the classical sense) does not have any zeros (see [Wiener, 1932]). Then, by using this and that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is non-polynomial, we conclude that  $l : X \to \mathbb{R}$  vanishes everywhere on X, which contradicts the initial assumption that  $l : X \to \mathbb{R}$  is non-zero. Hence,  $\mathcal{NN}_{U,d}^{\rho}$  must be dense in X with respect to  $\|\cdot\|_X$ .

To be able to prove Theorem 2.6 as outlined above, we now first generalize the Riesz representation theorem in [Dörsek and Teichmann, 2010, Theorem 2.7] to this vector-valued case with derivatives. Hereby, we define  $\mathcal{M}_{\gamma}(\mathbb{R}^m)$  as the vector space of finite signed Radon measures  $\eta : \mathcal{B}(\mathbb{R}^m) \to \mathbb{R}$  with  $\int_{\mathbb{R}^m} (1 + ||u||)^{\gamma} |\eta| (du) < \infty$ , where  $|\eta| : \mathcal{B}(\mathbb{R}^m) \to [0, \infty)$  denotes the corresponding total variation measure. Moreover, we denote by  $Z^*$  the dual space of a Banach space  $(Z, ||\cdot||_Z)$  which consists of continuous linear functionals  $l : Z \to \mathbb{R}$  and is equipped with the norm  $||l||_{Z^*} := \sup_{z \in Z, ||z||_Z \leq 1} |l(z)|$ . **Proposition 7.1** (Riesz representation). For  $k \in \mathbb{N}_0$  and  $\gamma \in (0, \infty)$ , let  $l : \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  be a continuous linear functional. Then, there exist some signed Radon measures  $(\eta_{\alpha,i})_{\alpha \in \mathbb{N}_{0,k}^m, i=1,...,d} \subseteq \mathcal{M}_{\gamma}(\mathbb{R}^m)$  such that for every  $f = (f_1, ..., f_d)^{\top} \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  it holds that

$$l(f) = \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \int_{\mathbb{R}^{m}} \partial_{\alpha} f_{i}(u) \eta_{\alpha,i}(du)$$

*Proof.* First, we show the conclusion for k = 0 and d = 1. Indeed, by defining  $\mathbb{R}^m \ni u \mapsto \psi(u) := (1 + ||u||)^{\gamma} \in (0, \infty)$ , the tuple  $(\mathbb{R}^m, \psi)$  is a weighted space in the sense of [Dörsek and Teichmann, 2010, p. 5]. Hence, the conclusion follows from [Dörsek and Teichmann, 2010, Theorem 2.4].

Now, for the general case of  $k \ge 1$  and  $d \ge 2$ , we fix a continuous linear functional  $l : \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  and define the number  $M := |\mathbb{N}_{0,k}^m| \cdot d$  and the map

$$\overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma} \ni f \quad \mapsto \quad \Xi(f) := \left(\partial_{\alpha} f_i\right)_{\alpha \in \mathbb{N}_{0,k}^m, i=1,\dots,d}^{\top} \in \overline{C_b^0(\mathbb{R}^m;\mathbb{R}^M)}^{\gamma}.$$

Moreover, we denote by  $\operatorname{Img}(\Xi) := \left\{ \Xi(f) : f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \right\} \subseteq \overline{C_b^0(\mathbb{R}^m; \mathbb{R}^M)}^{\gamma}$  the image vector subspace. Then, by using that  $\Xi : \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \operatorname{Img}(\Xi)$  is by definition bijective, there exists an inverse map  $\Xi^{-1} : \operatorname{Img}(\Xi) \to \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$ . Moreover, we conclude for every  $f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  that

$$\begin{split} \left\| \Xi^{-1}((\partial_{\alpha}f_{i})_{\alpha \in \mathbb{N}_{0,k}^{m}, i=1,...,d}) \right\|_{C_{pol,\gamma}^{k}(\mathbb{R}^{m};\mathbb{R}^{d})} &= \|f\|_{C_{pol,\gamma}^{k}(\mathbb{R}^{m};\mathbb{R}^{d})} = \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \mathbb{R}^{m}} \frac{\|\partial_{\alpha}f(u)\|}{(1+\|u\|)^{\gamma}} \\ &= \sup_{u \in \mathbb{R}^{m}} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \frac{\|\partial_{\alpha}f(u)\|}{(1+\|u\|)^{\gamma}} \leqslant \sup_{u \in \mathbb{R}^{m}} \frac{\|(\partial_{\alpha}f_{i})_{\alpha \in \mathbb{N}_{0,k}^{m}, i=1,...,d}\|}{(1+\|u\|)^{\gamma}} \\ &= \|f\|_{C_{pol,\gamma}^{k}(\mathbb{R}^{m};\mathbb{R}^{M})}, \end{split}$$

which shows that  $\Xi^{-1}$ : Img $(\Xi) \to \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  is continuous. Hence, the concatenation  $l \circ \Xi^{-1}$ : Img $(\Xi) \to \mathbb{R}$  is a continuous linear functional on Img $(\Xi)$ , which can be extended by using the Hahn-Banach theorem to a continuous linear functional  $l_0$ :  $\overline{C_b^0(\mathbb{R}^m; \mathbb{R}^M)}^{\gamma} \to \mathbb{R}$  such that for every  $f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  it holds that

$$l_0((\partial_\alpha f_i)_{\alpha\in\mathbb{N}_{0,k}^m,\,i=1,\dots,d}) = \left(l\circ\Xi^{-1}\right)\left((\partial_\alpha f_i)_{\alpha\in\mathbb{N}_{0,k}^m,\,i=1,\dots,d}\right) = l(f).$$
<sup>(29)</sup>

Now, for every fixed  $\alpha \in \mathbb{N}_{0,k}^m$  and i = 1, ..., d, we define the linear map  $\overline{C_b^0(\mathbb{R}^m)}^{\gamma} \ni g \mapsto l_{\alpha,i}(g) := l_0(ge_{\alpha,i}) \in \mathbb{R}$ , where  $e_{\alpha,i} \in \mathbb{R}^M := \mathbb{R}^{|\mathbb{N}_{0,k}^m| \cdot d} \cong \mathbb{R}^{|\mathbb{N}_{0,k}^m|} \times \mathbb{R}^d$  denotes the  $(\alpha, i)$ -th unit vector of  $\mathbb{R}^M := \mathbb{R}^{|\mathbb{N}_{0,k}^m| \cdot d} \cong \mathbb{R}^{|\mathbb{N}_{0,k}^m|} \times \mathbb{R}^d$ . Then, for every  $g \in \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$ , it follows with  $Z := \overline{C_b^0(\mathbb{R}^m;\mathbb{R}^M)}^{\gamma}$  that

$$|l_{\alpha,i}(g)| = |l_0(ge_{\alpha,i})| \leq ||l_0||_{Z^*} ||ge_{\alpha,i}||_{C^0_{pol,\gamma}(\mathbb{R}^m;\mathbb{R}^M)} = ||l_0||_{Z^*} ||g||_{C^0_{pol,\gamma}(\mathbb{R}^m)},$$

which shows that  $l_{\alpha,i} : \overline{C_b^0(\mathbb{R}^m)}^{\gamma} \to \mathbb{R}$  is a continuous linear functional. Hence, by using (29) and by applying for every  $\alpha \in \mathbb{N}_{0,k}^m$  and i = 1, ..., d the case with k = 0 and d = 1, there exist some Radon measures  $(\eta_{\alpha,i})_{\alpha \in \mathbb{N}_{0,k}^m; i=1,...,d} \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  such that for every  $f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  it holds that

$$\begin{split} \mathcal{I}(f) &= \left(l \circ \Xi^{-1}\right) \left( (\partial_{\alpha} f_{i})_{\alpha \in \mathbb{N}_{0,k}^{m}, i=1,\dots,d} \right) \\ &= l_{0}((\partial_{\alpha} f_{i})_{\alpha \in \mathbb{N}_{0,k}^{m}, i=1,\dots,d}) \\ &= \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} l_{\alpha,i} (\partial_{\alpha} f_{i} e_{\alpha,i}) \\ &= \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \int_{\mathbb{R}^{m}} \partial_{\alpha} f_{i}(u) \eta_{\alpha,i}(du), \end{split}$$

which completes the proof.

**Proposition 7.2.** For  $\gamma \in (0, \infty)$ , let  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  be a signed Radon measure and assume that  $\rho \in \overline{C_b^0(\mathbb{R})}^{\gamma}$  is non-polynomial. If for every  $a \in \mathbb{R}^m$  and  $b \in \mathbb{R}$  it holds that

$$\int_{\mathbb{R}^m} \rho\left(a^\top u - b\right) \eta(du) = 0,\tag{30}$$

then it follows that  $\eta = 0 \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ .

*Proof.* We follow the proof of [Cuchiero et al., 2023, Proposition 4.4 (A3)] and assume that  $\rho \in \overline{C_b^0(\mathbb{R})}^{\gamma}$  is non-polynomial. Then, by using e.g. [Rudin, 1991, Examples 7.16], there exists a non-zero point  $t_0 \in \mathbb{R} \setminus \{0\}$  which belongs to the support of  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$ . Moreover, let  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  satisfy (30) and assume by contradiction that  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  is non-zero.

Now, for every  $a \in \mathbb{R}^m$ , we define the push-forward measure  $\eta_a := \eta \circ (a^\top \cdot)^{-1} : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$  by  $\eta_a(B) := \eta (\{u \in \mathbb{R}^m : a^\top u \in B\})$ , for  $B \in \mathcal{B}(\mathbb{R})$ . Moreover, for every fixed  $\lambda \in \mathbb{R} \setminus \{0\}$ , we define the function  $\mathbb{R} \ni s \mapsto \rho_\lambda(s) := \rho(\lambda s) \in \mathbb{R}$ . Then, by applying [Bogachev, 2007, Theorem 3.6.1] (to the positive and negative part of  $\eta \in \mathcal{M}_\gamma(\mathbb{R}^m)$ ) and by using the assumption (30) (with  $\lambda a \in \mathbb{R}^m$  and  $\lambda b \in \mathbb{R}$  instead of  $a \in \mathbb{R}^m$  and  $b \in \mathbb{R}$ , respectively), it follows for every  $a \in \mathbb{R}^m$  and  $b \in \mathbb{R}$  that

$$\int_{\mathbb{R}} \rho_{\lambda}(s-b)\eta_a(ds) = \int_{\mathbb{R}^m} \rho\left(\lambda a^{\top}u - \lambda b\right)\eta(du) = 0.$$
(31)

Since  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  is non-zero, there exists some  $a \in \mathbb{R}^m$  such that  $\eta_a : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$  is non-zero. Hence, there exists some  $h \in \mathcal{S}(\mathbb{R};\mathbb{C})$  such that  $(z \mapsto f(z) := (h * \eta_a)(-z) := \int_{\mathbb{R}} h(-z - s)\eta_a(ds)) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), du; \mathbb{C})$  is also non-zero. Then, by using that the Fourier transform is injective,  $\hat{f} : \mathbb{R} \to \mathbb{C}$  is non-zero, too, i.e. there exists some  $t_1 \in \mathbb{R} \setminus \{0\}$  such that  $\hat{f}(t_1) \neq 0$ . Hence, by using [Folland, 1992, Table 7.2.2], the function  $(z \mapsto f_0(z) := f(z)e^{-it_1z}) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), du; \mathbb{C})$  satisfies  $\hat{f}_0(0) = \hat{f}(t_1) \neq 0$ . Moreover, we choose  $\lambda := \frac{t_1}{t_0} \in \mathbb{R} \setminus \{0\}$  and define the function  $\mathbb{R} \ni z \mapsto \rho_0(z) := \rho_\lambda(z)e^{-it_1z} \in \mathbb{C}$ . Next, we use [Bogachev, 2007, Theorem 3.6.1] (applied to  $|\eta| : \mathcal{B}(\mathbb{R}^m) \to [0, \infty)$ ), the inequality

Next, we use [Bogachev, 2007, Theorem 3.6.1] (applied to  $|\eta| : \mathcal{B}(\mathbb{R}^m) \to [0, \infty)$ ), the inequality  $1 + |\lambda a^\top u - b| \leq 1 + |\lambda| ||a|| ||u|| + |\lambda| ||b| \leq \max(1, |\lambda|)(1 + ||a||)(1 + ||b|)(1 + ||u||)$  for any  $a, u \in \mathbb{R}^m$  and  $b, y \in \mathbb{R}$ , the inequality  $(1 + |b|)^{\gamma} \leq 2^{\gamma} (1 + |b|^2)^{\gamma/2} \leq 2^{\gamma} (1 + |b|^2)^{[\gamma/2]}$  for any  $b \in \mathbb{R}$ , and that for every  $y \in \mathbb{R}$  the reflected translation  $\mathbb{R} \ni b \mapsto \tilde{h}_y(b) := h(-y - b) \in \mathbb{R}$  of the Schwartz function  $h \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  is again a Schwartz function (see [Folland, 1992, p. 331]) to conclude for every  $y \in \mathbb{R}$  that

$$\begin{split} \int_{\mathbb{R}} \int_{\mathbb{R}} |h(-y-b)| |\rho_{\lambda}(s-b)| |\eta_{a}|(ds)db &= \int_{\mathbb{R}^{m}} |h(-y-b)| \int_{\mathbb{R}} \left| \rho \left( \lambda a^{\top} u - \lambda b \right) \right| |\eta|(du)db \\ &\leq \int_{\mathbb{R}} |h(-y-b)| \left( \sup_{u \in \mathbb{R}^{m}} \frac{|\rho \left( \lambda a^{\top} u - \lambda b \right) \right)^{\gamma}}{(1+|\lambda a^{\top} u - \lambda b|)^{\gamma}} \right) \int_{\mathbb{R}} \left( 1+|\lambda a^{\top} u - \lambda b \right) \right)^{\gamma} |\eta|(du)db \\ &\leq \max(1,|\lambda|)^{\gamma} (1+\|a\|)^{\gamma} \left( \sup_{s \in \mathbb{R}} \frac{|\rho(s)|}{(1+|s|)^{\gamma}} \right) \left( \int_{\mathbb{R}} |h(-y-b)|(1+|b|)^{\gamma} db \right) \int_{\mathbb{R}} (1+\|u\|)^{\gamma} |\eta|(du) \\ &\leq \max(1,|\lambda|)^{\gamma} (1+\|a\|)^{\gamma} \|\rho\|_{C^{0}_{pol,\gamma}(\mathbb{R})} \left( \sup_{y \in \mathbb{R}} \left| \widetilde{h}_{y}(b) \right| \left( 1+|b|^{2} \right)^{\lceil \gamma/2 \rceil + 1} \right) \\ &\quad \cdot \left( \int_{\mathbb{R}} \frac{1}{1+b^{2}} db \right) \int_{\mathbb{R}} (1+\|u\|)^{\gamma} |\eta|(du) < \infty. \end{split}$$

Then, by using the substitution  $z \mapsto s - b$  and the identity (31), it follows for every  $y \in \mathbb{R}$  that

$$(f_0 * \rho_0)(y) = \int_{\mathbb{R}} f(y-z)e^{it_1(y-z)}\rho_\lambda(z)e^{-it_1z}dz = e^{it_1y}\int_{\mathbb{R}} (h*\eta_a)(z-y)\rho_\lambda(z)dz$$
$$= e^{it_1y}\int_{\mathbb{R}}\int_{\mathbb{R}} h(z-y-s)\rho_\lambda(z)\eta_a(ds)dz = e^{it_1y}\int_{\mathbb{R}} h(-y-b)\int_{\mathbb{R}} \rho_\lambda(s-b)\eta_a(ds)db = 0,$$
(33)

where (32) ensures that the convolution  $f_0 * \rho_0 : \mathbb{R} \to \mathbb{R}$  is well-defined.

Moreover, let  $\phi \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  such that  $\hat{\phi}(\xi) = 1$ , for all  $\xi \in [-1, 1]$ , and  $\hat{\phi}(\xi) = 0$ , for all  $\xi \in \mathbb{R} \setminus [-2, 2]$ . In addition, for every  $n \in \mathbb{N}$ , we define  $(s \mapsto \phi_n(s) := \frac{1}{n}\phi(\frac{1}{n})) \in \mathcal{S}(\mathbb{R};\mathbb{C})$ . Then, by following the proof of [Korevaar, 1965, Theorem A], there exists some large enough  $n \in \mathbb{N}$  and  $w \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), du)$  such that  $w * f_0 = \phi_{2n} \in \mathcal{S}(\mathbb{R};\mathbb{C})$ , Hence, by using (32), we conclude for every  $g \in \mathcal{S}(\mathbb{R};\mathbb{C})$  that

$$(T_{\rho_0} * \phi_{2n})(g) := T_{\rho_0}(\phi_{2n}(-\cdot) * g) = (g * \phi_{2n} * \rho_0)(0) = (g * w * f_0 * \rho_0)(0) = 0, \quad (34)$$

where  $\phi_{2n}(-\cdot)$  denotes the function  $\mathbb{R} \ni s \mapsto \phi_{2n}(-s) \in \mathbb{R}$ . Thus, by using [Folland, 1992, Equation 9.32] together with (34), i.e. that  $\widehat{\phi_{2n}}\widehat{T_{\rho_0}} = \widehat{T_{\rho_0}} * \widehat{\phi_{2n}} = 0 \in \mathcal{S}'(\mathbb{R};\mathbb{C})$ , and that  $\widehat{\phi_{2n}}(\xi) = \widehat{\phi}(2n\xi) = 1$  for any  $\xi \in [-\frac{1}{2n}, \frac{1}{2n}]$ , it follows that  $\widehat{T_{\rho_0}} \in \mathcal{S}'(\mathbb{R};\mathbb{C})$  vanishes on  $(-\frac{1}{2n}, \frac{1}{2n})$ . Finally, for some fixed  $g \in C^{\infty}((t_0 - \frac{1}{2n|\lambda|}, t_0 + \frac{1}{2n|\lambda|});\mathbb{C})$ , we define  $(z \mapsto g_0(z) := g(\frac{z}{\lambda} + t_0)) \in \mathbb{C}$ 

Finally, for some fixed  $g \in C^{\infty}((t_0 - \frac{1}{2n|\lambda|}, t_0 + \frac{1}{2n|\lambda|}); \mathbb{C})$ , we define  $(z \mapsto g_0(z) := g(\frac{z}{\lambda} + t_0)) \in C_c^{\infty}((-\frac{1}{2n}, \frac{1}{2n}); \mathbb{C})$ . Hence, by using the definition of  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$ , the substitution  $\zeta \mapsto \xi/\lambda$ , [Folland, 1992, Table 9.2.2], and that  $\widehat{T_{\rho_0}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  vanishes on  $(-\frac{1}{2n}, \frac{1}{2n})$ , we conclude that

$$\widehat{T_{\rho}}(g) = T_{\rho}(\widehat{g}) = \int_{\mathbb{R}} \rho(\xi)\widehat{g}(\xi)d\xi = \lambda \int_{\mathbb{R}} \rho(\lambda\zeta)\widehat{g}(\lambda\zeta)d\zeta = \int_{\mathbb{R}} \rho_0(\zeta)e^{it_1\zeta}\widehat{g(\cdot/\lambda)}(\zeta)dz$$

$$= \int_{\mathbb{R}} \rho_0(\zeta)\widehat{g_0}(\zeta)d\zeta = T_{\rho_0}(\widehat{g_0}) = \widehat{T_{\rho_0}}(g_0) = 0,$$
(35)

where  $\widehat{g(\cdot/\lambda)}$  denotes the Fourier transform of the function  $(s \mapsto g(s/\lambda)) \in \mathcal{S}(\mathbb{R}; \mathbb{C})$ . Since the function  $g \in C_c^{\infty}((t_0 - \frac{1}{2n|\lambda|}, t_0 + \frac{1}{2n|\lambda|}); \mathbb{C})$  was chosen arbitrary, (35) shows that  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  vanishes on the set  $\left(t_0 - \frac{1}{2n|\lambda|}, t_0 + \frac{1}{2n|\lambda|}\right)$ . This however contradicts the assumption that  $t_0 \in \mathbb{R} \setminus \{0\}$  belongs to the support of  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  and shows that  $\eta = 0 \in \mathcal{M}_{\gamma}(\mathbb{R})$ .

Next, we show some properties of measures  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ ,  $\gamma \in (0, \infty)$ , whenever they are convoluted with a bump function. For this purpose, we introduce the smooth bump function  $\phi : \mathbb{R}^m \to \mathbb{R}$  defined by

$$\phi(u) := \begin{cases} Ce^{-\frac{1}{1-\|u\|^2}}, & u \in \mathbb{B}_1(0), \\ 0, & u \in \mathbb{R}^m \setminus \mathbb{B}_1(0). \end{cases}$$

where C > 0 is a normalizing constant such that  $\|\phi\|_{L^1(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),du)} = 1$ . From this, we define for every fixed  $\delta > 0$  the mollifier  $\mathbb{R}^m \ni u \mapsto \phi_{\delta}(u) := \frac{1}{\delta^m} \phi\left(\frac{u}{\delta}\right) \in \mathbb{R}$ . Moreover, for any  $\gamma \in (0,\infty)$  and  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ , we define the function  $\mathbb{R}^m \ni u \mapsto (\phi_{\delta} * \eta)(u) := \int_{\mathbb{R}^m} \phi_{\delta}(u-v)\eta(dv) \in \mathbb{R}$ .

**Lemma 7.3.** For  $\gamma \in (0, \infty)$ , let  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  and  $f \in \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$ . Then, the following holds true:

- (i) For every  $\delta > 0$  the function  $\phi_{\delta} * \eta : \mathbb{R}^m \to \mathbb{R}$  is smooth with  $\partial_{\alpha}(\phi_{\delta} * \eta)(u) = (\partial_{\alpha}\phi_{\delta} * \eta)(u)$ for all  $\alpha \in \mathbb{N}_0^m$  and  $u \in \mathbb{R}^m$ .
- (ii) For every  $\delta > 0$  and  $\alpha \in \mathbb{N}_0^m$  it holds that

$$\lim_{r \to \infty} \sup_{u \in \mathbb{R}^m \setminus \overline{\mathbb{B}_r(0)}} |f(u)\partial_{\alpha}(\phi_{\delta} * \eta)(u)| = 0.$$

- (iii) For every  $\delta > 0$  and  $\alpha \in \mathbb{N}_0^m$  it holds that  $\partial_\alpha(\phi_\delta * \eta)(u) du|_{\mathcal{B}(\mathbb{R}^m)} \in \mathcal{M}_\gamma(\mathbb{R}^m)$ .
- (iv) For every  $\delta > 0$  and  $\alpha \in \mathbb{N}_0^m$  the map

$$(\overline{C_b^0(\mathbb{R}^m)}^{\gamma}, \|\cdot\|_{C^0_{pol,\gamma}(\mathbb{R}^m)}) \ni f \quad \mapsto \quad \int_{\mathbb{R}^m} f(u)\partial_\alpha(\phi_\delta * \eta)(u)du \in \mathbb{R}$$

is a continuous linear functional.

(v) For every  $\delta > 0$  it holds that

$$\int_{\mathbb{R}^m} f(u)(\phi_{\delta} * \eta)(u) du = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(u+y)\eta(du)\phi_{\delta}(y) dy.$$

(vi) It holds that

$$\lim_{\delta \to 0} \int_{\mathbb{R}^m} f(u)(\phi_{\delta} * \eta)(u) du = \int_{\mathbb{R}^m} f(u)\eta(du).$$

*Proof.* Fix some  $\gamma \in (0, \infty)$ ,  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ ,  $f \in \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$ ,  $\delta > 0$ , and  $\alpha \in \mathbb{N}_0^m$ . For (i), we first show that  $\partial_{\alpha}\phi_{\delta} * \eta : \mathbb{R}^m \to \mathbb{R}$  is continuous. Indeed, we observe that for every  $u, u_0, v \in \mathbb{R}^m$ , it holds that

$$\max\left(\left|\partial_{\alpha}\phi_{\delta}(u-v)\right|,\left|\partial_{\alpha}\phi_{\delta}(u_{0}-v)\right|\right) \leqslant C_{11} := \sup_{u_{1}\in\mathbb{R}^{m}}\left|\partial_{\alpha}\phi_{\delta}(u_{1})\right| < \infty.$$
(36)

Then, the dominated convergence theorem (with (36) and that  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  is finite) implies that

$$\lim_{u \to u_0} (\partial_\alpha \phi_\delta * \eta)(u) = \lim_{u \to u_0} \int_{\mathbb{R}^m} \partial_\alpha \phi_\delta(u - v) \eta(dv) = \int_{\mathbb{R}^m} \partial_\alpha \phi_\delta(u_0 - v) \eta(dv) = (\partial_\alpha \phi_\delta * \eta)(u_0),$$

which shows that  $\partial_{\alpha}\phi_{\delta} * \eta : \mathbb{R}^m \to \mathbb{R}$  is continuous. Moreover, for every fixed  $\beta \in \mathbb{N}_0^m$  and l = 1, ..., m(with  $e_l \in \mathbb{R}^m$  denoting the *l*-th unit vector of  $\mathbb{R}^m$ ), we use the mean-value theorem to conclude for every  $u, v \in \mathbb{R}^m$  and  $h \in \mathbb{R}$  that

$$\max\left(\left|\frac{\partial_{\beta}\phi_{\delta}(u+he_{l}-v)-\partial_{\beta}\phi_{\delta}(u-v)}{h}\right|, |\partial_{\beta+e_{l}}\phi_{\delta}(u-v)|\right) \\ \leqslant C_{12} := \sup_{u_{1}\in\mathbb{R}^{m}} |\partial_{\beta+e_{l}}\phi_{\delta}(u_{1})| < \infty.$$
(37)

Then, the dominated convergence theorem (with (37) and that  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  is finite) implies that

$$\partial_{e_l}(\partial_{\alpha}\phi_{\delta}*\eta)(u) = \lim_{h \to 0} \frac{(\partial_{\beta}\phi_{\delta}*\eta)(u+he_l) - (\partial_{\beta}\phi_{\delta}*\eta)(u)}{h}$$
$$= \lim_{h \to 0} \int_{\mathbb{R}^m} \frac{\partial_{\beta}\phi_{\delta}(u+he_l-v) - \partial_{\beta}\phi_{\delta}(u-v)}{h} \eta(dv)$$
$$= \int_{\mathbb{R}^m} \partial_{\beta+e_l}\phi_{\delta}(u-v)\eta(dv) = (\partial_{\beta+e_l}\phi_{\delta}*\eta)(u).$$

Hence, by induction on  $\beta \in \mathbb{N}_0^m$ , it follows that  $\partial_\alpha(\phi_\delta * \eta)(u) = (\partial_\alpha \phi_\delta * \eta)(u)$  for any  $u \in \mathbb{R}^m$ . This together with the previous step shows (i).

For (ii), we use (i), that  $\operatorname{supp}(\phi_{\delta}) = B_{\delta}(0)$  implies  $\operatorname{supp}(\partial_{\alpha}\phi_{\delta}) \subseteq B_{\delta}(0)$ , the inequality  $1 + x + y \leq (1 + x)(1 + y)$  for any  $x, y \geq 0$ , that the constant  $C_{13} := \operatorname{sup}_{y \in \mathbb{R}^m} |\partial_{\alpha}\phi_{\delta}(y)| > 0$  is finite, and that  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  to conclude that

$$C_{14} := \sup_{u \in \mathbb{R}^m} \left( (1 + \|u\|)^{\gamma} \left| (\phi_{\delta} * \eta)(u) \right| \right) \leq \sup_{u \in \mathbb{R}^m} \int_{\mathbb{R}^m} (1 + \|u\|)^{\gamma} \left| \partial_{\alpha} \phi_{\delta}(u - v) \right| |\eta|(dv) du$$
  
$$\leq \sup_{u \in \mathbb{R}^m} \int_{\mathbb{R}^m} (1 + \underbrace{\|u - v\|}_{\leq \delta} + \|v\|)^{\gamma} \left| \partial_{\alpha} \phi_{\delta}(u - v) \right| |\eta|(dv) \leq C_{13}(1 + \delta)^{\gamma} \int_{\mathbb{R}^m} (1 + \|v\|)^{\gamma} |\eta|(dv) < \infty.$$

Hence, by using this and that  $f \in \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$  together with Lemma 6.1, it follows that

$$\lim_{r \to \infty} \sup_{u \in \mathbb{R}^m \setminus \overline{\mathbb{B}_r(0)}} |f(u)\partial_{\alpha}(\phi_{\delta} * \eta)(u)| = \lim_{r \to \infty} \sup_{u \in \mathbb{R}^m \setminus \overline{\mathbb{B}_r(0)}} \left( \frac{|f(u)|}{(1 + \|u\|)^{\gamma}} (1 + \|u\|)^{\gamma} |\partial_{\alpha}(\phi_{\delta} * \eta)(u)| \right)$$
$$= C_{14} \lim_{r \to \infty} \sup_{u \in \mathbb{R}^m \setminus \overline{\mathbb{B}_r(0)}} \frac{|f(u)|}{(1 + \|u\|)^{\gamma}} = 0,$$

which shows (ii).

For (iii), we first prove that  $\partial_{\alpha}(\phi_{\delta} * \eta)(u)du|_{\mathcal{B}(\mathbb{R}^m)} : \mathcal{B}(\mathbb{R}^m) \to \mathbb{R}$  is a signed Radon measure. For this purpose, we denote its positive and negative part by  $\eta_{\delta,\pm} := \pm (\partial_{\alpha}(\phi_{\delta} * \eta)(u))_{\pm} du|_{\mathcal{B}(\mathbb{R}^m)} : \mathcal{B}(\mathbb{R}^m) \to [0,\infty]$  satisfying  $\eta_{\delta,+} - \eta_{\delta,-} = \partial_{\alpha}(\phi_{\delta} * \eta)(u)du|_{\mathcal{B}(\mathbb{R}^m)}$ , where  $s_+ := \max(s,0)$  and  $s_- := -\min(s,0)$ , for any  $s \in \mathbb{R}$ . Moreover, we define the finite constant  $C_{15} := \sup_{u \in \mathbb{R}^m} |\partial_{\alpha}\phi_{\delta}(u)| > 0$ . Then, for every  $u \in \mathbb{R}^m$ , we choose a compact subset  $K \subset \mathbb{R}^m$  with  $u \in K$  and use that  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  is finite to conclude that

$$\eta_{\delta,\pm}(K) = \pm \int_{K} (\partial_{\alpha}(\phi_{\delta} * \eta)(u))_{\pm} du \leq \left( \underbrace{\int_{K} du}_{=:|K|} \right) \sup_{u \in K} |(\partial_{\alpha}\phi_{\delta} * \eta)(u)|$$
$$\leq |K| \sup_{u \in K} \int_{\mathbb{R}^{m}} |\partial_{\alpha}\phi_{\delta}(u-v)| |\eta|(dv) \leq C_{15}|K||\eta|(\mathbb{R}^{m}) < \infty.$$

This shows that both measures  $\eta_{\delta,\pm} : \mathcal{B}(\mathbb{R}^m) \to [0,\infty]$  are locally finite. In addition, it holds for every  $B \in \mathcal{B}(\mathbb{R}^m)$  that

$$\begin{split} \eta_{\delta,\pm}(B) &= \pm \int_B \left( \partial_\alpha (\phi_\delta * \eta)(u) \right)_{\pm} du \\ &= \inf \left\{ \pm \int_U \left( \partial_\alpha (\phi_\delta * \eta)(u) \right)_{\pm} du : U \subseteq \mathbb{R}^m \text{ open with } B \subseteq U \right\} \\ &= \inf \left\{ \eta_{\delta,\pm}(U) : U \subseteq \mathbb{R}^m \text{ open with } B \subseteq U \right\}, \end{split}$$

which shows that both measures  $\eta_{\delta,\pm} : \mathcal{B}(\mathbb{R}^m) \to [0,\infty]$  are outer regular. Moreover, it holds for every  $B \in \mathcal{B}(\mathbb{R}^m)$  that

$$\eta_{\delta,\pm}(B) = \pm \int_{B} \left( \partial_{\alpha} (\phi_{\delta} * \eta)(u) \right)_{\pm} du$$
  
=  $\sup \left\{ \pm \int_{K} \left( \partial_{\alpha} (\phi_{\delta} * \eta)(u) \right)_{\pm} du : K \subset B \text{ relatively compact} \right\}$   
=  $\sup \left\{ \eta_{\delta,\pm}(K) : K \subset B \text{ relatively compact} \right\},$ 

which shows that both measures  $\eta_{\delta,\pm}: \mathcal{B}(\mathbb{R}^m) \to [0,\infty]$  are inner regular. Hence, both measures  $\eta_{\delta,\pm}: \mathcal{B}(\mathbb{R}^m) \to [0,\infty]$  are Radon measures and  $\partial_{\alpha}(\phi_{\delta}*\eta)(u)du|_{\mathcal{B}(\mathbb{R}^m)} = \eta_{\delta,+} - \eta_{\delta,-}: \mathcal{B}(\mathbb{R}^m) \to [0,\infty]$  is thus a signed Radon measure. Furthermore, by using the triangle inequality, that  $\operatorname{supp}(\phi_{\delta}) = B_{\delta}(0)$  implies  $\operatorname{supp}(\partial_{\alpha}\phi_{\delta}) \subseteq B_{\delta}(0)$ , the inequality  $1+x+y \leq (1+x)(1+y)$  for any  $x, y \geq 0$ , the substitution  $y \mapsto u - v$  together with  $\|\partial_{\alpha}\phi_{\delta}\|_{L^1(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),du)} < \infty$ , and that  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ , we have

$$\begin{split} \int_{\mathbb{R}^m} (1+\|u\|)^{\gamma} \left|\partial_{\alpha}(\phi_{\delta}*\eta)(u)\right| du &\leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (1+\|u\|)^{\gamma} \left|\partial_{\alpha}\phi_{\delta}(u-v)\right| du |\eta| (dv) \\ &\leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (1+\underbrace{\|u-v\|}_{\leq \delta}+\|v\|)^{\gamma} \left|\partial_{\alpha}\phi_{\delta}(u-v)\right| du |\eta| (dv) \\ &\leq (1+\delta)^{\gamma} \left(\sup_{v\in\mathbb{R}^m} \int_{\mathbb{R}^m} \left|\partial_{\alpha}\phi_{\delta}(u-v)\right| du\right) \left(\int_{\mathbb{R}^m} (1+\|v\|)^{\gamma} |\eta| (dv)\right) \\ &\leq (1+\delta)^{\gamma} \|\partial_{\alpha}\phi_{\delta}\|_{L^1(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),du)} \left(\int_{\mathbb{R}^m} (1+\|v\|)^{\gamma} |\eta| (dv)\right) < \infty. \end{split}$$

This shows that  $\partial_{\alpha}(\phi_{\delta} * \eta)(u) du|_{\mathcal{B}(\mathbb{R}^m)} \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  is a finite signed Radon measure.

For (iv), we use (iii) to conclude that the constant  $C_{16} := \int_{\mathbb{R}^m} (1 + ||u||)^{\gamma} |(\phi_{\delta} * \eta)(u)| du > 0$  is finite. Then, it follows for every  $f \in \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$  that

$$\left| \int_{\mathbb{R}^m} f(u) \partial_\alpha (\phi_\delta * \eta)(u) du \right| \leq \left( \sup_{u \in \mathbb{R}^m} \frac{|f(u)|}{(1+\|u\|)^{\gamma}} \right) \int_{\mathbb{R}^m} (1+\|u\|)^{\gamma} \left| \partial_\alpha (\phi_\delta * \eta)(u) \right| du$$
$$= C_{16} \|f\|_{C^0_{pol,\gamma}(\mathbb{R}^m)},$$

which shows that  $\overline{C_b^0(\mathbb{R}^m)}^{\gamma} \ni f \mapsto \int_{\mathbb{R}^m} f(u) \partial_{\alpha}(\phi_{\delta} * \eta)(u) du \in \mathbb{R}$  is a continuous linear functional. For (v), we use the substitution  $u \mapsto v + y$  to conclude that

$$\int_{\mathbb{R}^m} f(u)(\phi_{\delta} * \eta)(u) du = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(u)\phi_{\delta}(u-v)\eta(dv) du$$
$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(v+y)\eta(dv)\phi_{\delta}(y) dy.$$

For (vi), we define for every  $\delta \in (0, 1)$  the function  $\mathbb{R}^m \ni u \mapsto (\phi_{\delta} * f)(u) := \int_{\mathbb{R}^m} \phi_{\delta}(u-v)f(v)dv \in \mathbb{R}$ . Then, by using the triangle inequality, that  $\operatorname{supp}(\phi_{\delta}) = B_{\delta}(0)$ , the substitution  $y \mapsto u - v$  together with  $\int_{\mathbb{R}^m} |\phi_{\delta}(y)| dy = \|\phi_{\delta}\|_{L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)} = \|\phi\|_{L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)} = 1$ , the inequality  $1 + x + y \leq 1$ 

(1+x)(1+y) for any  $x, y \ge 0$ , and that  $f \in \overline{C_b^0(\mathbb{R}^m)}^{\gamma}$ , it follows for every  $u \in \mathbb{R}^m$  that

$$\begin{aligned} |(\phi_{\delta} * f)(u)| &\leq \int_{\mathbb{R}^{m}} |\phi_{\delta}(u-v)| \frac{|f(v)|}{(1+\|v\|)^{\gamma}} (1+\|v\|)^{\gamma} dv \\ &\leq \int_{\mathbb{R}^{m}} |\phi_{\delta}(u-v)| \frac{|f(v)|}{(1+\|v\|)^{\gamma}} (1+\|u\|+\underbrace{\|u-v\|}{\leq \delta})^{\gamma} dv \\ &\leq \left(\int_{\mathbb{R}^{m}} |\phi_{\delta}(u-v)| dv\right) \left(\sup_{v \in \mathbb{R}^{m}} \frac{|f(v)|}{(1+\|v\|)^{\gamma}}\right) (1+\|u\|+\delta)^{\gamma} \\ &\leq \left(\int_{\mathbb{R}^{m}} |\phi_{\delta}(y)| dy\right) \|f\|_{C^{0}_{pol,\gamma}(\mathbb{R}^{m})} (1+\delta)^{\gamma} (1+\|u\|)^{\gamma} \\ &\leq 2^{\gamma} \|f\|_{C^{0}_{pol,\gamma}(\mathbb{R}^{m})} (1+\|u\|)^{\gamma}. \end{aligned}$$
(38)

Moreover, by using that  $f \in \overline{C_h^0(\mathbb{R}^m)}^{\gamma}$ , we conclude for every  $u \in \mathbb{R}^m$  that

$$|f(u)| \leq \left(\sup_{u \in \mathbb{R}^m} \frac{|f(u)|}{(1+\|u\|)^{\gamma}}\right) (1+\|u\|)^{\gamma} \leq \|f\|_{C^0_{pol,\gamma}(\mathbb{R}^m)} (1+\|u\|)^{\gamma}.$$
(39)

Hence, by using (v), Fubini's theorem, the substitution  $u \mapsto v+y$ , and the dominated convergence theorem (with (38), (39),  $(1 + ||u||)^{\gamma} \in L^1(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), |\eta|)$  as  $\eta \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ , and [Evans, 2010, Theorem C.7], i.e. that  $\phi_{\delta} * f : \mathbb{R}^m \to \mathbb{R}$  converges a.e. to  $f : \mathbb{R}^m \to \mathbb{R}$ , as  $\delta \to 0$ ), it follows that

$$\lim_{\delta \to 0} \int_{\mathbb{R}^m} f(u)(\phi_{\delta} * \eta)(u) du = \lim_{\delta \to 0} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(v + y)\eta(dv)\phi_{\delta}(y) dy$$
$$= \lim_{\delta \to 0} \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(v + y)\phi_{\delta}(y) dy \right) \eta(dv)$$
$$= \lim_{\delta \to 0} \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \phi(v - u)f(u) du \right) \eta(dv)$$
$$= \lim_{\delta \to 0} \int_{\mathbb{R}^m} f(v)\eta(dv),$$
tes the proof.

which completes the proof.

Finally, we provide the proof of Theorem 2.6, i.e. the universal approximation property of deterministic neural networks  $\mathcal{NN}_{U,d}^{\rho}$  in any  $(k, U, \gamma)$ -approximable function space  $(X, \|\cdot\|_X)$ , where  $k \in \mathbb{N}_0, U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , and  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is the activation function.

Proof of Theorem 2.6. First, we use that  $(X, \|\cdot\|_X)$  is an  $(k, U, \gamma)$ -approximable function space together with Lemma 2.5 (ii) to conclude that  $\mathcal{NN}_{\mathbb{R}^m,d}^{\rho} \subseteq \overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}$  and that  $\mathcal{NN}_{U,d}^{\rho} \subseteq X$ .

Now, we assume by contradiction that  $\mathcal{NN}_{U,d}^{\rho}$  is not dense in X with respect to  $\|\cdot\|_X$ . Then, by using that  $(X, \|\cdot\|_X)$  is  $(k, U, \gamma)$ -approximable, i.e. that the restriction map in (4) is a continuous dense embedding, it follows from Remark 2.4 that  $\mathcal{NN}_{\mathbb{R}^m,d}^{\rho}$  cannot be dense in  $\overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}$  with respect to  $\|\cdot\|_{C_{pol,\gamma}^k(\mathbb{R}^m;\mathbb{R}^d)}$ . Hence, by applying the Hahn-Banach theorem, there exists a non-zero continuous linear functional  $l: \overline{C_b^k(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  such that for every  $\varphi \in \mathcal{NN}_{\mathbb{R}^m,d}^{\rho}$  it holds that  $l(\varphi) = 0$ .

Next, we use the Riesz representation result in Proposition 7.1 to conclude that there exist some signed Radon measures  $(\eta_{\alpha,i})_{\alpha \in \mathbb{N}_{0,k}^m, i=1,...,d} \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  such that for every  $f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  it holds that

$$l(f) = \sum_{\alpha \in \mathbb{N}_{0,k}^m} \sum_{i=1}^d \int_{\mathbb{R}^m} \partial_\alpha f_i(u) \eta_{\alpha,i}(du).$$

Since  $l(\varphi) = 0$  for any  $\varphi \in \mathcal{NN}_{\mathbb{R}^m,d}^{\rho}$ , it follows for every  $a \in \mathbb{R}^m$ ,  $b \in \mathbb{R}$ , and i = 1, ..., d that

$$l\left(e_{i}\rho\left(\lambda a^{\top}\cdot-b\right)\right) = \sum_{\alpha\in\mathbb{N}_{0,k}^{m}} \int_{\mathbb{R}^{m}} \rho^{\left(|\alpha|\right)}\left(a^{\top}u-b\right) a^{\alpha}\eta_{\alpha,i}(du) = 0,\tag{40}$$

where  $e_i \rho \left( \lambda a^\top \cdot -b \right)$  denotes the function  $\mathbb{R}^m \ni u \mapsto e_i \rho \left( \lambda a^\top u - b \right) \in \mathbb{R}^d$  with  $e_i \in \mathbb{R}^d$  being the *i*-th unit vector of  $\mathbb{R}^d$ , and where  $a^\alpha := \prod_{l=1}^m a_l^{\alpha_l}$  for  $a := (a_1, ..., a_m) \in \mathbb{R}^m$  and  $\alpha := (\alpha_1, ..., \alpha_m) \in \mathbb{N}_{0,k}^m$ .

Now, we define for every fixed  $\delta > 0$  the linear map  $l_{\delta} : \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  by

$$l_{\delta}(f) := \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \int_{\mathbb{R}^{m}} \partial_{\alpha} f_{i}(u) (\phi_{\delta} * \eta)(u) du,$$

for  $f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$ . Then, Lemma 7.3 (iv) shows that  $l_{\delta} : \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  is a continuous linear functional as it is a finite sum of the continuous linear functionals  $\overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \ni f \mapsto \int_{\mathbb{R}^m} \partial_{\alpha} f_i(u) (\phi_{\delta} * \eta)(u) du \in \mathbb{R}$  taken over  $\alpha \in \mathbb{N}_{0,k}^m$  and i = 1, ..., d. Moreover, for every fixed i = 1, ..., d, we define

$$\mathbb{R}^m \ni u \mapsto h_{\delta,i}(u) := \sum_{\alpha \in \mathbb{N}_{0,k}^m} (-1)^{|\alpha|} \partial_\alpha \left( \phi_\delta * \eta_{\alpha,i} \right)(u) \in \mathbb{R},$$

which satisfies  $h_{\delta,i}(u)du \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  as it is a finite linear combination of finite signed Radon measures  $\partial_{\alpha} (\phi_{\delta} * \eta_{\alpha,i}) (u)du \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  taken over  $\alpha \in \mathbb{N}_{0,k}^m$  (see Lemma 7.3 (iii)). Hence, integration by parts together with Lemma 7.3 (ii) shows that

$$l_{\delta}(f) = \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \int_{\mathbb{R}^{m}} \partial_{\alpha} f_{i}(u) \left(\phi_{\delta} * \eta_{\alpha,i}\right)(u) du$$
$$= \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} (-1)^{|\alpha|} \int_{\mathbb{R}^{m}} f_{i}(u) \partial_{\alpha} \left(\phi_{\delta} * \eta_{\alpha,i}\right)(u) du$$
$$= \sum_{i=1}^{d} \int_{\mathbb{R}^{m}} f_{i}(u) h_{\delta,i}(u) du.$$

Thus, by using this, Lemma 7.3 (v), and (40) (with  $b - a^{\top}y \in \mathbb{R}$  instead of  $b \in \mathbb{R}$ ), it follows for every  $a \in \mathbb{R}^m$ ,  $b \in \mathbb{R}$ , and i = 1, ..., d that

$$\int_{\mathbb{R}^m} \rho\left(a^\top u - b\right) h_{\delta,i}(u) du = \sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_{\mathbb{R}^m} \rho^{(|\alpha|)} \left(a^\top u - b\right) a^\alpha \left(\phi_\delta * \eta_{\alpha,i}\right)(u) du$$
$$= \sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_{\mathbb{R}^m} \rho^{(|\alpha|)} \left(a^\top (u + y) - b\right) a^\alpha \eta_{\alpha,i}(du) \phi_\delta(y) du$$
$$= \int_{\mathbb{R}^m} \underbrace{l\left(e_i \rho\left(a^\top \cdot - \left(b - a^\top y\right)\right)\right)}_{=0} \phi_\delta(y) dy = 0.$$

Now, for every i = 1, ..., d, we apply Proposition 7.2 with  $h_{\delta,i}(u)du \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$  to conclude that  $h_{\delta,i}(u)du = 0 \in \mathcal{M}_{\gamma}(\mathbb{R}^m)$ , and thus  $h_{\delta,i}(u) = 0$  for a.e.  $u \in \mathbb{R}^m$ . Hence, it follows for every  $f \in \overline{C_h^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  that

$$l_{\delta}(f) = \sum_{i=1}^{d} \int_{\mathbb{R}^m} f_i(u) h_{\delta,i}(u) du = 0,$$

which shows that  $l_{\delta}: C_b^k(\mathbb{R}^m; \mathbb{R}^d) \to \mathbb{R}$  vanishes everywhere on  $C_b^k(\mathbb{R}^m; \mathbb{R}^d)$ .

Finally, we use Lemma 7.3 (vi) to conclude for every  $f \in \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma}$  that

$$l(f) = \sum_{\alpha \in \mathbb{N}_{0,k}^m} \sum_{i=1}^d \int_{\mathbb{R}^m} \partial_\alpha f_i(u) \eta_{\alpha,i}(du) = \lim_{\delta \to \infty} \sum_{\alpha \in \mathbb{N}_{0,k}^m} \sum_{i=1}^d \int_{\mathbb{R}^m} f_i(u) (\phi_\delta * \eta)(u) du = \lim_{\delta \to \infty} l_\delta(f) = 0,$$

which shows that  $l: \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  vanishes everywhere. This however contradicts the assumption that  $l: \overline{C_b^k(\mathbb{R}^m; \mathbb{R}^d)}^{\gamma} \to \mathbb{R}$  is non-zero. Hence,  $\mathcal{NN}_{U,d}^{\rho}$  is dense in X with respect to  $\|\cdot\|_X$ .

7.3. **Proof of Example 2.8+2.9.** For the proof of Example 2.8 (v), we first generalize the approximation result for unweighted Sobolev spaces in [Adams, 1975, Theorem 3.18] to weighted Sobolev spaces  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$  introduced in Notation (xii).

**Proposition 7.4** (Approximation in Weighted Sobolev Spaces). For  $k \in \mathbb{N}$ ,  $p \in [1, \infty)$ , and  $U \subseteq \mathbb{R}^m$ open and having the segment property, let  $w : U \to [0, \infty)$  be a bounded weight such that for every bounded subset  $B \subseteq U$  it holds that  $\inf_{u \in B} w(u) > 0$ . Then,  $\{f|_U : U \to \mathbb{R}^d : f \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d)\}$  is dense in  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)}$ .

*Proof.* First, we follow the proof of [Adams, 1975, Theorem 3.18] to show that every fixed function  $f \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  can be approximated by elements from  $\{f|_U : U \to \mathbb{R}^d : f \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d)\}$  with respect to  $\|\cdot\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}$ . For this purpose, we choose some  $h \in C_c^{\infty}(\mathbb{R}^m)$  which satisfies h(u) = 1 for all  $u \in \overline{\mathbb{B}}_1(0)$ , h(u) = 0 for all  $u \in \mathbb{R}^m \setminus \mathbb{B}_2(0)$ , and that there exists a constant  $C_h > 0$  such that for every  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in \mathbb{R}^m$  it holds that  $|\partial_{\alpha}h(u)| \leq C_h$ . In addition, we define for every fixed r > 1 the functions  $\mathbb{R}^m \ni u \mapsto h_r(u) := h(u/r) \in \mathbb{R}$  and  $U \ni u \mapsto f_r(u) := f(u)h_r(u) \in \mathbb{R}^d$ , which both have bounded support. Then, by using the Leibniz product rule together with the triangle inequality, that  $|\partial_{\alpha}h_r(u)| = |\partial_{\alpha}h(u/r)| r^{-|\alpha|} \leq C_h$  for any  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in \mathbb{R}^m$ , and the inequality (26), it follows for every  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in U$  that

$$\begin{aligned} \|\partial_{\alpha}f_{r}(u)\|^{p} &\leqslant \left(\sum_{\substack{\beta_{1},\beta_{2}\in\mathbb{N}_{0}^{m}\\\beta_{1}+\beta_{2}=\alpha}} \frac{\alpha!}{\beta_{1}!\beta_{2}!} |\partial_{\beta_{1}}h_{r}(u)| \|\partial_{\beta_{2}}f(u)\|\right)^{p} \\ &\leqslant 2^{kp}C_{h}^{p}\max_{\beta_{2}\in\mathbb{N}_{0,k}^{m}} \|\partial_{\beta_{2}}f(u)\|^{p} \\ &\leqslant 2^{kp}C_{h}^{p}\sum_{\beta_{2}\in\mathbb{N}_{0,k}^{m}} \|\partial_{\beta_{2}}f(u)\|^{p}. \end{aligned}$$

Hence, by using this, it follows for every  $V \in \mathcal{L}(U)$  that

$$\begin{split} \|f_r\|_{W^{k,p}(V,\mathcal{L}(V),w;\mathbb{R}^d)} &= \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_V \|\partial_\alpha f_r(u)\|^p w(u) du\right)^{\frac{1}{p}} \\ &\leqslant \left|\mathbb{N}_{0,k}^m\right|^{\frac{1}{p}} \left(\max_{\alpha \in \mathbb{N}_{0,k}^m} \int_V \|\partial_\alpha f_r(u)\|^p w(u) du\right)^{\frac{1}{p}} \\ &\leqslant 2^k C_h \left|\mathbb{N}_{0,k}^m\right|^{\frac{1}{p}} \left(\sum_{\beta_2 \in \mathbb{N}_{0,k}^m} \int_V \|\partial_{\beta_2} f(u)\|^p w(u) du\right)^{\frac{1}{p}} \\ &\leqslant 2^k C_h \left|\mathbb{N}_{0,k}^m\right|^{\frac{1}{p}} \|f\|_{W^{k,p}(V,\mathcal{L}(V),w;\mathbb{R}^d)} < \infty. \end{split}$$
(41)

Thus, by taking V := U in (41), we conclude that  $f_r \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ . Similarly, by using the triangle inequality, that  $\partial_{\alpha} f_r(u) = \partial_{\alpha} (f(u)h_r(u)) = \partial_{\alpha} f(u)$  for any  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in U \cap \mathbb{B}_r(0)$  (as  $h_r(u) = 1$  for any  $u \in \mathbb{B}_r(0)$ ), and (41) with  $V := U \setminus \overline{\mathbb{B}_r(0)}$ , it follows that

$$\begin{split} \|f - f_r\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} &\leqslant \underbrace{\|f - f_r\|_{W^{k,p}(U\cap\mathbb{B}_r(0),\mathcal{L}(U\cap\mathbb{B}_r(0)),w;\mathbb{R}^d)}_{=0} + \|f - f_r\|_{W^{k,p}(U\setminus\overline{\mathbb{B}_r(0)},\mathcal{L}(U\setminus\overline{\mathbb{B}_r(0)}),w;\mathbb{R}^d)}_{\leqslant} \\ &\leqslant \|f\|_{W^{k,p}(U\setminus\overline{\mathbb{B}_r(0)},\mathcal{L}(U\setminus\overline{\mathbb{B}_r(0)}),w;\mathbb{R}^d)} + \|f_r\|_{W^{k,p}(U\setminus\overline{\mathbb{B}_r(0)},\mathcal{L}(U\setminus\overline{\mathbb{B}_r(0)}),w;\mathbb{R}^d)}_{\leqslant \left(1 + 2^k C_h \left|\mathbb{N}_{0,k}^m\right|^{\frac{1}{p}}\right) \|f\|_{W^{k,p}(U\setminus\overline{\mathbb{B}_r(0)},\mathcal{L}(U\setminus\overline{\mathbb{B}_r(0)}),w;\mathbb{R}^d)}. \end{split}$$

Since the right-hand side tends to zero, as  $r \to \infty$ , this shows that  $f \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  can be approximated by elements of  $\{f \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d) : \operatorname{supp}(f) \subseteq U \text{ is bounded}\}$  with respect to  $\| \cdot \|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}$ . Hence, we only need to show the approximation of the latter by elements from  $\{f|_U : U \to \mathbb{R}^d : f \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d)\}$  with respect to  $\| \cdot \|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}$ . Therefore, we now fix some  $f \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with bounded support  $\operatorname{supp}(f) \subseteq U$  and some  $\varepsilon > 0$ . Moreover, by recalling that  $w : U \to [0, \infty)$  is bounded, we can define the finite constant  $C_w := \sup_{u \in U} w(u) > 0$ . Then, by using that f(u) = 0 for any  $u \in U \setminus \operatorname{supp}(f)$ , thus  $\partial_{\alpha} f(u) = 0$  for any  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in U \setminus \operatorname{supp}(f)$ , and the assumption that  $C_{f,w} := \inf_{u \in \operatorname{supp}(f)} w(u) > 0$ , we have

$$\begin{split} \|f\|_{W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^d)} &= \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \|\partial_\alpha f(u)\|^p du\right)^{\frac{1}{p}} = \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_{\mathrm{supp}(f)} \|\partial_\alpha f(u)\|^p du\right)^{\frac{1}{p}} \\ &\leq C_{f,w}^{-1} \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_{\mathrm{supp}(f)} \|\partial_\alpha f(u)\|^p w(u) du\right)^{\frac{1}{p}} = C_{f,w}^{-1} \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \|\partial_\alpha f(u)\|^p w(u) du\right)^{\frac{1}{p}} \\ &= C_{f,w}^{-1} \|f\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} < \infty. \end{split}$$

This shows that  $f \in W^{k,p}(U, \mathcal{L}(U), du; \mathbb{R}^d)$ . Hence, by applying [Adams, 1975, Theorem 3.18] (with  $U \subseteq \mathbb{R}^m$  having the segment property) componentwise, there exists some  $g \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d)$  such that

$$\|f-g\|_{W^{k,p}(U,\mathcal{L}(U),du,\mathbb{R}^d)} = \left(\sum_{\alpha\in\mathbb{N}_{0,k}^m}\int_U \|\partial_\alpha f(u) - \partial_\alpha g(u)\|^p du\right)^{\frac{1}{p}} < \frac{\varepsilon}{C_w}.$$

Thus, by using that  $w: U \to [0, \infty)$  is bounded with  $C_w := \sup_{u \in U} w(u) < \infty$ , it follows that

$$\begin{split} \|f - g\|_{W^{k,p}(U,\mathcal{L}(U),du,\mathbb{R}^d)} &= \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \|\partial_\alpha f(u) - \partial_\alpha g(u)\|^p w(u) du\right)^{\frac{1}{p}} \\ &\leq C_w \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \|\partial_\alpha f(u) - \partial_\alpha g(u)\|^p du\right)^{\frac{1}{p}} \\ &< C_w \frac{\varepsilon}{C_w} = \varepsilon. \end{split}$$

Since  $f \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with bounded support  $\operatorname{supp}(f) \subseteq U$  and  $\varepsilon > 0$  were chosen arbitrarily, it follows together with the first step that  $\{f|_U : U \to \mathbb{R}^d : f \in C_c^\infty(\mathbb{R}^m; \mathbb{R}^d)\}$  is dense in  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)}$ .  $\Box$ 

Proof of Example 2.8. For (i), we use that  $U \subset \mathbb{R}^m$  is bounded to define the finite constant  $C_{21} := \sup_{u \in U} (1 + ||u||)^{\gamma}$ . Then, it follows for every  $f \in C_b^0(\mathbb{R}^m; \mathbb{R}^d)$  that

$$\begin{split} \|f|_U\|_{C_b^k(U;\mathbb{R}^d)} &= \max_{\alpha \in \mathbb{N}_{0,k}^m} \sup_{u \in U} \|\partial_\alpha f(u)\| \\ &\leqslant \left( \sup_{u \in U} (1+\|u\|)^\gamma \right) \max_{\alpha \in \mathbb{N}_{0,k}^m} \sup_{u \in U} \frac{\|\partial_\alpha f(u)\|}{(1+\|u\|)^\gamma} \\ &\leqslant C_{21} \|f\|_{C_{pol,\gamma}^k(\mathbb{R}^m;\mathbb{R}^d)}. \end{split}$$

Moreover, by using that  $\{f|_U : f \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)\} = C_b^k(U; \mathbb{R}^d)$ , the image  $\{f|_U : f \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)\}$  of the continuous embedding (4) is dense in  $C_b^k(U; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{C_b^k(U; \mathbb{R}^d)}$ .

For (ii), the restriction map in (4) is by definition continuous. Moreover, by using that  $\overline{C_b^k(U;\mathbb{R}^d)}^{\gamma}$  is defined as the closure of  $C_b^k(U;\mathbb{R}^d)$  with respect to  $\|\cdot\|_{C_{pol,\gamma}^k(U;\mathbb{R}^d)}$ , the image  $\{g|_U : g \in C_b^k(\mathbb{R}^m;\mathbb{R}^d)\} = C_b^k(U;\mathbb{R}^d)$  of the continuous embedding (4) is dense in  $\overline{C_b^k(U;\mathbb{R}^d)}^{\gamma}$  with respect to  $\|\cdot\|_{C_{pol,\gamma}^k(U;\mathbb{R}^d)}$ .

For (iii), we first recall that k = 0. Then, we use that  $f \in C_b^0(\mathbb{R}^m; \mathbb{R}^d)$  is continuous to conclude that its restriction  $f|_U : U \to \mathbb{R}^d$  is  $\mathcal{B}(U)/\mathcal{B}(\mathbb{R}^d)$ -measurable. Moreover, we define the finite constant  $C_{22} := \int_U (1 + ||u||)^{\gamma p} \mu(du) > 0$ , which implies that  $\mu : \mathcal{B}(U) \to [0, \infty)$  is finite as  $\mu(U) = \int_U \mu(du) \leq 0$ .

 $C_{22} < \infty$ . Then, it follows for every  $f \in C_b^0(\mathbb{R}^m; \mathbb{R}^d)$  that

$$\begin{split} \|f\|_{U}\|_{L^{p}(U,\mathcal{B}(U),\mu;\mathbb{R}^{d})} &= \left(\int_{U} \|f(u)\|^{p} \mu(du)\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_{U} (1+\|u\|)^{\gamma p} \mu(du)\right)^{\frac{1}{p}} \sup_{u \in U} \frac{\|f(u)\|}{(1+\|u\|)^{\gamma}} \\ &\leqslant C_{22}^{\frac{1}{p}} \|f\|_{C^{0}_{pol,\gamma}(\mathbb{R}^{m};\mathbb{R}^{d})}, \end{split}$$

which shows that the restriction map in (4) is continuous. In order to show that its image is dense, we fix some  $f \in L^p(U, \mathcal{B}(U), \mu; \mathbb{R}^d)$  and  $\varepsilon > 0$ . Then, we extend  $f : U \to \mathbb{R}^d$  to the function

$$\mathbb{R}^m \ni u \quad \mapsto \quad \overline{f}(u) := \begin{cases} f(u), & u \in U, \\ 0, & u \in \mathbb{R}^m \setminus U. \end{cases}$$

Moreover, we extend  $\mu : \mathcal{B}(U) \to [0, \infty)$  to the Borel measure  $\mathcal{B}(\mathbb{R}^m) \ni E \mapsto \overline{\mu}(E) := \mu(U \cap E) \in [0, \infty)$ , which implies that  $\overline{f} \in L^p(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \overline{\mu}; \mathbb{R}^d)$ . Hence, by applying [Bogachev, 2007, Corollary 2.2.2] componentwise (with  $\overline{\mu}(B) = \mu(U \cap B) \leq \mu(U) \leq C_{22} < \infty$  for any bounded  $B \in \mathcal{B}(\mathbb{R}^m)$ ), there exists some  $g \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d) \subseteq C_b^0(\mathbb{R}^m; \mathbb{R}^d)$  with  $\|\overline{f} - g\|_{L^p(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \overline{\mu}; \mathbb{R}^d)} < \varepsilon$ , which implies

$$\|f-g|_U\|_{L^p(U,\mathcal{B}(U),\mu;\mathbb{R}^d)} = \|\overline{f}-g\|_{L^p(\mathbb{R}^m,\mathcal{B}(\mathbb{R}^m),\overline{\mu};\mathbb{R}^d)} < \varepsilon.$$

Since  $f \in C^0(U; \mathbb{R}^d)$  and  $\varepsilon > 0$  were chosen arbitrarily, the image  $\{f|_U : f \in C_b^0(\mathbb{R}^m; \mathbb{R}^d)\}$  of the continuous embedding (4) is dense in  $L^p(U, \mathcal{B}(U), \mu; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{L^p(U, \mathcal{B}(U), \mu; \mathbb{R}^d)}$ .

For (iv), we first use that  $f \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)$  is k-times differentiable to conclude for every  $\alpha \in \mathbb{N}_{0,k}^m$  that  $\partial_\alpha f|_U : U \to \mathbb{R}^d$  is  $\mathcal{L}(U)/\mathcal{B}(\mathbb{R}^d)$ -measurable. Moreover, we use that  $U \subset \mathbb{R}^m$  is bounded to define the finite constant  $C_{23} := \int_U (1 + ||u||)^{\gamma p} du > 0$ . Then, it follows for every  $f \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)$  that

$$\begin{split} \|f\|_{W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^{d})} &= \left(\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \int_{U} \|\partial_{\alpha}f(u)\|^{p} du\right)^{\frac{1}{p}} \\ &\leq \left(\left|\mathbb{N}_{0,k}^{m}\right| \int_{U} (1+\|u\|)^{\gamma p} du\right)^{\frac{1}{p}} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U} \frac{\|\partial_{\alpha}f(u)\|}{(1+\|u\|)^{\gamma p}} \\ &\leq \left(C_{23} \left|\mathbb{N}_{0,k}^{m}\right|\right)^{\frac{1}{p}} \|f\|_{C^{k}_{pol,\gamma}(\mathbb{R}^{m};\mathbb{R}^{d})}, \end{split}$$

which shows that the restriction map in (4) is continuous. In addition, by applying [Adams, 1975, Theorem 3.18] componentwise,  $\{g|_U : g \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d)\}$  is dense in  $W^{k,p}(U, \mathcal{L}(U), du; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^d)}$ . Since  $C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d) \subseteq C_b^k(\mathbb{R}^m; \mathbb{R}^d)$ , the image  $\{g|_U : g \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)\}$  of the continuous embedding (4) is dense in  $W^{k,p}(U, \mathcal{L}(U), du; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{W^{k,p}(U,\mathcal{L}(U),du;\mathbb{R}^d)}$ .

For (v), we use that  $f \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)$  is k-times differentiable to conclude for every  $\alpha \in \mathbb{N}_{0,k}^m$ that  $\partial_\alpha f|_U : U \to \mathbb{R}^d$  is  $\mathcal{L}(U)/\mathcal{B}(\mathbb{R}^d)$ -measurable. Moreover, by using the finite constant  $C_{24} := \int_U (1 + ||u||)^{\gamma p} w(u) du > 0$ , it follows for every  $f \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)$  that

$$\begin{split} \|f\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})} &= \left(\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \int_{U} \|\partial_{\alpha}f(u)\|^{p} w(u) du\right)^{\frac{1}{p}} \\ &\leq \left(\left|\mathbb{N}_{0,k}^{m}\right| \int_{U} (1+\|u\|)^{\gamma p} w(u) du\right)^{\frac{1}{p}} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in U} \frac{\|\partial_{\alpha}f(u)\|}{(1+\|u\|)^{\gamma}} \\ &\leq \left(C_{24} \left|\mathbb{N}_{0,k}^{m}\right|\right)^{\frac{1}{p}} \|f\|_{C^{k}_{pol,\gamma}(\mathbb{R}^{m};\mathbb{R}^{d})}. \end{split}$$

which shows that the restriction map in (4) is continuous. In addition, we apply Proposition 7.4 to conclude that  $\{g|_U : g \in C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d)\}$  is dense in  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)}$ . Since  $C_c^{\infty}(\mathbb{R}^m; \mathbb{R}^d) \subseteq C_b^k(\mathbb{R}^m; \mathbb{R}^d)$ , it follows that the image  $\{g|_U : g \in C_b^k(\mathbb{R}^m; \mathbb{R}^d)\}$  of the continuous embedding (4) is dense in  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with respect to  $\|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)}$ . Proof of Example 2.9. First, since  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is of polynomial growth, in each case (i)-(iv), it induces the tempered distribution  $(g \mapsto T_{\rho}(g) := \int_{\mathbb{R}} \rho(s)g(s)ds) \in \mathcal{S}'(\mathbb{R};\mathbb{C})$  (see [Folland, 1992, p. 332]). For (ii), we recall that  $\tanh'(\xi) = \cosh(\xi)^{-2}$  holds true for all  $\xi \in \mathbb{R}$ . Moreover, the Fourier transform

For (ii), we recall that  $\tanh'(\xi) = \cosh(\xi)^{-2}$  holds true for all  $\xi \in \mathbb{R}$ . Moreover, the Fourier transform of the function  $\left(s \mapsto h(s) := \frac{\pi s}{\sinh(\pi s/2)}\right) \in L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), du)$  is for every  $\xi \in \mathbb{R}$  given by

$$\hat{h}(\xi) = \frac{2\pi}{\cosh(\xi)^2} = 2\pi \tanh'(\xi).$$
(42)

Then, by using  $\left(g \mapsto \left(p_1 \cdot \widehat{T_{\text{tanh}}}\right)(g) := \widehat{T_{\text{tanh}}}(p_1 \cdot g)\right) \in \mathcal{S}'(\mathbb{R};\mathbb{C})$ , [Folland, 1992, Equation 9.31] with  $\mathbb{R} \ni s \mapsto p_1(s) := s \in \mathbb{R}$ , the definition of  $\widehat{T_{\text{tanh}}} \in \mathcal{S}'(\mathbb{R};\mathbb{C})$ , the identity (42), and the Plancherel theorem in [Folland, 1992, p. 222], it follows for every  $g \in C_c^{\infty}(\mathbb{R} \setminus \{0\};\mathbb{C})$  that

$$\widehat{T_{\text{tanh}}}(p_1 \cdot g) = \left(p_1 \cdot \widehat{T_{\text{tanh}}}\right)(g) = \frac{1}{i} \widehat{T_{\text{tanh}'}}(g) = (-i) T_{\text{tanh}'}(\widehat{g})$$
$$= (-i) \int_{\mathbb{R}} \tanh'(\xi) \widehat{g}(\xi) d\xi = \frac{-i}{2\pi} \int_{\mathbb{R}} \overline{\widehat{h}(\xi)} \widehat{g}(\xi) d\xi$$
$$= (-i) \int_{\mathbb{R}} \overline{h(\xi)} g(\xi) d\xi = \int_{\mathbb{R}} \frac{-i\pi}{\sinh(\pi\xi/2)} (p_1 \cdot g)(\xi) d\xi.$$
(43)

Hence,  $\widehat{T_{\text{tanh}}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  coincides on  $\mathbb{R} \setminus \{0\}$  with  $\left(\xi \mapsto \widehat{f_{T_{\text{tanh}}}}(\xi) := \frac{i\pi}{\sinh(\pi\xi/2)}\right) \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C}).$ 

For (i), we denote by  $(s \mapsto \sigma(s) := \frac{1}{1 + \exp(-s)}) \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  the sigmoid function and observe that  $\sigma(s) = \frac{1}{2} (\tanh(\frac{s}{2}) + 1)$  for all  $s \in \mathbb{R}$ . Then, by using the linearity of the Fourier transform on  $\mathcal{S}'(\mathbb{R}; \mathbb{C})$ , [Folland, 1992, Equation 9.30], that  $\widehat{T_1}(g) = 2\pi\delta(g) := 2\pi g(0)$  for any  $g \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  (see [Folland, 1992, Equation 9.35]), the identity (43), and the substitution  $\xi \mapsto \widetilde{\xi}/2$ , it follows for every  $g \in C_c^{\infty}(\mathbb{R} \setminus \{0\}; \mathbb{C})$  that

$$\widehat{T_{\sigma}}(g) = \frac{1}{2} \widehat{T_{\tanh(\frac{i}{2})}}(g) + \frac{1}{2} \widehat{T_{1}}(g) = \frac{1}{2} \widehat{T_{\tanh}}\left(g\left(\frac{i}{2}\right)\right) + \frac{2\pi}{2} g(0)$$
$$= \frac{1}{2} \int_{\mathbb{R}} \frac{-i\pi}{\sinh\left(\pi\tilde{\xi}/2\right)} g\left(\tilde{\xi}/2\right) d\tilde{\xi} = \int_{\mathbb{R}} \frac{-i\pi}{\sinh(\pi\xi)} g(\xi) d\xi.$$
(44)

Hence,  $\widehat{T_{\sigma}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  coincides on  $\mathbb{R} \setminus \{0\}$  with  $\left(\xi \mapsto f_{\widehat{T_{\sigma}}}(\xi) := \frac{-i\pi}{\sinh(\pi\xi)}\right) \in L^{1}_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C}).$ 

For (iii), we denote by  $(s \mapsto \sigma^{(-1)}(s) := \ln(1 + e^s)) \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  the softplus function and observe that  $\frac{d}{ds}\sigma^{(-1)}(s) = \sigma(s)$  for all  $s \in \mathbb{R}$ . Then, by using [Folland, 1992, Equation 9.31] with  $\mathbb{R} \ni s \mapsto p_1(s) := s \in \mathbb{R}$  and the identity (44), it follows for every  $g \in C_c^{\infty}(\mathbb{R} \setminus \{0\}; \mathbb{C})$  that

$$\widehat{T_{\sigma^{(-1)}}(p_1 \cdot g)} = \left(p_1 \cdot \widehat{T_{\sigma^{(-1)}}}\right)(g) = \frac{1}{i}\widehat{T_{\sigma}}(g) = \frac{1}{i}\int_{\mathbb{R}} \frac{-i\pi}{\sinh(\pi\xi)}g(\xi)d\xi = \int_{\mathbb{R}} \frac{-\pi}{\xi\sinh(\pi\xi)}(p_1 \cdot g)(\xi)d\xi.$$
Hence  $\widehat{T_{\sigma^{(-1)}}} \in S'(\mathbb{R};\mathbb{C})$  coincides on  $\mathbb{R}\setminus\{0\}$  with  $\left(\xi \mapsto f_{\sigma^{(-1)}}(\xi) := -\frac{\pi}{2}\right) \in I^1$  ( $\mathbb{R}\setminus\{0\}:\mathbb{C}$ ).

Hence,  $\widehat{T_{\sigma^{(-1)}}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  coincides on  $\mathbb{R}\setminus\{0\}$  with  $\left(\xi \mapsto f_{\widehat{T_{\sigma^{(-1)}}}}(\xi) := \frac{-\pi}{\xi \sinh(\pi\xi)}\right) \in L^1_{loc}(\mathbb{R}\setminus\{0\}; \mathbb{C}).$ 

For (iv), we denote by  $(s \mapsto \operatorname{ReLU}(s) := \max(s, 0)) \in \overline{C_b^0(\mathbb{R})}^{\gamma}$  the ReLU function and observe that  $\operatorname{ReLU}(s) = \max(s, 0) = \frac{s+|s|}{2}$  for all  $s \in \mathbb{R}$ . Moreover, the absolute value  $\mathbb{R} \ni s \mapsto |s| \in \mathbb{R}$  is weakly differentiable with  $\frac{d}{ds}|s| = \operatorname{sgn}(s)$  for all  $s \in \mathbb{R}$ , where  $\operatorname{sgn}(s) := 1$  if s > 0,  $\operatorname{sgn}(0) := 0$ , and  $\operatorname{sgn}(s) := -1$  if s < 0. Then, by using the linearity of the Fourier transform on  $\mathcal{S}'(\mathbb{R}; \mathbb{C})$ , that  $\widehat{T_{p_1}}(g) = 2\pi i \delta'(g) := 2\pi i g'(0)$  for any  $g \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  with  $\mathbb{R} \ni s \mapsto p_1(s) := s \in \mathbb{R}$  (see [Folland, 1992, Equation 9.35]), [Folland, 1992, Equation 9.31], and [Folland, 1992, Example 9.4.4], i.e. that  $\widehat{T_{\operatorname{sgn}}}(g) = -2i \int_{\mathbb{R}} \frac{g(\xi)}{\xi} d\xi$  for any  $g \in C_c^{\infty}(\mathbb{R} \setminus \{0\}; \mathbb{C})$ , it follows for every  $g \in C_c^{\infty}(\mathbb{R} \setminus \{0\}; \mathbb{C})$  that

$$\widehat{T_{\text{ReLU}}}(p_1 \cdot g) = \frac{1}{2} \widehat{T_{p_1}}(p_1 \cdot g) + \frac{1}{2} \widehat{T_{|\cdot|}}(p_1 \cdot g) = \frac{2\pi i}{2} (p_1 \cdot g)'(0) + \frac{1}{2} \left( p_1 \cdot \widehat{T_{|\cdot|}} \right) (g)$$
$$= \frac{1}{2i} \widehat{T_{\text{sgn}}}(g) = \frac{-2i}{2i} \int_{\mathbb{R}} \frac{g(\xi)}{\xi} d\xi = \int_{\mathbb{R}} \frac{-1}{\xi^2} (p_1 \cdot g)(\xi) d\xi.$$

Hence,  $\widehat{T_{\text{ReLU}}} \in \mathcal{S}'(\mathbb{R}; \mathbb{C})$  coincides on  $\mathbb{R} \setminus \{0\}$  with  $\left(\xi \mapsto f_{\widehat{T_{\text{ReLU}}}}(\xi) := -\frac{1}{\xi^2}\right) \in L^1_{loc}(\mathbb{R} \setminus \{0\}; \mathbb{C}).$ 

### 8. PROOFS OF RESULTS IN SECTION 3.

In this section, we provide the proofs of the results in Section 3. First, we give a short introduction into the notion of Bochner spaces in Section 8.1. Subsequently, we show in Section 8.2 that every random neural network is a strongly measurable map. Finally, in Section 8.3, we prove the main result of Section 3, i.e. the universal approximation property of random neural networks formulated in Theorem 3.5.

8.1. **Introduction to Bochner Spaces.** In this section, we give a short introduction into the notion of Bochner spaces over a probability space, which allows us to consider random functions as Banach space-valued random variables. To this end, we follow the textbook [Hytönen et al., 2016]. Readers who are familiar with this topic may skip this section.

Throughout this paper, we fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and assume that  $(X, \|\cdot\|_X)$  is a separable Banach space. Moreover, for a sub- $\sigma$ -algebra  $\mathcal{F}_0 \subseteq \mathcal{F}$ , we define the set of  $\mathcal{F}_0$ -simple functions as

$$\mathcal{I}_{\mathcal{F}_0} \otimes X := \left\{ \Omega \ni \omega \mapsto \sum_{i=1}^{I} \mathbb{1}_{E_i}(\omega) f_i \in X : I \in \mathbb{N}, \ E_i \in \mathcal{F}_0, \ f_i \in X \right\}.$$

Using this, a map  $F : \Omega \to X$  is called  $\mathcal{F}_0$ -strongly measurable if there exists a sequence of simple functions  $(S_M)_{M \in \mathbb{N}} \subseteq \mathcal{I}_{\mathcal{F}_0} \otimes X$  converging  $\mathbb{P}$ -a.s. to  $F : \Omega \to X$ , i.e. there exists some  $A \in \mathcal{F}_0$  with  $\mathbb{P}[A] = 1$  such that for every  $\omega \in A$  it holds that

$$\lim_{M \to \infty} \|F(\omega) - S_M(\omega)\|_X = 0.$$

Now, for any  $\mathcal{F}_0$ -simple function  $S = \sum_{i=1}^{I} \mathbb{1}_{E_i} f_i \in \mathcal{I}_{\mathcal{F}_0} \otimes X$ , we define the *Bochner integral* of S as

$$\mathbb{E}[S] := \int_{\Omega} S(\omega) \mathbb{P}[d\omega] := \sum_{i=1}^{I} \mathbb{P}[E_i] f_i.$$

Then, one can introduce the Bochner space  $L^r(\Omega, \mathcal{F}_0, \mathbb{P}; X)$ , for  $r \in [1, \infty)$ .

**Definition 8.1.** For  $r \in [1, \infty)$  and  $\mathcal{F}_0 \subseteq \mathcal{F}$ , the Bochner space  $L^r(\Omega, \mathcal{F}_0, \mathbb{P}; X)$  is defined as the vector space of all (equivalence classes of)  $\mathcal{F}_0$ -strongly measurable maps  $F : \Omega \to X$  such that

$$\mathbb{E}\left[\|F\|_X^r\right] := \int_{\Omega} \|F(\omega)\|_X^r \mathbb{P}\left[d\omega\right] < \infty.$$

Moreover, we equip  $L^r(\Omega, \mathcal{F}_0, \mathbb{P}; X)$  with the  $L^r$ -norm given by  $\|F\|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)} := \mathbb{E}\left[\|F\|_X^r\right]^{\frac{1}{r}}$ .

One can show that the expectation (i.e. the Bochner integral) of every  $F \in L^r(\Omega, \mathcal{F}_0, \mathbb{P}; X)$  exists as a limit of  $\mathcal{F}_0$ -simple functions. Moreover, for every  $r \in [1, \infty)$ , it follows analogously to the real-valued case that  $(L^r(\Omega, \mathcal{F}_0, \mathbb{P}; X), \| \cdot \|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)})$  is a Banach space. In addition, the usual properties of  $L^p$ -spaces are satisfied, e.g., Jensen's inequality, Minkowski's inequality, and Fubini's theorem (see [Hytönen et al., 2016, Section 1.2]). Furthermore, we set  $L^r(\Omega, \mathcal{F}, \mathbb{P}) := L^r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ .

8.2. Preliminary Results: Strong Measurability of Random Neural Networks. In this section, we show that every random neural network  $\Phi \in \mathcal{RN}_{U,d}^{\rho}$  is  $\mathcal{F}_{A,B}$ -strongly measurable with values in an  $(k, U, \gamma)$ -approximable function space  $(X, \| \cdot \|_X)$ , where  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ . For this purpose, we first show that the convergence of weight vectors, bias, and linear readouts implies the convergence of the corresponding neurons as functions in  $(X, \| \cdot \|_X)$ .

**Lemma 8.2.** For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ approximable function space and let  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ . Then, for every sequence  $(y_M, a_M, b_M)_{M \in \mathbb{N}} \subseteq \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  converging to  $(y, a, b) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$ , we have that

$$\lim_{M \to \infty} \left\| y\rho \left( a^{\top} \cdot -b \right) - y_M \rho \left( a_M^{\top} \cdot -b_M \right) \right\|_X = 0,$$
(45)

where  $y\rho(a^{\top}\cdot -b)$  denotes the function  $U \ni u \mapsto y\rho(a^{\top}u - b) \in \mathbb{R}^d$ .

*Proof.* Let  $(y_M, a_M, b_M)_{M \in \mathbb{N}} \subseteq \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  be a sequence converging to  $(y, a, b) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$ and fix some  $\varepsilon > 0$ . Then, by using that  $y_M a_M^{\alpha}$  converges uniformly in  $\alpha \in \mathbb{N}_{0,k}^m$  to  $ya^{\alpha}$ , the constant  $C_{y,a} := 1 + \max_{\alpha \in \mathbb{N}_{0,k}^m} \|ya^{\alpha}\| + \sup_{M \in \mathbb{N}} \max_{\alpha \in \mathbb{N}_{0,k}^m} \|y_M a_M^{\alpha}\| > 0$  is finite, where  $a^{\alpha} := \prod_{l=1}^m a_l^{\alpha_l}$  for  $a := (a_1, ..., a_m)^\top \in \mathbb{R}^m$  and  $(\alpha_1, ..., \alpha_m) \in \mathbb{N}_{0,k}^m$ . Moreover, by using that  $(a_M, b_M)_{M \in \mathbb{N}}^\top \subseteq \mathbb{R}^m \times \mathbb{R}$  converges to  $(a, b) \in \mathbb{R}^m \times \mathbb{R}$ , the constant  $C_{a,b} := 1 + ||(a, b)|| + \sup_{M \in \mathbb{N}} ||(a_M, b_M)|| > 0$  is finite. In addition, by using that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ , there exists by definition of  $\overline{C_b^k(\mathbb{R})}^{\gamma}$  some  $\tilde{\rho} \in C_b^k(\mathbb{R})$  with

$$\|\rho - \widetilde{\rho}\|_{C^k_{pol,\gamma}(\mathbb{R})} := \max_{\alpha \in \mathbb{N}^m_{0,k}} \sup_{s \in \mathbb{R}} \frac{\left|\rho^{(|\alpha|)}(s) - \widetilde{\rho}^{(|\alpha|)}(s)\right|}{(1+|s|)^{\gamma}} < \frac{\varepsilon}{6C_{y,a}C_{a,b}}.$$
(46)

Now, we choose r > 0 large enough such that  $(1 + r)^{\gamma} \ge 6\varepsilon^{-1}C_{y,a}\|\widetilde{\rho}\|_{C_b^k(\mathbb{R})}$ . Then, the inequality  $1 + |a_M^\top u - b_M| \le 1 + ||a_M|| ||u|| + |b_M| \le (1 + ||a_M|| + |b_M|)(1 + ||u||)$  for any  $u \in \mathbb{R}^m$  and (46) imply

$$\max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \mathbb{R}^{m} \setminus \overline{\mathbb{B}_{r}(0)}} \frac{\|y_{M}\rho^{(|\alpha|)}(a_{M}^{\top}u - b_{M})a_{M}^{\alpha}\|}{(1 + \|u\|)^{\gamma}} \leqslant C_{y,a} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \mathbb{R}^{m} \setminus \overline{\mathbb{B}_{r}(0)}} \frac{|\rho^{(|\alpha|)}(a_{M}^{\top}u - b_{M})|}{(1 + \|u\|)^{\gamma}} \\
\leqslant C_{y,a} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \mathbb{R}^{m} \setminus \overline{\mathbb{B}_{r}(0)}} \frac{|\rho^{(|\alpha|)}(a_{M}^{\top}u - b_{M}) - \tilde{\rho}^{(|\alpha|)}(a_{M}^{\top}u - b_{M})|}{(1 + \|u\|)^{\gamma}} \\
+ C_{y,a} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \mathbb{R}^{m} \setminus \overline{\mathbb{B}_{r}(0)}} \frac{|\tilde{\rho}^{(|\alpha|)}(a_{M}^{\top}u - b_{M})|}{(1 + \|u\|)^{\gamma}} \\
\leqslant C_{y,a}(1 + \|a_{M}\| + \|b_{M}\|)^{\gamma} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \mathbb{R}^{m}} \frac{|\rho^{(|\alpha|)}(a_{M}^{\top}u - b_{M}) - \tilde{\rho}^{(|\alpha|)}(a_{M}^{\top}u - b_{M})|}{(1 + |a_{M}^{\top}u - b_{M}|)^{\gamma}} \\
+ C_{y,a}\frac{\|\tilde{\rho}\|_{C_{b}^{k}(\mathbb{R})}}{(1 + r)^{\gamma}} \\
\leqslant C_{y,a}C_{a,b} \max_{j=0,\dots,k} \sup_{s \in \mathbb{R}} \frac{|\rho^{(j)}(s) - \tilde{\rho}^{(j)}(s)|}{(1 + |s|)^{\gamma}} + C_{y,a}\frac{\varepsilon}{6C_{y,a}C_{a,b}}} \\
< C_{y,a}C_{a,b}\frac{\varepsilon}{6C_{y,a}C_{a,b}} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$
(47)

Analogously, we conclude that

$$\max_{\alpha \in \mathbb{N}_{0,k}^m} \sup_{u \in \mathbb{R}^m \setminus \overline{\mathbb{B}_r(0)}} \frac{\left\| y \rho^{(|\alpha|)} \left( a^\top u - b \right) a^\alpha \right\|}{(1 + \|u\|)^\gamma} < \frac{\varepsilon}{3}.$$
(48)

Moreover, we define the compact subset  $K := \left\{ x^{\top}u - y : u \in \overline{\mathbb{B}_r(0)}, \|x\| + \|y\| \leq C_{a,b} \right\} \subseteq \mathbb{R}$ . Then, by using that  $\rho, \rho', \dots, \rho^{(k)} \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  are by Lemma 6.1 continuous, thus uniformly continuous on K, there exists some  $\delta > 0$  such that for every  $j = 0, \dots, k$  and  $s_1, s_2 \in K$  with  $|s_1 - s_2| < \delta$  it holds that

$$\left|\rho^{(j)}(s_1) - \rho^{(j)}(s_2)\right| < \frac{\varepsilon}{6C_{y,a}}.$$
(49)

Now, we define the constant  $C_{r,\rho} := 1 + \max_{j=0,\dots,k} \sup_{u \in \overline{\mathbb{B}_r(0)}} |\rho^{(j)} (a^\top u - b)| > 0$ . Moreover, we choose some  $M_0 \in \mathbb{N}$  such that for every  $M \in \mathbb{N} \cap [M_0, \infty)$  it holds that  $||(a - a_M, b - b_M)|| < \delta/(1 + r)$  and that

$$\max_{\alpha \in \mathbb{N}_{0,k}^m} \|ya^{\alpha} - y_M a_M^{\alpha}\| < \frac{\varepsilon}{6C_{r,\rho}}.$$
(50)

Then, it follows for every  $M \ge M_0$  that

$$|(a^{\top}u - b) - (a_{M}^{\top}u - b_{M})| \leq |(a - a_{M})^{\top}u - (b - b_{M})|$$
  
$$\leq ||a - a_{M}|| ||u|| + |b - b_{M}|$$
  
$$\leq (||a - a_{M}|| + |b - b_{M}|) (1 + r)$$
  
$$\leq ||(a - a_{M}, b - b_{M})|| (1 + r) < \delta.$$
(51)

Hence, by using (50) and by combining (49) with (51), it follows for every  $M \in \mathbb{N} \cap [M_0, \infty)$  that

$$\max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \overline{\mathbb{B}_{r}(0)}} \left\| y \rho^{(|\alpha|)} \left( a^{\top} u - b \right) a^{\alpha} - y_{M} \rho^{(|\alpha|)} \left( a^{\top}_{M} u - b_{M} \right) a^{\alpha}_{M} \right\| \\
\leq \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \overline{\mathbb{B}_{r}(0)}} \left\| y \rho^{(|\alpha|)} \left( a^{\top} u - b \right) a^{\alpha} - y_{M} \rho^{(|\alpha|)} \left( a^{\top} u - b \right) a^{\alpha}_{M} \right\| \\
+ \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \sup_{u \in \overline{\mathbb{B}_{r}(0)}} \left\| y_{M} \rho^{(|\alpha|)} \left( a^{\top} u - b \right) a^{\alpha}_{M} - y_{M} \rho^{(|\alpha|)} \left( a^{\top}_{M} u - b_{M} \right) a^{\alpha}_{M} \right\| \\
\leq \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \left\| y a^{\alpha} - y_{M} a^{\alpha}_{M} \right\| \max_{j=0,\dots,k} \sup_{u \in \overline{\mathbb{B}_{r}(0)}} \left| \rho^{(j)} \left( a^{\top} u - b \right) \right| \\
+ \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \left\| y_{M} a^{\alpha}_{M} \right\| \max_{j=0,\dots,k} \sup_{u \in \overline{\mathbb{B}_{r}(0)}} \left| \rho^{(j)} \left( a^{\top}_{M} u - b_{M} \right) - \rho^{(j)} \left( a^{\top} u - b \right) \right| \\
\leq \frac{\varepsilon}{6C_{r,\rho}} C_{r,\rho} + C_{y,a} \frac{\varepsilon}{6C_{y,a}} = \frac{\varepsilon}{3}.$$
(52)

Thus, by combining (47) and (48) with (52), we conclude that

$$\begin{split} \|y\rho\left(a^{\top}\cdot-b\right)-y_{M}\rho\left(a_{M}^{\top}\cdot-b_{M}\right)\|_{C_{pol,\gamma}^{k}(\mathbb{R}^{m};\mathbb{R}^{d})} \\ &= \max_{\alpha\in\mathbb{N}_{0,k}^{m}}\sup_{u\in\mathbb{R}^{m}}\frac{\|y\rho^{(|\alpha|)}\left(a^{\top}u-b\right)a^{\alpha}-y_{M}\rho^{(|\alpha|)}\left(a_{M}^{\top}u-b_{M}\right)a_{M}^{\alpha}\|}{(1+\|u\|)^{\gamma}} \\ &\leqslant \max_{\alpha\in\mathbb{N}_{0,k}^{m}}\sup_{u\in\mathbb{R}_{r}(0)}\left\|y\rho^{(|\alpha|)}\left(a^{\top}u-b\right)a^{\alpha}-y_{M}\rho^{(|\alpha|)}\left(a_{M}^{\top}u-b_{M}\right)a_{M}^{\alpha}\right\|} \\ &+ \max_{\alpha\in\mathbb{N}_{0,k}^{m}}\sup_{u\in\mathbb{R}^{m}\setminus\overline{\mathbb{B}_{r}(0)}}\frac{\|y\rho^{(|\alpha|)}\left(a^{\top}u-b\right)a^{\alpha}\|}{(1+\|u\|)^{\gamma}} \\ &+ \max_{\alpha\in\mathbb{N}_{0,k}^{m}}\sup_{u\in\mathbb{R}^{m}\setminus\overline{\mathbb{B}_{r}(0)}}\frac{\|y_{M}\rho^{(|\alpha|)}\left(a_{M}^{\top}u-b_{M}\right)a_{M}^{\alpha}\|}{(1+\|u\|)^{\gamma}} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{split}$$

Since  $\varepsilon > 0$  was chosen arbitrarily, we obtain (45) with respect to  $\|\cdot\|_{C^k_{pol,\gamma}(\mathbb{R}^m;\mathbb{R}^d)}$  instead of  $\|\cdot\|_X$ . Finally, by using that  $(X, \|\cdot\|_X)$  is  $(k, U, \gamma)$ -approximable, i.e. that the restriction map  $(\overline{C^k_b(\mathbb{R}^m;\mathbb{R}^d)}^{\gamma}, \|\cdot\|_{C^k_{pol,\gamma}(\mathbb{R}^m;\mathbb{R}^d)}) \ni f \mapsto f|_U \in (X, \|\cdot\|_X)$  is by Remark 2.4 continuous, the convergence in (45) follows also with respect to  $\|\cdot\|_X$ .

Now, we can use Lemma 8.2 to show that every random neural networks  $\Phi \in \mathcal{RN}_{U,d}^{\rho}$  is well-defined as  $\mathcal{F}_{A,B}$ -strongly measurable map with values in an  $(k, U, \gamma)$ -approximable function space  $(X, \|\cdot\|_X)$ .

**Proposition 8.3.** For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , and  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ , let  $(X, \|\cdot\|_X)$  be an  $(k, U, \gamma)$ -approximable function space. Then, every random neural network  $\Phi \in \mathcal{RN}_{U,d}^{\rho}$  is well-defined as  $\mathcal{F}_{A,B}$ -strongly measurable map  $\Phi : \Omega \to X$  with values in the separable Banach space  $(X, \|\cdot\|_X)$ .

*Proof.* First, we show that every  $\Phi \in \mathcal{RN}_{U,d}^{\rho}$  takes values in the separable Banach space  $(X, \|\cdot\|_X)$ . Indeed, since  $(X, \|\cdot\|_X)$  is an  $(k, U, \gamma)$ -approximable function space, Lemma 2.5 (ii) implies that  $\Phi(\omega) \in \mathcal{NN}_{U,d}^{\rho} \subseteq X$  for all  $\omega \in \Omega$ . Moreover, Lemma 2.5 (i) shows that  $(X, \|\cdot\|_X)$  is separable.

Now, by using that  $\mathcal{RN}_{U,d}^{\rho}$  is defined as vector space of maps of the form  $\Omega \ni \omega \mapsto R_n(\omega) := W_n(\omega)\rho\left(A_n(\omega)^\top \cdot -B_n(\omega)\right) \in X$ , with  $n \in \mathbb{N}$  and  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable  $W_n : \Omega \to \mathbb{R}^d$ , it suffices to show that  $R_n : \Omega \to X$  is  $\mathcal{F}_{A,B}$ -strongly measurable. To this end, we use that  $W_n : \Omega \to \mathbb{R}^d$  is  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable and that  $\mathbb{R}^d$  is finite dimensional to conclude that there exists a sequence  $(W_{M,n})_{M\in\mathbb{N}}$  of  $\mathcal{F}_{A,B}$ -simple functions  $W_{M,n} : \Omega \to \mathbb{R}^d$  converging pointwise to  $W_n : \Omega \to \mathbb{R}^d$  in  $\mathbb{R}^d$ . Then, for every  $M \in \mathbb{N}$ , we define the map

$$\Omega \ni \omega \mapsto R_{M,n}(\omega) := W_{M,n}(\omega) \sum_{(\alpha,\beta) \in ([-M^2,M^2] \cap \mathbb{Z})^{m+1}} \mathbb{1}_{E_{M,n,(\alpha,\beta)}}(\omega) \rho\left(\frac{\alpha' \cdot}{M} - \frac{\beta}{M}\right) \in X,$$

where

$$E_{M,n,(\alpha,\beta)} := \left\{ \omega \in \Omega : A_n(\omega) \in \bigotimes_{l=1}^m \left[ \frac{\alpha_l}{M}, \frac{\alpha_l+1}{M} \right] \text{ and } B_n(\omega) \in \left[ \frac{\beta}{M}, \frac{\beta+1}{M} \right] \right\} \in \mathcal{F}_{A,B}$$

and where  $(\alpha, \beta) := (\alpha_1, ..., \alpha_m, \beta)^\top \in \mathbb{Z}^{m+1}$ . Since  $W_{M,n} : \Omega \to \mathbb{R}^d$  is an  $\mathcal{F}_{A,B}$ -simple function, the product of  $W_{M,n}(A_n(\omega), B_n(\omega))$  with the indicators  $\mathbb{1}_{E_{M,n,(\alpha,\beta)}}$ , for  $(\alpha, \beta) \in ([-M^2, M^2] \cap \mathbb{Z})^{m+1}$ , is again an indicator function, which implies that  $(R_{M,n})_{M \in \mathbb{N}} \subseteq \mathcal{I}_{\mathcal{F}_{A,B}} \otimes \mathcal{NN}_{U,d}^{\rho}$ .

Next, for any  $a := (a_1, ..., a_m)^\top \in \mathbb{R}^m$ , we use the notation  $[a] := ([a_1], ..., [a_m])^\top \in \mathbb{Z}^m$ . Then, for every  $\omega \in \Omega$ ,  $u \in U$ , and  $M \in \mathbb{N} \cap [M_{0,\omega}, \infty)$  with  $M_{0,\omega} := \max(||A_n(\omega)||, |B_n(\omega)|)$ , it follows that

$$R_{M,n}(\omega)(u) = W_{M,n}(\omega)\rho\left(\frac{\lfloor MA_n(\omega)\rfloor^{\top}u}{M} - \frac{\lfloor MB_n(\omega)\rfloor}{M}\right).$$
(53)

This provides us with an expilict expression for  $R_{M,n}(\omega) \in \mathcal{NN}_{U,d}^{\rho}$  once  $\omega \in \Omega$  is fixed.

Finally, we show that  $(R_{M,n})_{M \in \mathbb{N}} : \Omega \to X$  converges pointwise to  $R_n : \Omega \to X$  with respect to  $\|\cdot\|_X$ . For every fixed  $\omega \in \Omega$ , we use Lemma 8.2 with  $\left(W_{M,n}(A_n(\omega), B_n(\omega)), \frac{|MA_n(\omega)|}{M}, \frac{|MB_n(\omega)|}{M}\right)_{M \in \mathbb{N}} \subseteq \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  converging to  $(W_n(A_n(\omega), B_n(\omega)), A_n(\omega), B_n(\omega)) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  to conclude that  $\lim_{M \to \infty} \|R_{M,n}(\omega) - R_n(\omega)\|_X$ 

$$= \lim_{M \to \infty} \left\| W_{M,n}(\omega) \rho \left( \frac{\left[ M A_n(\omega) \right]^\top \cdot}{M} - \frac{\left[ M B_n(\omega) \right]}{M} \right) - W_n(\omega) \rho \left( A_n(\omega)^\top \cdot - B_n(\omega) \right) \right\|_X = 0.$$

This shows that the map  $R_n : \Omega \to X$  is strongly measurable as pointwise limit of the  $\mathcal{F}_{A,B}$ -simple functions  $(R_{M,n})_{M \in \mathbb{N}} : \Omega \to X$ .

8.3. **Proof of Theorem 3.5.** In this section, we provide the proof of the main result of Section 3, i.e. the universal approximation property of random neural networks. For this purpose, we assume that  $(X, \|\cdot\|_X)$  is an  $(k, U, \gamma)$ -approximable function space, where  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $\gamma \in (0, \infty)$ .

Let us first briefly sketch the main idea of the proof. Fix some  $r \in [0, \infty)$  and  $F \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$ . Then, we first apply [Hytönen et al., 2016, Lemma 1.2.19 (i)] to approximate  $F \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  by an  $\mathcal{F}_{A,B}$ -simple function of the form  $\sum_{i=1}^{I} \mathbb{1}_{E_i} f_i$ , with  $I \in \mathbb{N}, E_1, ..., E_I \in \mathcal{F}_{A,B}$ , and  $f_1, ..., f_I \in X$ , i.e.

$$F \approx \sum_{i=1}^{I} \mathbb{1}_{E_i} f_i \quad \text{in } L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X).$$
(54)

Now, for every fixed i = 1, ..., I, we use that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is non-polynomial to conclude from the universal approximation result for deterministic neural networks in Theorem 2.6 that there exists some  $\varphi_i = \sum_{j=1}^{J_i} y_{i,j} \rho\left(a_{i,j}^{\top} \cdot -b_{i,j}\right) \in \mathcal{NN}_{U,d}^{\rho}$ , with  $J_i \in \mathbb{N}, a_{i,j} \in \mathbb{R}^m$ ,  $b_{i,j} \in \mathbb{R}$ , and  $y_{i,j} \in \mathbb{R}^d$ , such that

$$\mathbb{1}_{E_i} f_i \approx \mathbb{1}_{E_i} \varphi_i = \sum_{j=1}^{J_i} \mathbb{1}_{E_i} y_{i,j} \rho \left( a_{i,j}^\top \cdot -b_{i,j} \right) \quad \text{in } L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X).$$
(55)

Next, for every fixed  $j = 1, ..., J_i$ , we use that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is by Lemma 6.1 k-times differentiable to conclude for every  $n \in \mathbb{N}$  and small enough  $\delta > 0$  that

$$\mathbb{1}_{E_i} y_{i,j} \rho \left( a_{i,j}^\top \cdot -b_{i,j} \right) \approx \mathbb{1}_{E_i} \mathbb{E} \left[ W_{n,i,j} \rho \left( A_n^\top \cdot -B_n \right) \right] \text{ in } L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X),$$
(56)

where  $\Omega \ni \omega \mapsto W_{n,i,j}(\omega) := C_{\delta}^{-1} y_{i,j} \mathbb{1}_{\{\|(A_n,B_n)-(a_{i,j},b_{i,j})\| < \delta\}} \in \mathbb{R}^d$  is  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable, with  $C_{\delta} := \mathbb{P}\left[\{\omega \in \Omega : \|(A_1,B_1)-(a_{i,j},b_{i,j})\| < \delta\}\right] > 0$ . Finally, we apply the strong law of large numbers for Banach space-valued random variables in [Hytönen et al., 2016, Theorem 3.3.10] to conclude that

$$\mathbb{1}_{E_i} \mathbb{E}\left[W_{n,i,j}\rho\left(A_1^\top \cdot -B_1\right)\right] \approx \Phi_{i,j} := \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{E_i} W_{n,i,j}\rho\left(A_n^\top \cdot -B_n\right) \text{ in } L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X),$$
(57)

where  $\Phi_{i,j} \in \mathcal{RN}_{U,d}^{\rho}$  is a random neural network. Hence, for Theorem 3.5 (i), we combine (54)-(57) to approximate  $F \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  by the random neural network  $\Phi := \sum_{i=1}^{I} \sum_{j=1}^{J_i} \Phi_{i,j} \in \mathcal{RN}_{U,d}^{\rho}$ . Moreover, Theorem 3.5 (ii) follows from Theorem 3.5 (i) and Chebyshev's inequality.

Now, let us first prove the approximation steps in (56)+(57) together in the following proposition.

**Proposition 8.4.** For  $k \in \mathbb{N}_0$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , let  $(X, \|\cdot\|_X)$  be  $(k, U, \gamma)$ approximable function space and let  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$ . Moreover, let  $r \in [1, \infty)$  and let  $(A_1, B_1)$  satisfy Assumption 3.1. Then, for every  $\varepsilon > 0$ ,  $E \in \mathcal{F}_{A,B}$ ,  $y \in \mathbb{R}^d$ ,  $a \in \mathbb{R}^m$ , and  $b \in \mathbb{R}$  there exists some  $\Phi \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  such that

$$\left\|\mathbb{1}_{E} y\rho\left(a^{\top}\cdot-b\right)-\Phi\right\|_{L^{r}\left(\Omega,\mathcal{F},\mathbb{P};X\right)}:=\mathbb{E}\left[\left\|\mathbb{1}_{E} y\rho\left(a^{\top}\cdot-b\right)-\Phi\right\|^{r}\right]^{\frac{1}{r}}<\varepsilon$$

*Proof.* Fix some  $r \in [1, \infty)$ ,  $\varepsilon > 0$ ,  $E \in \mathcal{F}_{A,B}$ ,  $y \in \mathbb{R}^d$ ,  $a \in \mathbb{R}^m$ , and  $b \in \mathbb{R}$ . Then, for every  $M, n \in \mathbb{N}$ , we define the map

$$\Omega \ni \omega \quad \mapsto \quad R_{M,n}(\omega) := W_M(\omega)\rho\left(A_n(\omega)^\top \cdot -B_n(\omega)\right) \in X$$

with  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable random variable

$$\Omega \ni \omega \quad \mapsto \quad W_M(\omega) := \frac{1}{C_M} y \mathbb{1}_{G_M}(A_n(\omega), B_n(\omega)) \in \mathbb{R}^d$$

where

$$G_M := \{(x, y) \in \mathbb{R}^m \times \mathbb{R} : \|(x, y) - (a, b)\| \leq 1/M\} \in \mathcal{B}(\mathbb{R}^m \times \mathbb{R})$$

and  $C_M := \mathbb{P}\left[\{\omega \in \Omega : (A_1(\omega), B_1(\omega)) \in G_M\}\right] > 0$  due to Assumption 3.1. Hereby, we recall that  $(A_n, B_n)_{n \in \mathbb{N}}$  is an i.i.d. sequence, which implies that  $(A_n, B_n) \sim (A_1, B_1)$  is identically distributed. Moreover, since  $W_M : \Omega \to \mathbb{R}^d$  is by definition  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable, it follows that  $R_{M,n} \in \mathcal{RN}_{U,d}^{\rho}$ . Hence, by using that  $(X, \|\cdot\|_X)$  is an  $(k, U, \gamma)$ -approximable function space, it follows from Proposition 8.3 that  $R_{M,n} : \Omega \to X$  is  $\mathcal{F}_{A,B}$ -strongly measurable for all  $M, n \in \mathbb{N}$ .

Now, we show that the sequence  $(\mathbb{1}_{G_M}(A_n(\omega), B_n(\omega))y\rho(a^\top \cdot -b) - C_M R_{M,n}(\omega))_{M \in \mathbb{N}}$  converges uniformly in  $\omega \in \Omega$  and  $n \in \mathbb{N}$  to  $0 \in X$  with respect to  $\|\cdot\|_X$ . For this purpose, we use Lemma 8.2 to conclude that the map

$$\mathbb{R}^m \times \mathbb{R} \ni (x, y) \quad \mapsto \quad y\rho\left(a^\top \cdot -b\right) - y\rho\left(x^\top \cdot -y\right) \in (X, \|\cdot\|_X)$$

is continuous. Hence, by using that the norm  $X \ni f \mapsto ||f||_X \in \mathbb{R}$  is continuous, it follows that

$$\mathbb{R}^m \times \mathbb{R} \ni (x, y) \quad \mapsto \quad \left\| y\rho\left( a^\top \cdot - b \right) - y\rho\left( x^\top \cdot - y \right) \right\|_X \in \mathbb{R}$$

is continuous as concatenation of continuous maps. Thus, for every  $M \in \mathbb{N}$ , we use that  $G_M \subseteq \mathbb{R}^m \times \mathbb{R}$  is compact to conclude from the extreme value theorem that there exists some  $(a_M, b_M) \in G_M$  such that

$$\sup_{(x,y)\in G_M} \left\| y\rho\left(a^{\top}\cdot -b\right) - y\rho\left(x^{\top}\cdot -y\right) \right\|_X = \left\| y\rho\left(a^{\top}\cdot -b\right) - y\rho\left(a_M^{\top}\cdot -b_M\right) \right\|_X.$$

Moreover, by using that  $G_{M+1} \subset G_M$  for all  $M \in \mathbb{N}$  and that  $\bigcap_{M \in \mathbb{N}} G_M = \{(a, b)\}$ , the sequence  $(y, a_M, b_M)_{M \in \mathbb{N}}^{\top} \subseteq \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  converges to  $(y, a, b) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$ . Hence, by using Lemma 8.2, it follows that

$$\begin{split} \lim_{M \to \infty} \sup_{\omega \in \Omega} \sup_{n \in \mathbb{N}} \left\| \mathbbm{1}_{G_M}(A_n(\omega), B_n(\omega)) y \rho \left( a^\top \cdot -b \right) - C_M R_{M,n}(\omega) \right\|_X \\ &= \lim_{M \to \infty} \sup_{\omega \in \Omega} \sup_{n \in \mathbb{N}} \left\| \mathbbm{1}_{G_M}(A_n(\omega), B_n(\omega)) \left( y \rho \left( a^\top \cdot -b \right) - y \rho \left( A_n(\omega)^\top \cdot -B_n(\omega) \right) \right) \right\|_X \\ &= \lim_{M \to \infty} \sup_{(x,y) \in G_M} \left\| y \rho \left( a^\top \cdot -b \right) - y \rho \left( x^\top \cdot -y \right) \right\|_X \\ &= \lim_{M \to \infty} \left\| y \rho \left( a^\top \cdot -b \right) - y \rho \left( a^\top_M \cdot -b_M \right) \right\|_X = 0. \end{split}$$

This shows that  $(\mathbb{1}_{G_M}(A_n(\omega), B_n(\omega))y\rho(a^\top \cdot -b) - C_M R_{M,n}(\omega))_{M \in \mathbb{N}}$  converges uniformly in  $\omega \in \Omega$ and  $n \in \mathbb{N}$  to  $0 \in X$  with respect to  $\|\cdot\|_X$ . Next, we show that  $R_{M,n} \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  for all  $M, n \in \mathbb{N}$ . For this purpose, we recall that  $R_{M,n} : \Omega \to X$  is  $\mathcal{F}_{A,B}$ -strongly measurable as shown above. Moreover, we use the previous step to conclude that the sequence  $(\mathbb{1}_{G_M}(A_n(\omega), B_n(\omega))y\rho(a^\top \cdot -b) - C_M R_{M,n}(\omega))_{M \in \mathbb{N}}$  is uniformly bounded in  $\omega \in \Omega$  and  $n \in \mathbb{N}$ , i.e. that

$$C := \sup_{n \in \mathbb{N}} \sup_{\omega \in \Omega} \left\| \mathbb{1}_{G_M}(A_n(\omega), B_n(\omega)) y \rho\left(a^\top \cdot -b\right) - C_M R_{M,n}(\omega) \right\|_X < \infty.$$

Hence, by using this and Minkowski's inequality, it follows for every  $M, n \in \mathbb{N}$  that

$$\begin{split} \mathbb{E} \left[ \|R_{M,n}\|_{X}^{r} \right]^{\frac{1}{r}} &= \frac{1}{C_{M}} \mathbb{E} \left[ \|C_{M}R_{M,n}\|_{X}^{r} \right]^{\frac{1}{r}} \\ &\leq \frac{1}{C_{M}} \mathbb{E} \left[ \|\mathbb{1}_{G_{M}}(A_{n}, B_{n})y\rho\left(a^{\top} \cdot -b\right)\|_{X}^{r} \right]^{\frac{1}{r}} \\ &\quad + \frac{1}{C_{M}} \mathbb{E} \left[ \|\mathbb{1}_{G_{M}}(A_{n}, B_{n})y\rho\left(a^{\top} \cdot -b\right) - C_{M}R_{M,n}\|_{X}^{r} \right]^{\frac{1}{r}} \\ &\leq \frac{1}{C_{M}} \|y\rho\left(a^{\top} \cdot -b\right)\|_{X} + \frac{1}{C_{M}} \sup_{\omega \in \Omega} \|\mathbb{1}_{G_{M}}(A_{n}(\omega), B_{n}(\omega))y\rho\left(a^{\top} \cdot -b\right) - C_{M}R_{M,n}(\omega)\|_{X} \\ &\leq \frac{1}{C_{M}} \|y\rho\left(a^{\top} \cdot -b\right)\|_{X} + \frac{C}{C_{M}} < \infty, \end{split}$$

which shows that  $R_{M,n} \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  for all  $M, n \in \mathbb{N}$ .

Now, we show that there exists some  $M_3 \in \mathbb{N}$  such that the constant maps  $(\omega \mapsto y\rho(a^{\top} \cdot -b)) \in L^r(\Omega, \mathcal{F}, \mathbb{P}; X)$  and  $(\omega \mapsto \mathbb{E}[R_{M_3,1}]) \in L^r(\Omega, \mathcal{F}, \mathbb{P}; X)$  are  $\frac{\varepsilon}{2}$ -close with respect to  $\|\cdot\|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)}$ . Indeed, by using the previous step, i.e. that  $(\mathbb{1}_{G_M}(A_n(\omega), B_n(\omega))y\rho(a^{\top} \cdot -b) - C_M R_{M,n}(\omega))_{M \in \mathbb{N}}$  converges uniformly in  $\omega \in \Omega$  and  $n \in \mathbb{N}$  to  $0 \in X$  with respect to  $\|\cdot\|_X$ , it follows that there exists some  $M_3 \in \mathbb{N}$  such that

$$\sup_{n\in\mathbb{N}}\sup_{\omega\in\Omega}\left\|\mathbb{1}_{E_{M_3}}(A_n(\omega), B_n(\omega))y\rho\left(a^{\top}\cdot -b\right) - C_{M_3}R_{M_3,n}(\omega)\right\|_X < \frac{\varepsilon}{2}.$$
(58)

Hence, by using that  $\Omega \ni \omega \mapsto S(\omega) - \mathbb{E}[R_{M_3,1}] \in X$  is constant, the identities  $\mathbb{E}[\mathbbm{1}_{G_{M_3}}(A_1, B_1)] = \mathbb{P}[\{\omega \in \Omega : (A_1(\omega), B_1(\omega)) \in G_{M_3}\}] = C_{M_3}$ , and [Hytönen et al., 2016, Proposition 1.2.2], we have

$$\begin{split} \|y\rho\left(a^{\top}\cdot-b\right)-\mathbb{E}[R_{M_{3},1}]\|_{L^{r}(\Omega,\mathcal{F},\mathbb{P};X)} &= \mathbb{E}\left[\|y\rho\left(a^{\top}\cdot-b\right)-\mathbb{E}[R_{M_{3},1}]\|_{X}^{r}\right]^{\frac{1}{r}} \\ &= \|y\rho\left(a^{\top}\cdot-b\right)-\mathbb{E}[R_{M_{3},1}]\|_{X} \\ &= \left\|\mathbb{E}\left[\frac{1}{C_{M_{3}}}\mathbb{1}_{G_{M_{3}}}(A_{1},B_{1})y\rho\left(a^{\top}\cdot-b\right)-R_{M_{3},1}\right]\right\|_{X} \\ &\leqslant \mathbb{E}\left[\mathbb{1}_{G_{M_{3}}}(A_{1},B_{1})\left\|\frac{1}{C_{M_{3}}}\mathbb{1}_{G_{M_{3},1}}y\rho\left(a^{\top}\cdot-b\right)-R_{M_{3},1}\right\|_{X}\right] \\ &\leqslant \underbrace{\mathbb{E}\left[\mathbb{1}_{G_{M_{3}}}(A_{1},B_{1})\right]}_{=1} \sup_{\omega\in\Omega}\left\|\mathbb{1}_{G_{M_{3}}}(A_{1}(\omega),B_{1}(\omega))y\rho\left(a^{\top}\cdot-b\right)-C_{M_{3}}R_{M_{3},1}(\omega)\right\|_{X} < \frac{\varepsilon}{2}. \end{split}$$
(59)

This shows that the constant maps  $(\omega \mapsto y\rho(a^{\top} \cdot -b)) \in L^r(\Omega, \mathcal{F}, \mathbb{P}; X)$  and  $(\omega \mapsto \mathbb{E}[R_{M_3,1}]) \in L^r(\Omega, \mathcal{F}, \mathbb{P}; X)$  are  $\frac{\varepsilon}{2}$ -close with respect to  $\|\cdot\|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)}$ .

Finally, we approximate the constant random variable  $(\omega \mapsto \mathbb{E}[R_{M_3,1}]) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; X)$  by the average of the i.i.d. sequence  $(R_{M_3,n})_{n \in \mathbb{N}} \subseteq L^1(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$ . Indeed, by applying the strong law of large numbers for Banach space-valued random variables in [Hytönen et al., 2016, Theorem 3.3.10] with Banach space  $(X, \|\cdot\|_X)$ , we conclude that

$$\frac{1}{N}\sum_{n=1}^{N}R_{M_{3},n} \xrightarrow{N \to \infty} \mathbb{E}\left[R_{M_{3},1}\right] \text{ in } L^{1}(\Omega,\mathcal{F},\mathbb{P};X) \text{ and } \mathbb{P}\text{-a.s. with respet to } \|\cdot\|_{X}.$$
(60)

Moreover, if r > 1, we generalize in (60) the convergence to  $L^r(\Omega, \mathcal{F}, \mathbb{P}; X)$ . To this end, we define the sequence of random variables  $(Z_N)_{N \in \mathbb{N}}$  by  $Z_N(\omega) := \left\| \mathbb{E} \left[ R_{M_3,1} \right] - \frac{1}{N} \sum_{n=1}^N R_{M_3,n} \right\|_X^r$ , for  $\omega \in \Omega$  and  $N \in \mathbb{N}$ . Then, [Hytönen et al., 2016, Proposition 1.2.2] and (58) imply for every  $N \in \mathbb{N}$  that

$$\begin{split} \sup_{\omega \in \Omega} Z_N(\omega) &\leq \sup_{\omega \in \Omega} \left( \left\| \mathbb{E} \left[ R_{M_3,1} \right] \right\|_X + \frac{1}{N} \sum_{n=1}^N \left\| R_{M_3,n}(\omega) \right\|_X \right)^r \\ &\leq \sup_{n \in \mathbb{N}} \sup_{\omega \in \Omega} \left( \mathbb{E} \left[ \left\| R_{M_3,1} \right\|_X \right] + \left\| R_{M_3,n}(\omega) \right\|_X \right)^r \\ &\leq \frac{2^r}{C_{M_3}^r} \sup_{n \in \mathbb{N}} \sup_{\omega \in \Omega} \left\| C_{M_3} R_{M_3,n}(\omega) \right\|_X^r \\ &\leq \frac{2^r}{C_{M_3}^r} \sup_{n \in \mathbb{N}} \sup_{\omega \in \Omega} \left( \left\| \mathbbm{1}_{G_{M_3}} (A_n(\omega), B_n(\omega)) y\rho \left( a^\top \cdot -b \right) \right\|_X + \\ &+ \left\| \mathbbm{1}_{G_{M_3}} (A_n(\omega), B_n(\omega)) y\rho \left( a^\top \cdot -b \right) - C_{M_3} R_{M_3,n}(\omega) \right\|_X \right)^r \\ &\leq \frac{2^r}{C_{M_3}^r} \left( \left\| y\rho \left( a^\top \cdot -b \right) \right\|_X + \frac{\varepsilon}{2} \right)^r =: C_Z < \infty. \end{split}$$

From this, we observe that  $\sup_{N \in \mathbb{N}} \mathbb{E} \left[ |Z_N| \mathbb{1}_{\{|Z_N| > C_Z\}} \right] = 0$ , which shows that the family of random variables  $(Z_N)_{N \in \mathbb{N}}$  is uniformly integrable (see [Hytönen et al., 2016, Definition A.3.1]). Thus, by using (60), i.e. that  $Z_N \to 0$ ,  $\mathbb{P}$ -a.s., as  $N \to \infty$ , together with Vitali's convergence theorem in [Hytönen et al., 2016, Proposition A.3.5], it follows that

$$\lim_{N \to \infty} \mathbb{E}\left[ \left\| \mathbb{E}\left[ R_{M_{3},1} \right] - \frac{1}{N} \sum_{n=1}^{N} R_{M_{3},n} \right\|_{X}^{r} \right] = \lim_{N \to \infty} \mathbb{E}[Z_{N}] = 0.$$
(61)

Hence, either by (60) (if r = 1) or (61) (if  $r \in (1, \infty)$ ) there exists some  $N_0 \in \mathbb{N}$  such that

$$\mathbb{E}\left[\left\|\mathbb{E}\left[R_{M_{3},1}\right] - \frac{1}{N_{0}}\sum_{n=1}^{N_{0}}R_{M_{3},n}\right\|_{X}^{r}\right]^{\frac{1}{r}} < \frac{\varepsilon}{2}.$$
(62)

Thus, by defining  $\Phi := \left(\omega \mapsto \frac{1}{N_0} \sum_{n=1}^{N_0} \mathbb{1}_E(\omega) R_{M_3,n}(\omega)\right) \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  and by combining (59) and (62) with Minkowski's inequality, it follows that

$$\begin{split} \|\mathbf{1}_{E}y\rho\left(a^{\top}\cdot-b\right)-\Phi\|_{L^{r}(\Omega,\mathcal{F},\mathbb{P};X)} &= \mathbb{E}\left[\left\|\mathbf{1}_{E}y\rho\left(a^{\top}\cdot-b\right)-\Phi\right\|^{r}\right]^{\frac{1}{r}} \\ &= \mathbb{E}\left[\underbrace{\mathbf{1}_{E}}_{\leqslant 1}\left\|y\rho\left(a^{\top}\cdot-b\right)-\frac{1}{N_{0}}\sum_{n=1}^{N_{0}}R_{M_{3},n}\right\|^{r}\right]^{\frac{1}{r}} \\ &\leqslant \mathbb{E}\left[\left\|y\rho\left(a^{\top}\cdot-b\right)-\frac{1}{N_{0}}\sum_{n=1}^{N_{0}}R_{M_{3},n}\right\|^{r}\right]^{\frac{1}{r}} \\ &\leqslant \|y\rho\left(a^{\top}\cdot-b\right)-\mathbb{E}\left[R_{M_{3},n}\right]\|_{X} + \mathbb{E}\left[\left\|\mathbb{E}\left[R_{M_{3},n}\right]-\frac{1}{N_{0}}\sum_{n=1}^{N_{0}}R_{M_{3},n}\right\|^{r}\right]^{\frac{1}{r}} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \end{split}$$

which completes the proof.

Proof of Theorem 3.5. In order to show (i), we fix some  $F \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  and  $\varepsilon > 0$ . Then, by using [Hytönen et al., 2016, Lemma 1.2.19 (i)], i.e. that the set of  $\mathcal{F}_{A,B}$ -simple functions  $\mathcal{I}_{\mathcal{F}_{A,B}} \otimes X := \left\{ \sum_{i=1}^{I} \mathbb{1}_{E_i} f_i : I \in \mathbb{N}, E_i \in \mathcal{F}_{A,B}, f_i \in X \right\}$  is dense in  $L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  with respect to  $\|\cdot\|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)}$ , there exists some  $I \in \mathbb{N}, E_1, ..., E_I \in \mathcal{F}_{A,B}$ , and  $f_1, ..., f_I \in X$  such that

$$\left\|F - \sum_{i=1}^{I} \mathbb{1}_{E_i} f_i\right\|_{L^r(\Omega, \mathcal{F}, \mathbb{P}; X)} := \mathbb{E}\left[\left\|F - \sum_{i=1}^{I} \mathbb{1}_{E_i} f_i\right\|_X^r\right]^{\frac{1}{r}} < \frac{\varepsilon}{3}.$$
(63)

Now, for every i = 1, ..., I, we use that  $\rho \in \overline{C_b^k(\mathbb{R})}^{\gamma}$  is non-polynomial to conclude from Theorem 2.6 that there exists a deterministic neural network  $\varphi = \sum_{j=1}^{J_i} y_{i,j} \rho\left(a_{i,j}^{\top} \cdot -b_{i,j}\right) \in \mathcal{NN}_{U,d}^{\rho}$ , with  $J_i \in \mathbb{N}$ ,  $a_{i,1}, ..., a_{i,J_i} \in \mathbb{R}^m$ ,  $b_{i,1}, ..., b_{i,J_i} \in \mathbb{R}$ , and  $y_{i,1}, ..., y_{i,J_i} \in \mathbb{R}^d$ , such that

$$\|f_i - \varphi_i\|_X < \frac{\varepsilon}{3I}.$$
(64)

Moreover, for every i = 1, ..., I and  $j = 1, ..., J_i$ , we apply Proposition 8.4 to conclude that there exists some  $\Phi_{i,j} \in \mathcal{RN}_{U,d}^{\rho}$  such that

$$\mathbb{E}\left[\left\|\mathbb{1}_{E_{i}}y_{i,j}\rho\left(a_{i,j}^{\top}\cdot-b_{i,j}\right)-\Phi_{i,j}\right\|_{X}^{r}\right]^{\frac{1}{r}} < \frac{\varepsilon}{3\left(J_{1}+\ldots+J_{I}\right)}.$$
(65)

Hence, by defining the random neural network  $\Phi := \sum_{i=1}^{I} \sum_{j=1}^{J_i} \Phi_{i,j} \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$ and by combining (63), (64), and (65) with Minkowski's inequality, it follows that

$$\begin{split} \|F - \Phi\|_{L^{r}(\Omega,\mathcal{F},\mathbb{P};X)} &:= \mathbb{E}\left[\|F - \Phi\|_{X}^{r}\right]^{\bar{r}} \\ &\leqslant \mathbb{E}\left[\left\|F - \sum_{i=1}^{I} \mathbbm{1}_{E_{i}}f_{i}\right\|_{X}^{r}\right]^{\frac{1}{r}} + \mathbb{E}\left[\left\|\sum_{i=1}^{I} \mathbbm{1}_{E_{i}}f_{i} - \sum_{i=1}^{I} \mathbbm{1}_{E_{i}}\varphi_{i}\right\|_{X}^{r}\right]^{\frac{1}{r}} \\ &+ \mathbb{E}\left[\left\|\sum_{i=1}^{I} \mathbbm{1}_{E_{i}}\varphi_{i} - \sum_{i=1}^{I} \sum_{j=1}^{J_{i}} \Phi_{i,j}\right\|_{X}^{r}\right]^{\frac{1}{r}} \\ &\leqslant \mathbb{E}\left[\left\|F - \sum_{i=1}^{I} \mathbbm{1}_{E_{i}}f_{i}\right\|_{X}^{r}\right]^{\frac{1}{r}} + \sum_{i=1}^{I} \mathbb{E}\left[\underbrace{\mathbbm{1}_{E_{i}}}_{\leq 1} \|f_{i} - \varphi_{i}\|_{X}^{r}\right]^{\frac{1}{r}} \\ &+ \sum_{i=1}^{I} \sum_{j=1}^{J_{i}} \mathbb{E}\left[\left\|\mathbbm{1}_{E_{i}}y_{i,j}\rho\left(a_{i,j}^{\top} \cdot -b_{i,j}\right) - \Phi_{i,j}\right\|_{X}^{r}\right]^{\frac{1}{r}} \\ &\leqslant \mathbb{E}\left[\left\|F - \sum_{i=1}^{I} \mathbbm{1}_{E_{i}}f_{i}\right\|_{X}^{r}\right]^{\frac{1}{r}} + \sum_{i=1}^{I} \|f_{i} - \varphi_{i}\|_{X} \\ &+ \sum_{i=1}^{I} \sum_{j=1}^{J_{i}} \mathbb{E}\left[\left\|\mathbbm{1}_{E_{i}}y_{i,j}\rho\left(a_{i,j}^{\top} \cdot -b_{i,j}\right) - \Phi_{i,j}\right\|_{X}^{r}\right]^{\frac{1}{r}} \\ &< \frac{\varepsilon}{3} + I \frac{\varepsilon}{3I} + (J_{1} + \dots + J_{I}) \frac{\varepsilon}{3(J_{1} + \dots + J_{I})} \leqslant \varepsilon, \end{split}$$

which proves the inequality (i).

In order to show (ii), we fix some  $F \in L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  and  $\delta, \varepsilon > 0$ . Then, by using (i) with  $\delta \varepsilon^r > 0$  on the right-hand side instead of  $\varepsilon > 0$ , there exists some  $\Phi \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; X)$  such that

$$\mathbb{E}\left[\|F - \Phi\|_X^r\right] < \delta \varepsilon^r.$$

Hence, by using this and Chebyshev's inequality, it follows that

$$\mathbb{P}\left[\left\{\omega \in \Omega : \|F - \Phi(\omega)\|_X \ge \varepsilon\right\}\right] \le \frac{1}{\varepsilon^r} \mathbb{E}\left[\|F - \Phi\|_X^r\right] \le \frac{1}{\varepsilon^r} \delta \varepsilon^r = \delta_{\mathcal{F}}$$

which completes the proof.

In this section, we provide the proofs of the results in Section 4. First, we recall the Radon transform (see [Helgason, 1999]) and the ridgelet transform (see [Candès, 1998] and [Sonoda and Murata, 2017]) in Section 9.1. Then, we use the reconstruction formula of [Sonoda and Murata, 2017] in Section 9.2 to derive an integral representation of the function to be approximated. This is used in Section 9.3 to show the main result of Section 4, i.e. the approximation rates in Theorem 4.4 for learning a deterministic function. Finally, in Section 9.4, we prove the generalization error formulated in Theorem 4.10.

9.1. **Preliminary Results: Radon Transform and Ridgelet Transform.** In the following, we denote by  $\mathbb{S}^{m-1} := \{v \in \mathbb{R}^m : \|v\| = 1\}$  the unit sphere in  $\mathbb{R}^m$  and define for every  $(v, h) \in \mathbb{S}^{m-1} \times \mathbb{R}$  the hyperplane  $\mathbb{H}_{v,h}^m := \{u \in \mathbb{R}^m : u^\top v = h\}$ . Then, we follow [Sonoda and Murata, 2017, Section 2.5] and recall that the Radon transform of any function  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  is defined by

$$\mathbb{S}^{m-1} \times \mathbb{R} \ni (v,h) \quad \mapsto \quad (Rf)(v,h) = \int_{\mathbb{H}^m_{v,h}} f(u) du \in \mathbb{R}^d.$$
(66)

Now, we denote by  $L^{\infty,1}(\mathbb{S}^{m-1}\times\mathbb{R}, \mathcal{L}(\mathbb{S}^{m-1}\times\mathbb{R}), dv\otimes dh; \mathbb{R}^d)$  the vector space of  $\mathcal{L}(\mathbb{S}^{m-1}\times\mathbb{R})/\mathcal{B}(\mathbb{R}^d)$ -measurable functions  $Q: \mathbb{S}^{m-1}\times\mathbb{R} \to \mathbb{R}^d$  such that

$$\|Q\|_{L^{\infty,1}(\mathbb{S}^{m-1}\times\mathbb{R},\mathcal{L}(\mathbb{S}^{m-1}\times\mathbb{R}),dv\otimes dh;\mathbb{R}^d)} := \sup_{v\in\mathbb{S}^{m-1}} \int_{\mathbb{R}} \|Q(v,h)\| dh < \infty$$

For completeness, we show the following simple generalizations of the Radon transform's properties (including the Fourier slice theorem in [Helgason, 1999, Equation 4]) from the original one-dimensional setting to this multi-dimensional setting.

Lemma 9.1. The following holds true:

- (i) The Radon transform  $R : L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d) \to L^{\infty,1}(\mathbb{S}^{m-1} \times \mathbb{R}, \mathcal{L}(\mathbb{S}^{m-1} \times \mathbb{R}), dv \otimes dh; \mathbb{R}^d)$  defined in (66) is a continuous linear operator.
- (ii) For every  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$ ,  $v \in \mathbb{S}^{m-1}$ , and  $\xi \in \mathbb{R}$ , we have  $\widehat{(Rf)(v, \cdot)}(\xi) = \widehat{f}(\xi v)$ .

*Proof.* Fix some  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$ . For (i), we use that the function  $\mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^m \ni (v, h, u) \mapsto \mathbb{1}_{\mathbb{H}^m_{v, h}}(u) f(u) \in \mathbb{R}^d$  is  $\mathcal{L}(\mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^m) / \mathcal{B}(\mathbb{R}^d)$ -measurable to conclude that

$$\mathbb{S}^{m-1} \times \mathbb{R} \ni (v,h) \quad \mapsto \quad \int_{\mathbb{H}^m_{v,h}} f(u) du = \int_{\mathbb{R}^m} \mathbbm{1}_{\mathbb{H}^m_{v,h}}(u) f(u) du \in \mathbb{R}^d$$

is  $\mathcal{L}(\mathbb{S}^{m-1} \times \mathbb{R})/\mathcal{B}(\mathbb{R}^d)$ -measurable. Moreover, by using the definition of the Radon transform in (66) and that  $\bigcup_{h \in \mathbb{R}} \mathbb{H}_{v,h}^m = \mathbb{R}^m$ , we conclude that

$$\|Rf\|_{L^{\infty,1}(\mathbb{S}^{m-1}\times\mathbb{R},\mathcal{L}(\mathbb{S}^{m-1}\times\mathbb{R}),dv\otimes dh;\mathbb{R}^d)} = \sup_{v\in\mathbb{S}^{m-1}} \int_{\mathbb{R}} \|(Rf)(v,h)\| \, dh \leqslant \sup_{v\in\mathbb{S}^{m-1}} \int_{\mathbb{R}} \int_{\mathbb{H}^m_{v,h}} \|f(u)\| \, du dh$$
$$= \int_{\mathbb{R}^m} \|f(u)\| \, du = \|f\|_{L^1(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),du;\mathbb{R}^d)}.$$
(67)

This shows that  $R: L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d) \to L^{\infty,1}(\mathbb{S}^{m-1} \times \mathbb{R}, \mathcal{L}(\mathbb{S}^{m-1} \times \mathbb{R}), dv \otimes dh; \mathbb{R}^d)$  is bounded. Since the Radon transform is by definition linear, it follows that  $R: L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d) \to L^{\infty,1}(\mathbb{S}^{m-1} \times \mathbb{R}, \mathcal{L}(\mathbb{S}^{m-1} \times \mathbb{R}), dv \otimes dh; \mathbb{R}^d)$  is continuous.

In order to show (ii), we use the definition of the Fourier transform in (1), the definition of the Radon transform in (66), that  $h = v^{\top}u$  for any  $u \in \mathbb{H}_{v,h}^m$ , and that  $\bigcup_{h \in \mathbb{R}} \mathbb{H}_{v,h}^m = \mathbb{R}^m$  for any  $v \in \mathbb{S}^{m-1}$  to conclude for every  $v \in \mathbb{S}^{m-1}$  and  $\xi \in \mathbb{R}$  that

$$\begin{split} \widehat{(Rf)(v,\cdot)}(\xi) &= \int_{\mathbb{R}} (Rf)(v,h) e^{-i\xi h} dh = \int_{\mathbb{R}} \int_{\mathbb{H}_{v,h}^m} f(u) e^{-i\xi v^\top u} du dh \\ &= \int_{\mathbb{R}^m} f(u) e^{-i(\xi v)^\top u} du = \widehat{f}(\xi v), \end{split}$$

which completes the proof.

Next, we define the space  $\mathbb{Y}^{m+1} := \mathbb{S}^{m-1} \times (0, \infty) \times \mathbb{R}$ . Then, for any fixed  $\psi \in \mathcal{S}_0(\mathbb{R}; \mathbb{C})$ , we follow [Candès, 1998] and [Sonoda and Murata, 2017, Section 3.2] and recall that the ridgelet transform (in polar coordinates) of any function  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  is defined by

$$\mathbb{Y}^{m+1} \ni (v, s, t) \quad \mapsto \quad (\mathfrak{R}_{\psi} f)(v, s, t) := \int_{\mathbb{R}} (Rf)(v, h) \overline{\psi\left(\frac{h-t}{s}\right)} \frac{1}{s} dh \in \mathbb{C}^d, \tag{68}$$

For completeness, we show the following simple generalizations of the Ridgelet transform's properties from the original one-dimensional setting to this multi-dimensional setting (see also [Kostadinova et al., 2014] for continuity results between (Lizorkin) Schwartz spaces and distributions).

**Lemma 9.2.** Let  $\psi \in S_0(\mathbb{R}; \mathbb{C})$  and  $\mathbb{Y}^{m+1} \ni (v, s, t) \mapsto w_{\mathfrak{R}}(v, s, t) := s \in [0, \infty)$  be a weight. Then, the following holds true:

- (i) The ridgelet transform  $\mathfrak{R}_{\psi} : L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d) \to L^{\infty}(\mathbb{Y}^{m+1}, \mathcal{L}(\mathbb{Y}^{m+1}), w_{\mathfrak{R}}; \mathbb{C}^d)$  defined in (68) is a continuous linear operator.
- (ii) For every  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  and  $(v, s, t) \in \mathbb{Y}^{m+1}$  it holds that

$$(\mathfrak{R}_{\psi}f)(v,s,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi v) \overline{\widehat{\psi}(\xi s)} e^{i\xi t} d\xi.$$

*Proof.* Fix some  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$ . Then, by using Lemma 9.1 (i), i.e. that  $\mathbb{Y}^{m+1} \times \mathbb{R} \ni (v, s, t, h) \mapsto (Rf)(v, h) \overline{\psi(\frac{h-t}{s})} \frac{1}{s} \in \mathbb{C}^d$  is  $\mathcal{L}(\mathbb{Y}^{m+1} \times \mathbb{R})/\mathcal{B}(\mathbb{C}^d)$ -measurable, it follows that

$$\mathbb{Y}^{m+1} \ni (v, s, t) \quad \mapsto \quad (\mathfrak{R}_{\psi} f)(v, s, t) = \int_{\mathbb{R}} (Rf)(v, h) \overline{\psi\left(\frac{h-t}{s}\right)} \frac{1}{s} dh \in \mathbb{C}^d$$

is  $\mathcal{L}(\mathbb{Y}^{m+1})/\mathcal{B}(\mathbb{C}^d)$ -measurable. Moreover, by using the definition of the Ridgelet transform (see (68)), the inequality (67), and that  $\|\psi\|_{C^0(\mathbb{R};\mathbb{C})} := \sup_{z \in \mathbb{R}} |\psi(z)| < \infty$ , we conclude that

$$\begin{aligned} \|\mathfrak{R}_{\psi}f\|_{L^{\infty}(\mathbb{Y}^{m+1},\mathcal{L}(\mathbb{Y}^{m+1}),w_{\mathfrak{R}};\mathbb{C}^{d})} &= \sup_{(v,s,t)\in\mathbb{Y}^{m+1}} s \left\| \int_{\mathbb{R}} (Rf)(v,h)\overline{\psi\left(\frac{h-t}{s}\right)} \frac{1}{s} dh \right\| \\ &\leqslant \sup_{(v,s,t)\in\mathbb{Y}^{m+1}} \int_{\mathbb{R}} \|(Rf)(v,h)\| \left| \psi\left(\frac{h-t}{s}\right) \right| dh \\ &\leqslant \|\psi\|_{C^{0}(\mathbb{R};\mathbb{C})} \sup_{v\in\mathbb{S}^{m-1}} \int_{\mathbb{R}} \|(Rf)(v,h)\| dh \\ &= \|\psi\|_{C^{0}(\mathbb{R};\mathbb{C})} \|Rf\|_{L^{\infty,1}(\mathbb{S}^{m-1}\otimes\mathbb{R},\mathcal{L}(\mathbb{S}^{m-1}\otimes\mathbb{R}),1\otimes1;\mathbb{R}^{d})} \\ &\leqslant \|\psi\|_{C^{0}(\mathbb{R};\mathbb{C})} \|f\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du;\mathbb{R}^{d})}. \end{aligned}$$
(69)

This shows that  $\mathfrak{R}_{\psi} : L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d) \to L^{\infty}(\mathbb{Y}^{m+1}, \mathcal{L}(\mathbb{Y}^{m+1}), w_{\mathfrak{R}}; \mathbb{C}^d)$  is bounded. Hence, by using that the ridgelet transform in polar coordinates is by definition linear, we conclude that  $\mathfrak{R}_{\psi} : L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d) \to L^{\infty}(\mathbb{Y}^{m+1}, \mathcal{L}(\mathbb{Y}^{m+1}), w_{\mathfrak{R}}; \mathbb{C}^d)$  is continuous.

In order to show (ii), we use the definition of the Ridgelet transform (see (68)), the Plancherel theorem in [Folland, 1992, p. 222], Lemma 9.1 (ii), [Folland, 1992, Table 7.2.2], and [Folland, 1992, Table 7.2.4] to conclude for every  $(v, s, t) \in \mathbb{Y}^{m+1}$  that

$$(\mathfrak{R}_{\psi}f)(v,s,t) = \int_{\mathbb{R}} (Rf)(v,h)\psi\left(\frac{h-t}{s}\right)\frac{1}{s}dh$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\left(Rf\right)(v,\cdot)}(\xi)\overline{\frac{1}{s}\psi\left(\frac{\cdot-t}{s}\right)(\xi)}d\xi$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\xi v)\overline{\frac{1}{s}\psi\left(\frac{\cdot}{s}\right)(\xi)e^{-i\xi t}}d\xi$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\xi v)\overline{\widehat{\psi}(\xi s)}e^{i\xi t}d\xi,$$

which shows (ii) and completes the proof.

9.2. **Preliminary Results: Integral Representation and Bochner Norm.** In order to obtain the approximation rates in Theorem 4.4, we discretize an integral representation of the function to be approximated. This is a standard technique in approximation theory and has been also used for deterministic neural networks (see e.g. [Carroll and Dickinson, 1989], [Ito, 1991], [Barron, 1993], [Darken et al., 1993], [Kůrková, 2012], [Kainen et al., 2007], and [Sonoda and Murata, 2017]). In our context of random neural networks, we express the function to be approximated as expectation of a random neuron.

For this purpose, we use the reconstruction formula in [Sonoda and Murata, 2017, Theorem 5.6] to express any sufficiently regular and integrable function as the expectation of a random neuron. Hereby, we define the real part of a vector  $z := (z_1, ..., z_m)^\top \in \mathbb{C}^m$  as  $\operatorname{Re}(z) := (\operatorname{Re}(z_1), ..., \operatorname{Re}(z_m))^\top \in \mathbb{C}^m$ . Moreover, we follow [Sonoda and Murata, 2017, Definition 4.4] and recall that the dual ridgelet transform  $\mathfrak{R}_{\rho}^{\dagger}$  of any function  $Q : \mathbb{S}^{m-1} \times (0, \infty) \times \mathbb{R} \to \mathbb{C}$  satisfying  $Q(v, s, v^\top u - s \cdot) := (z \mapsto Q(v, s, v^\top u - sz)) \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  for all  $v \in \mathbb{S}^{m-1}$ ,  $s \in (0, \infty)$ , and  $u \in \mathbb{R}^m$  is defined by

$$\mathbb{R}^m \ni u \quad \mapsto \quad (\mathfrak{R}^{\dagger}_{\rho}Q)(u) := \lim_{\substack{\delta_1 \to 0\\ \delta_2 \to \infty}} \int_{\mathbb{S}^{m-1}} \int_{\delta_1}^{\delta_2} T_{\rho} \left( Q\left(v, s, v^{\top}u - s \cdot \right) \right) \frac{1}{s^m} ds dv \in \mathbb{R}^d$$

**Proposition 9.3** (Integral Representation). For  $\nu \in (0, \infty)$ , let  $(A_1, B_1) \sim \theta_A \otimes t_1(\nu)$  satisfy Assumption 4.1, and let  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times L^1_{loc}(\mathbb{R}; \mathbb{C})$  be *m*-admissible. Moreover, let  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $\hat{f} \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{C}^d)$  and define for every  $u \in \mathbb{R}^m$  the map

$$\Omega \ni \omega \quad \mapsto \quad R_n^f(\omega) := W_n^f(\omega) \rho \left( A_n(\omega)^\top u - B_n(\omega) \right) \in \mathbb{R}^d, \tag{70}$$

where

$$\Omega \ni \omega \quad \mapsto \quad W_n^f(\omega) := \begin{cases} \operatorname{Re}\left(\frac{(\mathfrak{R}_{\psi}f)(\frac{A_n(\omega)}{\|A_n(\omega)\|}, \frac{1}{\|A_n(\omega)\|}, \frac{B_n(\omega)}{\|A_n(\omega)\|})}{C_m^{(\psi,\rho)}\theta_A(A_n(\omega))\theta_B(B_n(\omega))}\right), & \text{if } A_n(\omega) \neq 0, \\ 0, & \text{if } A_n(\omega) = 0. \end{cases}$$
(71)

Then, for a.e.  $u \in \mathbb{R}^m$ , it holds that  $\mathbb{E}[R_n^f(u)] = f(u)$ .

*Proof.* Fix  $n \in \mathbb{N}$  and  $f = (f_1, ..., f_d)^\top \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $\hat{f} \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{C}^d)$ . Moreover, we define the map  $R_n^f : \Omega \to W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  as in (70) with  $W_n^f : \Omega \to \mathbb{R}^d$  as in (71). Then, by using that  $\hat{f} \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{C}^d)$ , it follows for every i = 1, ..., d that

$$\|\widehat{f}_i\|_{L^1(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),du)} = \int_{\mathbb{R}^m} \left|\widehat{f}_i(\xi)\right| d\xi \leqslant \int_{\mathbb{R}^m} \|\widehat{f}(\xi)\| d\xi = \|\widehat{f}\|_{L^1(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),du;\mathbb{C}^d)} < \infty.$$
(72)

Hence, by using that  $(A_n, B_n) \sim (A_1, B_1)$  is identically distributed with probability density functions  $\theta_A : \mathbb{R}^m \to (0, \infty)$  and  $\theta_B : \mathbb{R} \to (0, \infty)$ , respectively, that the left-hand side is real-valued, the substitution  $(\mathbb{R}^m \setminus \{0\}) \times \mathbb{R} \ni (a, b) \mapsto (v, s, z) := \left(\frac{a}{\|a\|}, \frac{1}{\|a\|}, a^\top u - b\right) \in \mathbb{S}^{m-1} \times (0, \infty) \times \mathbb{R}$  with Jacobi determinante  $dbda = s^{-m} dz ds dv$ , and [Sonoda and Murata, 2017, Theorem 5.6] applied to  $f_i \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$  with  $\hat{f}_i \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{C})$  by (72), it follows for a.e.  $u \in \mathbb{R}^m$  that

$$\begin{split} \mathbb{E}\left[R_{n}^{f}(u)\right] &= \int_{\mathbb{R}^{m}\setminus\{0\}} \int_{\mathbb{R}} \operatorname{Re}\left(\frac{\left(\mathfrak{R}_{\psi}f\right)\left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right)}{C_{m}^{(\psi,\rho)}\theta_{A}(a)\theta_{B}(b)}\right) \rho\left(a^{\top}u-b\right)\theta_{A}(a)\theta_{B}(b)dbda\\ &= \frac{1}{C_{m}^{(\psi,\rho)}} \int_{\mathbb{R}^{m}\setminus\{0\}} \int_{\mathbb{R}}(\mathfrak{R}_{\psi}f)\left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right)\rho\left(a^{\top}u-b\right)dbda\\ &= \frac{1}{C_{m}^{(\psi,\rho)}}\left(\int_{\mathbb{S}^{m-1}} \int_{0}^{\infty} \int_{\mathbb{R}}(\mathfrak{R}_{\psi}f_{i})\left(v,s,v^{\top}u-sz\right)\rho(z)\frac{1}{s^{m}}dzdsdv\right)_{i=1,\dots,d}^{\top}\\ &= \frac{1}{C_{m}^{(\psi,\rho)}}\left(\lim_{\substack{\delta_{1}\to0\\\delta_{2}\to\infty}} \int_{\mathbb{S}^{m-1}} \int_{\delta_{1}}^{\delta_{2}} T_{\rho}\left((\mathfrak{R}_{\psi}f_{i})\left(v,s,v^{\top}u-s\cdot\right)\right)\frac{1}{s^{m}}dsdv\right)_{i=1,\dots,d}^{\top}\\ &= \frac{1}{C_{m}^{(\psi,\rho)}}\left(\left(\mathfrak{R}_{\rho}^{\dagger}\mathfrak{R}_{\psi}f_{i}\right)\right)(u)\right)_{i=1,\dots,d}^{\top} = (f_{i}(u))_{i=1,\dots,d}^{\top} = f(u). \end{split}$$

This proves that  $\mathbb{E}[R_n^f(u)] = f(u)$  for a.e.  $u \in \mathbb{R}^m$ .

Note that Proposition 9.3 is not concerned about any measurability properties of the random neuron  $R_n^f$  in (70). To this end, we follow Proposition 8.3 to show that such random neurons are strongly measurable with values in a weighted Sobolev space  $W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  introduced in Notation (xii)+(xiii).

For this purpose, we first show that  $(W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)})$  is separable.

**Lemma 9.4.** Let  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $w : U \to [0, \infty)$  be a weight. Then, the Banach space  $(W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)})$  in Notation (xii)+(xiii) is separable.

*Proof.* First, we show the conclusion for k = 0, i.e. that the Banach space  $(W^{0,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \| \cdot \|_{W^{0,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}) := (L^p(U,\mathcal{L}(U),w(u)du;\mathbb{R}^d), \| \cdot \|_{L^p(U,\mathcal{L}(U),w(u)du;\mathbb{R}^d)})$  defined in Notation (xiii) is separable. For this purpose, we observe that  $\mathcal{B}(U)$  is generated by sets of the form  $U \cap \times_{l=1}^m [r_{l,1}, r_{l,2})$ , with  $r_{l,1}, r_{l,2} \in \mathbb{Q}, l = 1, ..., m$ . Moreover, by using that  $\mathcal{L}(U)$  and  $\mathcal{B}(U)$  coincide up to Lebesgue nullsets and that  $w : U \to [0, \infty)$  is a weight, i.e. that the measure spaces  $(U, \mathcal{L}(U), w(u)du)$  and  $(U, \mathcal{L}(U), du)$  share the same null sets, we conclude that  $(U, \mathcal{L}(U), w(u)du)$  is countably generated up to w(u)du-null sets. Hence, by applying [Doob, 1994, p. 92] componentwise, it follows that  $(W^{0,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \| \cdot \|_{W^{0,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)}) := (L^p(U, \mathcal{L}(U), w(u)du; \mathbb{R}^d), \| \cdot \|_{L^p(U, \mathcal{L}(U), w(u)du; \mathbb{R}^d)})$  is separable.

Now, for the general case of  $k \ge 1$ , we consider the Banach space  $(W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d), \| \cdot \|_{W^{k,p}(U,\mathcal{L}(U),w,\mathbb{R}^d)})$  introduced in Notation (xii). Then, we define the map

$$W^{k,p}(U,\mathcal{L}(U),w,\mathbb{R}^d) \ni f \quad \mapsto \quad \Xi(f) := (\partial_\alpha f)_{\alpha \in \mathbb{N}_{0,k}^m} \in \bigotimes_{\alpha \in \mathbb{N}_{0,k}^m} L^p(U,\mathcal{L}(U),w(u)du,\mathbb{R}^d) =: Z,$$

where Z is equipped with the norm  $||g||_Z := \sum_{\alpha \in \mathbb{N}_{0,k}^m} ||g_\alpha||_{L^p(U,\mathcal{L}(U),du,\mathbb{R}^d)}$ , for  $g := (g_\alpha)_{\alpha \in \mathbb{N}_{0,k}^m} \in Z$ . Then, by using the previous step, we conclude that the Banach space  $(Z, \|\cdot\|_Z)$  is separable as finite product of separable Banach spaces. Hence, by using that  $W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)$  is by definition isometrically isomorphic to the closed vector subspace  $\operatorname{Img}(\Xi) := \{\Xi(f) : f \in W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)\} \subseteq Z$ , it follows that  $(W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)})$  is separable.  $\Box$ 

Next, we apple the same arguments as in the proof of Proposition 8.3 to obtain strong measurability.

**Lemma 9.5.** For  $k \in \mathbb{N}_0$ ,  $p \in (1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in [0, \infty)$ , and  $\rho \in C^k_{pol,\gamma}(\mathbb{R})$ , let  $w : U \to [0, \infty)$  be a weight such that the constant  $C^{(\gamma, p)}_{U,w} > 0$  defined in (8) is finite. Moreover, for a sub- $\sigma$ -algebra  $\mathcal{F}_{A,B} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}$ , let  $W : \Omega \to \mathbb{R}^d$  be an  $\mathcal{F}_0/\mathbb{R}^d$ -measurable random vector and define

$$\Omega \ni \omega \quad \mapsto \quad R_n(\omega) := W(\omega)\rho\left(A_n(\omega)^\top \cdot -B_n(\omega)\right) \in W^{k,p}(U,\mathcal{L}(U),w,\mathbb{R}^d).$$

Then, the map  $R_n : \Omega \to W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)$  is  $\mathcal{F}_0$ -strongly measurable with values in the separable Banach space  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)}).$ 

*Proof.* First, we show that the map  $R_n : \Omega \to W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)$  takes values in the Banach space  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \| \cdot \|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$ , where the latter is by Lemma 9.5 separable. Indeed, since  $\rho \in C_{pol,\gamma}^k(\mathbb{R})$  is k-times differentiable, it follows for every fixed  $\omega \in \Omega$  and  $\alpha \in \mathbb{N}_{0,k}^m$  that  $U \ni u \mapsto \partial_{\alpha}R_n(\omega) = W_n^f(\omega)\rho^{(|\alpha|)}\left(A_n(\omega)^\top u - B_n(\omega)\right)A_n(\omega)^\alpha \in \mathbb{R}^d$  is  $\mathcal{L}(U)/\mathcal{B}(\mathbb{R}^d)$ -measurable. Moreover, by using that  $\rho \in C_{pol,\gamma}^k(\mathbb{R})$ , i.e. that  $|\rho^{(j)}(s)| \leq \|\rho\|_{C_{pol,\gamma}^k(\mathbb{R})}(1+|s|)^\gamma$  for any j = 0, ..., k and  $s \in \mathbb{R}$ , the inequality  $1 + |A_n(\omega)^\top u - B_n(\omega)| \leq 1 + \|A_n(\omega)\| \|u\| + |B_n(\omega)| \leq (1 + \|A_n(\omega)\| + |B_n(\omega)|)(1 + \|u\|)$  for any  $u \in \mathbb{R}^m$ , and that  $C_{U,w}^{(\gamma,p)} := \left(\int_U (1 + \|u\|)^{\gamma p} w(u) du\right)^{1/p} > 0$  is finite, we conclude that

$$\begin{aligned} \|R_{n}(\omega)\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{p} &= \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \int_{U} \left\|W_{n}^{f}(\omega)\rho^{(|\alpha|)}\left(A_{n}(\omega)^{\top}u - B_{n}(\omega)\right)A_{n}(\omega)^{\alpha}\right\|^{p}w(u)du \\ &\leqslant \left(\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \left\|W_{n}^{f}(\omega)A_{n}(\omega)^{\alpha}\right\|^{p}\right)\int_{U} \left(1 + \left|A_{n}(\omega)^{\top}u - B_{n}(\omega)\right|\right)^{\gamma p}w(u)du \\ &\leqslant \left(\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \left\|W_{n}^{f}(\omega)A_{n}(\omega)^{\alpha}\right\|^{p}\right)\left(1 + \left\|A_{n}(\omega)\right\| + \left|B_{n}(\omega)\right|\right)^{\gamma p}\int_{U} (1 + \|u\|)^{\gamma p}w(u)du < \infty. \end{aligned}$$

This shows that  $R_n(\omega) \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  for all  $\omega \in \Omega$ .

Now, in order to show that  $R_n : \Omega \to W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)$  is  $\mathcal{F}_0$ -strongly measurable, we aim to apply the same arguments as in Proposition 8.3. For this purpose, we first show that for every sequence  $(y_M, a_M, b_M)_{M \in \mathbb{N}} \subseteq \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  converging to  $(y, a, b) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$ , it holds that

$$\lim_{M \to \infty} \left\| y\rho\left(a^{\top} \cdot -b\right) - y_M\rho\left(a_M^{\top} \cdot -b_M\right) \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} = 0,\tag{73}$$

where  $y\rho\left(a^{\top}\cdot-b\right)$  denotes the function  $U \ni u \mapsto y\rho\left(a^{\top}u-b\right) \in \mathbb{R}^d$ . Indeed, fix a sequence  $(y_M, a_M, b_M)_{M\in\mathbb{N}} \subseteq \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$  converging to  $(y, a, b) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}$ . Then, by using that  $y_M a_M^{\alpha}(1+\|a_M\|+|b_M|)$  converges uniformly in  $\alpha \in \mathbb{N}_{0,k}^m$  to  $ya^{\alpha}(1+\|a\|+|b|)$ , the constant  $C_{y,a,b} := \max_{\alpha \in \mathbb{N}_{0,k}^m} \|ya^{\alpha}\| (1+\|a\|+|b|) + \sup_{M\in\mathbb{N}} \left(\max_{\alpha \in \mathbb{N}_{0,k}^m} \|y_M a_M^{\alpha}\| (1+\|a_M\|+|b_M|)\right) \ge 0$  is finite, where  $a^{\alpha} := \prod_{l=1}^m a_l^{\alpha_l}$  for  $a := (a_1, ..., a_m)^{\top} \in \mathbb{R}^m$  and  $\alpha \in \mathbb{N}_{0,k}^m$ . Hence, by using that  $\rho \in C_{pol,\gamma}^k(\mathbb{R})$ , i.e. that  $|\rho^{(j)}(s)| \le \|\rho\|_{C_{pol,\gamma}^k}(\mathbb{R})(1+|s|)^{\gamma}$  for any j = 0, ..., k and  $s \in \mathbb{R}$ , the inequality  $1+|a_M^{\top}u-b_M| \le 1+\|a_M\|\|u\|+|b_M| \le (1+\|a_M\|+|b_M|)(1+\|u\|)$  for any  $M \in \mathbb{N}$  and  $u \in \mathbb{R}^m$ , it follows for every  $\alpha \in \mathbb{N}_{0,k}^m$ ,  $u \in U$ , and  $M \in \mathbb{N}$  that

$$\begin{aligned} \left\| y_{M} \rho^{(|\alpha|)} \left( a_{M}^{\top} u - b_{M} \right) a_{M}^{\alpha} \right\| &\leq \left\| y_{M} a_{M}^{\alpha} \right\| \left| \rho^{(|\alpha|)} \left( a_{M}^{\top} u - b_{M} \right) \right| \\ &\leq \left\| y_{M} a_{M}^{\alpha} \right\| \left\| \rho \right\|_{C_{pol,\gamma}^{k}(\mathbb{R})} \left( 1 + \left| a_{M}^{\top} u - b_{M} \right| \right)^{\gamma} \\ &\leq \left\| y_{M} a_{M}^{\alpha} \right\| \left( 1 + \left\| a_{M} \right\| + \left| b_{M} \right| \right)^{\gamma} \left\| \rho \right\|_{C_{pol,\gamma}^{k}(\mathbb{R})} (1 + \left\| u \right\| \right)^{\gamma} \\ &\leq C_{y,a,b} \| \rho \|_{C_{pol,\gamma}^{k}(\mathbb{R})} (1 + \left\| u \right\| )^{\gamma}. \end{aligned}$$
(74)

Analogously, we conclude for every  $\alpha \in \mathbb{N}_{0,k}^m$  and  $u \in U$  that

$$\left\| y_{M} \rho^{(|\alpha|)} \left( a_{M}^{\top} u - b_{M} \right) a_{M}^{\alpha} \right\| \leq C_{y,a,b} \| \rho \|_{C^{k}_{pol,\gamma}(\mathbb{R})} (1 + \| u \|)^{\gamma}.$$
(75)

Hence, by using the triangle inequality together with the inequality  $(x + y)^p \leq 2^{p-1} (x^p + y^p)$  for any  $x, y \geq 0$  as well as the inequalities (74) and (75), it follows for every  $\alpha \in \mathbb{N}_{0,k}^m$ ,  $u \in U$ , and  $M \in \mathbb{N}$  that

$$\begin{split} \left\| y \rho^{(|\alpha|)} \left( a^{\top} u - b \right) a^{\alpha} - y_M \rho^{(|\alpha|)} \left( a^{\top}_M u - b_M \right) a^{\alpha}_M \right\|^p \\ &\leq 2^{p-1} \left( \left\| y \rho^{(|\alpha|)} \left( a^{\top} u - b \right) a^{\alpha} \right\|^p + \left\| y_M \rho^{(|\alpha|)} \left( a^{\top}_M u - b_M \right) a^{\alpha}_M \right\|^p \right) \\ &\leq 2^p C^p_{y,a,b} \| \rho \|^p_{C^k_{pol,\gamma}(\mathbb{R})} (1 + \| u \|)^{\gamma p}. \end{split}$$
(76)

Thus, by applying the  $\mathbb{R}^d$ -valued dominated convergence theorem in [Hytönen et al., 2016, Proposition 1.2.5] (with (76) and  $\int_U (1 + ||u||)^{\gamma p} w(u) du = \left(C_{U,w}^{(\gamma,p)}\right)^p < \infty$  by assumption), we have

$$\lim_{M \to \infty} \left\| y\rho\left(a^{\top} \cdot -b\right) - y_M\rho\left(a_M^{\top} \cdot -b_M\right) \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} \\ = \left( \sum_{\alpha \in \mathbb{N}_{0,k}^m} \lim_{M \to \infty} \int_U \left\| y\rho^{(|\alpha|)}\left(a^{\top}u - b\right)a^{\alpha} - y_M\rho^{(|\alpha|)}\left(a_M^{\top}u - b_M\right)a_M^{\alpha} \right\|^p w(u)du \right)^{\frac{1}{p}} = 0.$$

This shows the desired convergence in (73). Hence, by applying the same arguments as in the proof of Proposition 8.3 (with (73) instead of Lemma 8.2 and with  $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$ -measurable  $W : \Omega \to \mathbb{R}^d$ instead of an  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable  $W : \Omega \to \mathbb{R}^d$ , where it holds that  $\mathcal{F}_{A,B} \subseteq \mathcal{F}_0$ ), it follows that  $R_n : \Omega \to W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)$  is  $\mathcal{F}_0$ -strongly measurable.  $\Box$  Finally, for the approximation rates in Theorem 4.4, we compute the Bochner norm of the random neuron  $R_n^f: \Omega \to W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  with linear  $W_n^f: \Omega \to \mathbb{R}^d$  also used in Proposition 9.3.

**Proposition 9.6.** For  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ),  $\gamma \in (0, \infty)$ , let  $w : U \to [0, \infty)$ be a weight such that the constant  $C_{U,w}^{(\gamma,p)} > 0$  defined in (8) is finite. Moreover, for  $\nu \in (0, \infty)$ , let  $(A_1, B_1) \sim \theta_A \otimes t_1(\nu)$  satisfy Assumption 4.1, and let  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C_{pol,\gamma}^k(\mathbb{R})$  be madmissible. Then, there exists a constant  $C_{31} > 0$  (independent of  $m, d \in \mathbb{N}$ ) such that for every  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform satisfying

$$C_{f} := \max_{\substack{j \in \mathbb{N}_{0} \cap [0, [\gamma] + [\nu] + 1], \\ \beta \in \mathbb{N}_{0, [\gamma] + [\nu] + 1}^{m}}} \int_{\mathbb{R}} \frac{\left| \widehat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{r}}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \widehat{f}(\xi) \|^{r} \frac{(1 + \|\xi/\zeta\|)^{(k+2[\gamma] + [\nu] + 2)r}}{\theta_{A}(\xi/\zeta)^{r-1}} d\xi \right)^{\frac{1}{r}} d\zeta < \infty,$$
(77)

the map

$$\Omega \ni \omega \quad \mapsto \quad R_n^f(\omega) := W_n^f(\omega) \rho \left( A_n(\omega)^\top \cdot -B_n(\omega) \right) \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$$
(78)

with

$$\Omega \ni \omega \quad \mapsto \quad W_n^f(\omega) := \begin{cases} \operatorname{Re}\left(\frac{(\mathfrak{R}_{\psi}f)(\frac{A_n(\omega)}{\|A_n(\omega)\|}, \frac{1}{\|A_n(\omega)\|}, \frac{B_n(\omega)}{\|A_n(\omega)\|})}{C_m^{(\psi,\rho)}\theta_A(A_n(\omega))\theta_B(B_n(\omega))}\right), & \text{if } A_n(\omega) \neq 0, \\ 0, & \text{if } A_n(\omega) = 0, \end{cases}$$
(79)

satisfies  $R_n^f \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d))$  with

$$\left\|R_{n}^{f}\right\|_{L^{r}(\Omega,\mathcal{F},\mathbb{P};W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d}))} \leqslant C_{31}\frac{C_{U,w}^{(\gamma,p)}C_{f}}{\left|C_{m}^{(\psi,\rho)}\right|}m^{\frac{k}{p}+\lceil\gamma\rceil+\lceil\nu\rceil+1}.$$
(80)

Proof. Fix some  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform and finite constant  $C_f \ge 0$  defined in (77). Then, we define  $R_n^f : \Omega \to W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ as in (78) with linear readout  $W_n^f : \Omega \to \mathbb{R}^d$  as in (79). Hence, by using that  $W_n^f : \Omega \to \mathbb{R}^d$  is by definition  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^d)$ -measurable and Lemma 9.5 (with  $\mathcal{F}_0 := \mathcal{F}_{A,B}$ ), it follows that  $R_n^f : \Omega \to W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  is well-defined as  $\mathcal{F}_{A,B}$ -strongly measurable map in  $\mathcal{RN}_{U,d}^{\rho}$  with values in the separable Banach space  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$ .

In order to show the inequality (80), we fix some  $c \in \{0, [\gamma] + [\nu] + 1\}$ . Then, by using Lemma 9.2 (ii), *c*-times integration by parts, the Leibniz product rule together with the chain rule, the substitution  $\zeta \mapsto \xi s$ , and the inequality  $|(v/s)^{\beta}| \leq \prod_{l=1}^{\beta} |v_l|^{\beta_l}/s^{\beta_l} \leq \prod_{l=1}^{\beta} ||v||^{\beta_l}/s^{\beta_l} = (1 + 1/s)^{|\beta|} \leq (1 + 1/s)^c$  for any  $v \in \mathbb{S}^{m-1}$ ,  $s \in (0, \infty)$ , and  $\beta \in \mathbb{N}_{0,c}^m$ , it follows for every  $(v, s, t) \in \mathbb{Y}^{m+1}$  that

$$\begin{split} \frac{|t|^c}{s^c} \left\| (\mathfrak{R}_{\psi} f)(v,s,t) \right\| &= \frac{1}{2\pi} \frac{1}{s^c} \left\| \int_{\mathbb{R}} \widehat{f}(\xi v) \overline{\widehat{\psi}(\xi s)} \frac{\partial^c}{\partial \xi^c} \left( e^{i\xi t} \right) d\xi \right\| \\ &= \frac{1}{2\pi} \frac{1}{s^c} \left\| \int_{\mathbb{R}} \frac{\partial^c}{\partial \xi^c} \left( \widehat{f}(\xi v) \overline{\widehat{\psi}(\xi s)} \right) e^{i\xi t} d\xi \right\| \\ &= \frac{1}{2\pi} \frac{1}{s^c} \left\| \sum_{\beta \in \mathbb{N}_{0,c}^m} \frac{c!}{|\beta|!(c-|\beta|)!} \int_{\mathbb{R}} v^{\beta} \partial_{\beta} \widehat{f}(\xi v) \overline{\widehat{\psi}^{(c-|\beta|)}(\xi s)} s^{c-|\beta|} e^{i\xi t} d\xi \right\| \\ &\leqslant \frac{1}{2\pi} \sum_{\beta \in \mathbb{N}_{0,c}^m} \frac{c!}{|\beta|!(c-|\beta|)!} \int_{\mathbb{R}} \left| \left( \frac{v}{s} \right)^{\beta} \right| \left\| \partial_{\beta} \widehat{f}(\xi v) \right\| \left| \widehat{\psi}^{(c-|\beta|)}(\xi) \right| d\xi \\ &= \frac{1}{2\pi} \sum_{\beta \in \mathbb{N}_{0,c}^m} \frac{c!}{|\beta|!(c-|\beta|)!} \int_{\mathbb{R}} \left| \left( \frac{v}{s} \right)^{\beta} \right| \left\| \partial_{\beta} \widehat{f} \left( \frac{\zeta v}{s} \right) \right\| \left| \widehat{\psi}^{(c-|\beta|)}(\zeta) \right| \frac{1}{s} d\zeta \\ &\leqslant \frac{c!}{2\pi} \left( 1 + \frac{1}{s} \right)^{c+1} \sum_{\beta \in \mathbb{N}_{0,c}^m} \int_{\mathbb{R}} \left\| \partial_{\beta} \widehat{f} \left( \frac{\zeta v}{s} \right) \right\| \left| \widehat{\psi}^{(c-|\beta|)}(\zeta) \right| d\zeta. \end{split}$$

Hence, by using this, Minkowski's integral inequality (with measure spaces  $(\mathbb{R}^m \setminus \{0\}, \mathcal{L}(\mathbb{R}^m \setminus \{0\}), da)$ and  $(\mathbb{N}_{0,k}^m \times \mathbb{R}, \mathcal{P}(\mathbb{N}_{0,k}^m) \otimes \mathcal{B}(\mathbb{R}), \mu \otimes d\zeta)$ , where  $\mathcal{P}(\mathbb{N}_{0,k}^m)$  denotes the power set of  $\mathbb{N}_{0,k}^m$ , and where  $\mathcal{P}(\mathbb{N}_{0,k}^m) \ni E \mapsto \mu(E) := \sum_{\alpha \in \mathbb{N}_{0,k}^m} \mathbb{1}_E(\alpha) \in [0,\infty)$  is the counting measure), the substitution  $\xi \mapsto \zeta a$  with Jacobi determinant  $d\xi = |\zeta|^m da$ , and the constant  $C_f \ge 0$  in (77), it follows for every  $b \in \mathbb{R}$  that

$$\begin{split} \left( \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}}{\theta_{A}(a)^{r-1}} \left( |b|^{c} \left\| (\mathfrak{R}_{\psi}f) \left( \frac{a}{\|a\|}, \frac{1}{\|a\|}, \frac{b}{\|a\|} \right) \right\| \right)^{r} da \right)^{\frac{1}{r}} \\ & \leq \frac{c!}{2\pi} \left( \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k+c+1)r}}{\theta_{A}(a)^{r-1}} \left( \sum_{\beta\in\mathbb{N}_{0,c}^{m}} \int_{\mathbb{R}} \left\| \partial_{\beta}\widehat{f}(\zeta a) \right\| \left| \widehat{\psi}^{(c-|\beta|)}(\zeta) \right| d\zeta \right)^{r} da \right)^{\frac{1}{r}} \\ & \leq \frac{c!}{2\pi} \sum_{\beta\in\mathbb{N}_{0,c}^{m}} \int_{\mathbb{R}} \left| \widehat{\psi}^{(c-|\beta|)}(\zeta) \right| \left( \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{([\gamma]+k+c+1)r}}{\theta_{A}(a)^{r-1}} \| \partial_{\beta}\widehat{f}(\zeta a)\|^{r} da \right)^{\frac{1}{r}} d\zeta \end{split} \tag{81} \\ & = \frac{c!}{2\pi} \sum_{\beta\in\mathbb{N}_{0,c}^{m}} \int_{\mathbb{R}} \frac{\left| \widehat{\psi}^{(c-|\beta|)}(\zeta) \right|}{|\zeta|^{\frac{m}{r}}} \left( \int_{\mathbb{R}^{m}\setminus\{0\}} \| \partial_{\beta}\widehat{f}(\xi)\|^{r} \frac{(1+\|\xi/\zeta\|)^{([\gamma]+k+c+1)r}}{\theta_{A}(\xi/\zeta)^{r-1}} d\xi \right)^{\frac{1}{r}} d\zeta \\ & \leq \frac{c!}{2\pi} \left| \mathbb{N}_{0,c}^{m} \right| C_{f}. \end{split}$$

Thus, by using the inequality  $(x + y)^{[\gamma] + [\nu] + 1} \leq 2^{[\gamma + \nu]} (x^{[\gamma] + [\nu] + 1} + y^{[\gamma] + [\nu] + 1})$  for any  $x, y \geq 0$ , Minkowski's inequality, the inequality (81) with c = 0 and  $c = [\gamma] + [\nu] + 1$ , that  $|\mathbb{N}_{0,0}^m| = 1$ , and that  $|\mathbb{N}_{0,[\gamma] + [\nu] + 1}^m| = \sum_{j=0}^{[\gamma] + [\nu] + 1} m^j \leq 2m^{[\gamma] + [\nu] + 1}$ , we conclude for every  $b \in \mathbb{R}$  that

$$\begin{split} &\left(\int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{(\gamma+\nu+1)r} \left\| (\mathfrak{R}_{\psi}f) \left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right) \right\|^{r}}{\theta_{A}(a)^{r-1}} da \right)^{\frac{1}{r}} \\ & \leqslant \left(\int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{([\gamma]+k)r}(1+|b|)^{([\gamma]+[\nu]+1)r} \left\| (\mathfrak{R}_{\psi}f) \left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right) \right\|^{r}}{\theta_{A}(a)^{r-1}} da \right)^{\frac{1}{r}} \\ & \leqslant 2^{[\gamma+\nu]} \left(\int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r} \left(1+|b|^{[\gamma]+[\nu]+1}\right)^{r} \left\| (\mathfrak{R}_{\psi}f) \left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right) \right\|^{r}}{\theta_{A}(a)^{r-1}} da \right)^{\frac{1}{r}} \\ & \leqslant 2^{[\gamma+\nu]} \left( \left(\int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}}{\theta_{A}(a)^{r-1}} \left\| (\mathfrak{R}_{\psi}f) \left(\frac{a}{\|a\|},\frac{1}{\|a\|}\right) \right\|^{r} da \right)^{\frac{1}{r}} \\ & + \left(\int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}}{\theta_{A}(a)^{r-1}} \left(|b|^{[\gamma]+[\nu]+1} \left\| (\mathfrak{R}_{\psi}f) \left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right) \right\| \right)^{r} da \right)^{\frac{1}{r}} \right) \\ & \leqslant 2^{[\gamma+\nu]} \left(\frac{0!}{2\pi} \left\| \mathbb{N}_{0,0}^{m} \right\| C_{f} + \frac{([\gamma]+[\nu]+1)!}{2\pi} \right\| \mathbb{N}_{0,[\gamma]+[\nu]+1}^{m} \left\| C_{f} \right) \\ & \leqslant 2^{[\gamma+\nu]} \left(\frac{1}{2\pi} C_{f} + \frac{([\gamma]+[\nu]+1)!}{2\pi} 2m^{[\gamma]+[\nu]+1} C_{f} \right) \\ & \leqslant 2^{[\gamma+\nu]+2} ([\gamma]+[\nu]+1)!m^{[\gamma]+[\nu]+1} C_{f}. \end{split}$$

(82) Moreover, we use that  $\rho \in C_{pol,\gamma}^k(\mathbb{R})$ , i.e. that  $|\rho^{(j)}(s)| \leq \|\rho\|_{C_{pol,\gamma}^k(\mathbb{R})}(1+|s|)^{\gamma}$  for any j = 0, ..., k and  $s \in \mathbb{R}$ , the inequality  $1 + |a^{\top}u - b| \leq 1 + \|a\| \|u\| + |b| \leq (1 + \|a\| + |b|)(1 + \|u\|)$  for any  $a, u \in \mathbb{R}^m$  and  $b \in \mathbb{R}$ , and the finite constant  $C_{U,w}^{(\gamma,p)} > 0$  to conclude for every  $a \in \mathbb{R}^m$ ,  $b \in \mathbb{R}$ , and j = 0, ..., k that

$$\left( \int_{U} \left| \rho^{(j)} \left( a^{\top} u - b \right) \right|^{p} w(u) du \right)^{\frac{1}{p}} \leq \|\rho\|_{C^{k}_{pol,\gamma}(\mathbb{R})} \left( \int_{U} \left( 1 + \left| a^{\top} u - b \right| \right)^{\gamma p} w(u) du \right)^{\frac{1}{p}}$$

$$\leq \|\rho\|_{C^{k}_{pol,\gamma}(\mathbb{R})} (1 + \|a\| + |b|)^{\gamma} \left( \int_{U} \left( 1 + \|u\| \right)^{\gamma p} w(u) du \right)^{\frac{1}{p}}$$

$$= \|\rho\|_{C^{k}_{pol,\gamma}(\mathbb{R})} (1 + \|a\| + |b|)^{\gamma} C^{(\gamma,p)}_{U,w}.$$

$$(83)$$

In addition, by using the constant  $C_{\nu} := \frac{\Gamma((1+\nu)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}} > 0$  and the inequality  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$  for any  $x, y \geq 0$ , it follows for every  $b \in \mathbb{R}$  that

$$\frac{1}{\theta_B(b)} = C_{\nu}^{-1} \left( 1 + \frac{|b|^2}{\nu} \right)^{\frac{1+\nu}{2}} \leqslant C_{\nu}^{-1} \left( 1 + \frac{|b|}{\sqrt{\nu}} \right)^{1+\nu} \leqslant C_{\nu}^{-1} \min\left( 1, \sqrt{\nu} \right)^{-1} \left( 1 + |b| \right)^{1+\nu}.$$
 (84)

Hence, by using that  $(A_n, B_n) \sim (A_1, B_1) \sim \theta_A \otimes t_1(\nu)$  is identically distributed with probability density functions  $\theta_A : \mathbb{R}^m \to (0, \infty)$  and  $\theta_B : \mathbb{R} \to (0, \infty)$  (see Assumption 4.1), the inequality (83), that  $|a^{\alpha}| = \prod_{l=1}^{m} |a_l|^{\alpha_l} \leq \prod_{l=1}^{m} (1 + ||a||)^{\alpha_l} \leq (1 + ||a||)^k$  for any  $\alpha \in \mathbb{N}_{0,k}^m$  and  $a \in \mathbb{R}^m$ , that  $|\mathbb{N}_{0,k}^m| = \sum_{j=0}^k m^j \leq 2m^k$ , the inequality (84), the inequality (82), and the constant  $C_{31} := \frac{2^{1/p}}{C_{\nu}\min(1,\sqrt{\nu})} \|\rho\|_{C_{pol,\gamma}^k}(\mathbb{R}) \frac{2^{[\gamma+\nu]+2}}{2\pi} ([\gamma] + [\nu] + 1)! > 0$  (independent of  $m, d \in \mathbb{N}$ ), we have

$$\begin{split} & \left\| R_{n}^{f} \right\|_{L^{r}(\Omega,\mathcal{F},\mathbb{P};W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d}))} = \mathbb{E} \left[ \left\| W_{n}^{f}\rho\left(A_{n}^{\top}\cdot-B_{n}\right) \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{r} \right]^{\frac{1}{r}} \\ & = \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{m}\setminus\{0\}} \left( \sum_{\alpha\in\mathbb{N}_{0,k}^{m}} \int_{U} \left\| \frac{(\mathfrak{R}\psi f)\left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right)}{C_{m}^{(\psi,\rho)}\theta_{A}(a)\theta_{B}(b)} \rho^{(|\alpha|)}\left(a^{\top}u-b\right)a^{\alpha} \right\|^{p} w(u)du \right)^{\frac{r}{p}} \theta_{A}(a)\theta_{B}(b)dadb \right)^{\frac{1}{r}} \\ & \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{m}\setminus\{0\}} \left( \sum_{\alpha\in\mathbb{N}_{0,k}^{m}} \frac{|a^{\alpha}|^{p} \left\| (\mathfrak{R}\psi f)\left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right) \right\|^{p}}{|C_{m}^{(\psi,\rho)}|^{p}} \theta_{A}(a)^{p}\theta_{B}(b)^{p}} \int_{U} \left| \rho^{(|\alpha|)}\left(a^{\top}u-b\right) \right|^{p} w(u)du \right)^{\frac{r}{p}} \theta_{A}(a)\theta_{B}(b)dadb \right)^{\frac{1}{r}} \\ & \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{m}\setminus\{0\}} \left( \sum_{\alpha\in\mathbb{N}_{0,k}^{m}} \frac{|a^{\alpha}|^{p} \left(1+\|a\|+|b|\right)^{\gamma p}}{|C_{m}^{(\psi,\rho)}|^{p}} \left\| (\mathfrak{R}\psi f)\left(\frac{a}{\|a\|},\frac{1}{\|a\|},\frac{b}{\|a\|}\right) \right\|^{p}} \right)^{\frac{r}{p}} \theta_{B}(b)dadb \right)^{\frac{1}{r}} \\ & \leq \frac{\|\rho\|_{C_{pd,\gamma}^{k}(\mathbb{R})}C_{U,w}^{(\gamma,p)}}{|C_{m}^{(\psi,\rho)}|} \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{\gamma r}}{\theta_{A}(a)^{(r-1/r)p}\theta_{B}(b)^{r}} \right)^{\frac{r}{p}} \theta_{B}(b)dadb \right)^{\frac{1}{r}} \\ & \leq \frac{\left| \mathbb{N}_{0,k}^{m} \right|^{\frac{1}{p}} \|\rho\|_{C_{pd,\gamma}^{k}(\mathbb{R})}C_{U,w}^{(\gamma,p)}}{|C_{m}^{(\psi,\rho)}|} \sup_{b\in\mathbb{R}} \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{(\gamma+\nu+1)r}}{\theta_{A}(a)^{r-1}\theta_{B}(b)^{r}} da \right)^{\frac{1}{r}} \\ & \leq \frac{\left| \mathbb{N}_{0,k}^{m} \right|^{\frac{1}{p}} \|\rho\|_{C_{pd,\gamma}^{k}(\mathbb{R})}C_{U,w}^{(\gamma,p)}}{|C_{m}^{(\psi,\rho)}|} \sup_{b\in\mathbb{R}} \left( \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{(\gamma+\nu+1)r}}{\theta_{A}(a)^{r-1}} \right)^{\frac{1}{p}} da \right)^{\frac{1}{r}} \\ & \leq \frac{\left| \frac{1}{2^{m}} \frac{m}{p} \right| \left| \frac{1}{p} \right|_{C_{pd,\gamma}^{k}(\mathbb{R})}C_{U,w}^{(\gamma,p)}}{|C_{m}^{(\psi,\rho)}|}} \sup_{e\in\mathbb{R}} \int_{\mathbb{R}^{m}\setminus\{0\}} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{(\gamma+\nu+1)r}}{\theta_{A}(a)^{r-1}} \right| \left( \frac{1}{p} \right)^{\frac{1}{p}} da \right)^{\frac{1}{r}} \\ & \leq \frac{2^{\frac{1}{p}} \frac{m}{p} \right| \left| \frac{1}{p} \right|_{C_{pd,\gamma}^{k}(\mathbb{R})}C_{U,w}^{(\gamma,p)}}{|C_{m}^{(\psi,\rho)}|} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{(\gamma+\nu+1)r}}{2\pi} \left( \frac{1}{p} \right)^{\frac{1}{p}} da \right)^{\frac{1}{r}} da \right)^{\frac{1}{r}} \\ & \leq \frac{2^{\frac{1}{p}} \frac{m}{p} \right|_{C_{pd,\gamma}^{k}(\mathbb{R})}C_{U,w}^{(\gamma,p)}}{|C_{m}^{(\psi,\rho)}|} \frac{(1+\|a\|)^{(\gamma+k)r}(1+|b|)^{(\gamma+\nu+1)r}}{2\pi}$$

which completes the proof.

9.3. **Proof of Theorem 4.4.** In this section, we prove the approximation rates in Theorem 4.4. Let us first sketch the main ideas of the proof. To this end, we fix some  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$  with  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform and finite constant  $C_f \ge 0$  defined in (8). Then, we define the random neural network  $\Phi_N := \frac{1}{N} \sum_{n=1}^N R_n^f \in \mathcal{RN}_{U,d}^{\rho}$  as average of the random neurons  $\Omega \ni \omega \mapsto R_n^f(\omega) := W_n^f(\omega)\rho(A_n(\omega)^\top \cdot -B_n(\omega)) \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  defined in (78), where  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $w : U \to [0, \infty)$  is a weight. Then, by using the integral representation in Proposition 9.3, i.e. that  $f = \mathbb{E}[R_n^f]$ , a symmetrization argument with Rademacher averages, and the Banach space type of  $W^{k,p}(U, \mathcal{L}(U), w, \mathbb{R}^d)$ , we obtain the inequality

$$\begin{split} \mathbb{E}\left[\left\|f - \Phi_{N}\right\|_{W^{k,p}(U,\mathcal{L}(U),w,\mathbb{R}^{d})}^{r}\right]^{\frac{1}{r}} &= \frac{1}{N}\mathbb{E}\left[\left\|\sum_{n=1}^{N}\left(\mathbb{E}\left[R_{n}^{f}\right] - R_{n}^{f}\right)\right\|_{W^{k,p}(U,\mathcal{L}(U),w,\mathbb{R}^{d})}^{r}\right] \\ &\leqslant C\frac{\left\|R_{n}^{f}\right\|_{L^{r}(\Omega,\mathcal{F},\mathbb{P};W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d}))}}{N^{1-\frac{1}{\min(2,p,r)}}, \end{split}$$

where C > 0 is a constant and where  $||R_n^f||_{L^r(\Omega, \mathcal{F}, \mathbb{P}; W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d))}$  is given in Proposition 9.3.

First, we recall the notion of Banach space types and refer to [Albiac and Kalton, 2006, Section 6.2], [Ledoux and Talagrand, 1991, Chapter 9], and [Hytönen et al., 2016, Section 4.3.b] for more details.

**Definition 9.7** ([Hytönen et al., 2016, Definition 4.3.12 (1)]). A Banach space  $(X, \|\cdot\|_X)$  is called of type  $t \in [1, 2]$  if there exists a constant  $C_X > 0$  such that for every  $N \in \mathbb{N}$ ,  $(f_n)_{n=1,...,N} \subseteq X$ , and Rademacher sequence  $(\epsilon_n)_{n=1,...,N}$  (i.e. i.i.d. random variables  $(\epsilon_n)_{n=1,...,N}$  defined on a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  with  $\tilde{\mathbb{P}}[\epsilon_n = \pm 1] = 1/2$ ), it holds that

$$\widetilde{\mathbb{E}}\left[\left\|\sum_{n=1}^{N}\epsilon_{n}f_{n}\right\|_{X}^{t}\right]^{\frac{1}{t}} \leq C_{X}\left(\sum_{n=1}^{N}\|f_{n}\|_{X}^{t}\right)^{\frac{1}{t}}.$$

Then, by [Albiac and Kalton, 2006, Remark 6.2.11 (b)+(c)], every Banach space  $(X, \|\cdot\|_X)$  is of type t = 1 with constant  $C_X = 1$ , whereas only some Banach spaces have non-trivial type  $t \in (1, 2]$ , e.g., every Hilbert space  $(X, \|\cdot\|_X)$  is of type t = 2 with constant  $C_X = 1$ .

**Lemma 9.8.** Let  $(X, \|\cdot\|_X)$  be a Banach space of type  $t \in [1, 2]$  with constant  $C_X > 0$ , and let  $t' \in [1, t]$ . Then,  $(X, \|\cdot\|_X)$  is a Banach space of type t' with constant  $C_X > 0$ .

*Proof.* Let  $(X, \|\cdot\|_X)$  be a Banach space of type  $t \in [1, 2]$  with constant  $C_X > 0$ , and let  $t' \in [1, t]$ . Moreover, we fix some  $N \in \mathbb{N}$ ,  $(f_n)_{n=1,...,N} \subseteq X$ , and an i.i.d. sequence  $(\epsilon_n)_{n=1,...,N}$  defined on a probability space  $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$  such that  $\widetilde{\mathbb{P}}[\epsilon_n = \pm 1] = 1/2$ . Then, by applying Jensen's inequality and the inequality  $(\sum_{n=1}^N x_n)^{t'/t} \leq \sum_{n=1}^N x_n^{t'/t}$  for any  $x_1, ..., x_N \geq 0$ , it follows that

$$\widetilde{\mathbb{E}}\left[\left\|\sum_{n=1}^{N}\epsilon_{n}f_{n}\right\|_{X}^{t'}\right]^{\frac{1}{t'}} \leqslant \widetilde{\mathbb{E}}\left[\left\|\sum_{n=1}^{N}\epsilon_{n}f_{n}\right\|_{X}^{t}\right]^{\frac{1}{t}} \leqslant C_{X}\left(\sum_{n=1}^{N}\|f_{n}\|_{X}^{t}\right)^{\frac{1}{t}} \leqslant C_{X}\left(\sum_{n=1}^{N}\|f_{n}\|_{X}^{t'}\right)^{\frac{1}{t'}}$$

This shows that  $(X, \|\cdot\|_X)$  is also a Banach space of type  $t' \in [1, t]$  with the same constant  $C_X > 0$ .  $\Box$ 

Moreover, [Albiac and Kalton, 2006, Theorem 6.2.14] shows that  $(L^p(U, \Sigma, \mu; \mathbb{R}^d), \|\cdot\|_{L^p(U, \Sigma, \mu; \mathbb{R}^d)})$ introduced in Notation (x) is a Banach space of type  $t = \min(2, p)$  with constant  $C_{L^p(U, \Sigma, \mu; \mathbb{R}^d)} > 0$ depending only on  $p \in [1, \infty)$ . Now, we show that this still holds true for the weighted Sobolev space  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$  introduced in Notation (xii)+(xiii).

**Lemma 9.9.** Let  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty)$ ,  $U \subseteq \mathbb{R}^m$  (open, if  $k \ge 1$ ), and  $w : U \to [0, \infty)$  be a weight. Then, the Banach space  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \| \cdot \|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$  introduced in Notation (xii)+(xiii) is of type  $t = \min(2, p)$  with constant  $C_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)} > 0$  depending only on  $p \in [1, \infty)$ .

*Proof.* First, we recall that  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$  is a Banach space. Indeed, this follows from [Rudin, 1987, p. 96] (for k = 0) and [Adams, 1975, Theorem 3.2] (for  $k \ge 1$ ).

Now, we fix some  $N \in \mathbb{N}$ ,  $(f_n)_{n=1,\dots,N} \subseteq W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ , and an i.i.d. sequence  $(\epsilon_n)_{n=1,\dots,N}$  defined on a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P})$  such that  $\mathbb{P}[\epsilon_n = \pm 1] = 1/2$ . Then, by using Fubini's theorem and the classical Khintchine inequality in [Ledoux and Talagrand, 1991, Lemma 4.1] with constant  $C_p > 0$  depending only on  $p \in [1, \infty)$ , it follows that

$$\widetilde{\mathbb{E}}\left[\left\|\sum_{n=1}^{N}\epsilon_{n}f_{n}\right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{p}\right]^{\frac{1}{p}} = \widetilde{\mathbb{E}}\left[\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\int_{U}\left\|\sum_{n=1}^{N}\epsilon_{n}\partial_{\alpha}f_{n}(u)\right\|^{p}w(u)du\right]^{\frac{1}{p}} \\ = \left(\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\int_{U}\widetilde{\mathbb{E}}\left[\left\|\sum_{n=1}^{N}\epsilon_{n}\partial_{\alpha}f_{n}(u)\right\|^{p}\right]w(u)du\right)^{\frac{1}{p}} \\ \leqslant C_{p}\left(\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\int_{U}\left(\sum_{n=1}^{N}\|\partial_{\alpha}f_{n}(u)\|^{2}\right)^{\frac{p}{2}}w(u)du\right)^{\frac{1}{p}}.$$
(85)

If  $p \in [1,2]$ , we use (85) and the inequality  $\left(\sum_{n=1}^{N} x_n\right)^{p/2} \leq \sum_{n=1}^{N} x_n^{p/2}$  for any  $x_1, ..., x_N \geq 0$  to conclude that

$$\begin{split} \widetilde{\mathbb{E}} \left[ \left\| \sum_{n=1}^{N} \epsilon_n f_n \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^{\min(2,p)} \right]^{\frac{1}{\min(2,p)}} &= \widetilde{\mathbb{E}} \left[ \left\| \sum_{n=1}^{N} \epsilon_n f_n \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^p \right]^{\frac{1}{p}} \\ &\leq C_p \left( \sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \left( \sum_{n=1}^{N} \|\partial_\alpha f_n(u)\|^2 \right)^{\frac{p}{2}} w(u) du \right)^{\frac{1}{p}} \\ &\leq C_p \left( \sum_{n=1}^{N} \sum_{\alpha \in \mathbb{N}_{0,k}^m} \int_U \|\partial_\alpha f_n(u)\|^p w(u) du \right)^{\frac{1}{p}} \\ &= C_p \left( \sum_{n=1}^{N} \|f_n\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^{\min(2,p)} \right)^{\frac{1}{\min(2,p)}}. \end{split}$$

This shows for  $p \in [1, 2]$  that the Banach space  $(W^{k, p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \| \cdot \|_{W^{k, p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$  is of type  $t = \min(2, p)$ , where the constant  $C_p > 0$  depends only on  $p \in [1, \infty)$ .

Otherwise, if  $p \in (2, \infty)$ , we consider the measure spaces  $(\{1, ..., N\}, \mathcal{P}(\{1, ..., N\}), \eta)$  and  $(\mathbb{N}_{0,k}^m \times U, \mathcal{P}(\mathbb{N}_{0,k}^m) \otimes \mathcal{L}(U), \mu \otimes w)$ , where  $\mathcal{P}(\{1, ..., N\})$  and  $\mathcal{P}(\mathbb{N}_{0,k}^m)$  denote the power sets of  $\{1, ..., N\}$  and  $\mathbb{N}_{0,k}^m$ , respectively, and where  $\mathcal{P}(\{1, ..., N\}) \ni A \mapsto \eta(A) := \sum_{n=1}^N \mathbb{1}_A(n) \in [0, \infty)$  and  $\mathcal{P}(\mathbb{N}_{0,k}^m) \otimes \mathcal{L}(U) \ni (A, B) \mapsto (\mu \otimes w)(A, B) := \left(\sum_{\alpha \in \mathbb{N}_{0,k}^m} \mathbb{1}_A(\alpha)\right) \int_B w(u) du \in [0, \infty]$  are both measures. Then, by using the Minkowski inequality in [Hytönen et al., 2016, Proposition 1.2.22] with  $p \ge 2$ , it follows for every  $\mathbf{f} \in L^2(\{1, ..., N\}, \mathcal{P}(\{1, ..., N\}), \eta; L^p(\mathbb{N}_{0,k}^m \times U, \mathcal{P}(\mathbb{N}_{0,k}^m) \otimes \mathcal{L}(U), \mu \otimes w; \mathbb{R}^d))$  that

$$\mathbf{f}\|_{L^{p}(\mathbb{N}_{0,k}^{m}\times U,\mathcal{P}(\mathbb{N}_{0,k}^{m})\otimes\mathcal{L}(U),\mu\otimes w;L^{2}(\{1,...,N\},\mathcal{P}(\{1,...,N\}),\eta;\mathbb{R}^{d}))} \\ \leqslant \|\mathbf{f}\|_{L^{2}(\{1,...,N\},\mathcal{P}(\{1,...,N\}),\eta;L^{p}(\mathbb{N}_{0,k}^{m}\times U,\mathcal{P}(\mathbb{N}_{0,k}^{m})\otimes\mathcal{L}(U),\mu\otimes w;\mathbb{R}^{d}))}.$$

$$(86)$$

Now, we define the map  $\{1, ..., N\} \times (\mathbb{N}_{0,k}^m \times U) \ni (n; \alpha, u) \mapsto \mathbf{f}(n; \alpha, u) := \partial_{\alpha} f_n(u) \in \mathbb{R}^d$  satisfying

$$\|\mathbf{f}\|_{L^{2}(\{1,...,N\},\mathcal{P}(\{1,...,N\}),\eta;L^{p}(\mathbb{N}_{0,k}^{m}\times U,\mathcal{P}(\mathbb{N}_{0,k}^{m})\otimes\mathcal{L}(U),\mu\otimes w;\mathbb{R}^{d}))} = \left(\sum_{n=1}^{N}\left(\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\int_{U}\|\partial_{\alpha}f_{n}(u)\|^{p}w(u)du\right)^{\frac{2}{p}}\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{N}\|f_{n}\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{2}\right)^{\frac{1}{2}} < \infty,$$
(87)

which shows that  $\mathbf{f} \in L^2(\{1, ..., N\}, \mathcal{P}(\{1, ..., N\}), \eta; L^p(\mathbb{N}^m_{0,k} \times U, \mathcal{P}(\mathbb{N}^m_{0,k}) \otimes \mathcal{L}(U), \mu \otimes w; \mathbb{R}^d))$ . Hence, by using first Jensen's inequality and then by combining (85) and (86) with (87), we conclude that

$$\begin{split} \widetilde{\mathbb{E}} \left[ \left\| \sum_{n=1}^{N} \epsilon_{n} f_{n} \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{\min(2,p)} \right]^{\frac{1}{\min(2,p)}} &= \widetilde{\mathbb{E}} \left[ \left\| \sum_{n=1}^{N} \epsilon_{n} f_{n} \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{2} \right]^{\frac{1}{2}} \\ &\leq \widetilde{\mathbb{E}} \left[ \left\| \sum_{n=1}^{N} \epsilon_{n} f_{n} \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{p} \right]^{\frac{1}{p}} \\ &\leq C_{p} \left( \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \int_{U} \left( \sum_{n=1}^{N} \|\partial_{\alpha} f_{n}(u)\|^{2} \right)^{\frac{p}{2}} w(u) du \right)^{\frac{1}{p}} \\ &= C_{p} \|\mathbf{f}\|_{L^{p}(\mathbb{N}_{0,k}^{m} \times U,\mathcal{P}(\mathbb{N}_{0,k}^{m}) \otimes \mathcal{L}(U),\mu \otimes w;L^{2}(\{1,\ldots,N\},\mathcal{P}(\{1,\ldots,N\}),\eta;\mathbb{R}^{d}))} \\ &\leq C_{p} \|\mathbf{f}\|_{L^{2}(\{1,\ldots,N\},\mathcal{P}(\{1,\ldots,N\}),\eta;L^{p}(\mathbb{N}_{0,k}^{m} \times U,\mathcal{P}(\mathbb{N}_{0,k}^{m}) \otimes \mathcal{L}(U),\mu \otimes w;\mathbb{R}^{d}))} \\ &= C_{p} \left( \sum_{n=1}^{N} \|f\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d})}^{\min(2,p)} \right)^{\frac{1}{\min(2,p)}} . \end{split}$$
This shows for  $n \in (2,\infty)$  that the Banach space  $(W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^{d}) \parallel 1$  we show  $n \in \mathbb{R}^{n}$  is of

This shows for  $p \in (2, \infty)$  that the Banach space  $(W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d), \|\cdot\|_{W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)})$  is of type  $t = \min(2, p)$ , where the constant  $C_p > 0$  depends only on  $p \in [1, \infty)$ .

Proof of Theorem 4.4. Fix some  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform and finite constant  $C_f \ge 0$  defined in (9). Then, for every fixed  $n \in \mathbb{N}$ , we define the map  $R_n^f : \Omega \to W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  as in (78). Hence, by using Proposition 9.6, it follows that  $R_n^f \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d))$  with Bochner norm bounded by (80).

In order to show (i), we use that  $\psi \in S_0(\mathbb{R}; \mathbb{C})$  is necessarily non-zero (otherwise  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C_{pol,\gamma}^k(\mathbb{R})$  cannot be *m*-admissible), implying that  $\hat{\psi} \in S_0(\mathbb{R}; \mathbb{C})$  is also non-zero (by the injectivity of the Fourier transform) to conclude from  $C_f < \infty$  that there exists some  $\zeta \in \mathbb{R} \setminus \{0\}$  such that

$$\left(\int_{\mathbb{R}^m} \left\| \widehat{f}(\xi) \right\|^r \frac{(1 + \|\xi/\zeta\|)^{(k+2[\gamma] + [\nu] + 2)r}}{\theta_A(\xi/\zeta)^{r-1}} d\xi \right)^{\frac{1}{r}} < \infty.$$
(88)

Thus, by using Hölder's inequality, the substitution  $a \mapsto \xi/\zeta$ , and (88), it follows that

$$\|\widehat{f}\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du;\mathbb{C}^{d})} = \int_{\mathbb{R}^{m}} \left\|\widehat{f}(\xi)\right\| d\xi \leqslant \left(\int_{\mathbb{R}^{m}} \frac{\left\|\widehat{f}(\xi)\right\|^{r}}{\theta_{A}(\xi/z)^{r-1}} d\xi\right)^{\overline{r}} \left(\int_{\mathbb{R}^{m}} \theta_{A}(\xi/\zeta) d\xi\right)^{\frac{r-1}{r}} \\ \leqslant \left(\int_{\mathbb{R}^{m}} \left\|\widehat{f}(\xi)\right\|^{r} \frac{(1+\|\xi/\zeta\|)^{(k+2[\gamma]+[\nu]+2)r}}{\theta_{A}(\xi/\zeta)^{r-1}} d\xi\right)^{\frac{1}{r}} \left(|\zeta| \underbrace{\int_{\mathbb{R}^{m}} \theta_{A}(a) da}_{=1}\right)^{\frac{r-1}{r}} < \infty.$$

$$(89)$$

Hence, by using that  $(\psi, \rho) \in S_0(\mathbb{R}; \mathbb{C}) \times C_{pol,\gamma}^k(\mathbb{R})$  is *m*-admissible together with (89), we can apply Proposition 9.3 to conclude for a.e.  $u \in \mathbb{R}^m$  that  $\mathbb{E}[R_n^f](u) = \mathbb{E}[R_n^f(u)] = f(u)$ . Moreover, if  $k \ge 1$ , we use this, that  $\mathbb{E}[R_n^f] \in W^{k,p}(U, \mathcal{L}(U), du; \mathbb{R}^d)$  by Proposition 9.6, and integration by parts to conclude for every  $\alpha \in \mathbb{N}_{0,k}^m$  and  $g \in C_c^{\infty}(U)$  that

$$\begin{split} \int_{U} \partial_{\alpha} \mathbb{E} \left[ R_{n}^{f} \right](u) g(u) du &= (-1)^{|\alpha|} \int_{U} \mathbb{E} \left[ R_{n}^{f} \right](u) \partial_{\alpha} g(u) du = (-1)^{|\alpha|} \int_{U} f(u) \partial_{\alpha} g(u) du \\ &= \int_{U} \partial_{\alpha} f(u) g(u) du \end{split}$$

This shows for every  $\alpha \in \mathbb{N}_{0,k}^m$  and a.e.  $u \in U$  that  $\partial_{\alpha} \mathbb{E}[R_n^f](u) = \partial_{\alpha} \mathbb{E}[R_n^f(u)] = \partial_{\alpha} f(u)$ , which implies that  $f = \mathbb{E}[R_n^f] \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ .

In order to show (ii), we fix some  $N \in \mathbb{N}$ , Then, by using that  $f = \mathbb{E}[R_n^f] \in W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d)$ , the right-hand side of [Ledoux and Talagrand, 1991, Lemma 6.3] for the independent mean-zero random variables  $\left(\mathbb{E}\left[R_n^f\right] - R_n^f\right)_{n=1,...,N}$  (with i.i.d.  $(\epsilon_n)_{n=1,...,N}$  satisfying  $\mathbb{P}[\epsilon_n = \pm 1] = 1/2$  being independent of  $\left(\mathbb{E}\left[R_n^f\right] - R_n^f\right)_{n=1,...,N}$ ), the Kahane-Khintchine inequality in [Hytönen et al., 2016, Theorem 3.2.23] with constant  $\kappa_{p,r} := \kappa_{r,\min(r,\min(2,p))} > 0$  only depending on  $r \in (1,\infty)$  and  $\min(r,\min(2,p)) \in (1,2]$ , that  $(W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d), \|\cdot\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)})$  is by Lemma 9.9 a Banach space of type  $\min(2,p) \in (1,2]$  (with constant  $C_p := C_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} > 0$  depending only on  $p \in (1,2]$ ), thus by Lemma 9.8 of type  $\min(2,p,r) \in [0,\min(2,p)]$  (with the same constant  $C_p > 0$ ), Jensen's inequality, Minkowski's inequality, [Hytönen et al., 2016, Proposition 1.2.2], the inequality (80), and the constant  $C_1 := 4C_p\kappa_{p,r}C_{31} > 0$  (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ), it follows that

$$\begin{split} \mathbb{E} \left[ \|f - \Phi_N\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r \right]^{\frac{1}{r}} &= \frac{1}{N} \mathbb{E} \left[ \left\| \sum_{n=1}^N \left( \mathbb{E} \left[ R_n^f \right] - R_n^f \right) \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r \right]^{\frac{1}{r}} \\ &\leq \frac{2}{N} \mathbb{E} \left[ \left\| \sum_{n=1}^N \epsilon_n \left( \mathbb{E} \left[ R_n^f \right] - R_n^f \right) \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r \right]^{\frac{1}{r}} \\ &\leq \frac{2\kappa_{p,r}}{N} \mathbb{E} \left[ \left\| \sum_{n=1}^N \epsilon_n \left( \mathbb{E} \left[ R_n^f \right] - R_n^f \right) \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^{\min(r,\min(2,p))} \right]^{\frac{1}{\min(r,\min(2,p))}} \\ &\leq \frac{2C_p \kappa_{p,r}}{N} \mathbb{E} \left[ \left\| \mathbb{E} \left[ R_n^f \right] - R_n^f \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^{\min(2,p,r)} \right] \right)^{\frac{1}{\min(2,p,r)}} \\ &= \frac{2C_p \kappa_{p,r}}{N^{1-\frac{1}{\min(2,p,r)}}} \mathbb{E} \left[ \left\| \mathbb{E} \left[ R_n^f \right] - R_n^f \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^{\min(2,p,r)} \right]^{\frac{1}{r}} \\ &\leq \frac{2C_p \kappa_{p,r}}{N^{1-\frac{1}{\min(2,p,r)}}} \mathbb{E} \left[ \left\| \mathbb{E} \left[ R_n^f \right] - R_n^f \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r \right]^{\frac{1}{r}} \\ &\leq \frac{2C_p \kappa_{p,r}}{N^{1-\frac{1}{\min(2,p,r)}}} \mathbb{E} \left[ \left\| \mathbb{E} \left[ R_n^f \right] - R_n^f \right\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r \right]^{\frac{1}{r}} \\ &\leq \frac{4C_p \kappa_{p,r}}{N^{1-\frac{1}{\min(2,p,r)}}} \left\| R_n^f \right\|_{L^r(\Omega,\mathcal{F},\mathbb{P};W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d))} \\ &\leq \frac{4C_p \kappa_{p,r}}{N^{1-\frac{1}{\min(2,p,r)}}} \left\| R_n^f \right\|_{L^r(\Omega,\mathcal{F},\mathbb{P};W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d))} \\ &\leq C_1 \frac{C_{(\gamma,p)}^{(\gamma,p)} C_f}{(C_m^{(\psi,p)}]} \frac{m^{\frac{k}{p} + |\gamma| + |\nu| + 1}}{N^{1-\frac{1}{\min(2,p,r)}}}. \end{split}$$

This shows the approximation rate in (ii).

Finally, in order to show (iii), we fix some  $\delta, \varepsilon > 0$ . Then, for every  $N \in \mathbb{N}$  satisfying the inequality in (iii) there exists by (ii) some  $\Phi_N \in \mathcal{RN}_{U,d}^{\rho} \cap L^r(\Omega, \mathcal{F}, \mathbb{P}; W^{k,p}(U, \mathcal{L}(U), w; \mathbb{R}^d))$  with  $N \in \mathbb{N}$  neurons such that

$$\mathbb{E}\left[\|f - \Phi_N\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r\right]^{\frac{1}{r}} \leqslant C_1 \frac{C_{U,w}^{(\gamma,p)}C_f}{\left|C_m^{(\psi,\rho)}\right|} \frac{m^{\frac{k}{p} + \lceil\gamma\rceil + \lceil\nu\rceil + \lceil\nu\rceil + 1}}{N^{1 - \frac{1}{\min(2,p,r)}}} \leqslant \delta^{\frac{1}{r}}\varepsilon.$$

Hence, by applying Chebyshev's inequality, it follows that

$$\mathbb{P}\left[\left\{\omega \in \Omega : \|f - \Phi_N(\omega)\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)} \ge \varepsilon\right\}\right] \le \frac{1}{\varepsilon^r} \mathbb{E}\left[\|f - \Phi_N\|_{W^{k,p}(U,\mathcal{L}(U),w;\mathbb{R}^d)}^r\right] \le \frac{\delta\varepsilon^r}{\varepsilon^r} = \delta,$$
  
which completes the proof.

In the following, we also provide the proof of Example 4.6+4.7 and Remark 4.8.

*Proof of Example 4.6.* Fix some  $f := (f_1, ..., f_d)^\top \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + ||u||)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} du; \mathbb{R}^d) \cap W^{m+2k+4\lceil \gamma \rceil + 3\lceil \nu \rceil + 4, 2}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + ||\cdot||)^{2(\lceil \gamma \rceil + \lceil \nu \rceil + 1)}; \mathbb{R}^d)$ . Now, we first show that  $\widehat{f} : \mathbb{R}^m \to \mathbb{C}^d$  is  $([\gamma] + [\nu] + 1)$ -times weakly differentiable. Indeed, for every fixed i = 1, ..., d, we use the polynomial  $\mathbb{R}^{m} \ni u \mapsto u^{\beta} := \prod_{l=1}^{m} u_{l}^{\beta_{l}} \in \mathbb{R} \text{ for } \beta \in \mathbb{N}_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}^{m}, \text{ that } \left| u^{\beta} \right| = \prod_{l=1}^{m} \left| u_{l} \right|^{\beta_{l}} \leqslant \prod_{l=1}^{m} (1 + \|u\|)^{\beta_{l}} \leqslant (1 + \|u\|)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} \text{ for any } u \in \mathbb{R}^{m} \text{ and } \beta \in \mathbb{N}_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}^{m} \text{ to conclude for every } \beta \in \mathbb{N}_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}^{m} \text{ that } \mathbb{R}^{m}$ 

$$\begin{aligned} \|p_{\beta} \cdot f_i\|_{L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)} &= \int_{\mathbb{R}^m} \left| u^{\beta} f_i(u) \right| du \\ &\leqslant \int_{\mathbb{R}^m} |f_i(u)| (1 + \|u\|)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} du \\ &\leqslant \|f\|_{L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + \|u\|)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1} du; \mathbb{R}^d)} < \infty \end{aligned}$$

Hence, by iteratively applying [Folland, 1992, Theorem 7.8. (c)] to every component  $f_i : \mathbb{R}^m \to \mathbb{C}$ ,

i = 1, ..., d, the partial derivative  $\partial_{\beta} \hat{f} = (\partial_{\beta} \hat{f}_1, ..., \partial_{\beta} \hat{f}_d)^{\top} : \mathbb{R}^m \to \mathbb{C}^d$  exists for all  $\beta \in \mathbb{N}^m_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}$ . Next, we show that the constant  $C_f \ge 0$  defined in (9) is finite. To this end, we fix some  $\beta \in \mathbb{N}^m_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}$  and i = 1, ..., d, and define  $s := m + 2k + 4\lceil \gamma \rceil + 3\lceil \nu \rceil + 4 \in \mathbb{N}_0$ . Then, we observe that  $\mathbb{R}^m \ni \mathbb{N}^m_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}$  and i = 1, ..., d, and define  $s := m + 2k + 4\lceil \gamma \rceil + 3\lceil \nu \rceil + 4 \in \mathbb{N}_0$ .  $u \mapsto (p_{\beta} \cdot f_i)(u) := u^{\beta} f_i(u) \in \mathbb{R}$  is k-times weakly differentiable, where  $u^{\beta} := \prod_{l=1}^m u_l^{\beta_l}$ . Moreover, by using the inequality  $\left(\sum_{i=1}^{d} x_i\right)^{1/2} \leq \sum_{i=1}^{d} x^{1/2}$  for all  $x_1, ..., x_d \geq 0$ , the multivariate Leibniz product rule, the finite constant  $C_0 := \sum_{\alpha \in \mathbb{N}_{0,s}^m} \sum_{\lambda_1, \lambda_2 \in \mathbb{N}_{0,s}^m, \lambda_1 + \lambda_2 = \alpha} \frac{\alpha!}{\lambda_1! \lambda_2!} > 0$ , and the inequality  $|u^{\lambda}| = \prod_{l=1}^{m} |u_l|^{\beta_l} \leq \prod_{l=1}^{m} (1 + ||u||)^{\beta_l} \leq (1 + ||u||)^s$  for any  $u \in \mathbb{R}^m$ , it follows that

$$\begin{split} \|p_{\beta} \cdot f_{i}\|_{W^{s,2}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du)} &\leq \left(\sum_{\alpha \in \mathbb{N}_{0,s}^{m}} \int_{\mathbb{R}^{m}} \left|\partial_{\alpha} \left(u^{\beta}f_{i}(u)\right)\right|^{2} du\right)^{\frac{1}{2}} \\ &\leq \sum_{\alpha \in \mathbb{N}_{0,s}^{m}} \left(\int_{\mathbb{R}^{m}} \left|\sum_{\substack{\lambda_{1},\lambda_{2} \in \mathbb{N}_{0}^{m} \\ \lambda_{1}+\lambda_{2}=\alpha}} \frac{\alpha !}{\lambda_{1}!\lambda_{2}!} \partial_{\lambda_{1}} \left(u^{\beta}\right) \partial_{\lambda_{2}}f_{i}(u)\right|^{2} du\right)^{\frac{1}{2}} \\ &\leq \sum_{\alpha \in \mathbb{N}_{0,s}^{m}} \sum_{\substack{\lambda_{1},\lambda_{2} \in \mathbb{N}_{0,s}^{m} \\ \lambda_{1}+\lambda_{2}=\alpha}} \frac{\alpha !}{\lambda_{1}!\lambda_{2}!} \left(\int_{\mathbb{R}^{m}} \left|\frac{\beta !}{(\beta-\lambda_{1})!}u^{\beta-\lambda_{1}}\partial_{\lambda_{2}}f_{i}(u)\right|^{2} du\right)^{\frac{1}{2}} \\ &\leq C_{0}\beta ! \max_{\lambda_{2} \in \mathbb{N}_{0,s}^{m}} \left(\int_{\mathbb{R}^{m}} |\partial_{\lambda_{2}}f_{i}(u)|^{2} (1+\|u\|)^{2(\lceil\gamma\rceil+\lceil\nu\rceil+1)} du\right)^{\frac{1}{2}} \\ &\leq C_{0}\beta ! \left(\sum_{\lambda_{2} \in \mathbb{N}_{0,s}^{m}} \int_{\mathbb{R}^{m}} \|\partial_{\lambda_{2}}f(u)\|^{2} (1+\|u\|)^{2(\lceil\gamma\rceil+\lceil\nu\rceil+1)} du\right)^{\frac{1}{2}} \\ &= C_{0}\beta ! \|f\|_{W^{m+2k+4[\gamma]+3[\nu]+4,2}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|\cdot\|)^{2(\lceil\gamma\rceil+\lceil\nu\rceil+1)};\mathbb{R}^{d})} < \infty, \end{split}$$

which shows that  $p_{\beta} \cdot f_i \in W^{s,2}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$ . Hence, by using [Folland, 1992, Theorem 7.8. (c)], the inequality  $\left(\sum_{i=1}^{d} x_i\right)^{1/2} \leq \sum_{i=1}^{d} x^{1/2}$  for all  $x_1, \dots, x_d \geq 0$  and [Grubb, 2009, Lemma 6.8], i.e. that  $p_{\beta} \cdot f_i \in W^{s,2}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$  implies  $\int_{\mathbb{R}^m} |(\widehat{p_{\beta} \cdot f_i})(\xi)| (1 + ||\xi||^2)^{s/2} d\xi < \infty$ , we conclude that

$$\left(\int_{\mathbb{R}^m} \|\partial_{\beta}\widehat{f}(\xi)\|^2 \left(1 + \|\xi\|^2\right)^{\frac{s}{2}} d\xi\right)^{\frac{1}{2}} = \left(\sum_{i=1}^d \int_{\mathbb{R}^m} \left|(\widehat{p_{\beta} \cdot f})_i(\xi)\right|^2 \left(1 + \|\xi\|^2\right)^{\frac{s}{2}} d\xi\right)^{\frac{1}{2}} = \sum_{i=1}^d \left(\int_{\mathbb{R}^m} \left|(\widehat{p_{\beta} \cdot f}_i)(\xi)\right|^2 \left(1 + \|\xi\|^2\right)^{\frac{s}{2}} d\xi\right)^{\frac{1}{2}} < \infty.$$
(90)

Finally, by using that  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$  for some  $0 < \xi_1 < \xi_2 < \infty$ , that  $(x+y)^2 \leq 2(x^2+y^2)$ for any  $x, y \ge 0$ , the constant  $C_{41} := \max_{j \in \mathbb{N}_0 \cap [0, \lceil \gamma \rceil + \lceil \nu \rceil + 1]} \int_{\xi_1}^{\infty} \left| \hat{\psi}^{(j)}(\zeta) \right| d\zeta > 0$  (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ), that  $A_1 \sim t_m(\nu)$  with  $\theta_A(a)^{-1} = \frac{\Gamma(\nu/2)(\pi\nu)^{m/2}}{\Gamma((m+\nu)/2)} \left(1 + \|a\|^2/\nu\right)^{(m+\nu)/2} \le 1$  $\frac{\Gamma(\nu/2)(\pi\nu)^{m/2}}{\Gamma((m+\nu)/2)} \left(1 + \|a\|^2/\nu\right)^{(m+\lceil\nu\rceil)/2} \text{ for any } a \in \mathbb{R}^m \text{, the constant } C_{42} := C_{41} \left(\nu\xi_1^2\right)^{(2k+4\lceil\gamma\rceil+3\lceil\nu\rceil+4)/2} > C_{42} = C_{42} \left(\nu\xi_1^2\right)^{(2k+4\lceil\gamma\rceil+3\lceil\nu+4)/2} > C_{42} = C_{42} \left(\nu\xi_1^2\right)^{(2k+4\lceil\gamma+4)/2} > C_{42} \left(\nu\xi_1^2\right)^{(2k+4)/2} > C_{42} \left(\nu\xi_1^2\right)^{(2k+4\lceil\gamma+4)/2} > C_{42} \left(\nu\xi_1^2$ 0 (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ), and (90), it follows that

$$C_{f} = \max_{\substack{j \in \mathbb{N}_{0} \cap [0, |\gamma| + [\nu] + 1], \\ \beta \in \mathbb{N}_{0}^{m}, [\gamma] + [\nu] + 1}} \int_{\mathbb{R}} \frac{\left| \hat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{2}}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \hat{f}(\xi) \|^{2} \frac{(1 + \|\xi/\zeta\|)^{2(k+2[\gamma] + [\nu] + 2)}}{\theta_{A}(\xi/\zeta)} d\xi \right)^{\frac{1}{2}} d\zeta$$

$$\leq \max_{\substack{j \in \mathbb{N}_{0} \cap [0, [\gamma] + [\nu] + 1], \\ \beta \in \mathbb{N}_{0}^{m}, [\gamma] + [\nu] + 1}} \int_{\xi_{1}}^{\xi_{2}} \frac{\left| \hat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{2}}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \hat{f}(\xi) \|^{2} \left( 1 + \frac{\|\xi\|^{2}}{\zeta^{2}} \right)^{k+2[\gamma] + [\nu] + 2} \theta_{A} \left( \frac{\xi}{\zeta} \right)^{-1} d\xi \right)^{\frac{1}{2}} d\zeta$$

$$\leq \frac{C_{41}}{\xi_{1}^{\frac{m}{2}}} \max_{\beta \in \mathbb{N}_{0, [\gamma] + [\nu] + 1}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \hat{f}(\xi) \|^{2} \left( 1 + \frac{\|\xi\|^{2}}{\xi_{1}^{2}} \right)^{\frac{2k+4[\gamma] + 2[\nu] + 4}{2}} \frac{\Gamma\left( \frac{\nu}{2} \right) (\pi \nu)^{\frac{m}{2}}}{\Gamma\left( \frac{m+\nu}{2} \right)} \left( 1 + \frac{\|\xi\|^{2}}{\nu\xi_{1}^{2}} \right)^{\frac{m+\nu}{2}} d\xi \right)^{\frac{1}{2}}$$

$$\leq \frac{C_{42} \pi^{\frac{m}{4}} \max\left( 1/\xi_{1}, \sqrt{\nu} \right)^{m}}{\Gamma\left( \frac{m+\nu}{2} \right)^{\frac{1}{2}}} \max_{\beta \in \mathbb{N}_{0, [\gamma] + [\nu] + 1}^{m}} \left( \int_{\mathbb{R}^{m}} \| \partial_{\beta} \hat{f}(\xi) \|^{2} \left( 1 + \|\xi\|^{2} \right)^{\frac{m+\nu+\mu}{2}} \frac{1}{2} d\xi \right)^{\frac{1}{2}} < \infty,$$
(91)
which completes the proof.

which completes the proof.

*Proof of Example 4.7.* In order to show (a), let  $w: U \to [0, \infty)$  be a weight of separable form w(u) := $w_0(u_1)\cdots w_0(u_m)$  for all  $u:=(u_1,\ldots,u_m)^{\top}\in U$ , where  $w_0:\mathbb{R}\to[0,\infty)$  is another weight satisfying  $\int_{\mathbb{R}} w_0(s) ds = 1 \text{ and } C_{\mathbb{R},w_0}^{(\gamma,p)} := \left(\int_{\mathbb{R}} (1+|s|)^{\gamma p} w_0(s) ds\right)^{1/p} < \infty. \text{ Then, by using that } 1+\|u\| \leq 1+\sum_{l=1}^m |u_l| \leq \sum_{l=1}^m (1+|u_l|) \text{ for any } u := (u_1, \dots, u_m)^\top \in \mathbb{R}^m, \text{ the inequality } (x_1 + \dots + x_m)^{\gamma p} \leq m^{\gamma p} (x_1^{\gamma p} \dots + x_m^{\gamma p}) \text{ for any } x_1, \dots, x_m \geq 0, \text{ and Fubini's theorem, it follows that}$ 

$$\begin{split} C_{U,w}^{(\gamma,p)} &= \left( \int_{U} (1+\|u\|)^{\gamma p} w(u) du \right)^{\frac{1}{p}} \\ &\leqslant \left( \int_{U} \left( \sum_{l=1}^{m} (1+|u_{l}|) \right)^{\gamma p} w(u) du \right)^{\frac{1}{p}} \\ &\leqslant m^{\gamma} \left( \sum_{l=1}^{m} \int_{\mathbb{R}^{m}} (1+|u_{l}|)^{\gamma p} w_{0}(u_{1}) \cdots w_{0}(u_{m}) du \right)^{\frac{1}{p}} \\ &\leqslant m^{\gamma} \left( \sum_{l=1}^{m} \left( \underbrace{\int_{\mathbb{R}} (1+|u_{l}|)^{\gamma p} w_{0}(u_{l}) du_{l}}_{= \left( C_{\mathbb{R},w_{0}}^{(\gamma,p)} \right)^{p}} \right) \prod_{\substack{i=1\\i \neq l}}^{m} \underbrace{\int_{\mathbb{R}^{m}} w_{0}(u_{i}) du_{i}}_{=1} \right)^{\frac{1}{p}} \\ &\leqslant C_{\mathbb{R},w_{0}}^{(\gamma,p)} m^{\gamma + \frac{1}{p}}, \end{split}$$

which shows the inequality in (a).

In order to show (b), we first observe in each case (i)-(iv) that  $\rho \in C^k_{pol,\gamma}(\mathbb{R})$  is of polynomial growth and thus induces the tempered distribution  $(g \mapsto T_{\rho}(g) := \int_{\mathbb{R}} \rho(s)g(s)ds) \in \mathcal{S}'(\mathbb{R};\mathbb{C})$  (see [Folland, 1992, p. 332]). Now, we fix some  $m \in \mathbb{N}$  and  $\psi \in \mathcal{S}_0(\mathbb{R}; \mathbb{C})$  with non-negative  $\hat{\psi} \in C_c^{\infty}(\mathbb{R})$ such that  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$  for some  $0 < \xi_1 < \xi_2 < \infty$ . Then, by using Example 2.9, the Fourier transform  $\widehat{T_{\rho}} \in \mathcal{S}'(\mathbb{R};\mathbb{C})$  coincides on  $\mathbb{R}\setminus\{0\}$  with the function  $f_{\widehat{T_{\rho}}} \in L^1_{loc}(\mathbb{R}\setminus\{0\};\mathbb{C})$  given in the last column of (i)-(iv). Hence, in each case (i)-(iv), we use that  $\hat{\psi} \in C_c^{\infty}(\mathbb{R})$  is non-negative to conclude that

$$C_{m}^{(\psi,\rho)} = (2\pi)^{m-1} \int_{\mathbb{R}\setminus\{0\}} \frac{\widehat{\psi}(\xi) f_{\widehat{T_{\rho}}}(\xi)}{|\xi|^{m}} d\xi = (2\pi)^{m-1} \int_{\xi_{1}}^{\xi_{2}} \frac{\widehat{\psi}(\xi) f_{\widehat{T_{\rho}}}(\xi)}{|\xi|^{m}} d\xi \neq 0.$$

This shows that  $(\psi, \rho) \in \mathcal{S}_0(\mathbb{R}; \mathbb{C}) \times C^k_{pol,\gamma}(\mathbb{R})$  is *m*-admissible. Moreover, in each case (i)-(iv), we define the constant  $C_{\psi,\rho} := (2\pi)^{-1} \left| \int_{\xi_1}^{\xi_2} \overline{\widehat{\psi}(\xi)} f_{\widehat{T_{\rho}}}(\xi) d\xi \right|$  (independent of  $m \in \mathbb{N}$ ) to conclude that

$$\left|C_{m}^{(\psi,\rho)}\right| = (2\pi)^{m-1} \left| \int_{\xi_{1}}^{\xi_{2}} \frac{\overline{\widehat{\psi}(\xi)} f_{\widehat{T_{\rho}}}(\xi)}{|\xi|^{m}} d\xi \right| \ge \left| \int_{\xi_{1}}^{\xi_{2}} \frac{\overline{\widehat{\psi}(\xi)} f_{\widehat{T_{\rho}}}(\xi)}{2\pi} d\xi \right| \left(\frac{2\pi}{\xi_{2}}\right)^{m} = C_{\psi,\rho} \left(\frac{2\pi}{\xi_{2}}\right)^{m},$$
  
h completes the proof.

which completes the proof.

*Proof of Remark 4.8.* Assume the setting of Example 4.6+4.7 with a function  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1 + 1))$  $\|u\|)^{[\gamma]+[\nu]+1}du;\mathbb{R}^d) \cap W^{m+2k+4[\gamma]+3[\nu]+4,2}(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),(1+\|\cdot\|)^{2([\gamma]+[\nu]+1)};\mathbb{R}^d) \text{ and some } \psi \in \mathbb{R}^d$  $\mathcal{S}_0(\mathbb{R};\mathbb{C})$  such that  $\operatorname{supp}(\widehat{\psi}) = [\xi_1,\xi_2]$ , where  $0 < \xi_1 < \xi_2 < \infty$ . Then, by using Example 4.7 (a), i.e. that there exists a constant  $C_{\mathbb{R},w_0}^{(\gamma,p)} > 0$  (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ) such that  $C_{U,w}^{(\gamma,p)} \leqslant C_{\mathbb{R},w_0}^{(\gamma,p)} m^{\gamma+1/p}$ , Example 4.7 (b), i.e. that there exists a constant  $C_{\psi,\rho} > 0$  (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ) such that  $\left| C_m^{(\psi, \rho)} \right| \ge C_{\psi, \rho} (2\pi/\xi_2)^m$ , the inequality (91) (with constant  $C_{42} > 0$ , independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ), the constant  $C_2 := C_{\mathbb{R},w_0}^{(\gamma,p)} C_{\psi,\rho} C_{42} > 0$ (independent of  $m, d \in \mathbb{N}$  and  $f : \mathbb{R}^m \to \mathbb{R}^d$ ), and that  $\xi_1 \in (0, \nu^{-1/2}]$ , i.e. that  $\sqrt{\nu} \leq 1/\xi_1$ , we have

$$\begin{split} \frac{C_{U,w}^{(\gamma,p)}C_{f}}{\left|C_{m}^{(\psi,\rho)}\right|} &\leqslant C_{\mathbb{R},w_{0}}^{(\gamma,p)}m^{\gamma+1/p}C_{\psi,\rho}\frac{\xi_{2}^{m}}{(2\pi)^{m}}\frac{C_{42}\pi^{\frac{m}{4}}\max\left(1/\xi_{1},\sqrt{\nu}\right)^{m}}{\Gamma\left(\frac{m+\nu}{2}\right)^{\frac{1}{2}}} \\ &\quad \cdot \max_{\beta \in \mathbb{N}_{0,[\gamma]+[\nu]+1}^{m}}\left(\int_{\mathbb{R}^{m}}\|\partial_{\beta}\widehat{f}(\xi)\|^{2}\left(1+\|\xi\|^{2}\right)^{\frac{m+2k+4[\gamma]+3[\nu]+4}{2}}d\xi\right)^{\frac{1}{2}} \\ &= C_{2}\frac{m^{\gamma+1/p}\xi_{2}^{m}\pi^{\frac{m}{4}}}{(2\pi\xi_{1})^{m}\Gamma\left(\frac{m+\nu}{2}\right)^{\frac{1}{2}}}\max_{\beta \in \mathbb{N}_{0,[\gamma]+[\nu]+1}^{m}}\left(\int_{\mathbb{R}^{m}}\|\partial_{\beta}\widehat{f}(\xi)\|^{2}\left(1+\|\xi\|^{2}\right)^{\frac{m+2k+4[\gamma]+3[\nu]+4}{2}}d\xi\right)^{\frac{1}{2}}, \end{split}$$
 which completes the proof.

which completes the proof.

9.4. Proof of Proposition 4.9 and Theorem 4.10. In this section, we first prove the properties of Algorithm 1 stated in Proposition 4.9. Subsequently, we prove Theorem 4.10 which provides us with the generalization error of learning a deterministic function by a random neural network.

Proof of Proposition 4.9. Fix a k-times weakly differentiable function  $f: U \to \mathbb{R}^d$  and some  $J, N \in \mathbb{N}$ . For (i), we first observe that an  $\mathbb{R}^{d \times N}$ -valued random variable  $W^{(J)} = \left(W_{i,n}^{(J)}\right)_{\substack{i=1,...,N\\i=1,...,d}}^{n=1,...,N}$  satisfies (14) if and only if for every i = 1, ..., d the  $\mathbb{R}^N$ -valued random variable  $W_i^{(J)} := \left(W_{i,n}^{(J)}\right)_{n=1,...,N}^{\top}$ 

$$W_{i}^{(J)} = \arg\min_{\widetilde{W}\in\mathcal{W}_{N,1}} \left( \frac{1}{J} \sum_{j=1}^{J} \sum_{\alpha\in\mathbb{N}_{0,k}^{m}} c_{\alpha}^{2} \left| \partial_{\alpha}f_{i}(V_{j}) - \sum_{n=1}^{N} \widetilde{W}_{n}\rho^{(|\alpha|)} \left( A_{n}^{\top}V_{j} - B_{n} \right) A_{n}^{\alpha} \right|^{2} \right)$$

$$= \arg\min_{\widetilde{W}\in\mathcal{W}_{N,1}} \left\| Y_{i} - R\widetilde{W} \right\|^{2},$$
(92)

where  $\mathcal{W}_{N,1}$  consists of all  $\mathbb{R}^N$ -valued  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^N)$ -measurable random variables  $\widetilde{W} = \left(\widetilde{W}_n\right)_{n=1,\dots,N}^\top$ . Then, by using [Björck, 1996, Theorem 1.1.2], (92) forms for every fixed  $\omega \in \Omega$  the least squares problem  $R(\omega)^{\top}R(\omega)W_i^{(J)}(\omega) = R(\omega)^{\top}Y_i(\omega)$ . Hence, the least squares problem in Step 6 admits by

[Björck, 1996, Theorem 1.2.10] a solution  $W_i^{(J)}(\omega) \in \mathbb{R}^N$ , which shows that Algorithm 1 terminates. Next, we show that Algorithm 1 is correct. Indeed, by using the first step, (92) is equivalent to the condition  $R^{\top}RW_i^{(J)} = R^{\top}Y_i$  stated in Step 6 of Algorithm 1. Hence, the  $\mathbb{R}^{d \times N}$ -valued random variable  $W^{(J)} = \left(W_{i,n}^{(J)}\right)_{i=1,\dots,d}^{n=1,\dots,N}$  indeed solves (14). In the following, we now show that  $W^{(J)}$  is  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^{N \times d})$ -measurable. For this purpose, we define  $M := J \cdot |\mathbb{N}_{0,k}^m|$  and the set-valued function  $\mathbb{R}^{M \times N} \times \mathbb{R}^M \ni (r, y) \quad \mapsto \quad \Xi(r, y) := \left\{ w \in \mathbb{R}^N : w = \arg\min_{\widetilde{w} \in \mathbb{R}^N} \|y - r\widetilde{w}\|^2 \right\}.$ (93) Then, by using [Björck, 1996, Theorem 1.1.2], we have  $w \in \Xi(r, y)$  if and only if  $r^{\top}rw = r^{\top}y$ . Hence, it follows for every open subset  $U \subseteq \mathbb{R}^{d \times N}$  that the set

$$\begin{split} \big\{ (r,y) \in \mathbb{R}^{M \times N} \times \mathbb{R}^M : \Xi(r,y) \cap U \neq \emptyset \big\} \\ &= \big\{ (r,y) \in \mathbb{R}^{M \times N} \times \mathbb{R}^M : \big\{ w \in \mathbb{R}^N : r^\top r w = r^\top y \big\} \cap U \neq \emptyset \big\} \end{split}$$

is  $\mathcal{B}(\mathbb{R}^{M \times N} \times \mathbb{R}^M)$ -measurable. Thus, by using the Kuratowski-Ryll-Nardzewski measurable selection theorem in [Kuratowski and Ryll-Nardzewski, 1965], there exists an  $\mathcal{B}(\mathbb{R}^{M \times N} \times \mathbb{R}^M)/\mathcal{B}(\mathbb{R}^N)$ -measurable selection of (93), i.e. an  $\mathcal{B}(\mathbb{R}^{M \times N} \times \mathbb{R}^M)/\mathcal{B}(\mathbb{R}^N)$ -measurable function  $\chi : \mathbb{R}^{M \times N} \times \mathbb{R}^M \to \mathbb{R}^N$  such that  $\chi(r, y) = \arg \min_{\widetilde{w} \in \mathbb{R}^N} \|y - r\widetilde{w}\|^2$  for all  $(r, y) \in \mathbb{R}^{M \times N} \times \mathbb{R}^M$ . Since the  $\mathbb{R}^{M \times N}$ -valued random variable R in Step 3 is by definition  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^M)$ -measurable and the  $\mathbb{R}^{J \times |\mathbb{N}_{0,k}^m|}$ -valued random variable  $Y_i$  in Step 5 is by definition  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^M)$ -measurable, we conclude that

$$\Omega \ni \omega \quad \mapsto \quad W_i^{(J)}(\omega) = \arg \min_{\widetilde{W} \in \mathcal{W}_{N,1}} \left\| Y_i(\omega) - R(\omega)\widetilde{W}(\omega) \right\|^2 = \chi(R(\omega), Y(\omega)) \in \mathbb{R}^N$$

is  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^N)$ -measurable. This implies that the  $\mathbb{R}^{d \times N}$ -valued random  $W^{(J)} = \left(W_i^{(J)}\right)_{i=1,...,d}$  is  $\mathcal{F}_{A,B,V}/\mathcal{B}(\mathbb{R}^{d \times N})$ -measurable, which shows that  $W \in \mathcal{W}_N$  and that the algorithm is correct.

In order to show (ii), we compute the complexity  $\mathscr{C}_{m,d,k}(J,N)$  of Algorithm 1. In Step 1, we generate the random variables  $(A_n, B_n)_{n=1,...,N} \sim \theta_A \otimes t_1(\nu)$ , which needs N(m + 1) units. In Step 2, we generate the random variables  $(V_j)_{j=1,...,J} \sim w$ , which requires Jm units. In Step 1, we compute the  $\mathbb{R}^{(J \cdot |\mathbb{N}_{0,k}^m|) \times N}$ -valued random variable  $R = (R_{(j,\alpha),n})_{(j,\alpha) \in \{1,...,J\} \times \mathbb{N}_{0,k}^m}$  with components  $R_{(j,\alpha),n} := c_\alpha \rho^{(|\alpha|)} (A_n^\top V_j - B_n) A_n^\alpha$ , for  $(j,\alpha) \in \{1,...,J\} \times \mathbb{N}_{0,k}^m$  and n = 1,...,N, which needs

$$J\left|\mathbb{N}_{0,k}^{m}\right|N\left(\underbrace{m+(m-1)}_{A_{n}^{\top}V_{j}}+\underbrace{1}_{+B_{n}}+\underbrace{1}_{\rho^{\left(\left|\alpha\right|\right)}\left(\cdot\right)}+\underbrace{\left|\alpha\right|+1}_{\cdot A_{n}^{\alpha}}+\underbrace{1}_{c_{\alpha}\cdot}\right)\leqslant J\left|\mathbb{N}_{0,k}^{m}\right|N(2m+k+3)$$

units. In Step 5 (inside the for-loop), we compute for fixed i = 1, ..., d the  $\mathbb{R}^{J \times |\mathbb{N}_{0,k}^m|}$ -valued random variable  $Y_i := (c_\alpha \partial_\alpha f_i(V_j))_{(j,\alpha) \in \{1,...,J\} \times \mathbb{N}_{0,k}^m}$ , which requires  $2J |\mathbb{N}_{0,k}^m|$  units. In Step 6 (inside the for-loop), we solve for fixed i = 1, ..., d the least squares problem via Cholesky decomposition and forward/backward substitution, which needs  $\frac{1}{2}J|\mathbb{N}_{0,k}^m|N^2 + \frac{1}{6}N^3 + \mathcal{O}\left(J|\mathbb{N}_{0,k}^m|N\right)$  units (see [Björck, 1996, p. 45]). Hence, the complete for-loop in Step 4 executing d-times Step 5+6 requires

$$d\left(2J\left|\mathbb{N}_{0,k}^{m}\right| + \frac{1}{2}J\left|\mathbb{N}_{0,k}^{m}\right|N^{2} + \frac{1}{6}N^{3} + \mathcal{O}\left(J\left|\mathbb{N}_{0,k}^{m}\right|N\right)\right) = \mathcal{O}\left(dJ\left|\mathbb{N}_{0,k}^{m}\right|N^{2} + dN^{3}\right)$$

units. Thus, by combining these results and using that  $|\mathbb{N}_{0,k}^m| = \sum_{j=0}^k m^j \leq 2m^k$ , it follows that

$$\mathscr{C}_{m,d,k}(J,N) = N(m+1) + Jm + J \left| \mathbb{N}_{0,k}^m \right| N(2m+k+3) + \mathcal{O}\left( dJ \left| \mathbb{N}_{0,k}^m \right| N^2 + dN^3 \right) = \mathcal{O}\left( (k+1)dJm^{k+1}N^2 + dN^3 \right),$$

which completes the proof.

Next, we provide the proof of Theorem 4.10. Let us briefly outline the main ideas of the proof, where we here in the outline assume for simplicity that k = 0 and d = 1. Moreover, we fix some  $J, N \in \mathbb{N}$ and a function  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$  with  $|f(u)| \leq L$  for all  $u \in U$  and  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform such that the constant  $C_f \geq 0$  defined in (15) is finite. Then, we obtain from Algorithm 1 a random neural network  $\Phi_N^{W^{(J)}} \in \mathcal{RN}_{U,d}^{\rho,V}$  with linear readout  $W^{(J)} \in \mathcal{W}_N$  solving (14). From this, we define the  $L^2(U, \mathcal{L}(U), w)$ -valued random variable

$$\Omega \ni \omega \quad \mapsto \quad \Delta_L^{W^{(J)}}(\omega) := \left( u \mapsto f(u) - T_L\left(\Phi_N^{W^{(J)}}(\omega)(u)\right) \right) \in L^2(U, \mathcal{L}(U), w).$$

In addition, we define the empirical version  $\Omega \ni \omega \mapsto \|\Delta_L^{W^{(J)}}\|_J := \left(\frac{1}{J}\sum_{j=1}^J \|\Delta_L^{W^{(J)}}(\cdot)(V_j)\|^2\right)^{1/2} \in \mathbb{R}$  of  $\|\Delta_L^{W^{(J)}}\|_{L^2(U,\mathcal{L}(U),w)}$ . Then, by using the inequality  $(x+y)^2 \leq 2(x^2+y^2)$  for any  $x, y \geq 0$ , we have

$$\mathbb{E}\left[\int_{U} \left| f(u) - T_{L}\left(\Phi^{W^{(J)}}(\cdot)(u)\right) \right|^{2} w(u) du \right] = \mathbb{E}\left[ \left\| \Delta_{L}^{W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)}^{2} \right] 
= \mathbb{E}\left[ \left( \left\| \Delta_{L}^{W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{L}^{W^{(J)}} \right\|_{J} + 2 \left\| \Delta_{L}^{W^{(J)}} \right\|_{J} \right)^{2} \right] 
\leqslant 2\mathbb{E}\left[ \max\left( \left\| \Delta_{L}^{W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{L}^{W^{(J)}} \right\|_{J}, 0 \right)^{2} \right] + 8\mathbb{E}\left[ \left\| \Delta_{L}^{W^{(J)}} \right\|_{J}^{2} \right].$$
(94)

Now, for the first term on the right hand-side of (94), we follow the results on non-parametric function regression in [Györfi et al., 2002, Section 11.3] to derive an upper bound for the difference between  $\|\Delta_L^{W^{(J)}}\|_{L^2(U,\mathcal{L}(U),w)}$  and its empirical version  $\|\Delta_L^{W^{(J)}}\|_J$ . Hereby, we use in particular that  $W^{(J)} \in \mathcal{W}_N$  is the least squares solution of (14) and that  $(V_j)_{j=1,\dots,J} \sim w$  is i.i.d.

Moreover, for the second term on the right-hand side of (94), we use that the mean squared error (MSE) is minimized by  $W^{(J)} \in W_N$  and is thus smaller than the MSE of the random neural network  $\Phi^{W^f} \in \mathcal{RN}_{U,d}^{\rho}$  used in Theorem 4.4, and that  $(V_j)_{j\in\mathbb{N}}$  are independent of  $(A_n, B_n)_{n\in\mathbb{N}}$  to conclude that

$$\mathbb{E}\left[\left\|\Delta_{L}^{W^{(J)}}\right\|_{J}^{2}\right] = \mathbb{E}\left[\frac{1}{J}\sum_{j=1}^{J}\left|f(V_{j}) - \Phi_{N}^{W^{(J)}}(\omega)(V_{j})\right|^{2}\right]$$
$$\leq \mathbb{E}\left[\frac{1}{J}\sum_{j=1}^{J}\left|f(V_{j}) - \Phi_{N}^{W^{f}}(\omega)(V_{j})\right|^{2}\right]$$
$$= \mathbb{E}\left[\left\|f(u) - \Phi^{W^{f}}\right\|_{W^{k,2}(U,\mathcal{L}(U),du;\mathbb{R}^{d})}^{2}\right]$$

Hence, we can upper bound the second term on the right-hand side of (94) with Theorem 4.4 (ii).

Proof of Theorem 4.10. Fix some  $J, N \in \mathbb{N}, L > 0$ , and  $f \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du; \mathbb{R}^d)$  with  $|\partial_{\alpha} f_i(u)| \leq L$  for all  $\alpha \in \mathbb{N}_{0,k}^m$ , i = 1, ..., d, and  $u \in U$ , and  $([\gamma] + [\nu] + 1)$ -times weakly differentiable Fourier transform such that the constant  $C_f \geq 0$  defined in (15) is finite. Then, we apply Algorithm 1 to obtain some  $\Phi_N^{W(J)} \in \mathcal{RN}_{U,d}^{\rho,V}$  with  $\mathbb{R}^{d \times N}$ -valued random variable  $W^{(J)} = \left(W_n^{(J)}\right)_{n=1,...,N}^{\top} = \left(W_{i,n}^{(J)}\right)_{i=1,...,d}^{n=1,...,N} \in \mathcal{W}_N$  solving (14). Moreover, by using Lemma 9.5 (with  $\mathcal{F}_0 := \mathcal{F}_{A,B,V}$  satisfying  $\mathcal{F}_{A,B} \subseteq \mathcal{F}_{A,B,V} \subseteq \mathcal{F}$ ), it follows that  $\Phi_N^{W^{(J)}} : \Omega \to W^{k,2}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  is an  $\mathcal{F}_{A,B,V}$ -strongly measurable map with values in the separable Banach space  $(W^{k,2}(U, \mathcal{L}(U), du; \mathbb{R}^d), \|\cdot\|_{W^{k,2}(U, \mathcal{L}(U), du; \mathbb{R}^d)})$ .

In order to show (15), we adapt the proof of [Györfi et al., 2002, Theorem 11.3]. To this end, we define for every  $\alpha \in \mathbb{N}_{0,k}^m$  and i = 1, ..., d the  $L^2(U, \mathcal{L}(U), w)$ -valued random variable

$$\Omega \ni \omega \quad \mapsto \quad \Delta^{W^{(J)}}_{\alpha,i,L}(\omega) := \left( u \mapsto \partial_{\alpha} f_i(u) - T_L\left(\partial_{\alpha} \Phi^{W^{(J)}}_{N,i}(\omega)(u)\right) \right) \in L^2(U, \mathcal{L}(U), w).$$

Moreover, we define for every fixed  $\alpha \in \mathbb{N}_{0,k}^m$ , i = 1, ..., d,  $a := (a_1, ..., a_N)^\top \in \mathbb{R}^{N \times m}$ , and  $b := (b_1, ..., b_N)^\top \in \mathbb{R}^N$  the  $L^2(U, \mathcal{L}(U), w)$ -valued random variable

$$\Omega \ni \omega \quad \mapsto \quad \Delta_{\alpha,i,L}^{(a,b),W^{(J)}}(\omega) := \left( u \mapsto \partial_{\alpha} f_i(u) - T_L\left(\partial_{\alpha} \Phi_{N,i}^{(a,b),W^{(J)}}(\omega)(u)\right) \right) \in L^2(U,\mathcal{L}(U),w),$$

where  $\Omega \ni \omega \mapsto \Phi_{N,i}^{(a,b),W^{(J)}}(\omega) := \sum_{n=1}^{N} W_{i,n}^{(J)}(\omega) \rho^{(|\alpha|)} \left(a_n^{\top} \cdot -b_n\right) a_n^{\alpha} \in C_{pol,\gamma}^k(U)$ , where  $a_n^{\alpha} := \prod_{l=1}^{m} a_l^{\alpha_l} \in \mathbb{R}$ . In addition, we define the (random) empirical mean squared error  $\|\cdot\|_J$  of such an  $L^2(U, \mathcal{L}(U), w)$ -valued random variable as

$$\Omega \ni \omega \quad \mapsto \quad \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} := \left( \frac{1}{J} \sum_{j=1}^{J} \left\| \Delta_{\alpha,i,L}^{W^{(J)}}(\cdot)(V_{j}) \right\|^{2} \right)^{\frac{1}{2}} \in \mathbb{R},$$

and analogously for  $\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{J}$ . Then, by using the inequality  $(x+y)^2 \leq 2(x^2+y^2)$  for any  $x, y \geq 0$ , conditioning on  $\mathcal{F}_{A,B}$ , that  $|\mathbb{N}_{0,k}^m| = \sum_{j=0}^k m^j \leq 2m^k$ , that the random variables  $(V_j)_{j\in\mathbb{N}}$  are independent of  $(A_n, B_n)_{n\in\mathbb{N}}$ , and the notation  $(A, B) := (A_n, B_n)_{n=1,\dots,N}$  it follows that

$$\mathbb{E}\left[\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \int_{U} \left\| \partial_{\alpha} f(u) - T_{L} \left( \partial_{\alpha} \Phi_{N}^{W^{(J)}}(\cdot)(u) \right) \right\|^{2} w(u) du \right] \\
\leq \mathbb{E}\left[\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \int_{U} \left| \partial_{\alpha} f_{i}(u) - T_{L} \left( \partial_{\alpha} \Phi_{N,i}^{W^{(J)}}(\cdot)(u) \right) \right|^{2} w(u) du \right] \\
\leq \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \mathbb{E}\left[ \left( \left\| \Delta_{\alpha,i,L}^{W} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} + 2 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} \right)^{2} \right] \\
\leq \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \mathbb{E}\left[ \left( \max \left( \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} , 0 \right) + 2 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} \right)^{2} \right] \\
\leq 2\sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \mathbb{E}\left[ \max \left( \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} , 0 \right)^{2} + 4 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J}^{2} \right] \\
\leq 2 \left| \mathbb{N}_{0,k}^{m} \right| d \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \max_{i=1,\dots,d} \mathbb{E}\left[ \mathbb{E}\left[ \max \left( \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{\alpha,i,L}^{W^{(J)}} \right\|_{J} , 0 \right)^{2} \right| \mathcal{F}_{A,B} \right] \right] \\
+ 8\mathbb{E}\left[ \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \max_{i=1}^{d} \frac{1}{J} \sum_{j=1}^{J} \left\| \Delta_{\alpha,i,L}^{W^{(J)}} (\cdot)(V_{j}) \right\|^{2} \right] \\
+ 8\mathbb{E}\left[ \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \frac{1}{J} \sum_{j=1}^{J} \left\| \Delta_{\alpha,i,L}^{W^{(J)}} (\cdot)(V_{j}) \right\|^{2} \right].$$
(95)

Moreover, we define for every fixed  $\alpha \in \mathbb{N}_{0,k}^m$ , i = 1, ..., d,  $a := (a_1, ..., a_N)^\top \in \mathbb{R}^{N \times m}$ , and  $b := (b_1, ..., b_N)^\top \in \mathbb{R}^N$  the vector space of random functions

$$\mathcal{G}_{\alpha,i}^{(a,b)} := \left\{ \Omega \ni \omega \mapsto \sum_{n=1}^{N} \widetilde{W}_{i,n} \rho^{(|\alpha|)} \left( a_n^\top \cdot -b_n \right) a_n^\alpha \in C_{pol,\gamma}^0(U) : \widetilde{W} = \left( \widetilde{W}_{i,n} \right)_{i=1,\dots,d}^{n=1,\dots,N} \in \mathcal{W}_N \right\}.$$

Then, by following [Györfi et al., 2002, p. 193], i.e. by using [Györfi et al., 2002, Theorem 11.2] (with the set  $T_L \mathcal{G}_{\alpha,i}^{(a,b)} := \left\{ \Omega \ni \omega \mapsto (u \mapsto T_L(\Phi(\omega)(u))) \in C_{pol,\gamma}^0(U) : \Phi \in \mathcal{G}_{\alpha,i}^{(a,b)} \right\}$  and where  $\mathcal{G}_{\alpha,i}^{(a,b)}$  has for fixed  $a \in \mathbb{R}^{N \times m}$ ,  $b \in \mathbb{R}^N$ ,  $\alpha \in \mathbb{N}_{0,k}^m$ , and i = 1, ..., d the vector space dimension N in the sense of [Györfi et al., 2002, Theorem 11.1]) together with [Györfi et al., 2002, Lemma 9.2], [Györfi et al., 2002, Lemma 9.4], and [Györfi et al., 2002, Theorem 9.5], it follows for every  $u > 576L^2/J$  that

$$\mathbb{P}\left[\max\left(\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{J}, 0\right)^{2} > u\right] \\
\leq \mathbb{P}\left[\exists g \in T_{L}\mathcal{G}_{\alpha,i}^{(a,b)} : \|g\|_{L^{2}(U,\mathcal{L}(U),w)} - 2\|g\|_{J} > \frac{\sqrt{u}}{2}\right] \\
\leq 9(12eJ)^{2(N+1)}e^{-\frac{Ju}{2304L^{2}}}.$$
(96)

Hence, by using the constant  $v := \frac{2304L^2}{J} \ln \left(9(12eJ)^{2(N+1)}\right) > 576L^2/J$ , the inequality (96), and that  $\ln(108e) \ge 1$  together with  $2304 \le 9216 \ln(108e)$ , we conclude that

$$\mathbb{E}\left[\max\left(\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{L^{2}(U,\mathcal{L}(U),w)}-2\left\|\Delta_{\alpha,i,L}^{(a,b),W}\right\|_{J},0\right)^{2}\right] \\
=\int_{0}^{\infty}\mathbb{P}\left[\max\left(\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{L^{2}(U,\mathcal{L}(U),w)}-2\left\|\Delta_{\alpha,i,L}^{(a,b),W}\right\|_{J},0\right)^{2}>u\right]du \\
\leqslant v+\int_{v}^{\infty}\mathbb{P}\left[\max\left(\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{L^{2}(U,\mathcal{L}(U),w)}-2\left\|\Delta_{\alpha,i,L}^{(a,b),W^{(J)}}\right\|_{J},0\right)^{2}>u\right]du \\
\leqslant v+9(12eJ)^{2(N+1)}\int_{v}^{\infty}e^{-\frac{Ju}{2304L^{2}}}du \tag{97}\right) \\
=\frac{2304L^{2}}{J}\underbrace{\ln\left(9(12eJ)^{2(N+1)}\right)}_{\leqslant 4N\ln(108eJ)}\frac{2304L^{2}}{J}e^{-\frac{Jv}{2304L^{2}}} \\
=\frac{2304L^{2}}{J}4N\left(\ln(108e)+\ln(J)\right)+\frac{2304L^{2}}{J} \\
\leqslant 9216\ln(108e)L^{2}\frac{(\ln(J)+1)N}{J}.$$

On the other hand, for the second term on the right-hand side of (95), we use that  $|\partial_{\alpha} f_i(u)| \leq L$  for any  $\alpha \in \mathbb{N}_{0,k}^m$ , i = 1, ..., d, and  $u \in U$ , that  $||T_L(y)|| \leq ||y||$  for any  $y \in \mathbb{R}^d$ , that the  $\mathbb{R}^{d \times N}$ -valued random variable  $W^{(J)} = \left(W_{i,n}^{(J)}\right)_{i=1,...,d}^{n=1,...,N}$  solves (14), and Theorem 4.4 together with  $\mathcal{F}_{A,B} \subseteq \mathcal{F}_{A,B,V}$  (with constant  $C_1 > 0$  independent of  $m, d \in \mathbb{N}$ , where  $\Phi_N^f \in \mathcal{RN}_{U,d}^\rho \cap L^2(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; W^{k,2}(U, \mathcal{L}(U), w; \mathbb{R}^d)$  has  $\mathcal{F}_{A,B}/\mathcal{B}(\mathbb{R}^{d \times N})$ -measurable linear readout contained in  $\mathcal{W}_N$  as  $\mathcal{F}_{A,B} \subseteq \mathcal{F}_{A,B,V}$  to conclude that

$$\begin{split} \mathbb{E}\left[\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\sum_{i=1}^{d}\frac{1}{J}\sum_{j=1}^{J}\left\|\Delta_{\alpha,i,L}^{W^{(J)}}(\cdot)(V_{j})\right\|^{2}\right]^{\frac{1}{2}} &= \mathbb{E}\left[\frac{1}{J}\sum_{j=1}^{J}\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\left\|T_{L}\left(\partial_{\alpha}f(V_{j})-\partial_{\alpha}\Phi_{N}^{W^{(J)}}(\cdot)(V_{j})\right)\right\|^{2}\right]^{\frac{1}{2}} \\ &\leqslant \frac{1}{\min_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}}\mathbb{E}\left[\frac{1}{J}\sum_{j=1}^{J}\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}^{2}\left\|\partial_{\alpha}f(V_{j})-\partial_{\alpha}\Phi_{N}^{W^{(J)}}(\cdot)(V_{j})\right\|^{2}\right]^{\frac{1}{2}} \\ &= \frac{1}{\min_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}}\mathbb{E}\left[\min_{\widetilde{W}\in\mathcal{W}_{N}}\left(\frac{1}{J}\sum_{j=1}^{J}\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}^{2}\left\|\partial_{\alpha}f(V_{j})-\partial_{\alpha}\Phi_{N}^{\widetilde{W}}(\cdot)(V_{j})\right\|^{2}\right)\right]^{\frac{1}{2}} \\ &\leqslant \frac{1}{\min_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}}\inf_{\widetilde{W}\in\mathcal{W}_{N}}\mathbb{E}\left[\frac{1}{J}\sum_{j=1}^{J}\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}^{2}\left\|\partial_{\alpha}f(V_{j})-\partial_{\alpha}\Phi_{N}^{\widetilde{W}}(\cdot)(V_{j})\right\|^{2}\right]^{\frac{1}{2}} \\ &\leqslant \frac{\max_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}}{\min_{\alpha\in\mathbb{N}_{0,k}^{m}}c_{\alpha}}\inf_{\widetilde{W}\in\mathcal{W}_{N}}\mathbb{E}\left[\sum_{\alpha\in\mathbb{N}_{0,k}^{m}}\int_{U}\left\|\partial_{\alpha}f(u)-\partial_{\alpha}\Phi_{N}^{\widetilde{W}}(\cdot)(u)\right\|^{2}w(u)du\right]^{\frac{1}{2}} \\ &\leqslant \kappa\left((c_{\alpha})_{\alpha\in\mathbb{N}_{0,k}^{m}}\right)\mathbb{E}\left[\left\|f-\Phi_{N}^{f}\right\|_{W^{k,2}(U,\mathcal{L}(U),w,\mathbb{R}^{d})}^{2}\right]^{\frac{1}{2}} \\ &\leqslant \kappa\left((c_{\alpha})_{\alpha\in\mathbb{N}_{0,k}^{m}}\right)C_{1}\frac{C_{U,w}^{(\gamma,2)}C_{f}}{C_{w}^{(\psi,\rho)}}\frac{m^{\frac{k}{2}+[\gamma]+[\nu]+1}}{\sqrt{N}}. \end{split}$$

(98)

Hence, by inserting (97) and (98) into (95) with the inequality  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$  for any  $x, y \geq 0$ , and by using the constant  $C_3 := \max\left(2\sqrt{9216\ln(108e)}, \sqrt{8}C_1\right) > 0$  (independent of  $m, d \in \mathbb{N}$ ), we have

$$\begin{split} \mathbb{E} \left[ \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \int_{U} \left\| \partial_{\alpha} f(u) - T_{L} \left( \partial_{\alpha} \Phi_{N}^{W^{(J)}}(\cdot)(u) \right) \right\|^{2} w(u) du \right]^{\frac{1}{2}} \\ & \leq 2m^{\frac{k}{2}} \sqrt{d} \max_{\alpha \in \mathbb{N}_{0,k}^{m}} \max_{i=1,\dots,d} \mathbb{E} \left[ \mathbb{E} \left[ \max \left( \left\| \Delta_{\alpha,i,L}^{(a,b),W^{(J)}} \right\|_{L^{2}(U,\mathcal{L}(U),w)} - 2 \left\| \Delta_{\alpha,i,L}^{(a,b),W^{(J)}} \right\|_{J}, 0 \right)^{2} \right] \right|_{(a,b)=(A,B)} \right]^{\frac{1}{2}} \\ & + \sqrt{8} \mathbb{E} \left[ \sum_{\alpha \in \mathbb{N}_{0,k}^{m}} \sum_{i=1}^{d} \frac{1}{J} \sum_{j=1}^{J} \left\| \Delta_{\alpha,i,L}^{W^{(J)}}(\cdot)(V_{j}) \right\|^{2} \right]^{\frac{1}{2}} \\ & \leq 2m^{\frac{k}{2}} \sqrt{d} \sqrt{9216 \ln(108e)} L \frac{\sqrt{\ln(J) + 1} \sqrt{N}}{\sqrt{J}} + \sqrt{8} \kappa \left( (c_{\alpha})_{\alpha \in \mathbb{N}_{0,k}^{m}} \right) C_{1} \frac{C_{U,w}^{(\gamma,2)} C_{f}}{|C_{w}^{(\psi,\rho)}|} \frac{m^{\frac{k}{2} + [\gamma] + [\nu] + 1}}{\sqrt{N}} \\ & \leq C_{3} L \frac{\sqrt{m^{k} d} \sqrt{\ln(J) + 1} \sqrt{N}}{\sqrt{J}} + C_{3} \kappa \left( (c_{\alpha})_{\alpha \in \mathbb{N}_{0,k}^{m}} \right) \frac{C_{U,w}^{(\gamma,2)} C_{f}}{|C_{m}^{(\psi,\rho)}|} \frac{m^{\frac{k}{2} + [\gamma] + [\nu] + 1}}{\sqrt{N}}, \end{split}$$

which completes the proof.

### 10. PROOFS OF RESULTS IN SECTION 5.

In this section, we provide the proof of Lemma 5.1 in the numerical examples in Section 5.1. *Proof of Lemma 5.1.* In order to show (i), we fix some  $\lambda > 0$  and an initial condition  $g : \mathbb{R}^m \to \mathbb{R}$  that is a.e. continuous and a.e. bounded. Then, we first observe that (17) can be expressed as convolution of the kernel  $(0,\infty) \times \mathbb{R}^m \ni (t,y) \mapsto \phi_{\lambda,t}(t,y) := (4\pi\lambda t)^{-m/2} \exp\left(-\frac{\|y\|^2}{4\lambda t}\right) \in \mathbb{R}$  with the initial condition  $g: \mathbb{R}^m \to \mathbb{R}$ . Moreover, for every  $(t, y) \in (0, \infty) \times \mathbb{R}^m$ , it holds that that

$$\frac{\partial \phi_{\lambda,t}}{\partial t}(t,y) - \lambda \sum_{l=1}^{m} \frac{\partial^2 \phi_{\lambda,t}}{\partial y_l^2}(t,y) = \left(\frac{\|y\|^2}{4\lambda t^2} - \frac{m}{2t}\right) \frac{e^{-\frac{\|y\|^2}{4\lambda t}}}{(4\pi\lambda t)^{\frac{m}{2}}} - \lambda \sum_{l=1}^{m} \left(\frac{4y_l^2}{(4\lambda t)^2} - \frac{2}{4\lambda t}\right) \frac{e^{-\frac{\|y\|^2}{4\lambda t}}}{(4\pi\lambda t)^{\frac{m}{2}}} = 0.$$
(99)

Hence, by using [Hörmander, 1990, Theorem 1.3.1], i.e. that  $\frac{\partial f}{\partial t}(t, u) = \left(\frac{\partial \phi_{\lambda,t}}{\partial t} * g\right)(u)$  and  $\frac{\partial^2 f}{\partial u_t^2}(t, u) = \frac{\partial \phi_{\lambda,t}}{\partial u_t^2}(t, u)$  $\left(\frac{\partial^2 \phi_{\lambda,t}}{\partial^2 y_l} * g\right)(u)$  for any  $(t, u) \in (0, \infty) \times \mathbb{R}^m$  and l = 1, ..., m, and the identity (99), it follows for every  $(t, u) \in (0, \infty) \times \mathbb{R}^m$  that

$$\begin{split} \frac{\partial f}{\partial t}(t,u) &-\lambda \sum_{l=1}^{m} \frac{\partial^2 f}{\partial u_l^2}(t,u) = \frac{\partial (\phi_{\lambda,t} * g)}{\partial t}(t,u) - \lambda \sum_{l=1}^{m} \frac{\partial^2 (\phi_{\lambda,t} * g)}{\partial u_l^2}(t,u) \\ &= \int_{\mathbb{R}^m} \left( \frac{\partial \phi_{\lambda,t}}{\partial t}(t,u-v) - \lambda \sum_{l=1}^{m} \frac{\partial^2 \phi_{\lambda,t}}{\partial y_l^2}(t,u-v) \right) g(v) dv = 0. \end{split}$$

Moreover, by using the substitution  $y \mapsto \frac{u-v}{2\sqrt{\lambda t}}$  and the dominated convergence theorem (with the fact that  $g: \mathbb{R}^m \to \mathbb{R}$  is a.e. continuous, i.e. that  $\lim_{t\to 0} g\left(u + 2\sqrt{\lambda t}y\right) = g(u)$  for a.e.  $u, y \in \mathbb{R}^m$ , and that  $g: \mathbb{R}^m \to \mathbb{R}$  is a.e. bounded, i.e. that there exists some  $C_g > 0$  such that for a.e.  $u, y \in \mathbb{R}^m$  it holds that  $\max\left(\left|g\left(u+2\sqrt{\lambda t}y\right)\right|, |g(u)|\right) \leq C_g\right), \text{ we conclude for a.e. } u \in \mathbb{R}^m \text{ that}$ 

$$\begin{split} \lim_{t \to 0} f(t, u) &= \lim_{t \to 0} \frac{1}{(4\pi\lambda t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\frac{\|u-v\|^2}{4\lambda t}} g(v) dv = \lim_{t \to 0} \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\frac{\|y\|^2}{2}} g\left(u + 2\sqrt{\lambda t}y\right) dy \\ &= \left(\frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} e^{-\frac{\|y\|^2}{2}} dy\right) g(u) = g(u). \end{split}$$

This shows that the function  $f: (0, \infty) \times \mathbb{R}^m \to \mathbb{R}$  defined in (17) indeed solves the PDE (16).

In order to prove (ii), we fix some  $\gamma \in [0, \infty)$ ,  $p \in [1, \infty)$ ,  $\lambda, t, \nu \in (0, \infty)$ , and  $N \in \mathbb{N}$ , and let  $g \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), (1+\|u\|)^{[\gamma]+[\nu]+1}du)$  be a.e. continuous and a.e. bounded. Moreover, let  $w : \mathbb{R}^m \to [0, \infty)$  be as in Example 2.8 (a), and let  $(\psi, \rho) \in \mathcal{S}_0(\mathbb{R}; \mathbb{C}) \times C^0_{pol,\gamma}(\mathbb{R})$  be as in Example 2.8 (b) with  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$  for some  $0 < \xi_1 < \xi_2 < \infty$ . Now, we first show that  $f(t, \cdot) \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$ . Indeed, by using (i), i.e. that  $f(t, u) = (\phi_{\lambda, t} * g)(u) := \int_{\mathbb{R}^m} \phi_{\lambda, t}(u - y)g(y)dy$  for any  $u \in \mathbb{R}^m$  (with  $\phi_{\lambda, t} : \mathbb{R}^m \to \mathbb{R}$  defined above), and Young's convolutional inequality, it follows that

$$\begin{split} \|f(t,\cdot)\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du)} &= \|\phi_{\lambda,t} * g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du)} \leqslant \|\phi_{\lambda,t}\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du)} \|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du)} \\ &= \left(\underbrace{\frac{1}{(4\pi\lambda t)^{\frac{m}{2}}} \int_{\mathbb{R}^{m}} e^{-\frac{\|y\|^{2}}{4\lambda t}} dy}_{=1}\right) \|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|u\|)^{\lceil\gamma\rceil+\lceil\nu\rceil+1}du)} < \infty, \end{split}$$

which shows that  $f(t, \cdot) \in L^1(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), du)$ .

Next, we show that the Fourier transform  $f(t, \cdot) : \mathbb{R}^m \to \mathbb{C}$  is  $([\gamma] + [\nu] + 1)$ -weakly differentiable. To this end, we use Fubini's theorem, [Folland, 1992, Table 7.2.9], the substitution  $\zeta_l \mapsto \sqrt{2\lambda t} \xi_l$ , and the Hermite polynomials  $(h_n)_{n \in \mathbb{N}}$  in [Abramowitz and Stegun, 1970, Equation 22.2.15], to conclude for every  $\beta \in \mathbb{N}_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}^m$  and  $\xi \in \mathbb{R}^m$  that

$$\widehat{\partial}_{\beta}\widehat{\phi_{\lambda,t}}(\xi) = \widehat{\partial}_{\beta}\left(\prod_{l=1}^{m} \int_{\mathbb{R}} e^{-i\xi_{l}u_{l}} \frac{e^{-\frac{u_{l}^{2}}{4\lambda t}}}{\sqrt{4\pi\lambda t}} du_{l}\right) = \widehat{\partial}_{\beta}\left(\prod_{l=1}^{m} e^{-\lambda t\xi_{l}^{2}}\right) = \prod_{l=1}^{m} \frac{\widehat{\partial}^{\beta_{l}}}{\partial\xi_{l}^{\beta_{l}}} \left(e^{-\lambda t\xi_{l}^{2}}\right) \\
= (2\lambda t)^{\frac{|\beta|}{2}} \prod_{l=1}^{m} \frac{\widehat{\partial}^{\beta_{l}}}{\partial\zeta_{l}^{\beta_{l}}} \left(e^{-\frac{\zeta_{l}^{2}}{2}}\right) \Big|_{\zeta_{l}=\sqrt{2\lambda t}\xi_{l}} = (2\lambda t)^{\frac{|\beta|}{2}} \prod_{l=1}^{m} (-1)^{\beta_{l}} h_{\beta_{l}}(\zeta_{l}) e^{-\frac{\zeta_{l}^{2}}{2}} \Big|_{\zeta_{l}=\sqrt{2\lambda t}\xi_{l}}$$
(100)  

$$= (-1)^{|\beta|} (2\lambda t)^{\frac{|\beta|}{2}} \left(\prod_{l=1}^{m} h_{\beta_{l}} \left(\sqrt{2\lambda t}\xi_{l}\right)\right) e^{-\lambda t \|\xi\|^{2}}.$$

Moreover, we use the polynomial  $\mathbb{R}^m \ni u \mapsto u^{\beta} := \prod_{l=1}^m u_l^{\beta_l} \in \mathbb{R}$ , the inequality  $|u^{\beta}| = \prod_{l=1}^m |u_l|^{\beta_l} \leq \prod_{l=1}^m (1 + ||u||)^{\beta_l} = (1 + ||u||)^{|\beta|} \leq (1 + ||u||)^{\lceil \gamma \rceil + \lceil \nu \rceil + 1}$  for any  $\beta \in \mathbb{N}^m_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}$  and  $u \in \mathbb{R}^m$  to obtain for every  $\beta \in \mathbb{N}^m_{0, \lceil \gamma \rceil + \lceil \nu \rceil + 1}$  that

$$\|p_{\beta} \cdot g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),du)} = \int_{\mathbb{R}^{m}} \left| u^{\beta}g(u) \right| du \leq \int_{\mathbb{R}^{m}} |g(u)|(1+\|u\|)^{\lceil\gamma\rceil+\lceil\nu\rceil+1} du$$
  
=  $\|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|u\|)^{\lceil\gamma\rceil+\lceil\nu\rceil+1} du)} < \infty.$  (101)

Hence, by iteratively applying [Folland, 1992, Theorem 7.8. (c)], we conclude that the partial derivative  $\partial_{\beta}\hat{g} : \mathbb{R}^m \to \mathbb{C}$  exists, for all  $\beta \in \mathbb{N}_{0,[\gamma]+[\nu]+1}^m$ . Thus, by using [Folland, 1992, Theorem 7.8. (d)] and the Leibniz product rule, we conclude for every  $\beta \in \mathbb{N}_{0,[\gamma]+[\nu]+1}^m$  and  $\xi \in \mathbb{R}^m$  that

$$\partial_{\beta}\widehat{f(t,\cdot)}(\xi) = \partial_{\beta}\left(\widehat{\phi_{\lambda,t}}(\xi)\widehat{g}(\xi)\right) = \sum_{\substack{\beta_{1},\beta_{2}\in\mathbb{N}_{0}^{m}\\\beta_{1}+\beta_{2}=\beta}} \frac{\beta!}{\beta_{1}!\beta_{2}!}\partial_{\beta_{1}}\widehat{\phi_{\lambda,t}}(\xi)\partial_{\beta_{2}}\widehat{g}(\xi), \tag{102}$$

which shows that  $\overline{f}(t, \cdot) : \mathbb{R}^m \to \mathbb{C}$  is  $([\gamma] + [\nu] + 1)$ -weakly differentiable.

Now, we compute the constant  $C_{f(t,\cdot)} \ge 0$  defined in (9). To this end, we define the constant  $c := 1 + [\gamma] + [\nu] \in \mathbb{N}$  (independent of  $m \in \mathbb{N}$ ). Then, by using the explicit expression of the Hermite polynomials in [Abramowitz and Stegun, 1970, Equation 22.3.11] together with the triangle inequality, that  $|\zeta_l|^{\beta_l - 2j_l} \le (1 + \|\zeta\|)^{\beta_l - 2j_l} \le (1 + \|\zeta\|)^{\beta_l}$  for any  $l = 1, ..., m, \beta \in \mathbb{N}_0^m, j_l = 0, ..., \lfloor\beta_l/2\rfloor$ , and  $\zeta \in \mathbb{R}^m$ , that  $\sum_{j_l=1}^{\lfloor\beta_l/2\rfloor} \frac{\beta_{l!}}{2^{j_l}j_l!(\beta_l - 2j_l)!} \le \max_{j_l=1,...,\lfloor\beta_l/2\rfloor} \frac{(2j_l)!}{j_l!} \sum_{j_l=1}^{\lfloor\beta_l/2\rfloor} \frac{\beta_l!}{(2j_l)!(\beta_l - 2j_l)!} \le \beta_l! \sum_{k_l=1}^{\beta_l} \frac{\beta_l!}{k_l!(\beta_l - k_l)!} = 2^{\beta_l}\beta_l!$  for any l = 1, ..., m and  $\beta \in \mathbb{N}_0^m$ , the inequality  $\prod_{l=1}^m \beta_l! = \beta! \le |\beta|! \le c!$  for any  $\beta \in \mathbb{N}_{0,c}^m$ , and

the inequality  $(x+y)^2 \leq 2(x^2+y^2)$  for any  $x, y \geq 0$ , it follows for every  $\beta \in \mathbb{N}_{0,c}^m$  and  $\zeta \in \mathbb{R}^m$  that

$$\prod_{l=1}^{m} |h_{\beta_{l}}(\zeta_{l})| \leq \prod_{l=1}^{m} \left( \sum_{j_{l}=1}^{|\beta_{l}/2|} \frac{\beta_{l}! |\zeta_{l}|^{\beta_{l}-2j_{l}}}{2^{j_{l}} j_{l}! (\beta_{l}-2j_{l})!} \right) \leq \prod_{l=1}^{m} \left( (1 + \|\zeta\|)^{\beta_{l}} \sum_{j_{l}=1}^{|\beta_{l}/2|} \frac{\beta_{l}!}{2^{j_{l}} j_{l}! (\beta_{l}-2j_{l})!} \right) \\
\leq (1 + \|\zeta\|)^{|\beta|} \prod_{l=1}^{m} 2^{\beta_{l}} \beta_{l}! \leq 2^{c} c! (1 + \|\zeta\|)^{c} \leq 2^{c+1} c! (1 + \|\zeta\|^{2})^{\frac{c}{2}}.$$
(103)

Hence, by inserting (103) into (100) and by using the constant  $C_{51} := 2^c c! \max(1, 2\lambda t)^c > 0$ , we conclude for every  $\beta \in \mathbb{N}_{0,c}^m$  and  $\xi \in \mathbb{R}^m$  that

$$\left| \partial_{\beta} \widehat{\phi_{\lambda,t}}(\xi) \right| \leq (2\lambda t)^{\frac{|\beta|}{2}} 2^{c} c! \left( 1 + \left\| \sqrt{2\lambda t} \xi \right\|^{2} \right)^{\frac{c}{2}} e^{-\lambda t \|\xi\|^{2}} \leq C_{51} \left( 1 + \|\xi\|^{2} \right)^{\frac{c}{2}} e^{-\lambda t \|\xi\|^{2}}.$$
(104)

Moreover, for every  $b \in \mathbb{N}_0$ , we use that  $Y := ||Z||^2$  of  $Z \sim \mathcal{N}_m(0, I_m)$  follows a  $\chi^2(m)$ -distribution with probability density function  $[0, \infty) \ni y \mapsto \frac{y^{m/2-1}e^{-y/2}}{2^{m/2}\Gamma(m/2)} \in [0, \infty)$ , the substitution  $x \mapsto y/2$ , and the definition of the Gamma function in [Abramowitz and Stegun, 1970, Equation 6.1.1] to obtain that

$$\int_{\mathbb{R}^m} \|z\|^b \frac{e^{-\frac{\|z\|^2}{2}}}{(2\pi)^{\frac{m}{2}}} dz = \mathbb{E}\left[\|Z\|^b\right] = \mathbb{E}\left[Y^{\frac{b}{2}}\right] = \int_0^\infty y^{\frac{b}{2}} \frac{y^{\frac{m}{2}-1}e^{-\frac{y}{2}}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} dy$$

$$= \frac{2^{\frac{b+m}{2}}}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} \int_0^\infty x^{\frac{b+m}{2}-1} e^{-x} dx = 2^{\frac{b}{2}} \frac{\Gamma\left(\frac{m+b}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}.$$
(105)

Now, in order to ease notation, we define  $s_m := m + s_0 \in \mathbb{N}_0$  (depending on  $m \in \mathbb{N}$ ) with  $s_0 := 4\lceil \gamma \rceil + 3\lceil \nu \rceil + 4 \in \mathbb{N}_0$  (independent of  $m \in \mathbb{N}$ ). Then, by using the inequality (104) together with the constant  $C_{52} := C_{51} \max\left(1/\nu, \xi_1^2\right) > 0$  (independent of  $m \in \mathbb{N}$ ), that  $(x+y)^{s_m/2+c} \leq 2^{s_m/2+c} \left(x^{s_m/2+c} + y^{s_m/2+c}\right)$  for any  $x, y \ge 0$ , the substitution  $z \mapsto \sqrt{4\lambda t} \xi$ , the identity (105) with  $b := 0 \in \mathbb{N}_0$  and  $b := m + t \in \mathbb{N}_0$ , and the constants  $C_{53} := C_{52} 2^c \left(\frac{2\xi_2^4 \nu}{16\xi_1^2 \pi^3}\right)^{s_0/2} > 0$ ,  $C_{54} := C_{52} (2\lambda t)^{-c} \left(\frac{\xi_2^4}{32\xi_1^4 \pi^3 \lambda t}\right)^{s_0/2} > 0$ , and  $c_0 := s_0/2 + c \in [0, \infty)$  (independent of  $m \in \mathbb{N}$ ) to conclude that

$$\begin{split} \int_{\mathbb{R}^{m}} \left| \partial_{\beta} \widehat{\phi_{\lambda,t}}(\xi) \right|^{2} \left( \frac{\xi_{2}^{4}}{16\xi_{1}^{2}\pi^{3}} \left( \nu + \frac{\|\xi\|^{2}}{\xi_{1}^{2}} \right) \right)^{\frac{sm}{2}} d\xi &\leq C_{52} \left( \frac{\xi_{2}^{4}}{16\xi_{1}^{2}\pi^{3}} \right)^{\frac{sm}{2}} \int_{\mathbb{R}^{m}} \left( \nu + \frac{\|\xi\|^{2}}{\xi_{1}^{2}} \right)^{\frac{sm}{2}+c} e^{-2\lambda t \|\xi\|^{2}} d\xi \\ &\leq C_{52} 2^{\frac{sm}{2}+c} \left( \frac{\xi_{2}^{4}}{16\xi_{1}^{2}\pi^{3}} \right)^{\frac{sm}{2}} \left( \nu^{\frac{sm}{2}} \int_{\mathbb{R}^{m}} e^{-2\lambda t \|\xi\|^{2}} d\xi + \frac{1}{\xi_{1}^{sm}} \int_{\mathbb{R}^{m}} \|\xi\|^{sm+2c} e^{-2\lambda t \|\xi\|^{2}} d\xi \right) \\ &\leq C_{52} 2^{\frac{sm}{2}+c} \left( \frac{\xi_{2}^{4}}{16\xi_{1}^{2}\pi^{3}} \right)^{\frac{sm}{2}} \left( \nu^{\frac{sm}{2}} (2\pi)^{\frac{m}{2}} \int_{\mathbb{R}^{m}} \frac{e^{-\frac{\|z\|^{2}}{2}}}{(2\pi)^{\frac{m}{2}}} dz + \frac{(2\pi)^{\frac{m}{2}}}{\xi_{1}^{sm} (4\lambda t)^{\frac{sm}{2}+c}} \int_{\mathbb{R}^{m}} \|z\|^{sm+2c} \frac{e^{-\frac{\|z\|^{2}}{2}}}{(2\pi)^{\frac{m}{2}}} d\xi \right) \\ &\leq C_{52} 2^{\frac{sm}{2}+c} \left( \frac{\xi_{2}^{4}}{16\xi_{1}^{2}\pi^{3}} \right)^{\frac{sm}{2}} \nu^{\frac{sm}{2}} (2\pi)^{\frac{m}{2}} + C_{52} \left( \frac{\xi_{2}^{4}}{16\xi_{1}^{2}\pi^{3}} \right)^{\frac{sm}{2}} \frac{(2\pi)^{\frac{m}{2}}}{\xi_{1}^{sm} (4\lambda t)^{\frac{sm}{2}+c}} 2^{\frac{sm+2c}{2}} \frac{\Gamma\left(\frac{m+sm+2c}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \\ &= C_{53} \left( \frac{\xi_{2}^{2}\sqrt{\nu}}{2\pi\xi_{1}^{2}} \right)^{m} + C_{54} \left( \frac{\xi_{2}^{2}}{4\pi\sqrt{\lambda t}\xi_{1}^{2}} \right)^{m} \frac{\Gamma\left(m+c_{0}\right)}{\Gamma\left(\frac{m}{2}\right)}. \end{split}$$

Moreover, for the second term on the right-hand side of (106), we use [Gonon et al., 2021, Lemma 2.4], i.e. that  $\sqrt{2\pi/x}(x/e)^x \leq \Gamma(x) \leq \sqrt{2\pi/x}(x/e)^x e^{1/(12x)} \leq \sqrt{4\pi/x}(x/e)^x$  for any  $x \in [1/2, \infty)$ , and the constant  $C_{55} := 2^{\nu/2}(1+c_0)^{c_0} > 0$  (independent of  $m \in \mathbb{N}$ ) to obtain that

$$\frac{\Gamma(m+c_0)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m+\nu}{2}\right)} \leqslant \frac{\sqrt{\frac{8\pi}{m+c_0}}(m+c_0)^{m+c_0}e^{-m-c_0}}{\sqrt{\frac{4\pi}{m}}\left(\frac{m}{2}\right)^{\frac{m}{2}}e^{-\frac{m}{2}}\sqrt{\frac{4\pi}{m+\nu}}\left(\frac{m+\nu}{2}\right)^{\frac{m+\nu}{2}}e^{-\frac{m+\nu}{2}}} \leqslant \sqrt{m}2^{m+\frac{\nu}{2}}\frac{m^{m+c_0}(1+c_0)^{m+c_0}}{m^{\frac{m}{2}}m^{\frac{m+\nu}{2}}} \leqslant m^{c_0+\frac{1}{2}}(2+2c_0)^m.$$
(107)

Hence, by using the inequalities (106)+(107), the finite constant  $C_{55} := C_{53} \sup_{m \in \mathbb{N}} \left(\frac{\xi_2^2 \sqrt{\nu}}{2\pi \xi_1^2}\right)^m / \Gamma\left(\frac{m+\nu}{2}\right) > 0$  (independent of  $m \in \mathbb{N}$ ), and the inequality  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$  for any  $x, y \geq 0$ , it follows that

$$\left(\int_{\mathbb{R}^{m}} \frac{\left|\partial_{\beta_{1}}\widehat{\phi_{\lambda,t}}(\xi)\right|^{2} \left(1 + \|\xi\|^{2}\right)^{\frac{s_{m}}{2}}}{\Gamma\left(\frac{m+\nu}{2}\right)} d\xi\right)^{\frac{1}{2}} \leqslant \left(C_{53} \frac{\left(\frac{\xi_{2}^{2}\sqrt{\nu}}{2\pi\xi_{1}^{2}}\right)^{m}}{\Gamma\left(\frac{m+\nu}{2}\right)} + C_{54} \left(\frac{\xi_{2}^{2}}{4\pi\sqrt{\lambda t}\xi_{1}^{2}}\right)^{m} \frac{\Gamma\left(m+c_{0}\right)}{\Gamma\left(\frac{m+\nu}{2}\right)}\right)^{\frac{1}{2}} \\
\leqslant \left(C_{55} + C_{54} \left(\frac{\xi_{2}^{2}}{4\pi\sqrt{\lambda t}\xi_{1}^{2}}\right)^{m} m^{c_{0}+\frac{1}{2}} (2 + 2c_{0})^{m}\right)^{\frac{1}{2}} \leqslant \sqrt{C_{53}} + \sqrt{C_{54}} \left(\frac{(1+c_{0})\xi_{2}^{2}}{2\pi\sqrt{\lambda t}\xi_{1}^{2}}\right)^{\frac{m}{2}} m^{\frac{2c_{0}+1}{4}}.$$
(108)

Now, we use that  $\operatorname{supp}(\hat{\psi}) = [\xi_1, \xi_2]$ , the identity (102), Minkowski's inequality, the inequality  $(x + y)^2 \leq 2(x^2 + y^2)$  for any  $x, y \geq 0$ , the constant  $C_{56} := \max_{j \in \mathbb{N}_0 \cap [0, [\gamma] + [\nu] + 1]} \int_{\xi_1}^{\infty} |\hat{\psi}^{(j)}(\zeta)| d\zeta > 0$ (independent of  $m \in \mathbb{N}$ ), the inequality (26) for the sum of multinomial coefficients, that  $A_1 \sim t_m(\nu)$  with  $\theta_A(a)^{-1} = \frac{\Gamma(\nu/2)(\pi\nu)^{m/2}}{\Gamma((m+\nu)/2)} \left(1 + \|a\|^2/\nu\right)^{(m+\nu)/2} \leq \frac{\Gamma(\nu/2)(\pi\nu)^{m/2}}{\Gamma((m+\nu)/2)} \left(1 + \|a\|^2/\nu\right)^{(m+\lceil\nu\rceil)/2}$  for any  $a \in \mathbb{R}^m$ , the constant  $C_{57} := 2^c C_{56} \left(\frac{\xi_2^4 \nu}{16\xi_1^2 \pi^3}\right)^{(4\lceil\gamma\rceil + 3\lceil\nu\rceil + 4)/2} \max(1, 1/\nu)^{[\nu\rceil/2} > 0$  (independent of  $m \in \mathbb{N}$ ), [Folland, 1992, Theorem 7.8. (c)] componentwise, the inequality (108), that  $c_0 \leq 3[\gamma] + 3[\nu] + 3$ , the inequality (2), the inequality (101), and the constants  $C_{58} := C_{57} \max(C_{53}, C_{54})^{1/2}$  and  $C_{59} := (2c_0 + 1)/4 > 0$  (independent of  $m \in \mathbb{N}$ ) to conclude that

$$\begin{split} C_{f(t,\cdot)} &= \max_{\substack{j \in \mathbb{N}_{0} \cap [0,|+|+|\nu|+1], \\ \beta \in \mathbb{N}_{0}^{(1)}(\gamma|+|\nu|+1)}} \int_{\mathbb{R}} \frac{\left| \widehat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{2}}} \left( \int_{\mathbb{R}^{m}} \left| \partial_{\beta} \widehat{f(t,\cdot)}(\xi) \right|^{2} \frac{(1+\|\xi/\zeta\|)^{4[\gamma]+2[\nu]+4}}{\theta_{A}(\xi/\zeta)} d\xi \right)^{\frac{1}{2}} d\zeta \\ &= \max_{\substack{j \in \mathbb{N}_{0} \cap [0,-], \\ \beta \in \mathbb{N}_{0,c}^{(1)}} \int_{\xi_{1}} \frac{\left| \widehat{\psi}^{(j)}(\zeta) \right|}{|\zeta|^{\frac{m}{2}}} \left( \int_{\mathbb{R}^{m}} \left| \sum_{\substack{\beta_{1},\beta_{2} \in \mathbb{N}_{0}^{m}, \\ \beta_{1}+\beta_{2}=\beta}} \frac{\beta!}{\beta_{1}!\beta_{2}!} \partial_{\beta} \widehat{\phi_{\lambda,t}}(\xi) \partial_{\beta_{2}} \widehat{g}(\xi) \right|^{2} \frac{(1+\|\xi/\zeta\|)^{4[\gamma]+2[\nu]+4}}{\theta_{A}(\xi/\zeta)} d\xi \right)^{\frac{1}{2}} d\zeta \\ &\leq C_{56} \max_{\beta_{1},\beta_{2} \in \mathbb{N}_{0,c}^{m}}} \sum_{\substack{\beta_{1},\beta_{2} \in \mathbb{N}_{0,c}^{m}, \\ \beta_{1}+\beta_{2}=\beta}} \frac{\beta!}{\beta_{1}!\beta_{2}!} \left( \int_{\mathbb{R}^{m}} \frac{\left| \partial_{\beta_{1}} \widehat{\phi_{\lambda,t}}(\xi) \partial_{\beta_{2}} \widehat{g}(\xi) \right|^{2}}{\xi_{1}^{m}} \frac{(1+\|\xi\|^{2})^{4[\gamma]+2[\nu]+4}}{\theta_{A}(\xi)} d\xi \right)^{\frac{1}{2}} \\ &\leq C_{56} \max_{\beta_{1},\beta_{2} \in \mathbb{N}_{0,c}^{m}} \left( \int_{\mathbb{R}^{m}} \frac{\left| \partial_{\beta_{1}} \widehat{\phi_{\lambda,t}}(\xi) \partial_{\beta_{2}} \widehat{g}(\xi) \right|^{2}}{\xi_{1}^{m}} \left( 1+\frac{\|\xi\|^{2}}{\xi_{1}^{2}} \right)^{\frac{4[\gamma]+2[\nu]+4}{2}} \frac{\Gamma\left(\frac{\nu}{2}\right) (\pi\nu)^{\frac{m}{2}}}{\Gamma\left(\frac{m+\nu}{2}\right)} \left( 1+\frac{\|\xi\|^{2}}{\nu\xi_{1}^{2}} \right)^{\frac{m+|\nu|}{2}} d\xi \right)^{\frac{1}{2}} \\ &\leq C_{57} \frac{(2\pi)^{m}}{\xi_{2}^{m}} \max_{\beta \in \mathbb{N}_{0,c}^{m}} \left( \int_{\mathbb{R}^{m}} \frac{\left| \partial_{\beta_{1}} \widehat{\phi_{\lambda,t}}(\xi) \right|^{2} \left( \frac{\xi_{2}}{\xi_{1}^{2}} \frac{(\nu+\|\xi\|^{2})}{\Gamma\left(\frac{m+\nu}{2}\right)} \right)^{\frac{m+4[\gamma]+3[\nu]+4}{2}} d\xi \right)^{\frac{1}{2}} \max_{\beta_{2} \in \mathbb{N}_{0,c}^{m}} \xi_{2} \\ &\leq C_{57} \frac{(2\pi)^{m}}{\xi_{2}^{m}} \max_{\beta \in \mathbb{N}_{0,c}^{m}} \left( \int_{\mathbb{R}^{m}} \frac{\left| \partial_{\beta_{1}} \widehat{\phi_{\lambda,t}}(\xi) \right|^{2} \left( \frac{\xi_{2}}{\xi_{1}^{2}} \frac{(\nu+\|\xi\|^{2})}{\Gamma\left(\frac{m+\nu}{2}\right)} \right)^{\frac{m+4[\gamma]+3[\nu]+4}{2}} d\xi \right)^{\frac{1}{2}} \max_{\beta_{2} \in \mathbb{N}_{0,c}^{m}} \xi_{2} \\ &\leq C_{57} \frac{(2\pi)^{m}}{\xi_{2}^{m}} \left( \sqrt{C_{53}} + \sqrt{C_{54}} \left( \frac{(1+c_{0})\xi_{2}^{2}}{2\pi\sqrt{\lambda t\xi_{1}^{2}}} \right)^{\frac{m}{2}} m^{\frac{2c_{0}+1}{4}} \right) \|p_{\beta_{2}} \cdot g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|\mu\|)^{\frac{1}{2}|+1|d\mu|}} d\xi \\ &\leq C_{58} m^{C_{59}} \frac{(2\pi)^{m}}{\xi_{2}^{m}} \left( 1 + \left( \frac{(3[\gamma]+3[\nu]+4)\xi_{2}^{2}}{2\pi\sqrt{\lambda t\xi_{1}^{2}}} \right)^{\frac{m}{2}} \right) \|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|\mu\|)^{\frac{1}{2}|+1|d\mu|}} dx \\ &\leq C_{58} m^{C_{59}} \frac{(2\pi)^{m}}{\xi_{2}^{m}} \left( 1 + \frac{(2\pi)^{2}}{2\pi\sqrt{\lambda t\xi_{1}^{2}}} \right)^{\frac{m}{2}} \right)^$$

Since  $C_f \ge 0$  is finite, we can apply Theorem 4.4 (ii) (with constant  $C_1 > 0$ , independent of  $m \in \mathbb{N}$ ) to conclude that there exists some  $\Phi_N \in \mathcal{RN}_{\mathbb{R}^m,1}^{\rho} \cap L^2(\Omega, \mathcal{F}_{A,B}, \mathbb{P}; L^p(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m), w(u)du))$  such that

$$\mathbb{E}\left[\|f(t,\cdot) - \Phi_N\|_{L^p(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m),w(u)du)}^2\right]^{\frac{1}{2}} \leq C_1 \frac{C_{\mathbb{R}^m,w}^{(\gamma,p)}C_f}{\left|C_m^{(\psi,\rho)}\right|} \frac{m^{[\gamma]+[\nu]+1}}{N^{1-\frac{1}{\min(2,p)}}}$$

Hence, by using that the weight  $w : \mathbb{R}^m \to [0, \infty)$  is as in Example 4.7 (a), i.e. that there exists a constant  $C_{\mathbb{R},w_0}^{(\gamma,p)} > 0$  (independent of  $m \in \mathbb{N}$ ) such that  $C_{\mathbb{R}^m,w}^{(\gamma,p)} \leqslant C_{\mathbb{R},w_0}^{(\gamma,p)} m^{\gamma+1/p}$ , that  $(\psi,\rho) \in \mathcal{S}_0(\mathbb{R};\mathbb{C}) \times C_{pol,\gamma}^0(\mathbb{R})$  is as in Example 4.7 (b), i.e. that there exists a constant  $C_{\psi,\rho} > 0$  (independent of  $m \in \mathbb{N}$ ) such that  $|C_m^{(\psi,\rho)}| \ge C_{\psi,\rho}(2\pi/\xi_2)^m$ , the inequality (109), and the constants  $C_4 := C_1 C_{\mathbb{R},w_0}^{(\gamma,p)} C_{58}/C_{\psi,\rho} > 0$  and  $C_5 := \gamma + 1/p + [\gamma] + [\nu] + 1 > 0$  (independent of  $m \in \mathbb{N}$ ), it follows that

$$\begin{split} \mathbb{E}\left[\|f(t,\cdot) - \Phi_{N}\|_{L^{p}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),w(u)du)}^{2}\right]^{\frac{1}{2}} &\leq C_{1}\frac{C_{\mathbb{R}^{m},w}^{(\gamma,p)}C_{f}}{|C_{m}^{(\psi,\rho)}|}\frac{m^{[\gamma]+[\nu]+1}}{N^{1-\frac{1}{\min(2,p)}}} \\ &\leq C_{1}\frac{C_{\mathbb{R},w_{0}}^{(\gamma,p)}m^{\gamma+\frac{1}{p}}\xi_{2}^{m}}{C_{\psi,\rho}(2\pi)^{m}}\frac{m^{[\gamma]+[\nu]+1}}{N^{1-\frac{1}{\min(2,p)}}}C_{58}m^{C_{59}}\frac{(2\pi)^{m}}{\xi_{2}^{m}}\left(1+\left(\frac{(3[\gamma]+3[\nu]+3)\xi_{2}^{2}}{2\pi\sqrt{\lambda t}\xi_{1}^{2}}\right)^{\frac{m}{2}}\right) \\ &\cdot \|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|u\|)^{[\gamma]+[\nu]+1}du)} \\ &\leq \frac{C_{4}m^{C_{5}}\left(1+\left(\frac{(3[\gamma]+3[\nu]+3)\xi_{2}^{2}}{2\pi\sqrt{\lambda t}\xi_{1}^{2}}\right)^{\frac{m}{2}}\right)\|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|u\|)^{[\gamma]+[\nu]+1}du)} \\ &\lesssim \frac{N^{1-\frac{1}{\min(2,p)}}}{N^{1-\frac{1}{\min(2,p)}}}, \end{split}$$

which proves the inequality (18) in (ii).

Finally, in order to show (iii), we fix some R > 0 and  $\kappa \in [0, 1/2)$ . Then, the function  $\mathbb{R}^m \ni u \mapsto g(u) := \mathbb{1}_{\overline{\mathbb{B}_m \kappa_R(0)}}(u) \in \mathbb{R}$  is a.e. bounded and a.e. continuous. Moreover, by using the inequality  $||u||^2 = \sum_{l=1}^m u_l^2 \leq mR^2m^{2\kappa} = m^{2\kappa+1}R^2$  for any  $u \in \overline{\mathbb{B}_m \kappa_R(0)}$ , that the volume of the ball  $\overline{\mathbb{B}_r(0)}$  is equal to  $\frac{\pi^{m/2}r^m}{\Gamma(m/2+1)}$  for any  $r \ge 0$ , and [Gonon et al., 2021, Lemma 2.4], i.e. that  $\Gamma(x) \ge \sqrt{2\pi/x}(x/e)^x$  for any  $x \in (0, \infty)$ , it follows that

$$C_{g} := \|g\|_{L^{1}(\mathbb{R}^{m},\mathcal{L}(\mathbb{R}^{m}),(1+\|u\|)^{\lceil\gamma\rceil+\lceil\nu\rceil+1}du)} = \int_{\mathbb{R}^{m}} |g(u)|(1+\|u\|)^{\lceil\gamma\rceil+\lceil\nu\rceil+1}du$$

$$\leq \left(1+m^{\kappa+\frac{1}{2}}R\right)^{\lceil\gamma\rceil+\lceil\nu\rceil+1} \int_{\overline{\mathbb{B}}_{m^{\kappa}R}(0)} du = \left(1+m^{\kappa+\frac{1}{2}}R\right)^{\lceil\gamma\rceil+\lceil\nu\rceil+1} \frac{\pi^{\frac{m}{2}}(m^{\kappa}R)^{m}}{\Gamma\left(\frac{m}{2}+1\right)}$$

$$\leq 2e\left(1+m^{\kappa+\frac{1}{2}}R\right)^{\lceil\gamma\rceil+\lceil\nu\rceil+1} \frac{(2e\pi)^{\frac{m}{2}}R^{m}m^{\kappa m}}{\sqrt{\frac{4\pi}{m+2}}(m+2)^{\frac{m}{2}+1}} \leq \frac{2e\left(1+m^{\kappa+\frac{1}{2}}R\right)^{\lceil\gamma\rceil+\lceil\nu\rceil+1}(2e\pi)^{\frac{m}{2}}R^{m}}{m^{m(1/2-\kappa)}} < \infty.$$

Hence, by inserting this into the right-hand side of (18) and by using the constant

$$C_{60} := \sup_{m \in \mathbb{N}} \left( C_4 m^{C_5} \left( 1 + \left( \frac{3([\gamma] + [\nu] + 2)\xi_2^2}{2\pi\sqrt{\lambda t}\xi_1^2} \right)^{\frac{m}{2}} \right) \frac{2e \left( 1 + m^{\kappa + \frac{1}{2}} R \right)^{[\gamma] + [\nu] + 1} (2e\pi)^{\frac{m}{2}} R^m}{m^{m(1/2 - \kappa)}} \right) < \infty,$$

which is finite as  $\kappa \in [0, 1/2)$ , we conclude that

$$\frac{C_4 m^{C_5} \left(1 + \left(\frac{3([\gamma] + [\nu] + 2)\xi_2^2}{2\pi\sqrt{\lambda t}\xi_1^2}\right)^{\frac{m}{2}}\right) C_g}{N^{1 - \frac{1}{\min(2,p)}}} \\ \leqslant \frac{C_4 m^{C_5} \left(1 + \left(\frac{3([\gamma] + [\nu] + 2)\xi_2^2}{2\pi\sqrt{\lambda t}\xi_1^2}\right)^{\frac{m}{2}}\right) \frac{2e\left(1 + m^{\kappa + \frac{1}{2}}R\right)^{[\gamma] + [\nu] + 1} (2e\pi)^{\frac{m}{2}}R^m}{m^{m(1/2 - \kappa)}} \\ \leqslant \frac{C_60}{N^{1 - \frac{1}{\min(2,p)}}}.$$

This shows that the right-hand side of (18) grows polynomially in  $m \in \mathbb{N}$ .

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NANYANG TECHNOLOGICAL UNIVERSITY, DIVISION OF MATHEMATICAL SCIENCES, 21 NANYANG LINK, SINGAPORE *Email address*: ariel.neufeld@ntu.edu.sg

NANYANG TECHNOLOGICAL UNIVERSITY, DIVISION OF MATHEMATICAL SCIENCES, 21 NANYANG LINK, SINGAPORE *Email address*: philippt001@e.ntu.edu.sg