# ROBUST SGLD ALGORITHM FOR SOLVING NON-CONVEX DISTRIBUTIONALLY ROBUST OPTIMISATION PROBLEMS 

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#### Abstract

In this paper we develop a Stochastic Gradient Langevin Dynamics (SGLD) algorithm tailored for solving a certain class of non-convex distributionally robust optimisation problems. By deriving nonasymptotic convergence bounds, we build an algorithm which for any prescribed accuracy $\varepsilon>0$ outputs an estimator whose expected excess risk is at most $\varepsilon$. As a concrete application, we employ our robust SGLD algorithm to solve the (regularised) distributionally robust Mean-CVaR portfolio optimisation problem using real financial data. We empirically demonstrate that the trading strategy obtained by our robust SGLD algorithm outperforms the trading strategy obtained when solving the corresponding non-robust Mean-CVaR portfolio optimisation problem using, e.g., a classical SGLD algorithm. This highlights the practical relevance of incorporating model uncertainty when optimising portfolios in real financial markets.


## 1. Introduction

Given $\Xi \subseteq \mathbb{R}^{m}$, a distance $d_{c}(\cdot, \cdot)$ on the space of probability measures $\mathcal{P}(\Xi)$, a reference measure $\mu_{0} \in \mathcal{P}(\Xi)$, parameters $\eta_{1}, \eta_{2}>0$, and a possibly non-convex utility function $U: \mathbb{R}^{d} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$, we consider the following non-convex distributionally robust stochastic optimisation problem

$$
\begin{equation*}
\text { minimise } \mathbb{R}^{d} \ni \theta \mapsto u(\theta):=\left\{\sup _{\mu \in \mathcal{P}(\Xi)}\left(\int_{\Xi} U(\theta, x) \mathrm{d} \mu(x)-\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}\right)+\frac{\eta_{1}}{2}|\theta|^{2}\right\} . \tag{1}
\end{equation*}
$$

Here, $\mu_{0}$ represents an estimate for the true but unknown law of the environment of the optimisation problem while $\eta_{2}>0$ represents the level of model uncertainty an agent has in the environment. Indeed, the smaller $\eta_{2}$ is chosen the larger the penalty term $\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}$ becomes, hence the more certain the agent believes that his estimated measure $\mu_{0}$ actually represents the true law of the environment.

The goal of this paper is to construct an estimator $\hat{\theta}$ which minimises the expected excess risk associated with (1). More precisely, we aim to build an algorithm which for any prescribed accuracy $\varepsilon>0$ outputs a $d$-dimensional estimator $\hat{\theta}_{\varepsilon}$ defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{\varepsilon}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta)<\varepsilon \tag{2}
\end{equation*}
$$

Already in [71, 48], Knight and Ellsberg argued that an agent who is making decisions cannot have the precise knowledge of the true law characterising the environment and hence should take model uncertainty under consideration. In distributionally robust optimisation (DRO) problems, there are two approaches to overcome the problem of model uncertainty. In the first approach, one considers a set of probability measures representing all candidates for the true but unknown law of the environment and one optimises over the worst-case law among those candidate laws. A typical example for such an ambiguity set of laws would be a Wasserstein-ball of certain radius around a reference measure. In the second approach, like in our DRO problem (1), one starts with a reference measure $\mu_{0}$ representing the estimated law for the true but unknown law of the environment and then introduces a penalty function which penalises all probability measures the further they are away from that given reference measure. One then optimises robustly over all possible laws while the penalty function controls how much each law can contribute to the optimisation problem. Typically in both approaches, the corresponding reference measure has been estimated either from historical data by taking the empirical measure, or through experts insights.

Over the years, DRO problems became very popular in various fields. For applications in financial engineering, we refer to $[14,17,20,23,41,51,91,92,98,102,103,110,112,115]$ for portfolio optimisation, to $[2,18,26,30,31,36,37,43,44,45,49,76,97,99,101,111,122]$ for pricing of financial derivatives and its relation to robust no-arbitrage theory, and to $[24,25,47,73]$ for quantitative risk management. We also refer to [57, 59, 70, 89] for related applications in decision sciences in theoretical economics.

[^0]For applications of DRO problems in operations research, we refer to [131] for resource allocation, to [34, 72, 90, 109] for scheduling, to [56, 129, 134] for inventory management, to [104] for supply chain network design, to [105, 117] for facility location problem, to [21, 63, 127] for transportation, and to [126] for problems related to queueing. Moreover, for applications of DRO problems in computer science and statistics, we refer to $[74,75,121,125,128]$ for adversarial learning, e.g., in machine learning, to [ $8,9,58,119,123,124]$ for regression and classification, and to $[64,85,100]$ for robust reinforcement learning, to name but a few. We also refer to $[12,15,16,54,61,62,67,95,108]$ for the recent development on the sensitivity analysis of DRO problems in various fields.

In this paper, we develop a Stochastic Gradient Langevin Dynamics (SGLD) algorithm that can minimise the expected excess risk of a certain class of the DRO problems of the form (1) as described in (2). In Theorem 2.3, we obtain (under Assumption 1-5) non-asymptotic convergence bounds for our robust SGLD algorithm (13)-(14). As a consequence of the non-asymptotic convergence bounds, we can indeed develop an algorithm which for every prescribed acccuracy $\varepsilon>0$ outputs a $d$-dimensional estimator which minimises the expected excess risk as defined in (2). We refer to Algorithm 2 and its theoretical properties stated in Corollary 2.4.

SGLD algorithms are commonly used methodologies to solve (non-convex) stochastic optimisation problems $[35,42,50,69,96,114,133,141,143]$ as well as the sampling problem $[11,29,32,39,40$, 88, 130, 144]. Compared to stochastic gradient descent (SGD) algorithms, SGLD algorithms include an additional noise term in each iteration which allows them to better overcome local minima than SGD algorithms. We refer to [1, 78, 79, 83, 82, 84, 87, 132] for the development of SGLD based algorithms to solve stochastic optimisation problems involving the training of neural networks, to $[38,116]$ to solve portfolio optimisation problems, to [28] for deep hedging, to [66] for market risk dynamics, to [77] for pricing of financial instruments, to [22, 81, 93, 135] for dynamic topic models and information acquisition, to $[4,10,86,113,118]$ for time series prediction, to $[5,19,107,136,138]$ for uncertainty quantification, as well as to $[3,27,33,46,52,60,65,68,94,106,120,137,139,140]$ for large-scale Bayesian inference including, e.g., data classification, image recognition, Bayesian model selection, Bayesian probabilistic matrix factorisation, and variational inference.

However, so far, no SGLD algorithm has been developed tailored to solve general non-convex stochastic optimisation problems (1). In [80], the authors use a standard projected SGLD algorithm to solve robust Markov decision problems (MDP) defined on finite state and action spaces where the corresponding ambiguity set of probability measures does not need to be rectangular. Since their state space is finite, they can exploit the relation between the value function of the robust MDP and the sampling problem from the Gibbs distribution in order to show that a standard (i.e. not tailored to solve DRO problems) Langevin dynamics-based algorithm can minimise the expected excess risk arbitrarily well.

As a concrete application of the general framework (1), we apply our SGLD algorithm to solve the (regularised) distributionally robust Mean-CVaR portfolio optimisation problem using real financial data. We choose the reference measure $\mu_{0}$ to be the empirical distribution of the corresponding past returns of the financial assets under consideration, making the optimisation problem purely data-driven. We analyse the performance of our trading strategy obtained by our robust SGLD algorithm also in comparison with the trading strategy derived from the classical SGLD algorithm developed in [140] which solves the corresponding non-robust Mean-CVaR portfolio optimisation problem (see, e.g., [6]) defined with respect to the empirical measure. We refer to Section 3 for the precise formulation of our robust portfolio optimisation problem. We empirically demonstrate that our robust SGLD algorithm outperforms the non-robust SGLD algorithm when choosing the penalisation parameter $\eta_{2}>0$ in a suitable way (namely positive, but small enough). This highlights the practical relevance of incorporating model uncertainty when optimising portfolios in real financial markets.

The rest of this paper is organised as follows. In Section 2, we introduce the setting of our distributionally robust optimisation problem, the assumptions imposed, as well as present the main results of our paper. As a concrete application of our general setting in Section 2, we introduce in Section 3 the robust

Mean-CVaR portfolio optimisation problem and show that it fits into our general setting with the corresponding assumptions imposed in Section 2. In particular, the main results of our paper can be applied to this concrete DRO problem. In Section 4, we use real financial data to empirically demonstrate the applicability of our algorithm for the robust Mean-CVaR portfolio optimisation problem. In Section 5 we present an overview of the proofs of our main results, whereas in Sections 6-8 we present the remaining proofs of all results and statements presented in Sections 2-5.

Notation. We conclude this section by introducing some notation. Let $\mathbb{R}$ (respectively, $\mathbb{R}_{\geq 0}$ ) denote the set of (non-negative) real numbers. Let $(\Omega, \mathcal{F})$ be a measurable space. Given a random variable $Z$ and a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we denote by $\mathbb{E}_{\mathbb{Q}}[Z]:=\int_{\Omega} Z \mathrm{~d} \mathbb{Q}$ the expectation of $Z$ with respect to $\mathbb{Q}$. For $p \in[1, \infty), L^{p}(\Omega, \mathcal{F}, \mathbb{Q})$, or $L^{p}(\mathbb{Q})$ for short when the measurable space in consideration is clear from the context, is used to denote the space of $p$-integrable real-valued random variables on $\Omega$ with respect to $\mathbb{P}$. Fix integers $d, m \geq 1$. A random vector $\theta \in \mathbb{R}^{d}$ is always understood to be a column vector unless stated otherwise, with the exception of the gradient $\nabla f$ of a given function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ being a row vector as consistent with the interpretation of $\nabla$ acting as a linear operator from $\mathbb{R}^{d}$ to $\mathbb{R}$ and having matrix representation in $\mathbb{R}^{1 \times d}$. For an $\mathbb{R}^{d}$-valued random variable $Z$, its law on $\mathcal{B}\left(\mathbb{R}^{d}\right)$, i.e. the Borel sigma-algebra of $\mathbb{R}^{d}$, is denoted by $\mathcal{L}(Z)$. We denote by $I_{d}$ the $d$-dimensional identity matrix and by $\mathcal{N}\left(0, I_{d}\right)$ the $d$-dimensional standard normal distribution. For a positive real number $a$, we denote by $\lfloor a\rfloor$ its integer part, and $\lceil a\rceil=\lfloor a\rfloor+1$. The notation $\mathbb{1}$. is used to denote indicator functions. Given a normed space $\left(\Xi,\|\cdot\|_{\Xi}\right)$ and an element $x \in \Xi$, we denote the norm of $x$ by $\|x\|_{\Xi}$. In the particular case $\Xi=\mathbb{R}^{d}$ and $\|\cdot\|$ is the Euclidean norm, we understand the notation $|x|$ as referring to $|x|=\|x\|_{\mathbb{R}^{d}}$ for $x \in \mathbb{R}^{d}$. Similarly, for a real-valued $m \times d$ matrix $A \in \mathbb{R}^{m \times d}$, we understand $|A|$ as referring to the operator norm $|A|=\sup \left\{|A x|:|x| \leq 1, x \in \mathbb{R}^{d}\right\}$. The Euclidean scalar product is denoted by $\langle\cdot, \cdot\rangle$. For any normed space $\Xi$, let $\mathcal{P}(\Xi)$ denote the set of probability measures on $\mathcal{B}(\Xi)$. For $\mu, \mu^{\prime} \in \mathcal{P}(\Xi)$, let $\mathcal{C}\left(\mu, \mu^{\prime}\right)$ denote the set of couplings of $\mu, \mu^{\prime}$, that is, probability measures $\zeta$ on $\mathcal{B}(\Xi \times \Xi)$ such that its respective marginals are $\mu, \mu^{\prime}$. Given two Borel probability measures $\mu, \mu^{\prime} \in \mathcal{P}(\Xi)$ and a cost function $c: \Xi \times \Xi \rightarrow[0, \infty]$ in the sense of [13], the cost of transportation between $\mu$ and $\mu^{\prime}$ is defined by

$$
\begin{equation*}
d_{c}\left(\mu, \mu^{\prime}\right):=\inf _{\zeta \in \mathcal{C}\left(\mu, \mu^{\prime}\right)} \int_{\Xi \times \Xi} c\left(\theta, \theta^{\prime}\right) \mathrm{d} \zeta\left(\theta, \theta^{\prime}\right) . \tag{3}
\end{equation*}
$$

## 2. Assumptions and main results

2.1. Problem Statement. Let $\Xi$ be a compact subset of $\mathbb{R}^{m}$, let $U: \mathbb{R}^{d} \times \Xi \rightarrow \mathbb{R}$ be a measurable function, let $c: \Xi \times \Xi \rightarrow \mathbb{R}_{\geq 0}$ be defined as $c\left(x, x^{\prime}\right):=\left|x-x^{\prime}\right|^{p}$ for some $p \in[1, \infty)$, and let $\eta_{1}, \eta_{2}>0$ be regularisation parameters. Given a reference probability measure $\mu_{0} \in \mathcal{P}(\Xi)$, the main problem of interest is in the form of the following (regularised) distributionally robust stochastic optimisation problem

$$
\begin{equation*}
\text { minimise } \mathbb{R}^{d} \ni \theta \mapsto u(\theta):=\left\{\sup _{\mu \in \mathcal{P}(\Xi)}\left(\int_{\Xi} U(\theta, x) \mathrm{d} \mu(x)-\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}\right)+\frac{\eta_{1}}{2}|\theta|^{2}\right\} . \tag{4}
\end{equation*}
$$

2.2. Assumptions. In this section we present the assumptions imposed on the distributionally robust stochastic optimisation problem (4).
Assumption 1. $\Xi$ is a compact subset of $\mathbb{R}^{m}$. Denote, henceforth, $M_{\Xi}:=\max _{x \in \Xi}|x|<\infty$.
Assumption 2. For every $x \in \Xi$, the mapping $\theta \mapsto U(\theta, x)$ is continuously differentiable. Moreover, for every $\theta \in \mathbb{R}^{d}$, the mapping $x \mapsto U(\theta, x)$ is continuous.

Assumption 3. There exists constants $L_{\nabla}>0$ and $\nu \in \mathbb{N}_{0}$ such that for all $\theta_{1}, \theta_{2} \in \mathbb{R}^{d}$ and $x \in \Xi$,

$$
\begin{equation*}
\left|\nabla_{\theta} U\left(\theta_{1}, x\right)-\nabla_{\theta} U\left(\theta_{2}, x\right)\right| \leq L_{\nabla}(1+|x|)^{\nu}\left|\theta_{1}-\theta_{2}\right| . \tag{5}
\end{equation*}
$$

In addition, there exists a constant $K_{\nabla}>1$ such that for all $\theta \in \mathbb{R}^{d}$ and $x \in \Xi$,

$$
\begin{equation*}
\left|\nabla_{\theta} U(\theta, x)\right| \leq K_{\nabla}(1+|x|)^{\nu} . \tag{6}
\end{equation*}
$$

Remark 2.1. Under Assumption 3, it holds for all $\theta_{1}, \theta_{2} \in \mathbb{R}^{d}$ and $x \in \Xi$ that

$$
\begin{equation*}
\left|U\left(\theta_{1}, x\right)-U\left(\theta_{2}, x\right)\right| \leq K_{\nabla}(1+|x|)^{\nu}\left|\theta_{1}-\theta_{2}\right| . \tag{7}
\end{equation*}
$$

Moreover, under Assumptions 1, 2, 3, it holds for all $\theta \in \mathbb{R}^{d}$ and $x \in \Xi$ that

$$
\begin{equation*}
|U(\theta, x)| \leq \tilde{K}_{\nabla}(1+|x|)^{\nu}(1+|\theta|) \tag{8}
\end{equation*}
$$

where $\tilde{K}_{\nabla}:=\max \left\{K_{\nabla}, \max _{x \in \Xi}|U(0, x)|\right\}$.
Assumption 4. There exists a constant $J_{U}>0$ and $\chi \in \mathbb{N}_{0}$ such that for all $\theta \in \mathbb{R}^{d}$ and $x_{1}, x_{2} \in \Xi$,

$$
\begin{equation*}
\left|U\left(\theta, x_{1}\right)-U\left(\theta, x_{2}\right)\right| \leq J_{U}(1+|\theta|)\left(1+\left|x_{1}\right|+\left|x_{2}\right|\right)^{\chi}\left|x_{1}-x_{2}\right| \tag{9}
\end{equation*}
$$

Assumption 5. Let $\iota: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a surjective and continuously differentiable function such that its derivative $\iota^{\prime}$ is $L_{\iota}$-Lipschitz continuous and bounded by some constant $M_{\iota}>0$, and $\iota \cdot \iota^{\prime}$ is $\tilde{L}_{\iota}$-Lipschitz continuous. There exist constants $a_{\iota}, b_{\iota}>0$ such that for all $\alpha \in \mathbb{R}$, the following dissipativity condition holds:

$$
\begin{equation*}
\alpha \iota(\alpha) \iota^{\prime}(\alpha) \geq a_{\iota} \alpha^{2}-b_{\iota} . \tag{10}
\end{equation*}
$$

Remark 2.2. Assumption 5 is satisfied, e.g., with the choice of function $\iota(\alpha)=\log (\cosh \alpha)$.
2.3. Main Result. In this section, we define our robust SGLD algorithm constructed on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and state our main result, which is a non-asymptotic upper bound on the excess risk under $\mathbb{P}$ derived under the stated assumptions.

Given positive integers $\ell, \mathfrak{j}>0$, we define the set of dyadic rationals

$$
\begin{equation*}
\mathbb{K}_{\ell, \mathrm{j}}:=\left\{-2^{\ell-1},-2^{\ell-1}+\frac{1}{2^{\mathrm{j}}}, \cdots, 2^{\ell-1}-\frac{1}{2^{\mathrm{j}}}\right\} \tag{11}
\end{equation*}
$$

and fix, henceforth, an $\ell \in \mathbb{N}$ large enough such that $\Xi \subseteq\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}$. We also denote the finite set

$$
\begin{align*}
\left\{\xi_{j}^{\ell, \mathrm{j}}\right\}_{j=1, \cdots, N_{\ell, \mathrm{j}}} & :=\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m} \\
N_{\ell, \mathrm{j}} & :=2^{\ell+\mathrm{j}} \tag{12}
\end{align*}
$$

where $\mathbb{K}_{\ell, \mathrm{j}}^{m}$ denotes the $m$-th Cartesian power of the set $\mathbb{K}_{\ell, \mathrm{j}}$. In addition, we denote, for the ease of notation, $\xi_{j}:=\xi_{j}^{\ell, j}$ and $N:=N_{\ell, \mathrm{j}}$. That is, the dependence of the quantities $\xi_{j}$ and $N$ on $\ell$ and $\mathfrak{j}$ are suppressed for the sake of brevity. In addition, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\left(X_{n}\right)_{n \in \mathbb{N}_{0}}$, $\left(Z_{n}\right)_{n \in \mathbb{N}_{0}}$ are i.i.d. sequences with $\mathbb{P} \circ X_{0}^{-1}=\mu_{0} \in \mathcal{P}(\Xi)$ and $\mathbb{P} \circ Z_{0}^{-1} \sim \mathcal{N}\left(0, I_{d+1}\right)$.

Our SGLD algorithm yields a sequence of estimators $\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)_{n \in \mathbb{N}_{0}}$ with $\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}:=\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}, \hat{\alpha}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right) \in$ $\mathbb{R}^{d} \times \mathbb{R}$, which, for a given $\delta>0$, choice of step size $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, where the maximum step size restriction $\lambda_{\max , \delta}$ is given explicitly in (124), and $\mathfrak{j} \in \mathbb{N}$ controlling the grid mesh, is defined recursively as

$$
\begin{equation*}
\hat{\bar{\theta}}_{n+1}^{\lambda, \delta, \ell, \mathfrak{j}}:=\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}-\lambda H^{\delta, \ell, \mathfrak{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}, X_{n+1}\right)+\sqrt{2 \lambda \beta^{-1}} Z_{n+1}, \quad \hat{\bar{\theta}}_{0}^{\lambda, \delta, \ell, \mathrm{j}}=\bar{\theta}_{0} \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
H^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x) & :=\left(\eta_{1} \theta^{T}+\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathbf{j}}(\bar{\theta}, x) \nabla_{\theta} U\left(\theta, \xi_{j}\right)}{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}(\bar{\theta}, x)}, \quad \eta_{2} \iota(\alpha) \iota^{\prime}(\alpha)-\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathbf{j}}(\bar{\theta}, x) \iota^{\prime}(\alpha)\left|x-\xi_{j}\right|^{p}}{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}(\bar{\theta}, x)}\right)^{T} \\
F_{j}^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x) & :=\exp \left[\frac{1}{\delta}\left(U\left(\theta, \xi_{j}\right)-\iota(\alpha)\left|x-\xi_{j}\right|^{p}\right)\right] \tag{14}
\end{align*}
$$

with $\bar{\theta}=:(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}$ and $x \in \Xi \subset \mathbb{R}^{m}$. The following main result gives a non-asymptotic upper bound for the expected excess risk under $\mathbb{P}$ of the SGLD algorithm associated with (4).
Theorem 2.3. Let Assumptions 1, 2, 3, 4, and 5 hold. Let $\beta, \delta>0$, and let $\bar{\theta}_{0} \in L^{4}\left(\Omega, \mathcal{F}, \mathbb{P} ; \mathbb{R}^{d+1}\right)$. Moreover, let $\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\right)_{n \in \mathbb{N}}$ denote the first d components of the sequence of estimators obtained from the SGLD algorithm in (13) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exist explicit constants $a, c_{\delta, \beta}, C_{1, \delta, \ell, \mathfrak{j}, \beta}, C_{2, \delta, \ell, \mathrm{j}, \beta}, C_{3, \delta, \beta}, C_{4}, C_{5, \delta, \beta}, C_{6}>0$, defined in Appendix 8 , such that for each $n$, step size $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, and $\mathfrak{j} \in \mathbb{N}$,

$$
\begin{align*}
\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta) \leq & C_{1, \delta, \ell, \mathfrak{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta} \\
& +\delta(\ell+\mathfrak{j}) \log 2+\frac{\sqrt{m}}{2^{\mathfrak{j}}}\left(C_{4}+C_{5, \delta, \beta}+C_{6} e^{-a \lambda(n+1) / 2}\right) \tag{15}
\end{align*}
$$

Corollary 2.4. Let Assumptions 1, 2, 3, 4, and 5 hold, and let $\varepsilon>0$ be given. Then, Algorithm 1 outputs the estimator $\hat{\theta}_{n}^{\lambda, \delta, \ell, j}$ which satisfies

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta)<\varepsilon \tag{16}
\end{equation*}
$$

Proof. The proof of Theorem 2.3 and Corollary 2.4 can be found in Section 5.

```
Algorithm 1: SGLD Algorithm for DRO problem (4)
    Input: \(\varepsilon>0, d \in \mathbb{N}, m \in \mathbb{N}, p \in[1, \infty), \eta_{1}>0, \eta_{2}>0\), compact subset \(\Xi \in \mathbb{R}^{m}\), measurable
        function \(U: \mathbb{R}^{d} \times \mathbb{R}^{m} \rightarrow \mathbb{R}\), i.i.d. data \(\left(X_{\mathfrak{n}}\right)_{\mathfrak{n} \in \mathbb{N}_{0}} \subset \mathbb{R}^{m}\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) such that
        \(\mathbb{P} \circ X_{0}^{-1}=\mu_{0}\), initialisation \(\bar{\theta}_{0} \in \mathbb{R}^{d+1}\)
    Output: Estimator \(\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\)
    Set \(M_{\Xi}:=\max _{x \in \Xi}|x|\);
    Set \(L_{\nabla}, \nu, K_{\nabla}\) to be the constants given by Assumption 3;
    Set \(\tilde{K}_{\nabla}:=\max \left\{K_{\nabla}, \max _{x \in \Xi}|U(0, x)|\right\}\);
    Set \(J_{U}, \chi\) to be the constants given by Assumption 4;
    Set \(\iota: \mathbb{R} \rightarrow \mathbb{R}\) and \(a_{\iota}, b_{\iota}\) to be the function and constants given by Assumption 5, respectively;
    6 Set \(a:=\frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2}, b:=\eta_{2} b_{\iota}+\frac{2\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)^{2}}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}\);
```

    7 Set \(c_{\delta, \beta}, C_{1, \delta, \ell, \mathbf{j}, \beta}, C_{2, \delta, \ell, \mathbf{j}, \beta}, C_{3, \delta, \beta}\) to be the constants given in Theorem 2.3;
    8 Set \(\mathfrak{C}_{1}, \mathfrak{C}_{2}, \mathfrak{C}_{3}, \tilde{L}_{\delta}\) to be the constants defined in (125);
    9 Set \(\mathfrak{C}_{4}, \mathfrak{M}_{1}, \tilde{C}_{4}, C_{5, \delta, \beta}, C_{6}\) to be the constants defined in (142);
    10 Set $C_{4}:=(64)$;
11 Set $\lambda_{\max , \delta}:=(124)$;
Fix $\ell$ such that $\Xi \subset\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}$;
13 Fix $\mathfrak{j}>\log _{2}\left(\frac{5 \sqrt{m}\left(C_{4}+\mathfrak{C}_{4}\left(a^{-1}+2 b\right)\right)}{\varepsilon}\right)$;
14 Fix $\delta \in\left(0, \min \left\{\frac{\varepsilon}{10(\ell+\mathfrak{j}) \log 2}, \frac{\mathfrak{C}_{2}}{\sqrt{a \mathfrak{C}_{1}}}, \mathfrak{C}_{2} \sqrt{\frac{\varepsilon 2^{\mathfrak{j}}}{10 \mathfrak{C}_{1} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \sqrt{m}}}\right\}\right)$;
$5 \operatorname{Fix} \beta>\max \left\{\frac{100(d+1)}{\varepsilon^{2}}, \frac{10(d+1)\left(1+\log \left(\frac{\left(\tilde{L}_{\delta}-1\right) \mathbb{E}_{\mathbb{P}}\left[\left(1+\left|X_{0}\right|\right)^{2 p}\right]}{a}\right)\right)}{\varepsilon}, \frac{10 \sqrt{m} \mathfrak{C}_{4}(d+1)}{\varepsilon 2^{j}}\right\}$;
6 Fix $\lambda \in\left(0, \min \left\{\lambda_{\max , \delta}, \frac{\varepsilon^{4}}{625 C_{2, \delta, \ell, \mathrm{j}, \beta}^{4}}\right\}\right)$;
Fix $n>\max \left\{\frac{4}{c_{\delta, \beta} \lambda} \log \left(\frac{10 C_{1, \delta, \ell, \mathrm{j}, \beta}}{\varepsilon}\right), \frac{2}{a \lambda} \log \left(\frac{10 C_{6}}{\varepsilon}\right)-1\right\}$;
Set $H^{\delta, \ell, j}:=(14)$;
for $\mathfrak{n}=0, \cdots, n-1$ do
Draw $Z_{\mathfrak{n}+1} \sim \mathcal{N}\left(0, I_{d+1}\right)$;
Set $\hat{\bar{\theta}}_{\mathfrak{n}+1}^{\lambda, \delta, \ell, \mathfrak{j}}:=\hat{\bar{\theta}}_{\mathfrak{n}}^{\lambda, \delta, \ell, \mathfrak{j}}-\lambda H^{\delta, \ell, \mathfrak{j}}\left(\hat{\bar{\theta}}_{\mathfrak{n}}^{\lambda, \delta, \ell, \mathfrak{j}}, X_{\mathfrak{n}+1}\right)+\sqrt{2 \lambda \beta^{-1}} Z_{\mathfrak{n}+1} ;$
22 Set $\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}:=$ first $d$ components of $\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}$.

Remark 2.5. The assumption that $U: \mathbb{R}^{d} \times \Xi \rightarrow \mathbb{R}$ is continuously differentiable in order to obtain Theorem 2.3 can be relaxed in the following way. Assume that $\Xi$ satisfies Assumption 1 and that $\iota: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ satisfies Assumption 5. Moreover, let $U: \mathbb{R}^{d} \times \Xi \rightarrow \mathbb{R}$ be a measurable function satisfying the following. There exists a family of measurable functions $U_{\mathfrak{e}}: \mathbb{R}^{d} \times \Xi \rightarrow \mathbb{R}, \mathfrak{e} \in(0, \infty)$, such that:

- there exists $\left(a_{\mathfrak{e}}\right)_{\mathfrak{e} \in(0, \infty)},\left(b_{\mathfrak{e}}\right)_{\mathfrak{e} \in(0, \infty)} \subseteq[0, \infty)$ with $\lim _{\mathfrak{e} \rightarrow 0} a_{\mathfrak{e}}=0=\lim _{\mathfrak{e} \rightarrow 0} b_{\mathfrak{e}}$ such that for every $\mathfrak{e} \in(0, \infty)$

$$
U(\theta, x)-a_{\mathfrak{e}} \leq U_{\mathfrak{e}}(\theta, x) \leq U(\theta, x)+b_{\mathfrak{e}} \quad \text { for every } \theta \in \mathbb{R}^{d}, x \in \Xi
$$

- for every $\mathfrak{e} \in(0, \infty)$, the function $U_{\mathfrak{e}}: \mathbb{R}^{d} \times \Xi \rightarrow \mathbb{R}$ satisfies Assumptions 2-4.

Then we still obtain Theorem 2.3 (with $U \leftarrow U_{\mathfrak{e}}$ in (14) in the definition of our SGLD algorithm), but with the additional summand $a_{\mathfrak{e}}+b_{\mathfrak{e}}$ appearing in the non-asymptotic convergence bound.
We will exploit this fact in the next section when we apply our SGLD algorithm to solve the robust Mean-CVaR portfolio optimisation problem.

## 3. Application: Robust Mean-CVaR Portfolio Optimisation

As a concrete application of the general framework introduced in the previous section, we analyse in this section the robust Mean-CVaR portfolio optimisation problem.

Let $d \in \mathbb{N} \cap[2, \infty)$ and consider a portfolio of $m=d-1$ number of assets, and let $\left(X_{n}\right)_{n \in \mathbb{N}} \subseteq \Xi$ denote i.i.d. realisations of the returns of these assets over a single time period. Denote $f(y):=\max (y, 0)$ and the softmax function $s: \mathbb{R}^{d-1} \rightarrow[0,1]^{d-1}$ by $s(w)_{i}:=\frac{e^{w_{i}}}{\sum_{j=1}^{d-1} e^{w_{j}}}, i \in\{1, \cdots, d-1\}$. For any $w=\left(w_{1}, \cdots, w_{d-1}\right) \in \mathbb{R}^{d-1}$ which determines a corresponding long-only portfolio allocation $\tilde{w}:=s(w) \in[0,1]^{d-1}$, a vector $X \in \Xi \subset \mathbb{R}^{m}$ of asset returns being distributed according to $\mu \in \mathcal{P}(\Xi)$, and a confidence level $\gamma \in(0,1)$, the Conditional Value-at-Risk (CVaR) of the portfolio is defined as

$$
\begin{equation*}
\operatorname{CVaR}_{\gamma}(\tilde{w}, \mu):=\inf _{v \in \mathbb{R}}\left[v+\frac{1}{1-\gamma} \int_{\Xi} f(-\langle\tilde{w}, x\rangle-v) \mathrm{d} \mu(x)\right] . \tag{17}
\end{equation*}
$$

The classical mean-CVaR portfolio optimisation problem can then be formulated as

$$
\begin{equation*}
\inf _{w \in \mathbb{R}^{d-1}} \operatorname{CVaR}_{\gamma}(\tilde{w}, \mu)=\inf _{w \in \mathbb{R}^{d-1}} \inf _{v \in \mathbb{R}} \int_{\Xi}\left\{-\rho_{1}\langle\tilde{w}, x\rangle+\rho_{2}\left[v+\frac{1}{1-\gamma} f(-\langle\tilde{w}, x\rangle-v)\right]\right\} \mathrm{d} \mu(x) \tag{18}
\end{equation*}
$$

where $\rho_{1}, \rho_{2} \geq 0$ and $\tilde{w}=s(w)$, see, for example, [6]. The ratio $\frac{\rho_{2}}{\rho_{1}}$ measures the level of risk-aversion of the investor, where the case $\rho_{1}=0$ corresponds to finding the minimum CVaR portfolio at confidence level $\gamma$, while the case $\rho_{2}=0$ corresponds to the return maximisation problem.

We consider the following distributionally robust variant of the problem (18) given by minimise $\mathbb{R}^{d-1} \times \mathbb{R} \ni(w, v)=\theta \mapsto u^{\mathrm{CVaR}}(\theta) \in \mathbb{R}, \quad$ where

$$
\begin{equation*}
u^{\mathrm{CVaR}}(\theta):=\sup _{\mu \in \mathcal{P}(\Xi)}\left(\int_{\Xi}\left\{-\rho_{1}\langle s(w), x\rangle+\rho_{2}\left[v+\frac{f(-\langle s(w), x\rangle-v)}{1-\gamma}\right]\right\} \mathrm{d} \mu(x)-\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}\right)+\frac{\eta_{1}}{2}|(w, v)|^{2}, \tag{19}
\end{equation*}
$$

and where here the corresponding utility function in the notion of (4) is defined by

$$
\begin{equation*}
U(\theta, x)=-\rho_{1}\langle s(w), x\rangle+\rho_{2}\left[v+\frac{1}{1-\gamma} f(-\langle s(w), x\rangle-v)\right], \quad \theta=(w, v) \in \mathbb{R}^{d-1} \times \mathbb{R} \tag{20}
\end{equation*}
$$

However, $\theta \mapsto U(\theta, x)$ is not continuously differentiable due to the non-differentiability of $f$ at 0 . To apply our result, one may fix $\mathfrak{e}>0$ and replace $f$ by its smoothed version

$$
\begin{equation*}
f_{\mathfrak{e}}(y):=\frac{(y+\mathfrak{e})^{2}}{4 \mathfrak{e}} \mathbb{1}_{[-\mathfrak{e}, \mathfrak{e}]}(y)+y \mathbb{1}_{(\mathfrak{e}, \infty)}(y), \quad y \in \mathbb{R} \tag{21}
\end{equation*}
$$

in the definition of $U$ (which is also adopted in [6]). That is, we define

$$
\begin{equation*}
U_{\mathfrak{e}}(\theta, x):=-\rho_{1}\langle s(w), x\rangle+\rho_{2}\left[v+\frac{1}{1-\gamma} f_{\mathfrak{e}}(-\langle s(w), x\rangle-v)\right], \quad \theta=(w, v) \in \mathbb{R}^{d-1} \times \mathbb{R} \tag{22}
\end{equation*}
$$

and consider the distributionally robust optimisation problem (4) with choice of function $U_{\mathfrak{e}}$, that is,
minimise $\mathbb{R}^{d-1} \times \mathbb{R} \ni(w, v)=\theta \mapsto u_{\mathfrak{e}}^{\mathrm{CVaR}}(\theta) \in \mathbb{R}, \quad$ where

$$
\begin{align*}
u_{\mathfrak{e}}^{\mathrm{CVaR}}(\theta) & :=\sup _{\mu \in \mathcal{P}(\Xi)}\left(\int_{\Xi}\left\{-\rho_{1}\langle\tilde{w}, x\rangle+\rho_{2}\left[v+\frac{f_{\mathfrak{e}}(-\langle\tilde{w}, x\rangle-v)}{1-\gamma}\right]\right\} \mathrm{d} \mu(x)-\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}\right)+\frac{\eta_{1}}{2}|(w, v)|^{2} \\
& =\sup _{\mu \in \mathcal{P}(\Xi)}\left(\int_{\Xi} U_{\mathfrak{e}}(\theta, x) \mathrm{d} \mu(x)-\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}\right)+\frac{\eta_{1}}{2}|(w, v)|^{2} \tag{23}
\end{align*}
$$

Proposition 3.1. The function $U_{\mathfrak{e}}(\theta, x)$ as defined in (22) satisfies Assumptions 2, 3, and 4.
Proof. See Section 7.
As a corollary of Theorem 2.3, we hence obtain a non-asymptotic upper bound for the expected excess risk under $\mathbb{P}$ of the SGLD algorithm associated with $u^{\text {CVaR }}$ defined in (19), see also Remark 2.5.

Corollary 3.2. Let Assumptions 1 and 5 hold. Let $\beta, \delta>0$, and let $\bar{\theta}_{0} \in L^{4}\left(\Omega, \mathcal{F}, \mathbb{P} ; \mathbb{R}^{d+1}\right)$. Moreover, let $\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j, \mathfrak{e}}\right)_{n \in \mathbb{N}}$ denote the first $d$ components of the sequence of estimators obtained from the SGLD algorithm in (13), with $U \leftarrow U_{\mathrm{e}}$ in (14), defined on the probability space ( $\Omega, \mathcal{F}, \mathbb{P}$ ). Then, there exist explicit constants a, $c_{\delta, \beta, \mathfrak{e}}, C_{1, \delta, \ell, j, \beta, \mathfrak{e}}, C_{2, \delta, \ell, \mathfrak{j}, \beta, \mathfrak{e}}, C_{3, \delta, \beta, \mathfrak{e}}, C_{4, \mathfrak{e}}, C_{5, \delta, \beta, \mathfrak{e}}, C_{6, \mathfrak{e}}>0$, which explicit definitions can be derived from that of $c_{\delta, \beta}, C_{1, \delta, \ell, j, \beta}, C_{2, \delta, \ell, j, \beta}, C_{3, \delta, \beta}, C_{4}, C_{5, \delta, \beta}, C_{6}>0$ in Appendix 8 by setting $U \leftarrow U_{\mathrm{c}}$ in (14) in the SGLD algorithm (13), such that for each $n$, step size $\lambda \in\left(0, \lambda_{\text {max }, \delta}\right)$, and $\mathfrak{j} \in \mathbb{N}$,

$$
\begin{align*}
\mathbb{E}_{\mathbb{P}}\left[u^{\mathrm{CVaR}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}, \mathfrak{e}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u^{\mathrm{CVaR}}(\theta) \leq & C_{1, \delta, \ell, \mathfrak{j}, \beta, \mathfrak{e}} e^{-c_{\delta, \beta, \mathrm{e}} \lambda n / 4}+C_{2, \delta, \ell, j, \beta, \mathfrak{e}} \lambda^{1 / 4}+C_{3, \delta, \beta, \mathfrak{e}}+\delta(\ell+\mathfrak{j}) \log 2 \\
& +\frac{\sqrt{m}}{2^{\mathfrak{j}}}\left(C_{4, \mathfrak{e}}+C_{5, \delta, \beta, \mathfrak{e}}+C_{6, \mathfrak{e}} e^{-a \lambda(n+1) / 2}\right)+\frac{\rho_{2} \mathfrak{e}}{4(1-\gamma)} . \tag{24}
\end{align*}
$$

Corollary 3.3. Let Assumptions 1 and 5 hold and let $\bar{\varepsilon}>0$ be given. Then Algorithm 2 outputs the estimator $\hat{\theta}_{n}^{\lambda, \delta, \ell, j, \ell}$ which satisfies

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[u^{\mathrm{CVaR}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j, \mathrm{e}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u^{\mathrm{CVaR}}(\theta)<\bar{\varepsilon} . \tag{25}
\end{equation*}
$$

Proof. The proof of Corollary 3.2 and Corollary 3.3 can be found in Section 5 .

```
Algorithm 2: SGLD Algorithm for Robust Mean-CVaR Portfolio Optimisation (23)
    Input: \(\bar{\varepsilon}>0, d \in \mathbb{N}, p \in[1, \infty), \eta_{1}>0, \eta_{2}>0\), compact subset \(\Xi \in \mathbb{R}^{d-1}\), confidence level \(\gamma \in(0,1)\),
            realised returns \(\left(X_{\mathfrak{n}}\right)_{\mathfrak{n} \in \mathbb{N}} \subset \mathbb{R}^{d-1}\), initialisation \(\bar{\theta}_{0} \in \mathbb{R}^{d+1}\)
    Output: Estimator \(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}, \mathfrak{e}}\)
    Set \(m:=d-1\) and \(M_{\Xi}:=\max _{x \in \Xi}|x|\);
    Fix \(\ell\) such that \(\Xi \subset\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}\);
    Fix \(\mathfrak{e}<\frac{2(1-\gamma) \bar{\varepsilon}}{\rho_{2}}\);
    Set \(U_{\mathfrak{c}}:=(22)\), where \(s: R^{d-1} \rightarrow[0,1]^{d-1}\) is the softmax function.;
    Set \(L_{\nabla}, \nu, K_{\nabla}\) to be the constants given by Assumption 3, with \(U \leftarrow U_{\mathfrak{c}}\);
    Set \(\tilde{K}_{\nabla}:=\max \left\{K_{\nabla}, \max _{x \in \Xi}\left|U_{\mathfrak{e}}(0, x)\right|\right\}\);
    Set \(J_{U}, \chi\) to be the constants given by Assumption 4, with \(U \leftarrow U_{\mathfrak{e}}\);
    Set \(\iota: \mathbb{R} \rightarrow \mathbb{R}\) and \(a_{\iota}, b_{\iota}\) to be the function and constants given by Assumption 5, respectively, with
    \(U \leftarrow U_{\mathfrak{e}} ;\)
    Set \(a:=\frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2}, b:=\eta_{2} b_{\iota}+\frac{2\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}\right)^{2}}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}\);
    Set \(c_{\delta, \beta, \mathfrak{e}}, C_{1, \delta, \ell, \mathfrak{j}, \beta, \mathfrak{e}}, C_{2, \delta, \ell, \mathfrak{j}, \beta, \mathfrak{e}}, C_{3, \delta, \beta, \mathfrak{e}}\) to be the constants given in Theorem 2.3, with \(U \leftarrow U_{\mathfrak{e}}\);
    Set \(\mathfrak{C}_{1, \mathfrak{e}}, \mathfrak{C}_{2, \mathfrak{e}}, \mathfrak{C}_{3, \mathfrak{e}}, \tilde{L}_{\delta, \mathfrak{e}}\) to be the constants defined in (125), with \(U \leftarrow U_{\mathfrak{e}}\);
    Set \(\mathfrak{C}_{4, \mathfrak{e}}, \mathfrak{M}_{1, \mathfrak{e}}, \tilde{C}_{4, \mathfrak{e}}, C_{5, \delta, \beta, \mathfrak{e}}, C_{6, \mathfrak{e}}\) to be the constants defined in (142), with \(U \leftarrow U_{\mathfrak{e}}\);
    Set \(C_{4, \mathfrak{e}}:=(64)\), with \(U \leftarrow U_{\mathfrak{e}}\);
    Set \(\lambda_{\text {max }, \delta, \mathfrak{e}}:=(124)\), with \(U \leftarrow U_{\mathfrak{c}}\);
    Fix \(\mathfrak{j}>\log _{2}\left(\frac{10 \sqrt{m}\left(C_{4, \mathfrak{e}}+\mathfrak{C}_{4, \mathrm{e}}\left(a^{-1}+2 b, \mathfrak{e}\right)\right)}{\bar{\varepsilon}}\right)\);
    Fix \(\delta \in\left(0, \min \left\{\frac{\bar{\varepsilon}}{20(\ell+\mathfrak{j}) \log 2}, \frac{\mathfrak{C}_{2, \mathrm{e}}}{\sqrt{a \mathfrak{C}_{1, \mathrm{e}}}}, \mathfrak{C}_{2, \mathfrak{e}} \sqrt{\frac{\bar{\varepsilon} 2^{\mathrm{j}}}{20 \mathfrak{C}_{1, \mathfrak{e}} \mathfrak{C}_{4, \mathrm{e}}\left(2 \mathfrak{M}_{1, \mathrm{e}}+1\right) \sqrt{m}}}\right\}\right) ;\)
    \(\operatorname{Fix} \beta>\max \left\{\frac{400(d+1)}{\bar{\varepsilon}^{2}}, \frac{20(d+1)\left(1+\log \left(\frac{\left(\tilde{L}_{\delta, \mathrm{e}}-1\right) \mathbb{E}_{\mathrm{p}}\left[\left(1+\left|X_{0}\right|\right)^{2 p]}\right.}{a}\right)\right)}{\bar{\varepsilon}}, \frac{20 \sqrt{m} \mathfrak{C}_{4, \mathrm{e}}(d+1)}{\bar{\varepsilon} 2^{\mathrm{j}}}\right\} ;\)
    Fix \(\lambda \in\left(0, \min \left\{\lambda_{\max , \delta, \mathfrak{e}}, \frac{\bar{\varepsilon}^{4}}{10000 C_{2, \delta, \ell, j, \beta, \mathfrak{e}}^{4}}\right\}\right)\);
    Fix \(n>\max \left\{\frac{4}{c_{\delta, \beta, \mathrm{e} \lambda}} \log \left(\frac{20 C_{1, \delta, \ell,, \mathrm{j}, \beta, e}}{\bar{\varepsilon}}\right), \frac{2}{a \lambda} \log \left(\frac{20 C_{6, \mathrm{e}}}{\bar{\varepsilon}}\right)-1\right\}\);
    Set \(H^{\delta, \ell, \mathfrak{j}, \mathfrak{e}}:=(14)\), with \(U \leftarrow U_{\mathfrak{e}}\);
    for \(\mathfrak{n}=0, \cdots, n-1\) do
        Draw \(Z_{\mathfrak{n}+1} \sim \mathcal{N}\left(0, I_{d+1}\right)\);
        Set \(\hat{\bar{\theta}}_{\mathfrak{n}+1}^{\lambda, \delta, \mathfrak{j}, \mathfrak{e}}:=\hat{\bar{\theta}}_{\mathfrak{n}}^{\lambda, \delta, \ell, \mathfrak{j}, \mathfrak{e}}-\lambda H^{\delta, \ell, \mathfrak{j}, \mathfrak{e}}\left(\hat{\bar{\theta}}_{\mathfrak{n}}^{\lambda, \delta, \ell, \mathfrak{j}, \mathfrak{e}}, X_{\mathfrak{n}+1}\right)+\sqrt{2 \lambda \beta^{-1}} Z_{\mathfrak{n}+1}\);
    Set \(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}, \mathfrak{e}}:=\) first \(d\) components of \(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}, \mathfrak{e}}\).
```


## 4. Numerical Results

In this section, we present our numerical results. We apply our proposed robust SGLD algorithm on the distributionally robust mean-CVaR portofolio optimisation problem formulated explicitly in (23) and benchmark the results against that of applying the (non-robust) SGLD algorithm of [140] to the standard (i.e. non-robust) mean-CVaR portfolio optimisation problem formulated explicitly in (18). The results of both approaches are compared and assessed based on the cumulative values of the portfolios generated by the respective weights from each algorithm using a rolling window backtesting methodology which we describe here as follows.

As with the previous section, we denote by $m$ the number of assets in the portfolio. For a given time indexed by $t$, we denote by $X^{(t)} \in \Xi \subset \mathbb{R}^{m}$ the vector of realised returns of the assets in the portfolio from $t-1$ to $t$. Given a number of training windows $n_{w}$, a training window size $n_{\text {train }}$, and a test window size $n_{\text {test }}$, we organise historical asset returns for backtesting into rolling train and test windows as depicted in Table 1.

| Index | Train | Test |
| :--- | :--- | :--- |
| 1 | $X^{(1)}, \cdots, X^{\left(n_{\text {train }}\right)}$ | $X^{\left(n_{\text {train }}+1\right)}, \cdots, X^{\left(n_{\text {train }}+n_{\text {test }}\right)}$ |
| 2 | $X^{\left(n_{\text {test }}+1\right)}, \cdots, X^{\left(n_{\text {train }}+n_{\text {test }}\right)}$ | $X^{\left(n_{\text {train }}+n_{\text {test }}+1\right)}, \cdots, X^{\left(n_{\text {train }}+2 n_{\text {test }}\right)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $i$ | $X^{\left((i-1) n_{\text {test }}+1\right)}, \cdots, X^{\left((i-1) n_{\text {test }}+n_{\text {train }}\right)}$ | $X^{\left((i-1) n_{\text {test }}+n_{\text {train }}+1\right)}, \cdots, X^{\left(i n_{\text {test }}+n_{\text {train }}\right)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n_{w}$ | $X^{\left(\left(n_{w}-1\right) n_{\text {test }}+1\right)}, \cdots, X^{\left(\left(n_{w}-1\right) n_{\text {test }}+n_{\text {train }}\right)}$ | $X^{\left(\left(n_{w}-1\right) n_{\text {test }}+n_{\text {train }}+1\right)}, \cdots, X^{\left(n_{w} n_{\text {test }}+n_{\text {train }}\right)}$ |

Table 1: Rolling window backtesting methodology

Then, for each index $i$, we apply each algorithm on the returns in the training window by setting the reference probability measure $\mu_{0}$ of the asset returns to be the empirical distribution of the returns in the training window. This yields a vector of portfolio weights according to which we update the portfolio allocation over the periods corresponding to the returns in the test window. In this manner, each algorithm yields a portfolio allocation that is rebalanced in between each training and test window, and held constant over each test window, from which the cumulative portfolio values are computed.

We apply this rolling window backtesting methodology on real financial data using the weekly returns of a portfolio comprising $m=5$ assets - SPY, GDX, EEM, XLF, USD - on the adjusted close prices of each asset from 5 Jan 2015 to 1 Jan 2023. Over this period, there are 416 vectors of weekly asset returns which we divide into $n_{w}=14$ sets of rolling train and test windows with training window size $n_{\text {train }}=52$ and test window size $n_{\text {test }}=26$. For each portfolio, we start with an initial wealth of $V_{0}=10000$. On each training window, we apply both the SGLD algorithm of [140] to the standard (i.e. non-robust) mean-CVaR portfolio optimisation problem formulated explicitly in (18), and our robust SGLD algorithm defined in Algorithm 2 to the distributionally robust mean-CVaR portfolio optimisation problem formulated explicitly in (23) with $\eta_{2} \in\{0.01,0.05,0.075,0.1,0.5,1.0\}$ describing the penalty function, and compare the cumulative values of the resulting portfolios over time. In all runs of the algorithms, we solve the mean-CVaR problems with confidence level $\gamma=0.9$ and mix of return maximisation and CVaR minimisation specified by $\rho_{1}=\rho_{2}=1$, and we set the number of simulations and step size as $n=1000$ and $\lambda=0.01$, respectively. Since we mainly seek to assess whether introducing distributional ambiguity adds any value to the standard SGLD algorithm and $\eta_{2}$ directly controls this distributional ambiguity (with large values of $\eta_{2}$ corresponding to large ambiguity), we fix all other parameters to reasonable values and vary only $\eta_{2}$. We state all other parameters used for the runs of the standard SGLD and our robust SGLD in the numerical simulations and their respective interpretations
in Table 2. The comparison of the resulting portfolio values over time between the standard SGLD and our robust SGLD with different values of $\eta_{2}$ are displayed in Figure 1, while summary statistics of the portfolios generated by each algorithm are reported in Table 3. The code can be found under https://github.com/ng-cheng-en-matthew/robust_sgld

| Parameter | Value | Interpretation |
| :---: | :---: | :---: |
| $m$ | 5 | Number of assets |
| $d$ | 6 | Dimensionality of minimisation problem |
| $\mathfrak{e}$ | 0.01 | Smoothing parameter for ReLU approximation |
| $\gamma$ | 0.9 | Confidence level for CVaR |
| $\rho_{1}$ | 1 | The ratio $\frac{\rho_{2}}{\rho_{1}}$ is a measure of mean aversion. Setting $\rho_{1}=0$ corresponds to minimising CVaR. |
| $\rho_{2}$ | 1 | The ratio $\frac{\rho_{2}}{\rho_{1}}$ is a measure of mean aversion. Setting $\rho_{2}=0$ corresponds to maximising portfolio return. |
| $\mu_{0}$ | Empirical measure of returns on given training window | Reference probability measure for distribution of asset returns |
| $p$ | 2 | Controls convexity of cost of transportation between true distribution of asset returns and given reference distribution. |
| $\eta_{1}$ | 0.1 | Controls regularisation in robust minimisation problem. Smaller values of $\eta_{1}$ impose less regularisation. |
| $\eta_{2}$ | Various | Controls penalty imposed on the distance between any distribution of asset returns and given reference distribution. Larger values of $\eta_{2}$ impose smaller penalty. |
| $\Xi$ | $[-0.5,0.5]^{5}$ | Support of asset returns. |
| $\theta_{0}$ | 0 | Initial condition of algorithm. |
| $n$ | 1000 | Number of algorithm iterations. |
| $\lambda$ | 0.01 | Step size of algorithm in time space. |
| $\beta$ | 10 | "Mixing parameter" controlling amount of stochasticity in each algorithm iteration. Larger values of $\beta$ generate less randomness at each iteration. |
| $\delta$ | 0.1 | Nesterov's smoothing tolerance. |
| $\ell$ | 2 | Controls intersection between support of asset returns traversed in the algorithm and actual support of asset returns. Must be large enough relative to $\Xi$ to cover entire support. |
| j | 4 | Controls discretisation of support of asset returns as a finite grid. Larger values of $\mathfrak{j}$ yield finer meshes of order $\mathcal{O}\left(2^{-\mathfrak{j}}\right)$. |

Table 2: Parameter values used in the numerical simulations of the (robust) Mean-CVaR portfolio optimisation


Figure 1. Portfolio values over time

| Model | SGLD | Robust SGLD $\left(\eta_{2}=0.01\right)$ | Robust SGLD $\left(\eta_{2}=0.05\right)$ | Robust SGLD $\left(\eta_{2}=0.075\right)$ | Robust SGLD $\left(\eta_{2}=0.1\right)$ | Robust SGLD $\left(\eta_{2}=0.5\right)$ | Robust SGLD $\left(\eta_{2}=1.0\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Min. <br> Portfolio <br> Value | 9,109.03 | 9,363.92 | 9,550.08 | 9,371.87 | 9,222.90 | 9,379.02 | 9,352.85 |
| Max. <br> Portfolio <br> Value | 26,646.68 | 20,529.78 | 16,808.49 | 34,290.07 | 27,576.66 | 32,050.20 | 19,802.23 |
| Initial <br> Portfolio <br> Value | 10,000.00 | 10,000.00 | 10,000.00 | 10,000.00 | 10,000.00 | 10,000.00 | 10,000.00 |
| Terminal Portfolio Value | 21,389.07 | 17,267.83 | 13,379.95 | 30,100.18 | 22,267.14 | 26,900.63 | 16,290.16 |
| Min. <br> Weekly <br> Return <br> (\%) | -20.08 | -14.12 | -17.71 | -28.11 | -18.23 | -13.29 | -20.33 |
| Max. <br> Weekly <br> Return <br> (\%) | 14.42 | 14.37 | 7.90 | 14.31 | 12.85 | 7.17 | 7.89 |
| Mean <br> Weekly <br> Return <br> (\%) | 0.28 | 0.21 | 0.14 | 0.37 | 0.29 | 0.32 | 0.20 |
| Std. Dev. <br> Weekly <br> Return <br> (\%) | 3.00 | 2.81 | 2.89 | 3.00 | 2.96 | 2.54 | 2.90 |
| Sharpe Ratio | 0.09 | 0.07 | 0.05 | 0.12 | 0.10 | 0.13 | 0.07 |

Table 3: Summary statistics of portfolios generated by algorithms

From Table 3, we observe that across all choices of the parameter $\eta_{2}$ tested, the portfolios generated by the robust SGLD algorithm have standard deviation of weekly returns not more than that of the standard SGLD algorithm. Furthermore, for $\eta_{2} \in\{0.075,0.1,0.5\}$, the robust SGLD algorithm outperforms the standard SGLD algorithm in terms of mean weekly returns on both a non-risk adjusted as well as a risk-adjusted basis - the latter being measured by the Sharpe ratio which is computed as the ratio of the mean to standard deviation of weekly portfolio returns. This indicates that for suitable ranges of the parameter $\eta_{2}$, introducing distributional ambiguity to the mean-CVaR portfolio optimisation problem can reduce the risk and improve the performance of the resulting portfolio.

## 5. Proof Overview of Main Results

In this section, we present an overview of the proof for obtaining the non-asymptotic upper bound on the excess risk of our proposed robust SGLD algorithm stated in Theorem 2.3. The proof comprises three main steps. First, we make use of a duality result in [13] to express the distributionally robust optimisation problem of (4) in its dual form. After which, we reduce the problem from the compact support $\Xi$ of the observed data $X$ to the finite grid $\Xi \cap \mathbb{K}_{\ell, j}^{m}$. Finally, we obtain the convergence bound of the excess risk of the robust SGLD algorithm defined on this finite grid through Nesterov's smoothing technique and the duality result of Theorem 5.1.
5.1. Dual Problem Formulation. Recall that our main problem of interest was defined in (4) and can be stated as

$$
\begin{align*}
z_{P} & :=\inf _{\theta \in \mathbb{R}^{d}} u(\theta) \\
& =\inf _{\theta \in \mathbb{R}^{d}}\left\{\sup _{\mu \in \mathcal{P}(\Xi)}\left(\int_{\Xi} U(\theta, x) \mathrm{d} \mu(x)-\frac{d_{c}^{2}\left(\mu_{0}, \mu\right)}{2 \eta_{2}}\right)+\frac{\eta_{1}}{2}|\theta|^{2}\right\} . \tag{26}
\end{align*}
$$

The first step of our proof involves expressing the optimisation problem of (4) in dual form. To this end, we make use of the following duality result which is an immediate consequence of ${ }^{1}$ Theorem 2.4 of [13].

Theorem 5.1 ([13]). Let Assumption 1 hold and let $c: \Xi \times \Xi \rightarrow \mathbb{R}_{\geq 0}$ and $\varphi: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be cost and penalty functions in the sense of [13], respectively. Moreover, let $\mu_{0} \in \mathcal{P}(\Xi)$ and let $U: \mathbb{R}^{\bar{d}} \times \Xi \rightarrow \mathbb{R}$ be a measurable function such that for every $\theta \in \mathbb{R}^{d}, x \mapsto U(\theta, x)$ is in $L^{1}\left(\mu_{0}\right)$ and bounded from below. Then the following duality result holds: for every $\theta \in \mathbb{R}^{d}$ :

$$
\begin{equation*}
\sup _{\mu \in \mathcal{P}(\Xi)}\left\{\int_{\Xi} U(\theta, x) \mathrm{d} \mu(x)-\varphi\left(d_{c}\left(\mu_{0}, \mu\right)\right)\right\}=\inf _{\mathfrak{a} \geq 0}\left\{\varphi^{*}(\mathfrak{a})+\int_{\Xi} \sup _{y \in \Xi}\{U(\theta, y)-\mathfrak{a} c(x, y)\} \mathrm{d} \mu_{0}(x)\right\} \tag{27}
\end{equation*}
$$

where $\varphi^{*}$ denotes the convex conjugate of $\varphi$.
For our problem of interest as stated in (4), the choices of cost and penalty functions are $c\left(x, x^{\prime}\right):=$ $\left|x-x^{\prime}\right|^{p}$ and $\varphi(x):=\frac{x^{2}}{2 \eta_{2}}$, respectively, where $p \in[1, \infty)$ and $\eta_{2}>0$. By applying the duality result of Theorem 5.1, we obtain the equivalent dual formulation of the problem (4) as

$$
\begin{equation*}
z_{D}:=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \tag{28}
\end{equation*}
$$

such that strong duality $z_{P}=z_{D}$ holds. The purpose of obtaining this dual form of the problem is that, after applying the transformation $\mathfrak{a}=\iota(\alpha)$, where $\iota: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is a function satisfying Assumption 5, the dual problem can be expressed in the form of a standard, i.e. non-distributionally robust, stochastic optimisation problem to which the SGLD algorithm of [140] can be directly applied.

To see this clearly, we define for every $\bar{\theta}:=(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}$ and $x \in \Xi$

$$
\begin{equation*}
v(\bar{\theta}):=\int_{\Xi} \tilde{V}(\bar{\theta}, x) \mathrm{d} \mu_{0}(x), \quad \tilde{V}(\bar{\theta}, x):=\sup _{y \in \Xi}\left\{U(\theta, y)-\iota(\alpha)|x-y|^{p}\right\}+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\iota(\alpha)|^{2}, \tag{29}
\end{equation*}
$$

[^1]such that we have
\[

$$
\begin{equation*}
z_{D}=\inf _{\bar{\theta} \in \mathbb{R}^{d+1}} v(\bar{\theta})=z_{P} \tag{30}
\end{equation*}
$$

\]

by the surjectivity of $\iota$. The optimisation problem (30) is a standard stochastic optimisation problem over the whole domain $\mathbb{R}^{d+1}$ in $\bar{\theta}:=(\theta, \alpha)$ to which the SGLD algorithm of [140] can be applied.
5.2. Reduction to a Finite Grid. The dual problem $z_{D}$ as stated in (28) involves an observed data variable $X$ which has compact support $\Xi$ in $\mathbb{R}^{m}$. The next step of the proof involves reducing the dual problem $z_{D}$ to a discretised version $z_{D, \ell, \mathrm{j}}$, to be formulated subsequently, where the observed data has finite support in $\mathbb{R}^{m}$. The quadrature error $\left|z_{D}-z_{D, \ell, j}\right|$ is then controlled. To this end, we recall, given positive integers $\ell, \mathfrak{j}>0$, the definition of the set of dyadic rationals

$$
\begin{equation*}
\mathbb{K}_{\ell, \mathrm{j}}:=\left\{-2^{\ell-1},-2^{\ell-1}+\frac{1}{2^{\mathrm{j}}}, \cdots, 2^{\ell-1}-\frac{1}{2^{\mathrm{j}}}\right\} \tag{31}
\end{equation*}
$$

stated in (11), and that we have previously fixed aj $\mathfrak{N}$ and an $\ell \in \mathbb{N}$ large enough such that $\Xi \subseteq$ $\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}$. Note that the finite grid $\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}$ is the set on which the robust SGLD algorithm (13) is defined. In addition, we define, for each $\boldsymbol{i}=\left(i_{1}, \cdots, i_{m}\right) \in \mathbb{K}_{\ell, \mathrm{j}}^{m}$, the set

$$
\begin{equation*}
Q_{i, \mathrm{j}}:=\left[i_{1}, i_{1}+\frac{1}{2^{\mathrm{j}}}\right) \times\left[i_{2}, i_{2}+\frac{1}{2^{\mathrm{j}}}\right) \times \cdots \times\left[i_{m}, i_{m}+\frac{1}{2^{\mathrm{j}}}\right), \tag{32}
\end{equation*}
$$

such that

$$
\begin{equation*}
\bigcup_{i \in \mathbb{K}_{\ell, \mathrm{j}}^{m}} Q_{i, j}=\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m} \supseteq \Xi . \tag{33}
\end{equation*}
$$

The reference probability measure $\mu_{0} \in \mathcal{P}(\Xi)$ can then be extended to a probability measure $\mu_{0, \ell} \in$ $\mathcal{P}\left(\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}\right)$, defined by

$$
\begin{equation*}
\mu_{0, \ell}(B):=\mu_{0}(B \cap \Xi), \quad B \in \mathcal{B}\left(\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}\right) . \tag{34}
\end{equation*}
$$

Then, by applying a quadrature procedure, we can discretise $\mu_{0, \ell}$ to the finite grid $\mathbb{K}_{\ell, \mathrm{j}}^{m}$, that is, we define the discrete probability measure $\mu_{0, \ell, \mathrm{j}} \in \mathcal{P}\left(\mathbb{K}_{\ell, \mathrm{j}}^{m}\right)$ by

$$
\begin{equation*}
\mu_{0, \ell, j}(\{\boldsymbol{i}\}):=\mu_{0, \ell}\left(Q_{i, j}\right), \quad \boldsymbol{i} \in \mathbb{K}_{\ell, \mathrm{j}}^{m} . \tag{35}
\end{equation*}
$$

By defining the function $[\cdot]_{j}: \mathbb{R}^{m} \rightarrow\left(2^{-j} \mathbb{Z}\right)^{m}$ by

$$
\begin{equation*}
\left(x_{1}, \cdots, x_{m}\right)=: x \mapsto[x]_{\mathfrak{j}}=\left(\frac{\left\lfloor 2^{\mathfrak{j}} x_{1}\right\rfloor}{2^{\mathfrak{j}}}, \cdots, \frac{\left\lfloor 2^{\mathfrak{j}} x_{m}\right\rfloor}{2^{\mathfrak{j}}}\right), \tag{36}
\end{equation*}
$$

one may specify the discretised version of the primal problem (26) as

$$
\begin{equation*}
\operatorname{minimise} \mathbb{R}^{d} \ni \theta \mapsto u^{\ell, \mathrm{j}}(\theta):=\sup _{\mu \in \mathcal{P}\left(\Xi \cap \mathbb{K}_{\ell, j}^{m}\right)}\left(\int_{\Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}} U(\theta, x) \mathrm{d} \mu(x)-\frac{1}{2 \eta_{2}} d_{c}^{2}\left(\mu_{0, \ell, \mathfrak{j}}, \mu\right)\right)+\frac{\eta_{1}}{2}|\theta|^{2}, \tag{37}
\end{equation*}
$$

and

$$
\begin{align*}
z_{P, \ell, \mathfrak{j}} & :=\inf _{\theta \in \mathbb{R}^{d}} u^{\ell, \mathbf{j}}(\theta) \\
& =\inf _{\theta \in \mathbb{R}^{d}} \sup _{\mu \in \mathcal{P}\left(\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}\right)}\left(\int_{\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}} U(\theta, x) \mathrm{d} \mu(x)-\frac{1}{2 \eta_{2}} d_{c}^{2}\left(\mu_{0, \ell, \mathrm{j}}, \mu\right)\right)+\frac{\eta_{1}}{2}|\theta|^{2} . \tag{38}
\end{align*}
$$

We also define the discretised version of the dual problem (28) as

$$
\begin{equation*}
z_{D, \ell, \mathfrak{j}}:=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathrm{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} . \tag{39}
\end{equation*}
$$

The following lemma enables us to explicitly represent $z_{D, \ell, \mathrm{j}}$ as an optimisation problem that lives on a discrete probability space.

Lemma 5.2. The discretised version $z_{D, \ell, j}$ of the dual problem $z_{D}$ in (28), given by (39), has the equivalent representation

$$
\begin{equation*}
z_{D, \ell, \mathfrak{j}}=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\mathbb{K}_{\ell, \mathfrak{j}}^{m}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0, \ell, \mathfrak{j}}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \tag{40}
\end{equation*}
$$

Proof. See Section 8.
As a discrete analogue of (29), we define, for any $\bar{\theta}:=(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}$ and any $x \in \Xi$, the quantities

$$
\begin{align*}
v^{\ell, \mathfrak{j}}(\bar{\theta}) & :=\int_{\Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}} \tilde{V}^{\ell, \mathfrak{j}}(\bar{\theta}, x) \mathrm{d} \mu_{0, \ell, \mathfrak{j}}(x), \\
\tilde{V}^{\ell, \mathfrak{j}}(\bar{\theta}, x) & :=\max _{y \in \Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}}\left\{U(\theta, y)-\iota(\alpha)|x-y|^{p}\right\}+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\iota(\alpha)|^{2} . \tag{41}
\end{align*}
$$

Then, by the duality result of Theorem 5.1, we obtain for every $\theta \in \mathbb{R}^{d}$ that

$$
\begin{equation*}
u^{\ell, \mathrm{j}}(\theta)=\inf _{\alpha \in \mathbb{R}}\left(\int_{\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}} \tilde{V}^{\ell, \mathrm{j}}((\theta, \alpha), x) \mathrm{d} \mu_{0, \ell, \mathfrak{j}}(x)\right) \tag{42}
\end{equation*}
$$

This and the representation of $z_{D, \ell, \mathrm{j}}$ given by (40) in Lemma 5.2 hence imply the relation

$$
\begin{equation*}
z_{D, \ell, \mathrm{j}}=\inf _{\bar{\theta} \in \mathbb{R}^{d+1}} v^{\ell, \mathrm{j}}(\bar{\theta})=z_{P, \ell, \mathrm{j}} \tag{43}
\end{equation*}
$$

Note that (43) is a discrete analogue of (30).

Moreover, the compactness of the support $\Xi$ enables us to reduce the computation of $z_{D}$ as well as $z_{D, \ell, \mathrm{j}}$ to optimisation problems over compact subsets of $\mathbb{R}^{d+1}$ which do not depend on $\mathfrak{j}$. This is precisely stated in the next lemma and is paramount to obtaining the upper bound for the quadrature error $\left|z_{D}-z_{D, \ell, \mathrm{j}}\right|$.

Lemma 5.3. Let Assumptions 1, 2, and 3 hold. Then, there exists a compact $\mathcal{K}_{\Xi} \subset \mathbb{R}^{d} \times[0, \infty)$ such that

$$
\begin{align*}
z_{D} & :=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{(\theta, \mathfrak{a}) \in \mathcal{K}_{\Xi}}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} . \tag{44}
\end{align*}
$$

In addition, there exists a compact $\mathcal{K}_{\Xi, \ell} \subset \mathbb{R}^{d+1}$ not depending on $\mathfrak{j}$ such that

$$
\begin{align*}
z_{D, \ell, \mathfrak{j}} & =\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{(\theta, \mathfrak{a}) \in \mathcal{K}_{\Xi, \ell}}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \tag{45}
\end{align*}
$$

Proof. See Section 8.
Finally, we state in the following proposition an upper bound on the quadrature error, which implication is that, by varying $\mathfrak{j}$ to control the mesh $\frac{1}{2^{j}}$ of the grid, one can cause the discretised dual problem $z_{D, \ell, \mathfrak{j}}$ to be as close to the original dual problem $z_{D}$ as desired.
Proposition 5.4. Let Assumptions 1, 2, 3, and 4 hold. Then, given $\ell \in \mathbb{N}$ such that $\Xi \subset\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}$, there exists a compact set $\mathcal{K} \subset \mathbb{R}^{d+1}$ such that, for any given $\mathfrak{j} \in \mathbb{N}$, the following bound for the quadrature error $\left|z_{D}-z_{D, \ell, \mathrm{j}}\right|$ holds:

$$
\begin{equation*}
\left|z_{D}-z_{D, \ell, \mathrm{j}}\right| \leq \frac{\sqrt{m}\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+p\left(1+4 M_{\Xi}\right)^{p-1}\right)\left(1+\sup _{\bar{\theta} \in \mathcal{K}}|\bar{\theta}|\right)}{2^{\mathrm{j}}} \tag{46}
\end{equation*}
$$

Proof. See Section 8.
5.3. Nesterov's Smoothing Technique. Having obtained a non-asymptotic upper bound on the quadrature error $\left|z_{D}-z_{D, \ell, j}\right|$, the second step of the proof is to obtain a non-asymptotic upper bound on the expected excess risk of the algorithm over the optimal value of the discretised version of the dual problem - that is the quantity $\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\right)\right]-z_{D, \ell, j}$. To this end, we make use of the following result which can be obtained by applying Nesterov's smoothing technique to the maximum function, see, for example, Lemma 5 of [7].

Lemma 5.5 ([7]). Let $N \in \mathbb{N}$. Then, for any $\delta>0$, the following smooth approximation of the maximum function

$$
\begin{equation*}
\mathbb{R}^{N} \ni\left(x_{1}, \cdots, x_{N}\right) \mapsto \phi_{\delta}\left(x_{1}, \cdots, x_{N}\right):=\delta \log \left(\frac{1}{N} \sum_{j=1}^{N} e^{x_{j} / \delta}\right) \tag{47}
\end{equation*}
$$

satisfies, for any $\left(x_{1}, \cdots, x_{N}\right) \in \mathbb{R}^{N}$, the inequalities

$$
\begin{equation*}
\phi_{\delta}\left(x_{1}, \cdots, x_{N}\right) \leq \max \left\{x_{j}: j=1, \cdots, N\right\} \leq \phi_{\delta}\left(x_{1}, \cdots, x_{N}\right)+\delta \log N . \tag{48}
\end{equation*}
$$

Recall that we have fixed $\ell \in \mathbb{N}$ large enough such that $\Xi \subset\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}$, and that we have also previously denoted

$$
\begin{align*}
\left\{\xi_{j}^{\ell, j}\right\}_{j=1, \cdots, N_{\ell, j}}: & =\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}, \\
N_{\ell, \mathrm{j}} & :=2^{\ell+\mathrm{j}} . \tag{49}
\end{align*}
$$

Fixing also $\mathfrak{j} \in \mathbb{N}$, we denote $\xi_{j}:=\xi_{j}^{\ell, j}$ and $N:=N_{\ell, \mathrm{j}}$ thereby suppressing dependence of the quantities $\xi_{j}$ and $N$ on $\ell$ and $j$ for the sake of brevity.

An application of Lemma 5.5 to the representation of $z_{D, \ell, j}$ given in Lemma 5.2 and the surjectivity of $\iota: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ then yields the following result.

Corollary 5.6. Let Assumptions 1, 2, and 3 hold. For every $\delta>0$, define $V^{\delta, \ell, j}: \mathbb{R}^{d+1} \times \Xi \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
V^{\delta, \ell, j}(\bar{\theta}, x):=\delta \log \left(\frac{1}{N} \sum_{j=1}^{N} \exp \left[\frac{1}{\delta}\left(U\left(\theta, \xi_{j}\right)-\iota(\alpha)\left|x-\xi_{j}\right|^{p}\right)\right]\right), \quad \bar{\theta}=(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}, x \in \Xi . \tag{50}
\end{equation*}
$$

Moreover, define for every $\delta>0$

$$
\begin{equation*}
\tilde{V}^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x):=V^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\iota(\alpha)|^{2}, \quad v^{\delta, \ell, \mathrm{j}}(\bar{\theta}):=\int_{\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}} \tilde{V}^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x) \mathrm{d} \mu_{0, \ell, \mathrm{j}}(x), \tag{51}
\end{equation*}
$$

where $\bar{\theta}=(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}$ and $x \in \Xi$. Furthermore, define for every $\delta>0$

$$
\begin{equation*}
z_{D, \ell, \mathrm{j}, \delta}:=\inf _{\bar{\theta}=(\theta, \alpha) \in \mathbb{R}^{d+1}} v^{\delta, \ell, \mathbf{j}}(\bar{\theta}), \tag{52}
\end{equation*}
$$

Then, for every $\delta>0$ and $\bar{\theta} \in \mathbb{R}^{d+1}$ we have that
$v^{\delta, \ell, \mathrm{j}}(\bar{\theta}) \leq v^{\ell, \mathrm{j}}(\bar{\theta}) \leq v^{\delta, \ell, \mathrm{j}}(\bar{\theta})+\delta \log N$, which also implies that $z_{D, \ell, \mathrm{j}, \delta} \leq z_{D, \ell, \mathrm{j}} \leq z_{D, \ell, \mathrm{j}, \delta}+\delta \log N$,
where $v^{\ell, \mathrm{j}}(\bar{\theta})$ is defined in (41) and $z_{D, \ell, \mathrm{j}}$ is defined in (39).
Proof. This follows immediately from applying Lemma 5.5 to the definition of $\tilde{V}^{\ell, j}$ in (41).

The following is a summary of the definitions of the quantities $z_{P}, z_{P, \ell, \mathrm{j}}, z_{D}, z_{D, \ell, \mathrm{j}}, z_{D, \ell \mathrm{j}, \delta}$ and their relationship with each other:

| Quantity | Primal | Dual | Relation |
| ---: | :--- | :--- | :--- |
| Original | $z_{P}:=(26)$ | $z_{D}:=(28)$ | $z_{P}=z_{D}$ by (30) |
| Discretised | $z_{P, \ell, \mathrm{j}}:=(38)$ | $z_{D, \ell, \mathrm{j}}:=(39)$ | $z_{P, \ell, \mathrm{j}}=z_{D, \ell, \mathrm{j}}$ by $(43),\left\|z_{D}-z_{D, \ell, \mathrm{j}}\right\| \leq(46)$ |
| Discretised and Smoothed | - | $z_{D, \ell, \mathrm{j}, \delta}:=(52)$ | $z_{D, \ell, \mathrm{j}, \delta} \leq z_{D, \ell, \mathrm{j}} \leq z_{D, \ell, \mathrm{j}, \delta}+\delta \log N$ by (53) |

5.4. Applying the SGLD Algorithm. The final step of the proof is to obtain convergence bounds on the SGLD algorithm of [140] applied to the smoothed and discretised version of the dual problem $z_{D, \ell, \mathrm{j}, \delta}$ as defined in (52). Note that here, we apply the SGLD algorithm of [140] to the variable $\bar{\theta}=(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}$ which lies in the enlarged space $\mathbb{R}^{d+1}$, as opposed to the original variable $\theta \in \mathbb{R}^{d}$. The next two propositions establish the global Lipschitz and dissipativity conditions on the function $\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)$.

Proposition 5.7. Let Assumptions 1, 2, and 3 hold. Then, for every $\delta>0$, there exists $L_{\delta}>0$ such that for all $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and all $x \in \Xi$,

$$
\begin{equation*}
\left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right)-\nabla_{\bar{\theta}} V^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)\right| \leq L_{\delta}(1+|x|)^{2 p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| . \tag{54}
\end{equation*}
$$

Proof. See Section 8.
Proposition 5.8. Let Assumptions 1, 2, 3, and 5 hold. Then, there exist $a, b>0$ such that for all $\bar{\theta} \in \mathbb{R}^{d+1}$,

$$
\begin{equation*}
\left\langle\bar{\theta}, \nabla_{\bar{\theta}} \tilde{V}^{\delta \cdot \ell, \mathrm{j}}(\bar{\theta}, x)\right\rangle \geq a|\bar{\theta}|^{2}-b . \tag{55}
\end{equation*}
$$

Proof. See Section 8.
Since the global Lipschitz and dissipativity conditions are satisfied by $\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)$, the assumptions of the SGLD algorithm of [140], when applied to the discretised and smoothed version of the dual problem $z_{D, \ell, \mathrm{j}, \delta}$ as defined in (52), are satisfied. Hence, with the choice of the stochastic gradient $H^{\delta, \ell, \mathrm{j}}$ of the SGLD algorithm defined in (13) as

$$
\begin{equation*}
H^{\delta, \ell, \mathfrak{j}}:=\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathrm{j}} \tag{56}
\end{equation*}
$$

which is consistent with its definition previously given in (14), the following convergence bounds on the excess risk of the SGLD algorithm can be obtained.

Proposition 5.9. Let Assumptions 1, 2, 3, 4, and 5 hold. Let $\beta, \delta>0$ and $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, and let $\bar{\theta}_{0} \in L^{4}\left(\Omega, \mathcal{F}, \mathbb{P} ; \mathbb{R}^{d+1}\right)$. Moreover, let $\left(\overline{\hat{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)_{n \in \mathbb{N}}$ denote the sequence of estimators obtained from the $S G L D$ algorithm in (13) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exist constants $c_{\delta}, C_{1, \delta, \ell, \mathfrak{j}, \beta}, C_{2, \delta, \ell, \mathfrak{j}, \beta}, C_{3, \delta, \beta}>0$ such that for each n, step size $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, and $\mathfrak{j} \in \mathbb{N}$,

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[v^{\delta, \ell, \mathfrak{j}}\left(\overline{\hat{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{D, \ell, \mathfrak{j}, \delta} \leq C_{1, \delta, \ell, \mathrm{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta} \tag{57}
\end{equation*}
$$

where $z_{D, \ell, \mathbf{j}, \delta}$ and $v^{\delta, \ell, \mathfrak{j}}$ are defined in (52) and (51), respectively. Moreover, the constants $c_{\delta}, C_{1, \delta, \ell, \mathfrak{j}, \beta}$, $C_{2, \delta, \ell, \mathfrak{j}, \beta}, C_{3, \delta, \beta}>0$ do not depend on $n$ or $\lambda$, and their growth orders are specified as

$$
\begin{align*}
C_{1, \delta, \ell, \mathfrak{j}, \beta} & =\mathcal{O}\left(e^{\tilde{C}_{\delta}(1+d / \beta)(1+\beta)}\left(1+\frac{1}{1-e^{-c_{\delta} / 2}}\right)\right), \\
C_{2, \delta, \ell, \mathfrak{j}, \beta} & =\mathcal{O}\left(e^{\tilde{C}_{\delta}(1+d / \beta)(1+\beta)}\left(1+\frac{1}{1-e^{-c_{\delta} / 2}}\right)\right),  \tag{58}\\
C_{3, \delta, \beta} & =\mathcal{O}\left((d / \beta) \log \left(\tilde{C}_{\delta}(\beta / d+1)\right)\right)
\end{align*}
$$

with $\tilde{C}_{\delta}>0$ being a constant not depending on $\lambda, n, d, \beta$.
Proof. See Section 8.

We note that Proposition 5.9 gives the discretised excess risk of the SGLD algorithm (13) which is applied on the variable $\bar{\theta}=(\theta, \alpha)$ living in the extended space $\mathbb{R}^{d} \times \mathbb{R}$. Applying the duality result (43) immediately yields the corresponding bound on the excess risk of the discretised primal problem (37) which lives in the original space $\mathbb{R}^{d}$. To see this, observe that by (42) and (53) (with $N=2^{\ell+\mathrm{j}}$ )

$$
\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{P, \ell, \mathfrak{j}}=\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathfrak{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{D, \ell, \mathrm{j}} \\
& =\mathbb{E}_{\mathbb{P}}\left[\left.\left(\inf _{\alpha \in \mathbb{R}} \int_{\Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}} \tilde{V}^{\ell, \mathfrak{j}}(\bar{\theta}, x) \mathrm{d} \mu_{0, \ell, \mathrm{j}}(x)\right)\right|_{\theta=\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}}\right]-z_{D, \ell, \mathrm{j}} \\
& \leq \mathbb{E}_{\mathbb{P}}\left[\left.\left(\int_{\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}} \tilde{V}^{\ell, \mathrm{j}}(\bar{\theta}, x) \mathrm{d} \mu_{0, \ell, \mathrm{j}}(x)\right)\right|_{\bar{\theta}=\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathbf{j}}=\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}, \hat{\alpha}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)}\right]-z_{D, \ell, \mathrm{j}} \\
& =\mathbb{E}_{\mathbb{P}}\left[v^{\ell, \mathfrak{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{D, \ell, \mathfrak{j}} \\
& \leq \mathbb{E}_{\mathbb{P}}\left[v^{\delta, \ell, \mathfrak{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{D, \ell, \mathfrak{j}, \delta}+\delta(\ell+\mathfrak{j}) \log 2 .
\end{aligned}
$$

This hence indeed allows us to bound the excess risk of the discretised primal problem (37) directly using Proposition 5.9, as stated in the following corollary.

Corollary 5.10. Let Assumptions 1, 2, 3, 4, and 5 hold. Let $\beta, \delta>0$ and $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, and let $\bar{\theta}_{0} \in L^{4}\left(\Omega, \mathcal{F}, \mathbb{P} ; \mathbb{R}^{d+1}\right)$. Moreover, let $\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)_{n \in \mathbb{N}}$ denote the first d components of the sequence of estimators obtained from the $S G L D$ algorithm in (13) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then,

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathfrak{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{P, \ell, \mathfrak{j}} \leq C_{1, \delta, \ell, \mathfrak{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta}+\delta(\ell+\mathfrak{j}) \log 2 \tag{59}
\end{equation*}
$$

where $c_{\delta}, C_{1, \delta, \ell, \mathbf{j}, \beta}, C_{2, \delta, \ell, \mathbf{j}, \beta}, C_{3, \delta, \beta}>0$ are the constants given in Proposition 5.9.

## Proof. See Section 8.

Finally, the last piece required for the proof of the main results of this paper is an upper bound between the undiscretised and discretised expected risk of the first $d$ components of the SGLD algorithm (13), obtained from the primal problems (4) and (37), respectively. We state the bound in the following proposition.

Proposition 5.11. Let Assumptions 1, 2, 3, 4, and 5 hold. Let $\beta, \delta>0$ and $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, and let $\bar{\theta}_{0} \in L^{4}\left(\Omega, \mathcal{F}, \mathbb{P} ; \mathbb{R}^{d+1}\right)$. Moreover, let $\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)_{n \in \mathbb{N}}$ denote the first d components of the sequence of estimators obtained from the $S G L D$ algorithm in (13) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exists constants $\tilde{C}_{4}, C_{5, \delta, \beta}, C_{6}>0$, which explicit expressions are given in (142), such that for each $n$, step size $\lambda \in\left(0, \lambda_{\max , \delta}\right)$, and $\mathfrak{j} \in \mathbb{N}$,

$$
\begin{equation*}
\left|\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]\right| \leq \frac{\sqrt{m}\left(\tilde{C}_{4}+C_{5, \delta, \beta}+C_{6} e^{-a \lambda(n+1)}\right)}{2^{\mathfrak{j}}} . \tag{60}
\end{equation*}
$$

## Proof. See Section 8.

5.5. Proof of Main Results in Section 2. We have established sufficient machinery thus far to prove the main results of this paper.

Proof of Theorem 2.3. By the duality result of Theorem 5.1 and the triangle inequality, we obtain the following decomposition:

$$
\begin{align*}
\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta) & =\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{P} \\
& =\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{D} \\
& \leq\left|\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]\right|+\left|\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{D, \ell, \mathrm{j}}\right|+\left|z_{D, \ell, \mathfrak{j}}-z_{D}\right| \\
& =\left|\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathfrak{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]\right|+\left|\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-z_{P, \ell, \mathrm{j}}\right|+\left|z_{D, \ell, \mathrm{j}}-z_{D}\right| . \tag{61}
\end{align*}
$$

Observe that the first term on the RHS of the above decomposition has an upper bound given in Proposition 5.11, the second term has an upper bound given in Corollary 5.10, and the third term has an upper bound
given in Proposition 5.4. It follows that

$$
\begin{align*}
& \mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta) \\
\leq & \frac{\sqrt{m}\left(\tilde{C}_{4}+C_{5, \delta, \beta}+C_{6} e^{-a \lambda(n+1)}\right)}{2^{\mathfrak{j}}}+C_{1, \delta, \ell, \mathfrak{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta} \\
& +\delta(\ell+\mathfrak{j}) \log 2+\frac{\sqrt{m}\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+p\left(1+4 M_{\Xi}\right)^{p-1}\right)\left(1+\sup _{\bar{\theta} \in \mathcal{K}}|\bar{\theta}|\right)}{2^{\mathfrak{j}}} \\
= & C_{1, \delta, \ell, \mathfrak{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta} \\
& +\delta(\ell+\mathfrak{j}) \log 2+\frac{\sqrt{m}}{2^{\mathfrak{j}}}\left(C_{4}+C_{5, \delta, \beta}+C_{6} e^{-a \lambda(n+1) / 2}\right) . \tag{62}
\end{align*}
$$

Here, $c_{\delta, \beta}, C_{1, \delta, \ell, \mathrm{j}, \beta}, C_{2, \delta, \ell, \mathrm{j}, \beta}, C_{3, \delta, \beta}$ are as stated in Proposition 5.9 and given explicitly in Table 2 of [140], with

$$
\begin{equation*}
\dot{c} \leftarrow c_{\delta, \beta}, \quad C_{1}^{\#} \leftarrow C_{1, \delta, \ell, \mathbf{j}, \beta}, \quad C_{2}^{\#} \leftarrow C_{2, \delta, \ell, \mathbf{j}, \beta}, \quad C_{3}^{\#} \leftarrow C_{3, \delta, \beta} \tag{63}
\end{equation*}
$$

in the notation of [140]. The compact set $\mathcal{K} \subset \mathbb{R}^{d+1}$ is as specified in Proposition 5.4,

$$
\begin{equation*}
C_{4}:=\tilde{C}_{4}+\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+p\left(1+4 M_{\Xi}\right)^{p-1}\right)\left(1+\sup _{\bar{\theta} \in \mathcal{K}}|\bar{\theta}|\right), \tag{64}
\end{equation*}
$$

and $\tilde{C}_{4}, C_{5, \delta, \beta}, C_{6}$ are as specified in the proof of Proposition 5.11 as in (142). That is,

$$
\begin{align*}
\tilde{C}_{4} & :=J_{U}\left(1+M_{\Xi}\right)^{\chi}+\frac{4 p}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{p-1}\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\right)+\frac{2^{p+2} p M_{\Xi}}{\eta_{2}}\left(1+4 M_{\Xi}\right)^{p-1}, \\
C_{5, \delta, \beta} & :=\mathfrak{C}_{4} \mathfrak{c}_{1, \delta, \beta}^{1 / 2}\left(\lambda_{\max , \delta}+a^{-1}\right)^{1 / 2}, \\
C_{6} & :=\mathfrak{C}_{4}\left(\mathbb{E}_{\mathbb{P}}\left[\left|\hat{\bar{\theta}}_{0}\right|^{2}\right]\right)^{1 / 2}, \\
\mathfrak{C}_{4} & :=\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{\nu+p-1}\right), \\
\mathfrak{c}_{1, \delta, \beta} & :=\mathfrak{M}_{1} \lambda_{\max , \delta}+2 b+2(d+1) / \beta \\
\mathfrak{M}_{1} & :=\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\xi}+\eta_{2} \iota(0) \iota^{\prime}(0)\right)^{2} . \tag{65}
\end{align*}
$$

This completes the proof.

Proof of Corollary 2.4. Observe that

$$
\begin{align*}
C_{4}+C_{5, \delta, \beta} & =C_{4}+\mathfrak{C}_{4}\left(2 \mathfrak{M}_{1} \lambda_{\max , \delta}+2 b+2(d+1) / \beta\right)^{1 / 2}\left(\lambda_{\max , \delta}+a^{-1}\right)^{1 / 2} \\
& \leq C_{4}+\mathfrak{C}_{4}\left(\left(2 \mathfrak{M}_{1}+1\right) \lambda_{\max , \delta}+a^{-1}+2 b+2(d+1) / \beta\right) \\
& =\left(C_{4}+\mathfrak{C}_{4}\left(a^{-1}+2 b\right)\right)+\mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \lambda_{\max , \delta}+2 \mathfrak{C}_{4}(d+1) / \beta \tag{66}
\end{align*}
$$

Hence, it follows from the result of Theorem 2.3 that the upper bound on the excess risk of the algorithm can be decomposed as

$$
\begin{align*}
\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta) \leq & \frac{\sqrt{m}\left(C_{4}+\mathfrak{C}_{4}\left(a^{-1}+2 b\right)\right)}{2^{\mathfrak{j}}} \\
& +\delta(\ell+\mathfrak{j}) \log 2+\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \lambda_{\max , \delta} \\
& +\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}(d+1) / \beta+C_{3, \delta, \beta} \\
& +C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4} \\
& +C_{1, \delta, \ell, \mathfrak{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{6} e^{-a \lambda(n+1) / 2} \tag{67}
\end{align*}
$$

Let $\varepsilon>0$ be given. Fixing first $\ell$ such that $\Xi \subset\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}$, then fixing

$$
\begin{equation*}
\mathfrak{j}>\log _{2}\left(\frac{5 \sqrt{m}\left(C_{4}+\mathfrak{C}_{4}\left(a^{-1}+2 b\right)\right)}{\varepsilon}\right), \tag{68}
\end{equation*}
$$

one has

$$
\begin{equation*}
\frac{\sqrt{m}\left(C_{4}+\tilde{C}_{4}\left(a^{-1}+2 b\right)\right)}{2^{j}}<\frac{\varepsilon}{5} \tag{69}
\end{equation*}
$$

Next, we fix

$$
\begin{equation*}
\delta \in\left(0, \min \left\{\frac{\varepsilon}{10(\ell+\mathfrak{j}) \log 2}, \frac{\mathfrak{C}_{2}}{\sqrt{a \mathfrak{C}_{1}}}, \mathfrak{C}_{2} \sqrt{\frac{\varepsilon 2^{\mathfrak{j}}}{10 \mathfrak{C}_{1} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \sqrt{m}}}\right\}\right) \tag{70}
\end{equation*}
$$

so that

$$
\begin{equation*}
\delta(\ell+\mathfrak{j}) \log 2<\frac{\varepsilon}{10} . \tag{71}
\end{equation*}
$$

Then, since $\frac{\mathfrak{C}_{1}}{\tilde{L}_{\delta}^{2}}<\frac{\delta^{2} \mathfrak{C}_{1}}{\mathfrak{C}_{2}^{2}}<a^{-1}$, we have

$$
\begin{align*}
\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \lambda_{\max , \delta} & =\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \cdot \frac{\mathfrak{C}_{1}}{\tilde{L}_{\delta}^{2}} \\
& <\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \cdot \frac{\mathfrak{C}_{1}}{\mathfrak{C}_{2}^{2}} \cdot \delta^{2} \\
& <\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \cdot \frac{\mathfrak{C}_{1}}{\mathfrak{C}_{2}^{2}} \cdot\left(\mathfrak{C}_{2} \sqrt{\frac{\varepsilon 2^{\mathfrak{j}}}{10 \mathfrak{C}_{1} \mathfrak{C}_{4}\left(2 \mathfrak{M}_{1}+1\right) \sqrt{m}}}\right)^{2} \\
& =\frac{\varepsilon}{10} \tag{72}
\end{align*}
$$

We next fix

$$
\begin{equation*}
\beta>\max \left\{\frac{100(d+1)}{\varepsilon^{2}}, \frac{10(d+1)\left(1+\log \left(\frac{\left(\tilde{L}_{\delta}-1\right) \mathbb{E}_{\mathbb{P}}\left[\left(1+\left|X_{0}\right|\right)^{2 p}\right]}{a}\right)\right)}{\varepsilon}, \frac{10 \sqrt{m} \mathfrak{C}_{4}(d+1)}{\varepsilon 2^{j}}\right\} \tag{73}
\end{equation*}
$$

Then, from the explicit form of $C_{3, \delta, \beta}$ given in Table 2 of [140] with $C_{3}^{\#} \leftarrow C_{3, \delta, \beta}$ in the notation of [140], we obtain

$$
\begin{align*}
C_{3, \delta, \beta}= & \frac{d+1}{2 \beta} \log \left(1+\frac{b \beta}{d+1}\right)+\frac{d+1}{2 \beta}\left(1+\log \left(\frac{\left(\tilde{L}_{\delta}-1\right) \mathbb{E}_{\mathbb{P}}\left[\left(1+\left|X_{0}\right|\right)^{2 p}\right]}{a}\right)\right) \\
< & \frac{1}{2} \sqrt{\frac{d+1}{\beta}}+\frac{d+1}{2 \beta}\left(1+\log \left(\frac{\left(\tilde{L}_{\delta}-1\right) \mathbb{E}_{\mathbb{P}}\left[\left(1+\left|X_{0}\right|\right)^{2 p}\right]}{a}\right)\right) \\
< & \frac{\sqrt{d+1}}{2} \cdot \sqrt{\frac{\varepsilon^{2}}{100(d+1)}} \\
& +\frac{d+1}{2}\left(1+\log \left(\frac{\left(\tilde{L}_{\delta}-1\right) \mathbb{E}_{\mathbb{P}}\left[\left(1+\left|X_{0}\right|\right)^{2 p}\right]}{a}\right)\right) \cdot \frac{\varepsilon}{10(d+1)\left(1+\log \left(\frac{\left(\tilde{L}_{\delta}-1\right) \mathbb{E}_{\mathbb{P}}\left[\left(1+\left|X_{0}\right|\right)^{2 p}\right]}{a}\right)\right)} \\
< & \frac{\varepsilon}{20}+\frac{\varepsilon}{20} \\
= & \frac{\varepsilon}{10}, \tag{74}
\end{align*}
$$

as well as

$$
\begin{equation*}
\frac{\sqrt{m}}{2^{\mathfrak{j}}} \mathfrak{C}_{4}(d+1) / \beta<\frac{\varepsilon}{10} . \tag{75}
\end{equation*}
$$

Finally, we fix

$$
\begin{equation*}
\lambda \in\left(0, \min \left\{\lambda_{\max , \delta}, \frac{\varepsilon^{4}}{625 C_{2, \delta, \ell, \mathrm{j}, \beta}^{4}}\right\}\right) \tag{76}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
C_{2, \delta, \ell, \mathrm{j}, \beta} \lambda^{1 / 4}<\frac{\varepsilon}{5} \tag{77}
\end{equation*}
$$

and then fix

$$
\begin{equation*}
n>\max \left\{\frac{4}{c_{\delta, \beta} \lambda} \log \left(\frac{10 C_{1, \delta, \ell, j, \beta}}{\varepsilon}\right), \frac{2}{a \lambda} \log \left(\frac{10 C_{6}}{\varepsilon}\right)-1\right\}, \tag{78}
\end{equation*}
$$

implying that

$$
\begin{align*}
C_{1, \delta, \ell, \mathrm{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{6} e^{-a \lambda(n+1) / 2}< & C_{1, \delta, \ell, \mathrm{j}, \beta} \exp \left\{-\frac{c_{\delta, \beta} \lambda}{4} \cdot \frac{4}{c_{\delta, \beta} \lambda} \log \left(\frac{10 C_{1, \delta, \ell, \mathrm{j}, \beta}}{\varepsilon}\right)\right\} \\
& +C_{6} \exp \left\{-\frac{a \lambda}{2}\left(\frac{2}{a \lambda} \log \left(\frac{10 C_{6}}{\varepsilon}\right)-1+1\right)\right\} \\
= & \frac{\varepsilon}{10}+\frac{\varepsilon}{10} \\
= & \frac{\varepsilon}{5} . \tag{79}
\end{align*}
$$

Substituting (69), (71), (72), (74), (75), (77), and (79) into (67) thus yields

$$
\begin{align*}
\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u(\theta) & <\frac{\varepsilon}{5}+\frac{\varepsilon}{10}+\frac{\varepsilon}{10}+\frac{\varepsilon}{10}+\frac{\varepsilon}{10}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5} \\
& =\varepsilon, \tag{80}
\end{align*}
$$

which completes the proof.
5.6. Proof of Main Results in Section 3. Finally, we present the proofs of the main results regarding our Robust Mean-CVaR Portfolio Optimisation problem.

Proof of Corollary 3.2. By the definition (22) and the fact that $\left(f_{\mathfrak{c}}(y)-f(y)\right)$ is maximised at $y=0$, we have that

$$
U(\theta, x) \leq U_{\mathfrak{e}}(\theta, x) \leq U(\theta, x)+\frac{\rho_{2}}{1-\gamma} \cdot \frac{\mathfrak{e}}{4}, \quad \theta \in \mathbb{R}^{d}, x \in \Xi .
$$

Applying the inequality to the definition of $\theta \mapsto u_{\mathfrak{e}}^{\mathrm{CVaR}}(\theta)$ given in (23) yields

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[u^{\mathrm{CVaR}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}, \mathfrak{e}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u^{\mathrm{CVaR}}(\theta) \leq \mathbb{E}_{\mathbb{P}}\left[u_{\mathfrak{c}}^{\mathrm{CVaR}}\left(\hat{\theta}_{n}^{\lambda, \delta \ell, \mathrm{j}, \mathfrak{e}}\right)\right]-\inf _{\theta \in \mathbb{R}^{d}} u_{\mathfrak{e}}^{\mathrm{CVaR}}(\theta)+\frac{\rho_{2} \mathfrak{e}}{4(1-\gamma)} . \tag{81}
\end{equation*}
$$

The result then follows immediately by applying Theorem 2.3 with $U$ replaced by $U_{\mathrm{c}}$ in (14) since, by Proposition 3.1, Assumptions 1, 2, 3, 4, and 5 hold.

Proof of Corollary 3.3. This follows immediately by applying Corollary 3.2 and with $\varepsilon \leftarrow \bar{\varepsilon} / 2$ in the notation of Corollary 2.4.

## 6. Proof of Statements in Section 2

## Proof of Remark 2.1.

Proof. Fix $\theta_{1}, \theta_{2} \in \mathbb{R}^{d}, x \in \Xi$, and denote $f(t):=U\left(t \theta_{1}+(1-t) \theta_{2}, x\right)$ for any $t \in[0,1]$, such that $f^{\prime}(t)=\nabla_{\theta} U\left(t \theta_{1}+(1-t) \theta_{2}, x\right)\left(\theta_{1}-\theta_{2}\right)$. By Assumption 3, one obtains

$$
\begin{align*}
\left|U\left(\theta_{1}, x\right)-U\left(\theta_{2}, x\right)\right| & =|f(1)-f(0)| \\
& =\left|\int_{0}^{1} f^{\prime}(t) \mathrm{d} t\right| \\
& \leq \int_{0}^{1}\left|f^{\prime}(t)\right| \mathrm{d} t \\
& \leq \int_{0}^{1}\left|\nabla_{\theta} U\left(t \theta_{1}+(1-t) \theta_{2}, x\right)\right| \cdot\left|\theta_{1}-\theta_{2}\right| \mathrm{d} t \\
& \leq K_{\nabla}(1+|x|)^{\nu}\left|\theta_{1}-\theta_{2}\right| \tag{82}
\end{align*}
$$

which establishes the first part of the remark. It then follows for all $\theta \in \mathbb{R}^{d}$ and $x \in \Xi$ that

$$
\begin{align*}
|U(\theta, x)| & \leq|U(\theta, x)-U(0, x)|+|U(0, x)| \\
& \leq K_{\nabla}(1+|x|)^{\nu}|\theta|+|U(0, x)| \\
& \leq K_{\nabla}(1+|x|)^{\nu}|\theta|+|U(0, x)|(1+|x|)^{\nu} \\
& \leq \tilde{K}_{\nabla}(1+|x|)^{\nu}(1+|\theta|) \tag{83}
\end{align*}
$$

where $\tilde{K}_{\nabla}:=\max \left\{K_{\nabla}, \max _{x \in \Xi}|U(0, x)|\right\}$. This establishes the second part of the remark.

## Proof of Remark 2.2.

Proof. With the choice $\iota(\alpha)=\log (\cosh \alpha)$, one has $L_{\iota}=M_{\iota}=1$, since $\left|\iota^{\prime}(\alpha)\right|=|\tanh \alpha| \leq 1$ and $\left|\iota^{\prime \prime}(\alpha)\right|=\left|\operatorname{sech}^{2} \alpha\right| \leq 1$. Furthermore, note that $\iota(\alpha)-(|\alpha|-\log 2) \rightarrow 0$ and $\left(\iota^{\prime}(\alpha)-\operatorname{sgn} \alpha\right) \rightarrow 0$ as $|\alpha| \rightarrow \infty$. Therefore,

$$
\lim _{|\alpha| \rightarrow \infty}\left[\alpha \iota(\alpha) \iota^{\prime}(\alpha)-|\alpha|^{2}+(\log 2)|\alpha|\right]=0
$$

This implies that for any $\mathfrak{w}>0$, there exists an $R_{\mathfrak{w}}>0$ such that

$$
\begin{aligned}
\alpha \iota(\alpha) \iota^{\prime}(\alpha) & \geq\left[|\alpha|^{2}-(\log 2)|\alpha|-\mathfrak{w}\right] \mathbb{1}_{\left\{|\alpha|>R_{\mathfrak{w}}\right\}}+\alpha \iota(\alpha) \iota^{\prime}(\alpha) \mathbb{1}_{\left\{|\alpha| \leq R_{\mathfrak{w}}\right\}} \\
& \geq\left[\frac{1}{2}|\alpha|^{2} \mathbb{1}_{\{|\alpha|>2 \log 2\}}-\left(2 \log ^{2} 2\right) \mathbb{1}_{\{|\alpha| \leq 2 \log 2\}}-\mathfrak{w}\right] \mathbb{1}_{\left\{|\alpha|>R_{\mathfrak{w}}\right\}}+\alpha \iota(\alpha) \iota^{\prime}(\alpha) \mathbb{1}_{\left\{|\alpha| \leq R_{\mathfrak{w}}\right\}} \\
& =\left[\frac{1}{2}|\alpha|^{2}-\left(\frac{1}{2}|\alpha|^{2}+2 \log ^{2} 2\right) \mathbb{1}_{\{|\alpha| \leq 2 \log 2\}}-\mathfrak{w}\right] \mathbb{1}_{\left\{|\alpha|>R_{\mathfrak{w}}\right\}}+\alpha \iota(\alpha) \iota^{\prime}(\alpha) \mathbb{1}_{\left\{|\alpha| \leq R_{\mathfrak{w}}\right\}} \\
& \geq\left[\frac{1}{2}|\alpha|^{2}-4 \log ^{2} 2-\mathfrak{w}\right] \mathbb{1}_{\left\{|\alpha|>R_{\mathfrak{w}}\right\}}+\alpha \iota(\alpha) \iota^{\prime}(\alpha) \mathbb{1}_{\left\{|\alpha| \leq R_{\mathfrak{w}}\right\}} \\
& \geq\left[\frac{1}{2}|\alpha|^{2}-4 \log ^{2} 2-\mathfrak{w}\right] \mathbb{1}_{\left\{|\alpha|>R_{\mathfrak{w}}\right\}}-M_{\mathfrak{w}} \mathbb{1}_{\left\{|\alpha| \leq R_{\mathfrak{w}}\right\}} \\
& \geq\left[\frac{1}{2}|\alpha|^{2}-4 \log ^{2} 2-\mathfrak{w}\right]-\left[\frac{1}{2}|\alpha|^{2}+M_{\mathfrak{w}}\right] \mathbb{1}_{\left\{|\alpha| \leq R_{\mathfrak{w}}\right\}} \\
& \geq \frac{1}{2}|\alpha|^{2}-\left(4 \log ^{2} 2+\mathfrak{w}+\frac{1}{2} R_{\mathfrak{w}}^{2}+M_{\mathfrak{w}}\right),
\end{aligned}
$$

where $M_{\mathfrak{w}}:=\max _{|\alpha| \leq R_{\mathfrak{w}}}\left|\alpha \iota(\alpha) \iota^{\prime}(\alpha)\right|$. Therefore, by fixing a particular choice of $\mathfrak{w}>0$, the dissipativity condition holds with $a_{\iota}=\frac{1}{2}$ and $b_{\iota}=\left(4 \log ^{2} 2+\mathfrak{w}+\frac{1}{2} R_{\mathfrak{w}}^{2}+M_{\mathfrak{w}}\right)$, as desired.

## 7. Proof of Statements in Section 3

## Proof of Proposition 3.1.

Proof. Clearly Assumption 2 holds for $U_{\mathfrak{e}}(\theta, x)$ due to $y \mapsto f_{\mathfrak{e}}(y)$ being continuously differentiable.
To verify Assumption 3 , note that for any $i=1, \cdots, d-1, \theta \in \mathbb{R}^{d}$, and $x \in \Xi \subset \mathbb{R}^{m}$ where $m=d-1$,

$$
\begin{align*}
& \frac{\partial U_{\mathfrak{e}}}{\partial w_{i}}(\theta, x)=-s(w)_{i}\left[x_{i}-\frac{\sum_{j=1}^{d-1} x_{j} e^{w_{j}}}{\sum_{j=1}^{d-1} e^{w_{j}}}\right]\left[\rho_{1}+\frac{\rho_{2}}{1-\gamma} f_{\mathfrak{e}}^{\prime}(-\langle s(w), x\rangle-v)\right] \\
& \frac{\partial U_{\mathfrak{e}}}{\partial v}(\theta, x)=\rho_{2}\left[1-\frac{1}{1-\gamma} f_{\mathfrak{e}}^{\prime}(-\langle s(w), x\rangle-v)\right] . \tag{84}
\end{align*}
$$

Furthermore, $\left|f_{\mathfrak{e}}^{\prime}(y)\right| \leq 1$ and $\left|f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)-f_{\mathfrak{e}}^{\prime}\left(y_{2}\right)\right| \leq \frac{1}{2 \mathfrak{e}}\left|y_{1}-y_{2}\right|$ for all $y, y_{1}, y_{2} \in \mathbb{R}$. It immediately follows that

$$
\begin{align*}
\left|\nabla_{\theta} U_{\mathfrak{e}}(\theta, x)\right| & \leq(d-1) \cdot 2|x| \cdot\left(\rho_{1}+\frac{\rho_{2}}{1-\gamma}\right)+\rho_{2}\left(1+\frac{1}{1-\gamma}\right) \\
& \leq K_{\nabla}(1+|x|)^{2} \tag{85}
\end{align*}
$$

for all $\theta \in \mathbb{R}^{d}, x \in \mathbb{R}^{m}$, where $K_{\nabla}=\max \left\{\rho_{2}\left(1+\frac{1}{1-\gamma}\right), 2(d-1)\left(\rho_{1}+\frac{\rho_{2}}{1-\gamma}\right)\right\}$. This verifies the growth condition of $\nabla_{\theta} U_{\mathfrak{e}}$ in $\theta$. We verify next the global Lipschitz continuity of $\nabla_{\theta} U_{\mathfrak{c}}$ in $\theta$. Fix $\theta_{1}, \theta_{2} \in \mathbb{R}^{d}, x \in \mathbb{R}^{m}$, and denote $y_{k}=-\left\langle s\left(w_{k}\right), x\right\rangle-v_{k}, k=1,2$. For $i=1, \cdots, d-1$, one has the decomposition

$$
\begin{align*}
\left|\frac{\partial U_{\mathfrak{e}}}{\partial w_{i}}\left(\theta_{1}, x\right)-\frac{\partial U_{\mathfrak{e}}}{\partial w_{i}}\left(\theta_{2}, x\right)\right| \leq & \rho_{1} x_{i}\left|s\left(w_{1}\right)_{i}-s\left(w_{2}\right)_{i}\right|+\frac{\rho_{2}}{1-\gamma} x_{i}\left|s\left(w_{1}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)-s\left(w_{2}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{2}\right)\right| \\
& +\rho_{1}\left|s\left(w_{1}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{e_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-s\left(w_{2}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right| \\
& +\frac{\rho_{2}}{1-\gamma}\left|s\left(w_{1}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{1}\right) \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-s\left(w_{2}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{2}\right) \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right| . \tag{86}
\end{align*}
$$

It is well-known that the softmax function is 1-Lipschitz continuous - see, for example, [53], or the technique used in the proof of Proposition 5.7 for an alternative derivation. Thus, one has

$$
\begin{equation*}
\left|s\left(w_{1}\right)_{i}-s\left(w_{2}\right)_{i}\right| \leq\left|s\left(w_{1}\right)-s\left(w_{2}\right)\right| \leq\left|w_{1}-w_{2}\right| \leq\left|\theta_{1}-\theta_{2}\right| \tag{87}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|y_{1}-y_{2}\right| \leq\left|s\left(w_{1}\right)-s\left(w_{2}\right)\right| \cdot|x|+\left|v_{1}-v_{2}\right| \leq(1+|x|)\left|\theta_{1}-\theta_{2}\right| . \tag{88}
\end{equation*}
$$

The second term in the decomposition (86) is hence bounded by

$$
\begin{aligned}
\left|s\left(w_{1}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)-s\left(w_{2}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{2}\right)\right| & \leq\left|s\left(w_{1}\right)_{i}\right| \cdot\left|f_{\mathfrak{c}}^{\prime}\left(y_{1}\right)-f_{\mathfrak{e}}^{\prime}\left(y_{2}\right)\right|+\left|f_{\mathfrak{e}}^{\prime}\left(y_{2}\right)\right|\left|s\left(w_{1}\right)_{i}-s\left(w_{2}\right)_{i}\right| \\
& \leq \frac{1}{2 \mathfrak{e}}\left|y_{1}-y_{2}\right|+\left|\theta_{1}-\theta_{2}\right| \leq\left(1+\frac{1}{2 \mathfrak{e}}\right)(1+|x|)\left|\theta_{1}-\theta_{2}\right| .
\end{aligned}
$$

Applying the same argument used in (109) to (116) from the proof of Proposition 5.7 shows that

$$
\begin{align*}
&\left|\frac{\sum_{j=1}^{d-1} x_{j} e^{w_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-\frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right| \\
& \leq \frac{\sum_{1 \leq j<k \leq d-1}\left|e^{w_{1, j}+w_{2, k}}-e^{w_{2, j}+w_{1, k}}\right| \cdot\left|x_{j}\right|+\sum_{1 \leq j<k \leq d-1}\left|e^{w_{2, j}+w_{1, k}}-e^{w_{1, j}+w_{2, k}}\right| \cdot\left|x_{k}\right|}{\sum_{j, k=1}^{d-1} e^{w_{1, j}+w_{2, k}}} \\
& \leq 8\left|w_{1}-w_{2}\right| \cdot|x| \leq 8(1+|x|) \cdot\left|\theta_{1}-\theta_{2}\right| . \tag{89}
\end{align*}
$$

Therefore, the third term in the decomposition (86) can be bounded by

$$
\left|s\left(w_{1}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-s\left(w_{2}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right|
$$

$$
\begin{align*}
& \leq\left|s\left(w_{1}\right)_{i}\right| \cdot\left|\frac{\sum_{j=1}^{d-1} x_{j} e^{w_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-\frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right|+\left|\frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right| \cdot\left|s\left(w_{1}\right)_{i}-s\left(w_{2}\right)_{i}\right| \\
& \leq 8(1+|x|) \cdot\left|\theta_{1}-\theta_{2}\right|+|x| \cdot\left|\theta_{1}-\theta_{2}\right| \\
& \leq 9(1+|x|) \cdot\left|\theta_{1}-\theta_{2}\right| . \tag{90}
\end{align*}
$$

Lastly, the fourth term in the decomposition (86) can be bounded by

$$
\begin{align*}
&\left|s\left(w_{1}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{1}\right) \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-s\left(w_{2}\right)_{i} f_{\mathfrak{e}}^{\prime}\left(y_{2}\right) \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right| \\
& \leq\left|f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)\right| \cdot\left|s\left(w_{1}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{1, j}}}{\sum_{j=1}^{d-1} e^{w_{1, j}}}-s\left(w_{2}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right|+\left|s\left(w_{2}\right)_{i} \frac{\sum_{j=1}^{d-1} x_{j} e^{w_{2, j}}}{\sum_{j=1}^{d-1} e^{w_{2, j}}}\right| \cdot\left|f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)-f_{\mathfrak{e}}^{\prime}\left(y_{2}\right)\right| \\
& \leq 9(1+|x|) \cdot\left|\theta_{1}-\theta_{2}\right|+|x| \cdot \frac{1}{2 \mathfrak{e}} \cdot\left|y_{1}-y_{2}\right| \\
& \leq 9(1+|x|) \cdot\left|\theta_{1}-\theta_{2}\right|+\frac{1}{2 \mathfrak{e}} \cdot(1+|x|)^{2} \cdot\left|\theta_{1}-\theta_{2}\right| \\
& \leq\left(9+\frac{1}{2 \mathfrak{e}}\right)(1+|x|)^{2}\left|\theta_{1}-\theta_{2}\right| . \tag{91}
\end{align*}
$$

Therefore, substituting (87), (89), (90), and (91) into (86) yields, for $i=1, \cdots, d-1$,

$$
\begin{align*}
\left|\frac{\partial U_{\mathfrak{e}}}{\partial w_{i}}\left(\theta_{1}, x\right)-\frac{\partial U_{\mathfrak{e}}}{\partial w_{i}}\left(\theta_{2}, x\right)\right| \leq & \rho_{1}|x| \cdot\left|\theta_{1}-\theta_{2}\right|+\frac{\rho_{2}}{1-\gamma}|x| \cdot\left(1+\frac{1}{2 \mathfrak{e}}\right)(1+|x|)\left|\theta_{1}-\theta_{2}\right| \\
& +\rho_{1} \cdot 9(1+|x|) \cdot\left|\theta_{1}-\theta_{2}\right|+\frac{\rho_{2}}{1-\gamma} \cdot\left(9+\frac{1}{2 \mathfrak{e}}\right)(1+|x|)^{2}\left|\theta_{1}-\theta_{2}\right| \\
\leq & 10\left(\rho_{1}+\frac{\rho_{2}}{\mathfrak{e}(1-\gamma)}\right)(1+|x|)^{2}\left|\theta_{1}-\theta_{2}\right| . \tag{92}
\end{align*}
$$

It follows from (84) and (92) that for all $\theta_{1}, \theta_{2} \in \mathbb{R}^{d}, x \in \Xi$,

$$
\begin{align*}
& \left|\nabla_{\theta} U_{\mathfrak{e}}\left(\theta_{1}, x\right)-\nabla_{\theta} U_{\mathfrak{e}}\left(\theta_{2}, x\right)\right| \\
\leq & 10(d-1)\left(\rho_{1}+\frac{\rho_{2}}{\mathfrak{e}(1-\gamma)}\right)(1+|x|)^{2}\left|\theta_{1}-\theta_{2}\right|+\frac{\rho_{2}}{1-\gamma}\left|f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)-f_{\mathfrak{e}}^{\prime}\left(y_{1}\right)\right| \\
\leq & 10(d-1)\left(\rho_{1}+\frac{\rho_{2}}{\mathfrak{e}(1-\gamma)}\right)(1+|x|)^{2}\left|\theta_{1}-\theta_{2}\right|+\frac{\rho_{2}}{1-\gamma} \cdot \frac{1}{2 \mathfrak{e}}(1+|x|)\left|\theta_{1}-\theta_{2}\right| \\
\leq & L_{\nabla}(1+|x|)^{2}\left|\theta_{1}-\theta_{2}\right| \tag{93}
\end{align*}
$$

where $L_{\nabla}=10 d\left(\rho_{1}+\frac{\rho_{2}}{\mathfrak{c}(1-\gamma)}\right)$. This verifies Assumption 3 for $U_{\mathfrak{e}}(\theta, x)$.
Lastly, it remains to verify Assumption 4 . Fix $\theta \in \mathbb{R}^{d}$ and $x_{1}, x_{2} \in \Xi$, and denote $\tilde{y}_{k}:=-\left\langle s(w), x_{k}\right\rangle-v$, $k=1,2$. Applying again the fact that $\left|f_{\mathfrak{e}}^{\prime}(y)\right| \leq 1$ for all $y \in \mathbb{R}$ yields

$$
\begin{align*}
\left|U_{\mathfrak{e}}\left(\theta, x_{1}\right)-U_{\mathfrak{e}}\left(\theta, x_{2}\right)\right| & \leq \rho_{1}\left|\left\langle s(w), x_{1}-x_{2}\right\rangle\right|+\frac{\rho_{2}}{1-\gamma}\left|f_{\mathfrak{e}}\left(\tilde{y}_{1}\right)-f_{\mathfrak{e}}\left(\tilde{y}_{2}\right)\right| \\
& \leq \rho_{1}\left|\left\langle s(w), x_{1}-x_{2}\right\rangle\right|+\frac{\rho_{2}}{1-\gamma}\left|\tilde{y}_{1}-\tilde{y}_{2}\right| \\
& \leq \rho_{1}\left|\left\langle s(w), x_{1}-x_{2}\right\rangle\right|+\frac{\rho_{2}}{1-\gamma}\left|\left\langle s(w), x_{1}-x_{2}\right\rangle\right| \\
& \leq\left(\rho_{1}+\frac{\rho_{2}}{1-\gamma}\right) \cdot|s(w)| \cdot\left|x_{1}-x_{2}\right| \\
& \leq J_{U}\left|x_{1}-x_{2}\right|, \tag{94}
\end{align*}
$$

where $J_{U}:=\left(\rho_{1}+\frac{\rho_{2}}{1-\gamma}\right)$. This shows that Assumption 4 is satisfied, e.g., with this choice of $J_{U}$ and $\chi=0$. This completes the proof.

## 8. Proof of Statements in Section 5

Proof of Lemma 5.2. Indeed, one obtains from the definitions of $Q_{i, j}, \mu_{0, \ell, j},[\cdot]_{j}$, and the fact that $[x]_{j}=\boldsymbol{i}$ for all $x \in Q_{i, j}$ that

$$
\left.\begin{array}{rl}
z_{D, \ell, j} & :=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{j}\right)-\mathfrak{a}\left|[x]_{j}-[y]_{j}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\left[-2^{\ell-1}, 2^{\ell-1}\right)^{m}} \sup _{y \in \Xi \cap \mathbb{K}_{\ell, j}^{m}}\left\{U(\theta, y)-\mathfrak{a}\left|[x]_{j}-y\right|^{p}\right\} \mathrm{d} \mu_{0 . \ell}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\sum_{i \in \mathbb{K}_{\ell, j}^{m}} \int_{Q_{i, j}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, j}^{m}}\left\{U(\theta, y)-\mathfrak{a}\left|[x]_{j}-y\right|^{p}\right\} \mathrm{d} \mu_{0 . \ell}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\sum_{i \in \mathbb{K}_{\ell, j}^{m}} \int_{Q_{\boldsymbol{i}, j}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, j}^{m}}\left\{U(\theta, y)-\mathfrak{a}|\boldsymbol{i}-y|^{p}\right\} \mathrm{d} \mu_{0 . \ell}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\sum_{i \in \mathbb{K}_{\ell, j}^{m}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, j}^{m}}\left\{U(\theta, y)-\mathfrak{a}|\boldsymbol{i}-y|^{p}\right\} \mu_{0 . \ell}\left(Q_{\boldsymbol{i}, j}\right)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\sum_{i \in \mathbb{K}_{\ell, j}^{m}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, j}^{m}}\left\{U(\theta, y)-\mathfrak{a}|\boldsymbol{i}-y|^{p}\right\} \mu_{0, \ell, j}(\{\boldsymbol{i}\})+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\}
\end{array}\right\}
$$

as desired.

## Proof of Lemma 5.3.

Proof. We recall the definitions

$$
\begin{align*}
z_{D} & :=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
z_{D, \ell, \mathfrak{j}} & :=\inf _{\theta \in \mathbb{R}^{d}} \inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} . \tag{96}
\end{align*}
$$

To establish the first part of the lemma, it suffices to show that the coercivity condition

$$
\begin{equation*}
\liminf _{|((\theta, \mathfrak{a}))| \rightarrow \infty}\left\{\int_{\Xi} \sup _{y \in \Xi}\left(U(\theta, y)-\mathfrak{a}|x-y|^{p}\right) \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\}=\infty \tag{97}
\end{equation*}
$$

holds. Indeed, from Assumptions 1, 2, and 3, as well as Remark 2.1, one obtains

$$
\begin{align*}
& \int_{\Xi} \sup _{y \in \Xi}\left(U(\theta, y)-\mathfrak{a}|x-y|^{p}\right) \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}  \tag{98}\\
\geq & \int_{\Xi} \sup _{y \in \Xi}\left(U(\theta, y)-\mathfrak{a}|x-y|^{p}\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
= & \int_{\Xi}\left(U\left(\theta, y^{*}((\theta, \mathfrak{a}), x)\right)-\mathfrak{a}\left|x-y^{*}((\theta, \mathfrak{a}), x)\right|^{p}\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(-\left|U\left(\theta, y^{*}((\theta, \mathfrak{a}), x)\right)\right|-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(-\tilde{K}_{\nabla}\left(1+\left|y^{*}((\theta, \mathfrak{a}), x)\right|\right)^{\nu}(1+|(\theta, \mathfrak{a})|)-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|(\theta, \mathfrak{a})|)-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2}
\end{align*}
$$

$$
\begin{align*}
& \geq-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|(\theta, \mathfrak{a})|)-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
& \rightarrow \infty \text { as }|(\theta, \mathfrak{a})| \rightarrow \infty \tag{99}
\end{align*}
$$

where the inner supremum is attained at $y^{*}((\theta, \mathfrak{a}), x) \in \Xi$. This proves the first part of the lemma. To establish the second part of the lemma, we repeat again the same argument by applying Assumptions 1, 2, and 3, as well as Remark 2.1, to obtain the same lower bound

$$
\begin{align*}
& \int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2} \\
\geq & \int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(U\left(\theta, y_{\mathfrak{j}}^{*}((\theta, \mathfrak{a}), x)\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-y_{\mathfrak{j}}^{*}((\theta, \mathfrak{a}), x)\right|^{p}\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(-\left|U\left(\theta, y_{\mathfrak{j}}^{*}((\theta, \mathfrak{a}), x)\right)\right|-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(-\tilde{K}_{\nabla}\left(1+\left|y_{\mathfrak{j}}^{*}((\theta, \mathfrak{a}), x)\right|\right)^{\nu}(1+|(\theta, \mathfrak{a})|)-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & \int_{\Xi}\left(-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|(\theta, \mathfrak{a})|)-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|\right) \mathrm{d} \mu_{0}(x)+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\geq & -\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|(\theta, \mathfrak{a})|)-2^{p} M_{\Xi}^{p}|(\theta, \mathfrak{a})|+\frac{\min \left\{\eta_{1}, \eta_{2}\right\}}{2}|(\theta, \mathfrak{a})|^{2} \\
\rightarrow & \infty \text { as }|(\theta, \mathfrak{a})| \rightarrow \infty, \tag{100}
\end{align*}
$$

which does not depend on $\mathfrak{j}$. Here, $y_{\mathfrak{j}}^{*}((\theta, \mathfrak{a}), x) \in \Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}$ denotes, for given $\mathfrak{j}$, an optimiser for the inner supremum. Fix an $M^{\#}>z_{D, \ell, \mathrm{j}}$. The above lower bound shows that there exists a $K^{\#}>0$ not depending on $\mathfrak{j}$, such that, for all $\mathfrak{j}$, the inequality

$$
\begin{equation*}
\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2} \leq M^{\#} \tag{101}
\end{equation*}
$$

would imply $|(\theta, \mathfrak{a})| \leq K^{\#}$. Let $\epsilon>0$. Then, by the definition of $z_{D, \ell, \mathfrak{j}}$, there exists $(\theta, \mathfrak{a})_{\epsilon, \mathfrak{j}}=$ $\left(\theta_{\epsilon, \mathfrak{j}}, \mathfrak{a}_{\epsilon, \mathfrak{j}}\right) \in \mathbb{R}^{d} \times[0, \infty)$ such that

$$
\begin{align*}
\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta_{\epsilon, \mathfrak{j}},[y]_{\mathfrak{j}}\right)-\mathfrak{a}_{\epsilon, \mathfrak{j}} \mid[x]_{\mathfrak{j}}-[y] \mathrm{j}^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}\left|\theta_{\epsilon, \mathfrak{j}}\right|^{2}+\frac{\eta_{2}}{2}\left|\mathfrak{a}_{\epsilon, \mathfrak{j}}\right|^{2} & \leq z_{D, \ell, \mathfrak{j}}+\epsilon \\
& \leq M^{\#}+\epsilon \tag{102}
\end{align*}
$$

which by (101) implies that $\left|(\theta, \mathfrak{a})_{\epsilon, \mathrm{j}}\right| \leq K^{\#}$. Hence,

$$
\begin{align*}
& \inf _{|(\theta, \mathfrak{a})| \leq K^{\#}, \mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y] \mathfrak{j}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
\leq & \int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta_{\epsilon, \mathfrak{j}},[y]_{\mathfrak{j}}\right)-\mathfrak{a}_{\epsilon, \mathfrak{j}}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}\left|\theta_{\epsilon, \mathfrak{j}}\right|^{2}+\frac{\eta_{2}}{2}\left|\mathfrak{a}_{\epsilon, \mathfrak{j}}\right|^{2} \\
\leq & z_{D, \ell, \mathfrak{j}}+\epsilon \tag{103}
\end{align*}
$$

Since $\epsilon>0$ was arbitrary, this implies

$$
\begin{equation*}
\inf _{|(\theta, \mathfrak{a})| \leq K^{\#}, \mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta_{\epsilon, \mathfrak{j}},[y]_{\mathfrak{j}}\right)-\mathfrak{a}_{\epsilon, \mathfrak{j}}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}\left|\theta_{\epsilon, \mathfrak{j}}\right|^{2}+\frac{\eta_{2}}{2}\left|\mathfrak{a}_{\epsilon, \mathfrak{j}}\right|^{2}\right\} \leq z_{D, \ell, \mathfrak{j}} \tag{104}
\end{equation*}
$$

for any given $\mathfrak{j}$. Since the converse inequality holds trivially and $K^{\#}$ does not depend on $\mathfrak{j}$, choosing $\mathcal{K}_{\Xi, \ell}$ to be the intersection of the closed ball of radius $K^{\#}$ in $\mathbb{R}^{d+1}$ and $\mathbb{R}^{d} \times[0, \infty)$ concludes the proof.

Proof of Proposition 5.4. By Lemma 5.3, there exists a compact set $\mathcal{K} \subset \mathbb{R}^{d+1}$, not depending on $\mathfrak{j}$, such that the infimums in $z_{D}$ and $z_{D, \ell, \mathrm{j}}$ are both attained on $\mathcal{K}$. Applying the inequality

$$
\begin{equation*}
\max \left\{\left|\inf _{x \in A} f(x)-\inf _{x \in A} g(x)\right|,\left|\sup _{x \in A} f(x)-\sup _{x \in A} g(x)\right|\right\} \leq \sup _{x \in A}|f(x)-g(x)| \tag{105}
\end{equation*}
$$

for any functions $f, g$ and set $A$ contained within their domains, together with Assumption 4, yields

$$
\begin{align*}
& \left|z_{D}-z_{D, \ell, \mathfrak{j}}\right| \\
= & \left\lvert\, \inf _{\bar{\theta}=(\theta, \alpha) \in \mathcal{K}}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\}-\right. \\
& \left.\inf _{\bar{\theta}=(\theta, \alpha) \in \mathcal{K}}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathrm{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \right\rvert\, \\
\leq & \sup _{\bar{\theta} \in \mathcal{K}}\left|\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\}-\sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathrm{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)\right| \\
\leq & \sup _{\bar{\theta} \in \mathcal{K}} \int_{\Xi}\left|\sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\}-\sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathrm{j}}\right|^{p}\right\}\right| \mathrm{d} \mu_{0}(x) \\
\leq & \sup _{\bar{\theta} \in \mathcal{K}} \int_{\Xi} \sup _{y \in \Xi}\left|U(\theta, y)-U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\right| x-\left.y\right|^{p}+\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathrm{j}}\right|^{p} \mid \mathrm{d} \mu_{0}(x) \\
\leq & \sup _{\bar{\theta} \in \mathcal{K}} \int_{\Xi}\left(J_{U}(1+|\bar{\theta}|) \sup _{y \in \Xi}\left(1+|y|+\left|[y]_{\mathfrak{j}}\right|\right)^{\chi}\left|y-[y]_{\mathfrak{j}}\right|+\right. \\
& p|\bar{\theta}| \sup _{y \in \Xi}\left(1+|x-y|+\left|[x]_{\mathfrak{j}}-[y]_{\mathrm{j}}\right|\right)^{p-1}| | x-y\left|-\left|[x]_{\mathfrak{j}}-[y]_{\mathrm{j}}\right|\right) \mathrm{d} \mu_{0}(x) \\
\leq & \sup _{\bar{\theta} \in \mathcal{K}} \int_{\Xi}\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}(1+|\bar{\theta}|) \frac{\sqrt{m}}{2^{\mathfrak{j}}}+p|\bar{\theta}|\left(1+4 M_{\Xi}\right)^{p-1} \frac{2 \sqrt{m}}{2^{\mathfrak{j}}}\right) \mathrm{d} \mu_{0}(x) \\
\leq & \frac{\sqrt{m}\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+p\left(1+4 M_{\Xi}\right)^{p-1}\right)\left(1+\sup _{\bar{\theta} \in \mathcal{K}}|\bar{\theta}|\right)}{2^{\mathfrak{j}}}, \tag{106}
\end{align*}
$$

as desired.

## Proof of Proposition 5.7.

Proof. Fix any $\delta>0$. For $j \in\{1, \cdots, N\}$, denote $F_{j}^{\delta, \ell, j}(\bar{\theta}, x):=\exp \left[\frac{1}{\delta}\left(U\left(\theta, \xi_{j}\right)-\iota(\alpha)\left|x-\xi_{j}\right|^{p}\right)\right]$ with $\bar{\theta}=(\theta, \alpha) \in \mathbb{R}^{d} \times \mathbb{R}$, so that

$$
\begin{align*}
& \nabla_{\theta} V^{\delta, \ell, j}(\bar{\theta}, x)=\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell,}(\bar{\theta}, x) \nabla_{\theta} U\left(\theta, \xi_{j}\right)}{\sum_{j=1}^{N} F_{j}^{\delta, \ell j}(\bar{\theta}, x)}  \tag{107}\\
& \nabla_{\alpha} V^{\delta, \ell, j}(\bar{\theta}, x)=-\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell,}(\bar{\theta}, x) \iota^{\prime}(\alpha)\left|x-\xi_{j}\right|^{p}}{\sum_{j=1}^{N} F_{j}^{\delta,, j}(\bar{\theta}, x)} \tag{108}
\end{align*}
$$

Then, for all $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$, it holds that

$$
\begin{align*}
& \left|\nabla_{\theta} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right)-\nabla_{\theta} V^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)\right|  \tag{109}\\
= & \left|\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) \nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)}{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)}-\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) \nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)}{\sum_{j=1}^{N} F_{j}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right)}\right| \\
\leq & \frac{\left|\sum_{j, k=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right)\left(\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right)\right|}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta,, j}\left(\bar{\theta}_{2}, x\right)}
\end{align*}
$$

$$
\begin{align*}
& \left.=\frac{1}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)} \right\rvert\, \sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)\left(\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right) \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)\left(\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right) \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right)\left(\nabla_{\theta} U\left(\theta_{1}, \xi_{k}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right) \mid \\
& \left.=\frac{1}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \boldsymbol{j}}\left(\bar{\theta}_{2}, x\right)} \right\rvert\, \sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)\left(\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right) \\
& +\sum_{1 \leq j<k \leq N}\left(F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) \nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) \nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right) \\
& +\sum_{1 \leq j<k \leq N}\left(F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right) \nabla_{\theta} U\left(\theta_{1}, \xi_{k}\right)-F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right) \nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right) \mid \\
& \leq \frac{1}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)}\left(\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{j}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right|\right. \\
& +\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) \nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) \nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right| \\
& \left.+\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) \nabla_{\theta} U\left(\theta_{1}, \xi_{k}\right)-F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) \nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right|\right) \\
& \leq \frac{1}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)}\left(\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right|\right. \\
& +\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)-F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right)\right| \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right| \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right| \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{k}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right| \\
& \left.+\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right)-F_{j}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)\right| \cdot\left|\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right|\right) . \tag{110}
\end{align*}
$$

Let $j, k$ be such that $1 \leq j<k \leq N$. For $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$ such that $F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) \geq$ $F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right)$, one obtains, by the inequality $1-e^{-y} \leq y, y \geq 0$, that

$$
\begin{aligned}
& \frac{1}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)} \cdot\left|F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)-F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right)\right| \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right| \\
& =\frac{F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell \boldsymbol{j}}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)} \cdot\left(1-\frac{F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right)}{F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)}\right) \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right| \\
& =\frac{F_{j}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell, \mathfrak{j}}\left(\bar{\theta}_{2}, x\right)} \cdot\left(1-\exp \left[\frac { 1 } { \delta } \left(\left(U\left(\theta_{2}, \xi_{j}\right)-U\left(\theta_{1}, \xi_{j}\right)\right)-\left(\iota\left(\alpha_{2}\right)-\iota\left(\alpha_{1}\right)\right)\left|x-\xi_{j}\right|^{p}\right.\right.\right. \\
& \left.\left.\left.+\left(U\left(\theta_{1}, \xi_{k}\right)-U\left(\theta_{2}, \xi_{k}\right)\right)-\left(\iota\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right)\right)\left|x-\xi_{k}\right|^{p}\right)\right]\right) \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta,, j}\left(\bar{\theta}_{2}, x\right)}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)} \cdot\left[\frac { 1 } { \delta } \left(\left(U\left(\theta_{1}, \xi_{j}\right)-U\left(\theta_{2}, \xi_{j}\right)\right)-\left(\iota\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right)\right)\left|x-\xi_{j}\right|^{p}\right.\right. \\
& \left.\left.\left.+\left(U\left(\theta_{2}, \xi_{k}\right)-U\left(\theta_{1}, \xi_{k}\right)\right)-\left(\iota\left(\alpha_{2}\right)-\iota\left(\alpha_{1}\right)\right)\left|x-\xi_{k}\right|^{p}\right)\right] \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right|\right] \\
\leq & \frac{F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right)}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell}\left(\bar{\theta}_{2}, x\right)} \cdot \frac{1}{\delta}\left[\left|U\left(\theta_{1}, \xi_{j}\right)-U\left(\theta_{2}, \xi_{j}\right)\right|+\left|U\left(\theta_{1}, \xi_{k}\right)-U\left(\theta_{2}, \xi_{k}\right)\right|\right. \\
& \left.+\left|\iota\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right)\right|\left(\left|x-\xi_{j}\right|^{p}+\left|x-\xi_{k}\right|^{p}\right)\right] \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right| .
\end{aligned}
$$

Interchanging the roles of $\bar{\theta}_{1}$ and $\bar{\theta}_{2}$ in the above argument shows that for all $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$,

$$
\begin{align*}
& \frac{1}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell_{\mathrm{j}}}\left(\bar{\theta}_{2}, x\right)} \cdot\left|F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)-F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right)\right| \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right| \\
& \leq \frac{\max \left\{F_{j}^{\delta,, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, j}\left(\overline{( }_{2}, x\right), F_{F}^{\delta,, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)\right\}}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}, j, j}^{\delta, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right)} \cdot \frac{1}{\delta}\left[\left|U\left(\theta_{1}, \xi_{j}\right)-U\left(\theta_{2}, \xi_{j}\right)\right|+\left|U\left(\theta_{1}, \xi_{k}\right)-U\left(\theta_{2}, \xi_{k}\right)\right|\right. \\
& \left.+\left|\iota\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right)\right|\left(\left|x-\xi_{j}\right|^{p}+\left|x-\xi_{k}\right|^{p}\right)\right] \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right|, \tag{111}
\end{align*}
$$

and furthermore,

$$
\begin{align*}
& \frac{1}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell^{\prime}}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell \mathbf{j}}\left(\bar{\theta}_{2}, x\right)} \cdot\left|F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right)-F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)\right| \cdot\left|\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right| \\
& \leq \frac{\max \left\{F_{j}^{\delta, \ell j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right), F_{j}^{\delta,, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)\right\}}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, j, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell}\left(\bar{\theta}_{2}, x\right)} \cdot \frac{1}{\delta}\left[\left|U\left(\theta_{1}, \xi_{j}\right)-U\left(\theta_{2}, \xi_{j}\right)\right|+\left|U\left(\theta_{1}, \xi_{k}\right)-U\left(\theta_{2}, \xi_{k}\right)\right|\right. \\
& \left.+\left|\iota\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right)\right|\left(\left|x-\xi_{j}\right|^{p}+\left|x-\xi_{k}\right|^{p}\right)\right] \cdot\left|\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right| . \tag{112}
\end{align*}
$$

By Assumption 3, Remark 2.1, and the fact that $\iota^{\prime}$ is bounded by $M_{\iota}$, it holds that

$$
\begin{align*}
& \frac{1}{\delta}\left[\left|U\left(\theta_{1}, \xi_{j}\right)-U\left(\theta_{2}, \xi_{j}\right)\right|+\left|U\left(\theta_{1}, \xi_{k}\right)-U\left(\theta_{2}, \xi_{k}\right)\right|+\left|\iota\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right)\right|\left(\left|x-\xi_{j}\right|^{p}+\left|x-\xi_{k}\right|^{p}\right)\right] \\
& \cdot\left(\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)\right|+\left|\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right|\right) \\
\leq & \frac{1}{\delta}\left[K_{\nabla}\left(\left(1+\left|\xi_{j}\right|\right)^{\nu}+\left(1+\left|\xi_{k}\right|\right)^{\nu}\right)\left|\theta_{1}-\theta_{2}\right|+M_{\iota}\left|\alpha_{1}-\alpha_{2}\right|\left(\left|x-\xi_{j}\right|^{p}+\left|x-\xi_{k}\right|^{p}\right)\right] \cdot 2 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu} \\
\leq & \frac{1}{\delta}\left[2 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\left|\theta_{1}-\theta_{2}\right|+2^{p} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\left(1+|x|^{p}\right)\left|\alpha_{1}-\alpha_{2}\right|\right] \cdot 2 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu} \\
\leq & \frac{4 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta} \cdot(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| . \tag{113}
\end{align*}
$$

That is, combining (111) and (112), then summing over $1 \leq j<k \leq N$ yields

$$
\begin{align*}
& \leq \frac{4 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta} \cdot(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| \\
& \text {. } \frac{\sum_{1 \leq j<k \leq N} \max \left\{F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right), F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)\right\}}{\sum_{j^{\prime}, k^{\prime}=1}^{N_{j}^{\prime}} F_{j^{\prime}, k,}^{\delta, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, j, j}\left(\bar{\theta}_{2}, x\right)} \\
& \leq \frac{8 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta} \cdot(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| . \tag{114}
\end{align*}
$$

In addition, by Assumption 3 , for all $j, k \in\{1, \cdots, N\}, \bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$, it holds that

$$
\begin{aligned}
& \frac{1}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta,, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, j,}\left(\bar{\theta}_{2}, x\right)} \cdot F_{j}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{j}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{j}\right)\right| \\
& \leq \frac{F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell_{j}}\left(\bar{\theta}_{2}, x\right)} \cdot L_{\nabla}\left(1+\left|\xi_{j}\right|\right)^{\nu}\left|\theta_{1}-\theta_{2}\right| \\
& \leq \frac{F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell j}\left(\bar{\theta}_{1}, x\right)}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell j}\left(\bar{\theta}_{2}, x\right)} \cdot L_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| .
\end{aligned}
$$

This implies that

$$
\begin{align*}
& +\frac{\sum_{1 \leq j<k<N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) \cdot\left|\nabla_{\theta} U\left(\theta_{1}, \xi_{k}\right)-\nabla_{\theta} U\left(\theta_{2}, \xi_{k}\right)\right|}{\sum_{j^{\prime}, k^{\prime}=1}^{N} F_{j^{\prime}}^{\delta,, k}\left(\bar{\theta}_{1}, x\right) F_{k^{\prime}}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)} \\
& \leq 2 L_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| . \tag{115}
\end{align*}
$$

Therefore, substituting (114) and (115) into (110) yields, for all $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$,

$$
\begin{align*}
& \left|\nabla_{\theta} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right)-\nabla_{\theta} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)\right| \\
\leq & 2\left(1+M_{\Xi}\right)^{\nu}\left(\frac{4 K_{\nabla}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}+L_{\nabla}\right) \cdot(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| . \tag{116}
\end{align*}
$$

By a similar argument as in (110), it holds for all $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$ that

$$
\begin{aligned}
& \left|\nabla_{\alpha} V^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right)-\nabla_{\alpha} V^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)\right| \\
& \leq\left|\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) \iota^{\prime}\left(\alpha_{1}\right)\left|x-\xi_{j}\right|^{p}}{\sum_{j=1}^{N} F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right)}-\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) \iota^{\prime}\left(\alpha_{2}\right)\left|x-\xi_{j}\right|^{p}}{\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)}\right| \\
& \leq \frac{1}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right)}\left(\sum_{j=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{j}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right) \cdot\left|\iota^{\prime}\left(\alpha_{1}\right)-\iota^{\prime}\left(\alpha_{2}\right)\right| \cdot\left|x-\xi_{j}\right|^{p}\right. \\
& +\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)-F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)\right| \cdot\left|\iota^{\prime}\left(\alpha_{1}\right)\right| \cdot\left|x-\xi_{j}\right|^{p} \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta \ell, \boldsymbol{j}}\left(\bar{\theta}_{1}, x\right) \cdot\left|\iota^{\prime}\left(\alpha_{1}\right)-\iota^{\prime}\left(\alpha_{2}\right)\right| \cdot\left|x-\xi_{j}\right|^{p} \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) \cdot\left|\iota^{\prime}\left(\alpha_{1}\right)-\iota^{\prime}\left(\alpha_{2}\right)\right| \cdot\left|x-\xi_{k}\right|^{p} \\
& \left.+\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right)-F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)\right| \cdot\left|\iota^{\prime}\left(\alpha_{2}\right)\right| \cdot\left|x-\xi_{k}\right|^{p}\right) \\
& \leq \frac{2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\}(1+|x|)^{p}}{\sum_{j, k=1}^{N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right)}\left(\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) F_{j}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right) \cdot L_{\iota}\left|\alpha_{1}-\alpha_{2}\right|\right. \\
& +\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right)-F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, j}\left(\bar{\theta}_{1}, x\right)\right| \cdot M_{\iota} \\
& +\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell, j}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) \cdot L_{\iota}\left|\alpha_{1}-\alpha_{2}\right|+\sum_{1 \leq j<k \leq N} F_{j}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{1}, x\right) \cdot L_{\iota}\left|\alpha_{1}-\alpha_{2}\right| \\
& \left.+\sum_{1 \leq j<k \leq N}\left|F_{j}^{\delta, \ell,}\left(\bar{\theta}_{2}, x\right) F_{k}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right)-F_{j}^{\delta, \ell,}\left(\bar{\theta}_{1}, x\right) F_{k}^{\delta, \ell, \mathbf{j}}\left(\bar{\theta}_{2}, x\right)\right| \cdot M_{\iota}\right)
\end{aligned}
$$

$$
\begin{align*}
& \leq 2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\}(1+|x|)^{p}\left(2 L_{\iota}\left|\alpha_{1}-\alpha_{2}\right|+\frac{8 M_{\iota}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}(1+|x|)^{p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right|\right) \\
& \leq\left(2^{p} L_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}+\frac{2^{p+2} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}\right) \cdot(1+|x|)^{2 p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right| \tag{117}
\end{align*}
$$

where the second last inequality is obtained using the same arguments as in (111)-(114). Combining (116) and (117) thus yields

$$
\begin{equation*}
\left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right)-\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)\right| \leq L_{\delta}(1+|x|)^{2 p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right|, \tag{118}
\end{equation*}
$$

for all $\bar{\theta}_{1}, \bar{\theta}_{2} \in \mathbb{R}^{d+1}$ and $x \in \Xi$, where

$$
\begin{align*}
L_{\delta}:= & 2\left(1+M_{\Xi}\right)^{\nu}\left(\frac{4 K_{\nabla}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}+L_{\nabla}\right) \\
& +\left(2^{p} L_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}+\frac{2^{p+2} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}\right) . \tag{119}
\end{align*}
$$

## Proof of Proposition 5.8.

Proof. Recall the expressions for $\nabla_{\bar{\theta}} V^{\delta, \ell, j}$ given in (108). From Assumption 3, we derive the growth condition

$$
\begin{align*}
\left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathbf{j}}(\bar{\theta}, x)\right| & \leq\left|\nabla_{\theta} V^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)\right|+\left|\nabla_{\alpha} V^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)\right| \\
& =\frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)\left(\left|\nabla_{\theta} U\left(\theta, \xi_{j}\right)\right|+\left|\iota^{\prime}(\alpha)\right| \cdot\left|x-\xi_{j}\right|^{p}\right)}{\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)} \\
& \leq \frac{\sum_{j=1}^{N} F_{j}^{\delta, \ell, \mathfrak{j}}(\bar{\theta}, x)\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)}{\sum_{j=1}^{N} F_{j}^{\delta, \ell \mathfrak{j}}(\bar{\theta}, x)} \\
& =K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p} \tag{120}
\end{align*}
$$

which holds for all $\bar{\theta} \in \mathbb{R}^{d+1}$ and $x \in \Xi$. Hence, it follows from Assumption 5 that

$$
\begin{aligned}
& \left\langle\bar{\theta}, \nabla_{\bar{\theta}}\left(V^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\iota(\alpha)|^{2}\right)\right\rangle \\
\geq & -|\bar{\theta}| \cdot\left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x)\right|+\eta_{1}|\theta|^{2}+\eta_{2} \alpha \iota(\alpha) \iota^{\prime}(\alpha) \\
\geq & -|\bar{\theta}| \cdot\left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x)\right|+\eta_{1}|\theta|^{2}+\eta_{2} a_{\iota}|\alpha|^{2}-\eta_{2} b_{\iota} \\
\geq & -|\bar{\theta}| \cdot\left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}(\bar{\theta}, x)\right|+\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}|\bar{\theta}|^{2}-\eta_{2} b_{\iota} \\
\geq & -\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)|\bar{\theta}|+\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}|\bar{\theta}|^{2}-\eta_{2} b_{\iota} \\
\geq & \frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2}|\bar{\theta}|^{2} \mathbb{1}^{2}\left\{|\bar{\theta}|>\frac{2\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}\right\} \\
& \left.+\left(\frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2}|\bar{\theta}|^{2}-\frac{2\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)^{2}}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}\right) \mathbb{1}_{\left\{|\bar{\theta}| \leq \frac{2\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}\right.}\right\}-\eta_{2} b_{\iota}
\end{aligned}
$$

$$
\begin{equation*}
\geq a|\bar{\theta}|^{2}-b \tag{121}
\end{equation*}
$$

with $a:=\frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2}$ and $b:=\eta_{2} b_{\iota}+\frac{2\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)^{2}}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}$. This completes the proof.

## Proof of Proposition 5.9.

Proof. Clearly, Assumption 1 of [140] holds due to $\bar{\theta}_{0} \in L^{4}\left(\Omega, \mathcal{F}, \mathbb{P} ; \mathbb{R}^{d+1}\right)$ and the finiteness of the space $\Xi \cap \mathbb{K}_{\ell, j}^{m}$ which allows for exchange of order of differentiation and Lebesgue integration. Assumption 2 of [140] holds for the stochastic gradient $\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, j}(\bar{\theta}, x)$ due to Proposition 5.7 and $\iota \cdot \iota^{\prime}$ being Lipschitz continuous. Specifically, we have for all $\bar{\theta}_{1}=\left(\theta_{1}, \alpha_{2}\right), \bar{\theta}_{2}=\left(\theta_{2}, \alpha_{2}\right) \in \mathbb{R}^{d} \times \mathbb{R}$ and $x \in \Xi$,

$$
\begin{align*}
& \left|\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right)-\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)\right| \\
\leq & \left|\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{1}, x\right)-\nabla_{\bar{\theta}} V^{\delta, \ell, \mathrm{j}}\left(\bar{\theta}_{2}, x\right)\right|+\eta_{1}\left|\theta_{1}-\theta_{2}\right|+\eta_{2}\left|\iota\left(\alpha_{1}\right) \iota^{\prime}\left(\alpha_{1}\right)-\iota\left(\alpha_{2}\right) \iota^{\prime}\left(\alpha_{2}\right)\right| \\
\leq & L_{\delta}(1+|x|)^{2 p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right|+\eta_{1}\left|\theta_{1}-\theta_{2}\right|+\eta_{2} \tilde{L}_{\iota}\left|\alpha_{1}-\alpha_{2}\right| \\
\leq & \left(L_{\delta}+\eta_{1}+\eta_{2} \tilde{L}_{\iota}\right)(1+|x|)^{2 p}\left|\bar{\theta}_{1}-\bar{\theta}_{2}\right|, \tag{122}
\end{align*}
$$

and Assumption 2 of [140] with the correspondence of quantities between that in [140] and those in this paper being

$$
\begin{equation*}
d \leftarrow d+1, \quad \theta \leftarrow \bar{\theta}, \quad H \leftarrow \nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, j}, \quad L_{1} \leftarrow L_{\delta}+\eta_{1}+\eta_{2} \tilde{L}_{\iota}, \quad \eta(x) \leftarrow(1+|x|)^{2 p} \tag{123}
\end{equation*}
$$

where the LHS of the above assignments are in the notation of [140]. Furthermore, Assumption 3 of [140] holds for the stochastic gradient $\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, j}(\bar{\theta}, x)$ due to Proposition 5.8, and the constants $a, b$ in Remark 2.2 of [140] correspond exactly to the constants $a, b$ in Proposition 5.8 of this paper. One may obtain, from Equation (7) of [140], the maximum step size restriction

$$
\begin{equation*}
\lambda_{\max , \delta}=\min \left\{\frac{\mathfrak{C}_{1}}{\tilde{L}_{\delta}^{2}}, \frac{1}{a}\right\} \tag{124}
\end{equation*}
$$

for the algorithm, where the constants $a, \mathfrak{C}_{1}$ and $\tilde{L}_{\delta}:=1+L_{\delta}+\eta_{1}+\eta_{2} \tilde{L}_{\iota}$ are given explicitly as

$$
\begin{align*}
\mathfrak{C}_{1} & :=\frac{\min \left\{a, a^{1 / 3}\right\}}{16 \sqrt{\mathbb{E}_{\mathbb{P}}\left[\left(1+\left(1+\left|X_{0}\right|\right)^{2 p}\right)^{4}\right]}}, \\
a & :=\frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2}, \\
\tilde{L}_{\delta} & :=\frac{\mathfrak{C}_{2}}{\delta}+\mathfrak{C}_{3}, \\
\mathfrak{C}_{2} & :=\left(8 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p+2} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\right)\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\right), \\
\mathfrak{C}_{3} & :=2 L_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} L_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}+\eta_{1}+\eta_{2} \tilde{L}_{\iota}+1 . \tag{125}
\end{align*}
$$

(We note that the second condition in Assumption 2 of [140] was imposed by the authors to obtain sharper bounds for the Lipschitz constants. However, this second condition is not mandatory for the convergence bounds on the SGLD algorithm to hold, thus we do not verify it here.) Therefore, one may apply Corollary 2.8 of [140] with $\nabla_{\bar{\theta}} \tilde{V^{\delta}}(\bar{\theta}, x)$ as the stochastic gradient to obtain constants $c_{\delta, \beta}, C_{1, \delta, \ell, \mathfrak{j}, \beta}, C_{2, \delta, \ell, \mathbf{j}, \beta}, C_{3, \delta, \beta}>0$ not depending on $n$ or $\lambda$ and with growth orders as specified in (58) such that

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[v^{\delta, \ell, \mathrm{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]-\inf _{\bar{\theta} \in \mathbb{R}^{d+1}} v^{\delta, \ell, \mathrm{j}}(\bar{\theta}) \leq C_{1, \delta, \ell, \mathrm{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathrm{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta} \tag{126}
\end{equation*}
$$

where $v^{\delta, \ell, \mathfrak{j}}$ is defined as in (51). Note that $c_{\delta, \beta}, C_{1, \delta, \ell, \mathfrak{j}, \beta}, C_{2, \delta, \ell, \mathrm{j}, \beta}, C_{3, \delta, \beta}>0$ correspond to $\dot{c}, C_{1}^{\#}, C_{2}^{\#}, C_{3}^{\#}$ of Corollary 2.8 of [140], respectively, and that $c_{\delta, \beta}, C_{3, \delta, \beta}$ do not depend on $\ell$ and $\mathfrak{j}$. This completes the proof.

## Proof of Corollary 5.10.

Proof. Applying the duality result in (43) twice yields

$$
\mathbb{E}_{\mathbb{P}}\left[v^{\ell, \mathrm{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right]=\mathbb{E}_{\mathbb{P}}\left[\left.\left(\int_{\Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}} \tilde{V}^{\ell, \mathrm{j}}(\bar{\theta}, x) \mathrm{d} \mu_{0, \ell, \mathfrak{j}}(x)\right)\right|_{\bar{\theta}=\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathrm{j}}=\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}, \hat{\alpha}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)}\right]
$$

$$
\begin{align*}
& \geq \mathbb{E}_{\mathbb{P}}\left[\left.\left(\inf _{\alpha \in \mathbb{R}} \int_{\Xi \cap \mathbb{K}_{\ell, \mathrm{j}}^{m}} \tilde{V}^{\ell, \mathrm{j}}(\bar{\theta}, x) \mathrm{d} \mu_{0, \ell, \mathrm{j}}(x)\right)\right|_{\theta=\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}}\right] \\
& =\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)\right] \\
& \geq \inf _{\theta \in \mathbb{R}^{d}} u^{\ell, \mathfrak{j}}(\theta) \\
& =\inf _{\bar{\theta} \in \mathbb{R}^{d+1}} v^{\ell, \mathrm{j}}(\bar{\theta}) \\
& =z_{D, \ell, \mathrm{j}} \tag{127}
\end{align*}
$$

This, together with (126) and (43) as well as that $N=2^{\ell+\mathrm{j}}$ implies that

$$
\begin{align*}
\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathfrak{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{P, \ell, \mathfrak{j}} & =\mathbb{E}_{\mathbb{P}}\left[u^{\ell, \mathfrak{j}}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{D, \ell, \mathfrak{j}} \\
& \leq \mathbb{E}_{\mathbb{P}}\left[v^{\ell, \mathrm{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{D, \ell, \mathfrak{j}} \\
& \leq \mathbb{E}_{\mathbb{P}}\left[v^{\delta, \ell, \mathfrak{j}}\left(\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right)\right]-z_{D, \ell, \mathfrak{j}, \delta}+\delta \log N \\
& \leq C_{1, \delta, \ell, \mathfrak{j}, \beta} e^{-c_{\delta, \beta} \lambda n / 4}+C_{2, \delta, \ell, \mathfrak{j}, \beta} \lambda^{1 / 4}+C_{3, \delta, \beta}+\delta(\ell+\mathfrak{j}) \log 2, \tag{128}
\end{align*}
$$

where the second inequality is due to

$$
\begin{equation*}
v^{\delta, \ell, \mathfrak{j}}(\bar{\theta}) \leq v^{\ell, \mathfrak{j}}(\bar{\theta}) \leq v^{\delta, \ell, \mathfrak{j}}(\bar{\theta})+\delta \log N, \quad \bar{\theta} \in \mathbb{R}^{d+1} \tag{129}
\end{equation*}
$$

which follows from the definitions of $v^{\ell, \mathfrak{j}}$ and $v^{\delta, \ell, \mathfrak{j}}$ in (41) and (51), as well as the smoothing error given in Lemma 5.5. This completes the proof.

Proof of Proposition 5.11. Fix $\theta \in \mathbb{R}^{d}$. By the definition of $u^{\ell, j}$ in (37), the duality result in (27) due to Theorem 5.1, and by the proof of Lemma 5.2, one obtains the relation

$$
\begin{align*}
u^{\ell, \mathfrak{j}}(\theta) & =\inf _{\alpha \in \mathbb{R}}\left\{\int_{\Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}}\left\{U(\theta, y)-\iota(\alpha)|x-y|^{p}\right\} \mathrm{d} \mu_{0, \ell, \mathfrak{j}}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\iota(\alpha)|^{2}\right\} \\
& =\inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}} \max _{y \in \Xi \cap \mathbb{K}_{\ell, \mathfrak{j}}^{m}}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0, \ell, \mathfrak{j}}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \\
& =\inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} . \tag{130}
\end{align*}
$$

Similarly, by the duality result of Theorem 5.1, the relation

$$
\begin{equation*}
u(\theta)=\inf _{\mathfrak{a} \geq 0}\left\{\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\} \tag{131}
\end{equation*}
$$

holds. Observe that, by following the exact same argument in (98) to (100) from the proof of Lemma 5.3 , one obtains the following lower bound

$$
\begin{align*}
& \min \{ \\
& \int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}, \\
&\left.\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}\right\}  \tag{132}\\
& \geq-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)-\mathfrak{a} 2^{p} M_{\Xi}^{p}+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}
\end{align*}
$$

uniformly in $\mathfrak{j}$. Denote by $\mathfrak{K}_{\theta}$ the quantity

$$
\begin{equation*}
\mathfrak{K}_{\theta}:=\frac{2}{\sqrt{\eta}}\left(1+\sup _{x \in \Xi}|U(\theta, x)|+\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)\right)+\frac{2^{p+2} M_{\Xi}^{p}}{\eta_{2}} . \tag{133}
\end{equation*}
$$

Then, for all $\mathfrak{a}>\mathfrak{K}_{\theta}$, we obtain the inequality

$$
-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)-\mathfrak{a} 2^{p} M_{\Xi}+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}
$$

$$
\begin{align*}
& \geq-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)-\mathfrak{a} 2^{p} M_{\Xi}^{p} \cdot \frac{\mathfrak{a} \eta_{2}}{2^{p+2} M_{\Xi}^{p}}+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2} \\
& =-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{4}|\mathfrak{a}|^{2} \\
& >-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{4}\left|\mathfrak{K}_{\theta}\right|^{2} \\
& >-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{4}\left|\frac{2}{\sqrt{\eta}_{2}}\left(1+\sup _{x \in \Xi}|U(\theta, x)|+\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)\right)\right|^{2} \\
& >-\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)+\frac{\eta_{1}}{2}|\theta|^{2}+\left(1+\sup _{x \in \Xi}|U(\theta, x)|+\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)\right) \\
& =1+\sup _{x \in \Xi}|U(\theta, x)|+\frac{\eta_{1}}{2}|\theta|^{2} \\
& >\max \left\{u(\theta), u^{\ell, j}(\theta)\right\} . \tag{134}
\end{align*}
$$

This, in particular, implies from (132) that for all $\mathfrak{a}>\mathfrak{K}_{\theta}$,

$$
\begin{array}{r}
\int_{\Xi} \sup _{y \in \Xi}\left\{U(\theta, y)-\mathfrak{a}|x-y|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}>1+\sup _{x \in \Xi}|U(\theta, x)|+\frac{\eta_{1}}{2}|\theta|^{2}>u(\theta) \\
\int_{\Xi} \sup _{y \in \Xi}\left\{U\left(\theta,[y]_{\mathfrak{j}}\right)-\mathfrak{a}\left|[x]_{\mathfrak{j}}-[y]_{\mathfrak{j}}\right|^{p}\right\} \mathrm{d} \mu_{0}(x)+\frac{\eta_{1}}{2}|\theta|^{2}+\frac{\eta_{2}}{2}|\mathfrak{a}|^{2}>1+\sup _{x \in \Xi}|U(\theta, x)|+\frac{\eta_{1}}{2}|\theta|^{2}>u^{\ell, \mathfrak{j}}(\theta) \tag{135}
\end{array}
$$

Therefore, by the same argument in (101) to (104) from the proof of Lemma 5.3, the infimum in (130) and (131) are both attained in $\left[0, \mathfrak{K}_{\theta}\right]$. It follows by applying the same argument in the proof of Proposition 5.4 that

$$
\begin{align*}
\left|u(\theta)-u^{\ell, \mathfrak{j}}(\theta)\right| \leq & \sup _{\mathfrak{a} \in\left[0, \mathfrak{K}_{\theta}\right]} \int_{\Xi} \sup _{y \in \Xi}|U(\theta, y)-\mathfrak{a}| x-\left.y\right|^{p}-U\left(\theta,[y]_{j}\right)+\mathfrak{a}\left|[x]_{j}-[y]_{j}\right|^{p} \mid \mathrm{d} \mu_{0}(x) \\
\leq & J_{U}(1+|\theta|)\left(1+2 M_{\Xi}\right)^{\chi} \frac{\sqrt{m}}{2^{j}}+p \mathfrak{K}_{\theta}\left(1+4 M_{\Xi}\right)^{p-1} \frac{2 \sqrt{m}}{2^{j}} \\
= & J_{U}(1+|\theta|)\left(1+2 M_{\Xi}\right)^{\chi} \frac{\sqrt{m}}{2^{j}}+p\left(1+4 M_{\Xi}\right)^{p-1} \frac{2^{p+3} M_{\Xi} \sqrt{m}}{\eta_{2} 2^{j}} \\
& +p\left(1+\sup _{x \in \Xi}|U(\theta, x)|+\tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)\right)\left(1+4 M_{\Xi}\right)^{p-1} \frac{4 \sqrt{m}}{\sqrt{\eta_{2}} 2^{j}} \\
\leq & J_{U}(1+|\theta|)\left(1+2 M_{\Xi}\right)^{\chi} \frac{\sqrt{m}}{2^{j}}+p\left(1+4 M_{\Xi)^{p-1}} \frac{2^{p+3} M_{\Xi} \sqrt{m}}{\eta_{2} 2^{j}}\right. \\
& +p\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}(1+|\theta|)\right)\left(1+4 M_{\Xi}\right)^{p-1} \frac{4 \sqrt{m}}{\sqrt{\eta_{2} 2^{j}}} \\
\leq & \frac{\sqrt{m}}{2^{j}}\left[J_{U}\left(1+M_{\Xi}\right)^{\chi}+\frac{4 p}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi)^{p-1}\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\right)}\right.\right. \\
& \left.+\frac{2^{p+2} p M_{\Xi}}{\eta_{2}}\left(1+4 M_{\Xi}\right)^{p-1}+\left(J_{u}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{\nu+p-1}\right)|\theta|\right] . \tag{3}
\end{align*}
$$

An application of Lemma 4.2 of [140] yields, for $\lambda \in\left(0, \lambda_{\max , \delta}\right)$ where $\lambda_{\max , \delta}$ is as defined in (124), the second moment bound

$$
\begin{align*}
& \mathbb{E}_{\mathbb{P}}\left[\left|\hat{\bar{\theta}}_{n}^{\lambda, \delta, \ell, \mathfrak{j}}\right|^{2}\right] \\
\leq & e^{-a \lambda(n+1)} \mathbb{E}_{\mathbb{P}}\left[\left|\hat{\bar{\theta}}_{0}\right|^{2}\right]+\left(2 \lambda_{\max , \delta} \sup _{x \in \Xi}\left|\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathfrak{j}}(0, x)\right|^{2}+2 b+2(d+1) / \beta\right)\left(\lambda_{\max , \delta}+a^{-1}\right) . \tag{137}
\end{align*}
$$

Note that by the growth condition of (120), it holds that

$$
\sup _{x \in \Xi}\left|\nabla_{\bar{\theta}} \tilde{V}^{\delta, \ell, \mathfrak{j}}(0, x)\right|^{2} \leq\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\xi}+\eta_{2} \iota(0) \iota^{\prime}(0)\right)^{2}
$$

so that

$$
\begin{equation*}
\mathbb{E}_{\mathbb{P}}\left[\left|\hat{\vec{\theta}}_{n}^{\lambda, \delta,,, j}\right|^{2}\right] \leq e^{-a \lambda(n+1)} \mathbb{E}_{\mathbb{P}}\left[\left|\hat{\bar{\theta}}_{0}\right|^{2}\right]+\mathfrak{c}_{1, \delta, \beta}\left(\lambda_{\max , \delta}+a^{-1}\right), \tag{139}
\end{equation*}
$$

where

$$
\begin{align*}
\mathfrak{c}_{1, \delta, \beta} & :=2 \mathfrak{M}_{1} \lambda_{\max , \delta}+2 b+2(d+1) / \beta, \\
\mathfrak{M}_{1} & :=\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\xi}+\eta_{2} \iota(0) \iota^{\prime}(0)\right)^{2} . \tag{140}
\end{align*}
$$

Therefore, substituting (139) into (136) yields

$$
\begin{align*}
& \left|\mathbb{E}_{\mathbb{P}}\left[u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\right)\right]-\mathbb{E}_{\mathbb{P}}\left[u^{\ell, j}\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\right)\right]\right| \\
\leq & \mathbb{E}_{\mathbb{P}}\left[\left|u\left(\hat{\theta}_{n}^{\lambda, \delta, \ell, \mathrm{j}}\right)-u^{\ell, \mathrm{j}}\left(\hat{\theta}_{n}^{\lambda, \delta,, j}\right)\right|\right] \\
\leq & \frac{\sqrt{m}}{2^{\mathfrak{j}}}\left[J_{U}\left(1+M_{\Xi}\right)^{\chi}+\frac{4 p}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{p-1}\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\right)\right. \\
& \left.+\frac{2^{p+2} p M_{\Xi}}{\eta_{2}}\left(1+4 M_{\Xi}\right)^{p-1}+\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+M_{\Xi}\right)^{\nu+p-1}\right) \mathbb{E}_{\mathbb{P}}\left(\left[\left|\hat{\theta}_{n}^{\lambda, \delta, \ell, j}\right|^{2}\right]\right)^{1 / 2}\right] \\
\leq & \frac{\sqrt{m}}{2^{j}}\left[J_{U}\left(1+M_{\Xi}\right)^{\chi}+\frac{4 p}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{p-1}\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\right)\right. \\
& +\frac{2^{p+2} p M_{\Xi}}{\eta_{2}}\left(1+4 M_{\Xi}\right)^{p-1}+\left(J_{u}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{\nu+p-1}\right) \mathfrak{c}_{1, \delta, \beta}^{1 / 2}\left(\lambda_{\max , \delta}+a^{-1}\right)^{1 / 2} \\
& \left.+\left(J_{u}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{\nu+p-1}\right) e^{-a \lambda(n+1) / 2}\left(\mathbb{E}_{\mathbb{P}}\left[\left|\hat{\bar{\theta}}_{0}\right|^{2}\right]\right)^{1 / 2}\right] \\
= & \frac{\sqrt{m}\left(\tilde{C}_{4}+C_{5, \delta, \beta}+C_{6} e^{-a \lambda(n+1) / 2}\right)}{2^{j}}, \tag{141}
\end{align*}
$$

where

$$
\begin{align*}
\tilde{C}_{4} & :=J_{U}\left(1+M_{\Xi}\right)^{\chi}+\frac{4 p}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{p-1}\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\right)+\frac{2^{p+2} p M_{\Xi}}{\eta_{2}}\left(1+4 M_{\Xi}\right)^{p-1}, \\
C_{5, \delta, \beta} & :=\mathfrak{C}_{4} \mathfrak{c}_{1, \delta, \beta}^{1 / 2}\left(\lambda_{\max , \delta}+a^{-1}\right)^{1 / 2}, \\
C_{6} & :=\mathfrak{C}_{4}\left(\mathbb{E}_{\mathbb{P}}\left[\left|\hat{\theta}_{0}\right|^{2}\right]\right)^{1 / 2}, \\
\mathfrak{C}_{4} & :=\left(J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{\nu+p-1}\right), \\
\mathfrak{c}_{1, \delta, \beta} & :=2 \mathfrak{M}_{1} \lambda_{\max , \delta}+2 b+2(d+1) / \beta, \\
\mathfrak{M}_{1} & :=\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}+\eta_{2} \iota(0) \iota^{\prime}(0)\right)^{2} . \tag{142}
\end{align*}
$$

This completes the proof.

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## Appendix A. Analytic Expression of Constants

| Constant |  | Explicit Expression |
| :---: | :---: | :---: |
| Proposition 5.7 | $L_{\delta}$ | $\begin{aligned} & 2\left(1+M_{\Xi}\right)^{\nu}\left(\frac{4 K_{\nabla}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}+L_{\nabla}\right) . \\ & +\left(2^{p} L_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}+\frac{2^{p+2} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p-1} \max \left\{1, M_{\Xi}^{p}\right\} M_{\iota}\right)}{\delta}\right) \end{aligned}$ |
| Proposition 5.8 | $a$ $b$ | $\begin{aligned} & \frac{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}}{2} \\ & \eta_{2} b_{\iota}+\frac{2\left(K_{\mathrm{V}}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}^{p}\right)^{2}}{\min \left\{\eta_{1}, \eta_{2} a_{\iota}\right\}} \end{aligned}$ |
| Proposition 5.9 <br> Theorem 2.3 | $\mathfrak{C}_{1}$ | $\frac{\min \left\{a, a^{1 / 3}\right\}}{16 \sqrt{\mathbb{E}_{P}\left[\left(1+\left(1+\left\|X_{0}\right\|\right)^{2 p}\right)^{4}\right]}}$ |
|  | $\mathfrak{C}_{2}$ | $\begin{aligned} & \left(8 K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p+2} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\right)\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+\right. \\ & \left.2^{p-1} M_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}\right) \end{aligned}$ |
|  | $\mathfrak{C}_{3}$ | $2 L_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} L_{\iota} \max \left\{1, M_{\Xi}^{p}\right\}+\eta_{1}+\eta_{2} \tilde{L}_{\iota}+1$ |
|  | $\tilde{L}_{\delta}$ | $\frac{\min \left\{a, a^{1 / 3}\right\}}{16 \sqrt{\mathbb{E}_{P}\left[\left(1+\left(1+\left\|X_{0}\right\|\right)^{2 p}\right)^{4}\right]}}$ |
|  | $\lambda_{\text {max }, \delta}$ | $\min \left\{\frac{\mathfrak{c}_{1}}{L_{\delta}^{2}}, \frac{1}{a}\right\}$ |
|  | $c_{\delta, \beta}$ | See (123) and the explicit expression for $\dot{c}$ in Corollary 2.8 of [140]. |
|  | $C_{1, \delta, \ell, \mathrm{j}, \beta}$ | See (123) and the explicit expression for $C_{1}^{\#}$ in Corollary 2.8 of [140]. |
|  | $C_{2, \delta, \ell, \mathrm{j}, \beta}$ | See (123) and the explicit expression for $C_{2}^{\#}$ in Corollary 2.8 of [140]. |
|  | $C_{3, \delta, \beta}$ | See (123) and the explicit expression for $C_{3}^{\#}$ in Corollary 2.8 of [140]. |
| Proposition 5.11 <br> Theorem 2.3 | $\mathfrak{M}_{1}$ | $\left(K_{\nabla}\left(1+M_{\Xi}\right)^{\nu}+2^{p} M_{\iota} M_{\Xi}+\eta_{2} \iota(0) \iota^{\prime}(0)\right)^{2}$ |
|  | $\mathfrak{c}_{1, \delta, \beta}$ | $2 \mathfrak{M}_{1} \lambda_{\text {max }, \delta}+2 b+2(d+1) / \beta$ |
|  | $\mathfrak{C}_{4}$ | $J_{U}\left(1+2 M_{\Xi}\right)^{\chi}+\frac{8 p \tilde{K}_{\nabla}}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{\nu+p-1}$ |
|  | $\tilde{C}_{4}$ | $\begin{aligned} & J_{U}\left(1+M_{\Xi}\right)^{\chi}+\frac{4 p}{\sqrt{\eta_{2}}}\left(1+4 M_{\Xi}\right)^{p-1}\left(1+2 \tilde{K}_{\nabla}\left(1+M_{\Xi}\right)^{\nu}\right)+\frac{2^{p+2} p M_{\Xi}}{\eta_{2}}(1+ \\ & \left.4 M_{\Xi}\right)^{p-1} \end{aligned}$ |
|  | $C_{5, \delta, \beta}$ | $\mathfrak{C}_{4} \mathfrak{c}_{1, \delta, \beta}^{1 / 2}\left(\lambda_{\text {max }, \delta}+a^{-1}\right)^{1 / 2}$ |
|  | $C_{6}$ | $\mathfrak{C}_{4}\left(\mathbb{E}_{\mathbb{P}}\left[\left\|\hat{\hat{\theta}_{0}}\right\|^{2}\right]\right)^{1 / 2}$ |

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[^0]:    Key words and phrases. Stochastic Gradient Langevin Dynamics (SGLD), Distributionally Robust Optimisation (DRO), algorithms for stochastic optimisation, expected excess risk, (robust) Mean-CVaR portfolio optimisation, data-driven optimisation.

[^1]:    ${ }^{1}$ We also refer to [25], [55], [92], and [142] for similar duality results.

