# SENSITIVITY OF ROBUST OPTIMIZATION PROBLEMS UNDER DRIFT AND VOLATILITY UNCERTAINTY 

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#### Abstract

We examine optimization problems in which an investor has the opportunity to trade in $d$ stocks with the goal of maximizing her worst-case cost of cumulative gains and losses. Here, worst-case refers to taking into account all possible drift and volatility processes for the stocks that fall within a $\varepsilon$-neighborhood of predefined fixed baseline processes. Although solving the worst-case problem for a fixed $\varepsilon>0$ is known to be very challenging in general, we show that it can be approximated as $\varepsilon \rightarrow 0$ by the baseline problem (computed using the baseline processes) in the following sense: Firstly, the value of the worst-case problem is equal to the value of the baseline problem plus $\varepsilon$ times a correction term. This correction term can be computed explicitly and quantifies how sensitive a given optimization problem is to model uncertainty. Moreover, approximately optimal trading strategies for the worst-case problem can be obtained using optimal strategies from the corresponding baseline problem.


Key words. sensitivity analysis, robust stochastic optimization, model uncertainty, Itôsemimartingale, backward stochastic differential equations
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## 1. Introduction

Consider a semimartingale $S=\left(S_{t}\right)_{t \in[0, T]}$ representing the evolution of the value of a $d$ dimensional (discounted) stock price over time. We assume that a decision maker holds a financial position of the form $h\left(S_{T}\right)$ and aims to hedge against possible losses. To that end, she starts with an initial capital $x_{0} \in \mathbb{R}$ and has the opportunity to buy and sell the stock $S$ without transaction costs. If she invests according to the trading strategy $H$ (i.e., a predictable process), her capital at the terminal time $T$ equals her initial capital $x_{0}$ plus the cumulated sums of gains and losses from trading, i.e. the stochastic integral $(H \cdot S)_{T}:=\int_{0}^{T} H_{t}^{\top} d S_{t}$. The central objective the decision maker faces is to solve the following optimization problem

$$
\begin{equation*}
\inf _{H} \mathbb{E}\left[f\left(x_{0}+(H \cdot S)_{T}, h\left(S_{T}\right)\right)\right] \tag{1.1}
\end{equation*}
$$

where the infimum is taken over a suitable class of trading strategies $H$ and $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is the individual cost function of the decision maker. Notable examples that fall within this framework are utility maximization (e.g., $[17,18,38,44,48]$ ) in which case $f(x, h)=-U(-x-h)$ for a concave and increasing utility function $U: \mathbb{R} \rightarrow \mathbb{R}$, and mean-variance hedging (e.g., $[25,77,83]$ ) in which case $f(x, h)=(x-L-h)^{2}$ for a fixed target level $L>0$.

[^0]Arguably, one of the most common models for the underlying $S$ is that it follows the dynamics

$$
\begin{equation*}
d S_{t}:=d S_{t}^{b, \sigma}=b_{t} d t+\sigma_{t} d W_{t}, \quad S_{0}^{b, \sigma}=s_{0} \in \mathbb{R}^{d} \tag{1.2}
\end{equation*}
$$

where $W$ is a Brownian motion and the (possibly random and time-dependent) drift and volatility are given by the predictable processes $b$ and $\sigma$, respectively. We refer to [25, 36, 37, 44, 83] for a handful of the many articles analyzing problems similar to (1.1) in the setting (1.2).

A significant challenge arises when implementing this framework in practice: the true values of the parameters $b$ and $\sigma$ used to define $S$ in (1.2) are not perfectly known. Instead, they are typically estimated using e.g. historical market data and expert insights. Even though these estimation techniques strive to provide parameter values that closely approximate their actual counterparts, a margin for potential inaccuracies inherently exists. While this issue has always been present, it gained particularly prominent attention following the financial crisis in 2008. Since then, a substantial body of research in mathematical finance has been dedicated to developing methods that can accommodate potential model misspecification.

The most prominent approach can be traced back to the seminal papers $[15,24,30]$ and is widely recognized as the 'worst-case' approach, or Knightian approach to model uncertainty. In our current context, it consists in fixing an entire set $\mathcal{U}$ of parameters $(b, \sigma)$ that one would consider reasonable candidates for representing the actual drift and volatility. Subsequently, the objective is to address the worst-case optimization problem given by

$$
\begin{equation*}
\inf _{H} \sup _{(b, \sigma) \in \mathcal{U}} \mathbb{E}\left[f\left(x_{0}+\left(H \cdot S^{b, \sigma}\right)_{T}, h\left(S_{T}^{b, \sigma}\right)\right)\right] \tag{1.3}
\end{equation*}
$$

It is evident that (1.3) and (1.1) coincide when the set $\mathcal{U}$ is a singleton; however, in general, there exists a significant degree of latitude in selecting $\mathcal{U}$. One seemingly intuitive approach consists in starting with parameters $\left(b^{o}, \sigma^{o}\right)$ that are derived from some estimation procedure which one would typically employ in the non-robust problem (1.1), and then defining $\mathcal{U}$ by adding all small perturbations of these parameters, effectively creating a small neighborhood around $\left(b^{o}, \sigma^{o}\right)$.

Before examining that specific choice of $\mathcal{U}$ in more details, we note that problems of the form in (1.3) have conceived a considerable amount of attention in the mathematical finance community. In fact, most of the fundamental and often technically demanding mathematical questions therein are understood fairly well by now and genuinely have affirmative answers. We refer e.g. to $[10,22,61,80,82]$ for the existence of an optimal strategy of (1.3); to [11, 27,53,74] for an analysis of the dynamic programming principle; to $[3,23,33,63,66,76]$ for the relation of the present hedging problem to a dual pricing problem; to [21,59, 60, $62,69,70]$ for modeling uncertainty in semimartingale characteristics; to [ $5,7,8,67,68$ ] for extensions of (1.3) with optimal stopping time; and to $[16,55,78,79]$ for relations to second-order backward stochastic differential equations.

That being said, it is crucial to highlight that the mere existence of an optimal strategy, while theoretically intriguing, may not be particularly practical. Indeed, the primary interest of a decision maker often lies rather in finding methods to compute it. However, this is precisely where the worst-case approach encounters a notable limitation: The computation of both the value and the optimal strategy in (1.3) is notoriously difficult and except in a few exceptional cases (see $[10,27,53,61,66]$ ), explicit solutions are unknown.

The goal of this article is to overcome this limitation and show that in the setting where $\mathcal{U}$ is a small neighborhood of fixed parameters $\left(b^{o}, \sigma^{o}\right)$, the following hold.

- The value of the robust optimization problem (1.3) can be approximated accurately by (1.1).
- An almost optimal strategy for (1.3) can be derived using an optimal strategy for (1.1).
- Each optimization problem (1.1) has an associated number that quantifies how sensitive it is towards model uncertainty, i.e. by how much larger (1.3) is than (1.1). Moreover, that number can be computed explicitly.

Roughly put, the notion of sensitivity can be thought of as a robust variant of the so-called 'Vega' parameter in the Black-Scholes framework. For example, if the optimal strategy for the non-robust problem (1.1) computed with parameters ( $b^{o}, \sigma^{o}$ ) is 'greedy', even slight parameter variations can have a significant impact, leading to substantial differences between (1.1) and (1.3).

We proceed to describe our results more rigorously. For $p \geq 1$, denote by $\mathbb{L}^{p}$ and $\mathbb{H}^{p}$ the set of all predictable processes $Z$ with values in $\mathbb{R}^{d}$ and $\mathbb{R}^{d \times d}$, respectively. We endow $\mathbb{L}^{p}$ with the norm $\|Z\|_{\mathbb{L}^{p}}^{p}:=\mathbb{E}\left[\int_{0}^{T}\left|Z_{t}\right|^{p} d t\right]$ where $|\cdot|$ is the Euclidean norm, and $\mathbb{H}^{p}$ with the norm $\|Z\|_{\mathbb{H} p}^{p}:=\mathbb{E}\left[\left(\int_{0}^{T}\left\|Z_{t}\right\|_{\mathrm{F}}^{2} d t\right)^{p / 2}\right]$ where $\|\cdot\|_{\mathrm{F}}$ is the Frobenius / Hilbert-Schmidt norm.

Let $p>3$ and set $\gamma, \eta \geq 0$. Fix baseline parameters $b^{o} \in \mathbb{L}^{p}$ and $\sigma^{o} \in \mathbb{H}^{p}$, e.g. the 'estimators', and for $\varepsilon \geq 0$ denote by

$$
\begin{equation*}
\mathcal{B}^{\varepsilon}:=\left\{(b, \sigma) \in \mathbb{L}^{p} \times \mathbb{H}^{p}:\left\|b-b^{o}\right\|_{\mathbb{L}^{p}} \leq \gamma \varepsilon \text { and }\left\|\sigma-\sigma^{o}\right\|_{\mathbb{H}^{p}} \leq \eta \varepsilon\right\} \tag{1.4}
\end{equation*}
$$

the set of all parameters that fall in the $\varepsilon$-neighborhood of the baseline parameters, weighted by the 'aversion parameters' $\gamma, \eta$. We typically think of $\gamma=\eta=1$ (corresponding to drift and volatility uncertainty) or $\eta=0$ (drift uncertainty) or $\gamma=0$ (volatility uncertainty).

Consider the robust optimization problem (1.3) with the choice $\mathcal{U}=\mathcal{B}^{\varepsilon}$, that is,

$$
\begin{equation*}
V(\varepsilon):=\inf _{H} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[f\left(x_{0}+\left(H \cdot S^{b, \sigma}\right)_{T}, h\left(S_{T}^{b, \sigma}\right)\right)\right] \tag{1.5}
\end{equation*}
$$

where the infimum is taken over all predictable trading strategies $H$ taking values in some fixed convex compact subset of $\mathbb{R}^{d}$. The latter is a technical condition that is not too restrictive, see also Remark 2.6. In particular, $V(0)$ is the non-robust optimization problem computed using the baseline parameters $b^{o}$ and $\sigma^{o}$.

In Theorem 2.13 we show that if $f$ is strictly convex in its first argument, $f, h$ are twice continuously differentiable, $\sigma^{o}$ is non-degenerate, and $f, h, b^{o}, \sigma^{o}$ satisfy modest growth assumptions, then the following hold. As $\varepsilon \downarrow 0$,

$$
\begin{equation*}
V(\varepsilon)=V(0)+\varepsilon\left(\gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}\right)+O\left(\varepsilon^{2}\right) \tag{1.6}
\end{equation*}
$$

where $O$ denotes the Landau symbol, $H^{*}$ is the unique optimizer for $V(0)$, and $q=\frac{p}{p-1}$ is the conjugate Hölder exponent to $p$. Moreover, the processes $Y^{*}, \mathcal{Y}^{*}, Z^{*}, \mathcal{Z}^{*}$ appearing in (1.6) take the following form: set $S^{o}=S^{b^{o}, \sigma^{o}}$ to be stock following the baseline parameters and for
simplicity here in the introduction let $d=1$. Then

$$
\begin{aligned}
Y_{t}^{*} & =\mathbb{E}\left[\partial_{x} f\left(x_{0}+\left(H^{*} \cdot S^{o}\right)_{T}, h\left(S_{T}^{o}\right)\right) \mid \mathcal{F}_{t}\right] \\
\mathcal{Y}_{t}^{*} & =\mathbb{E}\left[\partial_{y} f\left(x_{0}+\left(H^{*} \cdot S^{o}\right)_{T}, h\left(S_{T}^{o}\right)\right) h^{\prime}\left(S_{T}^{o}\right) \mid \mathcal{F}_{t}\right] \\
Z_{t}^{*} & =\frac{d}{d t}\left\langle Y^{*}, W\right\rangle_{t}, \quad \text { and } \quad \mathcal{Z}_{t}^{*}=\frac{d}{d t}\left\langle\mathcal{Y}^{*}, W\right\rangle_{t},
\end{aligned}
$$

where $\langle M, N\rangle$ denotes their predictable covariation between two martingales $M$ and $N$.
Next, imposing the same assumption made before on $f, h, b^{o}, \sigma^{o}$, recall that $H^{*}$ is the unique optimizer for the non-robust problem $V(0)$, and set

$$
V^{*}(\varepsilon)=\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[f\left(x_{0}+\left(H^{*} \cdot S^{b, \sigma}\right)_{T}, h\left(S_{T}^{b, \sigma}\right)\right)\right]
$$

In other words, $V^{*}(\varepsilon)$ tracks how well the optimal strategy computed for the baseline parameters $\left(b^{o}, \sigma^{o}\right)$ performs in the worst case when nature is allowed to maliciously select parameter variations $(b, \sigma) \in \mathcal{B}^{\varepsilon}$. Hence one might actually argue that the quantity $V^{*}(\varepsilon)$ is practically more relevant than $V(\varepsilon)$. In Theorem 2.14 we show that as $\varepsilon \downarrow 0$,

$$
V^{*}(\varepsilon)=V(\varepsilon)+O\left(\varepsilon^{2}\right)
$$

In particular, $\left(V^{*}\right)^{\prime}(0)=V^{\prime}(0)$ and there is no need for the challenging computation of the robust strategy: up to a second order correction term, the unique optimal strategy $H^{*}$ computed for $\left(b^{o}, \sigma^{o}\right)$ will perform just as good as the strategy computed for the robust optimization problem $V(\varepsilon)$.

Related literature. The results that are closest to the present ones were obtained in the context of Wasserstein distributionally robust optimization, i.e. when $\mathcal{B}^{\varepsilon}$ consists in all probability laws (for $S^{b, \sigma}$ ) which are close in Wasserstein distance to the distribution of $S^{o}$. Starting with $[12,29,56,71]$, this branch of research has gained a significant amount of attention and in a one-period framework, a sensitivity analysis similar to ours was obtained in [2, 28, 58, 64]. In a multi-period framework the classical Wasserstein distance is not a suitable distance because it neglects the temporal structure of stochastic processes, see e.g. [1, 72, 73]. Instead, an adapted variant takes its role, and a sensitivity analysis similar to ours (but w.r.t the adapted Wasserstein distance) was established in $[4,42]$ in a discrete-time setting. It is important to highlight that although certain proof techniques in the present article bear similarities to those in the above settings, there are significant distinctions. Most notably, our choice of $\mathcal{B}^{\varepsilon}$ is far more rigid than its Wasserstein counterpart. Consequently, we cannot directly apply the duality between $L^{p}$ and $L^{q}$ spaces to deduce the value of $V^{\prime}(0)$ (e.g. as in $[2,4]$ ); instead, a substantial portion of our efforts revolves around establishing a suitable representation involving BSDE's which allow to express $V^{\prime}(0)$ as the supremum over certain linear functions.

In a continuous-time time framework, $[34,35]$ analyze the sensitivity of utility maximization problems to volatility-uncertainty in a somewhat different setting to ours. Roughly put, instead of taking the supremum only over those processes $\sigma$ that satisfy $\left\|\sigma-\sigma^{o}\right\|_{\mathbb{H}^{p}} \leq \varepsilon$, they allow for all $\sigma$ 's and penalize by $\frac{1}{\varepsilon}$ multiplied by 'far' $\sigma$ is from $\sigma^{o}$. The notion of being far takes a form similar to a KL-divergence but, crucially, with the specific choice of the utility function (i.e. $f$ in the present setting) appearing in its definition. In particular, the sensitivity to model uncertainty (corresponding to $V^{\prime}(0)$ in the present setting) obtained in [34,35] does not depend on the choice of the utility function nor on the optimal trading strategy $H^{*}$.

For completeness, let us also mention that analyzing the (quantitative) continuity of nonrobust optimization problems w.r.t. changes in the model is a classical topic. We refer e.g. to $[13,14,51,81]$ and the references therein for some general background, and to $[6,26,52,57]$ for works focusing on problems in mathematical finance.

## 2. Main ReSUlTS

2.1. Notation and preliminaries. Fix $d \in \mathbb{N}$. We endow $\mathbb{R}^{d}$ and $\mathbb{R}^{d \times d}$ with the Euclidean inner product $\langle\cdot, \cdot\rangle$ and the Frobenius inner product $\langle\cdot, \cdot\rangle_{\mathrm{F}}$, respectively. Let $\mathbb{S}_{+}^{d}$ be the set of all symmetric, positive semi-definite $d \times d$ matrices and let $\mathbb{S}_{++}^{d} \subset \mathbb{S}_{+}^{d}$ be the subset of strictly positive definite matrices. We denote by $I_{d}$ the identity matrix on $\mathbb{R}^{d \times d}$ and for two $d \times d$ matrices $A$ and $B$, we write $A \leq B$ if $B-A \in \mathbb{S}_{+}^{d}$.

Next, let us introduce the following function spaces:

- $C\left([0, T] ; \mathbb{R}^{d}\right)\left(\right.$ resp. $\left.C\left([0, T] ; \mathbb{R}^{d \times d}\right)\right)$ is the set of all $\mathbb{R}^{d}$-valued (resp. $\mathbb{R}^{d \times d}$-valued), continuous functions on $[0, T]$;
$C^{k}\left(\mathbb{R}^{d}\right)$ is the set of all real-valued, $k$-times continuously differentiable functions on $\mathbb{R}^{d}$. We denote by $\nabla g=\left(\partial_{s_{1}} g, \ldots, \partial_{s_{d}} g\right)^{\top}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ the gradient of $g$ and by $D^{2} g: \mathbb{R}^{d} \rightarrow \mathbb{S}^{d}$ its Hessian.

Let $\left(\Omega, \mathcal{F}, \mathbb{F}:=\left(\mathcal{F}_{t}\right)_{t \in[0, T]}, \mathbb{P}\right)$ be a filtered probability space that satisfies the usual conditions of right-continuity and completeness, where $T$ is a fixed finite time horizon. We assume that $\mathcal{F}_{0}$ is trivial. Fix a d-dimensional Brownian motion $W=\left(W_{t}\right)_{t \in[0, T]}$ on that filtered probability space. For any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we write $\mathbb{E}^{\mathbb{Q}}[\cdot]$ for the expectation under $\mathbb{Q}$ and set $\mathbb{E}[\cdot]:=\mathbb{E}^{\mathbb{P}}[\cdot]$. For sufficiently integrable $\mathbb{R}^{d}$-valued processes $Y$ and $Z$, let

$$
\langle Y, Z\rangle_{\mathbb{Q} \otimes d t}:=\mathbb{E}^{\mathbb{Q}}\left[\int_{0}^{T}\left\langle Y_{t}, Z_{t}\right\rangle d t\right] .
$$

In a similar manner, $\langle Y, Z\rangle_{\mathbb{Q} \otimes d t, \mathrm{~F}}:=\mathbb{E}^{\mathbb{Q}}\left[\int_{0}^{T}\left\langle Y_{t}, Z_{t}\right\rangle_{\mathrm{F}} d t\right]$ for $\mathbb{R}^{d \times d}$-valued processes. We denote by $\xrightarrow{\mathbb{Q}}$ the convergence in $\mathbb{Q}$-measure and adopt a similar notation for convergence in $\mathbb{Q} \otimes d t$ measure. Finally, for $p \geq 1$, denote by $C_{\mathrm{BDG}, p} \geq 1$ the constant appearing in the (upper) Burkholder-Davis-Gundy (BDG) inequality with exponent $p$ (see [20, Theorem 92, Chap. VII]).

For any real-valued semimartingales $M=\left(M_{t}\right)_{t \in[0, T]}$ and $N=(N)_{t \in[0, T]}$ with locally integrable quadratic variation, we denote by $\left([M, N]_{t}\right)_{t \in[0, T]}$ the quadratic co-variation and by $\left(\langle M, N\rangle_{t}\right)_{t \in[0, T]}$ the $\mathbb{F}$-predictable quadratic co-variation (i.e., the compensator of $\left.\left([M, N]_{t}\right)_{t \in[0, T]}\right)$. Note that by [41, Theorem 4.52, p. 55], for any $t \in[0, T]$, we have that $[M, M]_{t}:=\left\langle M^{c}, M^{c}\right\rangle_{t}+$ $\sum_{0 \leq s \leq t}\left|\Delta M_{s}\right|^{2}$ where $M^{c}$ is the continuous martingale part of $M$ and $\Delta M_{t}:=M_{t}-M_{t-}$.

Finally, for any $p \geq 1$, consider the following spaces.

- $L^{p}\left(\mathcal{F}_{T} ; \mathbb{R}^{d}\right)$ is the set of all $\mathbb{R}^{d}$-valued, $\mathcal{F}_{T}$-measurable random variables $X$ such that $\|X\|_{L^{p}}^{p}:=\mathbb{E}\left[|X|^{p}\right]<\infty$;
- $\mathscr{S}^{p}(\mathbb{R})$ is the set of all real-valued, $\mathbb{F}$-progressively measurable càdlàg (i.e., right-continuous with left-limits) processes $S=\left(S_{t}\right)_{t \in[0, T]}$ such that $\|S\|_{\mathscr{S}_{p}}^{p}:=\mathbb{E}\left[\sup _{t \in[0, T]}\left|S_{t}\right|^{p}\right]<\infty$;
$\mathscr{H}^{p}\left(\mathbb{R}^{d}\right)$ is the set of all $\mathbb{R}^{d}$-valued, $\mathbb{F}$-predictable processes $H=\left(H_{t}\right)_{t \in[0, T]}$ such that $\|H\|_{\mathscr{H}^{p}}^{p}:=\mathbb{E}\left[\left(\int_{0}^{T}\left|H_{t}\right|^{2} d t\right)^{p / 2}\right]<\infty ;$
- $\mathscr{M}^{p}(\mathbb{R})$ is the set of all real-valued, càdlàg $(\mathbb{F}, \mathbb{P})$-martingales $M=\left(M_{t}\right)_{t \in[0, T]}$ such that $\|M\|_{\mathscr{M}^{p}}^{p}:=\mathbb{E}\left[[M, M]_{T}^{p / 2}\right]<\infty ;$
$\mathbb{L}^{p}=\mathbb{L}^{p}\left(\mathbb{R}^{d}\right)$ and $\mathbb{H}^{p}=\mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right)$ are defined in the introduction.
2.2. The market model(s). For every (sufficiently integrable) predictable processes $b$ and $\sigma$ taking values in $\mathbb{R}^{d}$ and $\mathbb{R}^{d \times d}$ respectively, we define the Itô $(\mathbb{F}, \mathbb{P})$-semimartingale $S^{b, \sigma}$ by

$$
\begin{equation*}
S_{t}^{b, \sigma}:=s_{0}+\int_{0}^{t} b_{u} d u+\int_{0}^{t} \sigma_{u} d W_{u}, \quad t \in[0, T] \tag{2.1}
\end{equation*}
$$

where $s_{0} \in \mathbb{R}^{d}$ is fixed and does not depend on $b$ and $\sigma$, and we recall that $W=\left(W_{t}\right)_{t \in[0, T]}$ is a fixed $d$-dimensional Brownian motion. Moreover, recall that $b^{o}$ and $\sigma^{o}$ are two fixed baseline processes, that $\gamma, \eta \geq 0$ are fixed constants, and that

$$
\begin{equation*}
\mathcal{B}^{\varepsilon}=\left\{(b, \sigma) \in \mathbb{L}^{p}\left(\mathbb{R}^{d}\right) \times \mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right):\left\|b-b^{o}\right\|_{\mathbb{L}^{p}} \leq \gamma \varepsilon,\left\|\sigma-\sigma^{o}\right\|_{\mathbb{H}^{p}} \leq \eta \varepsilon\right\} \tag{2.2}
\end{equation*}
$$

The exact value of $p$ is specified in Assumption 2.1. For shorthand notation, set $S^{o}:=S^{b^{o}, \sigma^{o}}$.
Assumption 2.1. The following conditions hold:
(i) $b^{o} \in \mathbb{L}^{p}\left(\mathbb{R}^{d}\right)$ and $\sigma^{o} \in \mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right)$ for some $p>3$.
(ii) $\sigma^{o}$ is invertible, i.e., there is $\left(\sigma^{o}\right)^{-1}$ such that $\sigma_{t}^{o}\left(\sigma_{t}^{o}\right)^{-1}=I_{d} \mathbb{P} \otimes d t$-a.e.. Furthermore, the process $\mathcal{D}=\left(\mathcal{D}_{t}\right)_{t \in[0, T]}$ defined by

$$
\mathcal{D}_{t}:=\mathcal{E}\left(-\left(\left(\sigma^{o}\right)^{-1} b^{o}\right) \cdot W\right)_{t}=\exp \left(-\frac{1}{2} \int_{0}^{t}\left|\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o}\right|^{2} d u-\int_{0}^{t}\left(\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o}\right)^{\top} d W_{u}\right)
$$

is a $(\mathbb{F}, \mathbb{P})$-martingale satisfying $\mathcal{D}_{T} \in L^{\beta}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$ for every $\beta \geq 1$.
Remark 2.2. Assumption 2.1 (i) implies that for every $\varepsilon \geq 0, \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[\sup _{t \in[0, T]}\left|S_{t}^{b, \sigma}\right|^{p}\right]<$ $\infty$. In addition, Assumption 2.1 (ii) implies that the measure $\mathbb{Q}$ defined by

$$
\begin{equation*}
\left.\frac{d \mathbb{Q}}{d \mathbb{P}}\right|_{\mathcal{F}_{T}}:=\mathcal{D}_{T} \tag{2.3}
\end{equation*}
$$

is a probability measure equivalent to $\mathbb{P}$, and that $W^{\mathbb{Q}}:=W+\int_{0}^{\sim}\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o} d u$ is a $d$-dimensional Brownian motion under $\mathbb{Q}$ (by Girsanov's theorem). These observations will be used later.

Let us provide sufficient conditions for Assumption 2.1 to hold true. The proofs can be found in Section 4.

Lemma 2.3. Suppose that $\sigma^{o}$ is invertible. Then each of the following conditions is sufficient for Assumption 2.1 to hold:
(i) $b^{o}$ and $\sigma^{o}$ are uniformly bounded and $\left(\sigma^{o}\right)^{\top} \sigma^{o}$ satisfies a uniform ellipticity condition, i.e., there are $C_{b, \sigma}>0$ and $\underline{c} \in \mathbb{S}_{++}^{d}$ s.t. $\left|b_{t}^{o}\right|+\left\|\sigma_{t}^{o}\right\|_{\mathrm{F}} \leq C_{b, \sigma}$ and $\left(\sigma_{t}^{o}\right)^{\top} \sigma_{t}^{o} \geq \underline{c} \mathbb{P} \otimes d t$-a.e.
(ii) $b^{o}$ and $\sigma^{o}$ are of the following form:

- $b_{t}^{o}=\widetilde{b}^{o}\left(t, S_{t}^{o}\right), \sigma_{t}^{o}=\widetilde{\sigma}^{o}\left(t, S_{t}^{o}\right) \mathbb{P} \otimes d t$-a.e. $(\mathrm{SDE})$,
- $\left(\sigma_{t}^{o}\right)^{-1} b_{t}^{o}=\theta(t, W) \mathbb{P} \otimes d t$-a.e. (Beneš condition),
where $\widetilde{b}^{o}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\widetilde{\sigma}^{o}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ are Borel functions, there is $C_{\widetilde{b}, \widetilde{\sigma}}>0$ such that for every $t \in[0, T]$ and $x, y \in \mathbb{R}^{d}$

$$
\begin{aligned}
\left|\widetilde{b}^{o}(t, x)-\widetilde{b}^{o}(t, y)\right|+\left\|\widetilde{\sigma}^{o}(t, x)-\widetilde{\sigma}^{o}(t, y)\right\|_{\mathrm{F}} & \leq C_{\widetilde{b}, \widetilde{\sigma}}|x-y| \\
\left|\widetilde{b}^{o}(t, x)\right|+\left\|\widetilde{\sigma}^{o}(t, x)\right\|_{\mathrm{F}} & \leq C_{\widetilde{b}, \widetilde{\sigma}}(1+|x|)
\end{aligned}
$$

$\theta:[0, T] \times C\left([0, T] ; \mathbb{R}^{d}\right) \rightarrow \mathbb{R}^{d}$ is progressively measurable ${ }^{1}$, and there is $C_{\theta}>0$ s.t. for all $t \in[0, T]$ and $x=\left(x_{t}\right)_{t \in[0, T]} \in C\left([0, T] ; \mathbb{R}^{d}\right),|\theta(t, x)| \leq C_{\theta}\left(1+\sup _{s \in[0, t]}\left|x_{s}\right|\right)$.
Remark 2.4. In Lemma 2.3, the uniform boundedness condition of ( $b^{o}, \sigma^{o}$ ) and the SDEs with Lipschitz and linear growth conditions are commonly adopted in continuous-time robust optimization problems with model uncertainty (see, e.g., [3, 11, 27, 66, 67]). Moreover, the uniform ellipticity condition of $\sigma^{o}$ and the Beneš condition fit into classical utility maximization problems with a dual/martingale approach (see, e.g., [43, 46, 50]).

Having completed the description of the underlying processes, we can proceed to describe the decision maker's model.

Definition 2.5. Fix a convex, compact set $\mathcal{K} \subset \mathbb{R}^{d}$ and put $K:=\max _{x \in \mathcal{K}}|x|<\infty$. The set of admissible controls/strategies is given by

$$
\mathcal{A}:=\left\{H \mid \mathbb{R}^{d} \text {-valued, } \mathbb{F} \text {-predictable processes } H=\left(H_{t}\right)_{t \in[0, T]} \text { s.t. } H_{t} \in \mathcal{K} \mathbb{P} \otimes d t \text {-a.e. }\right\}
$$

Remark 2.6. The condition that $H_{t}$ takes its values in $\mathcal{K}$ has certain technical advantages in the proofs of our main results, and it appears in several works studying related topics $[4,17,50]$. While it is conceivable to relax this assumption (e.g., by relying on boundedness in $\mathbb{L}^{\beta}$ for suitable $\beta$ ), doing so would introduce greater notational and technical complexity and we believe that the added complexity would not provide a commensurate benefit.

Fix $x_{0} \in \mathbb{R}$, the initial capital of the market participant. For $(b, \sigma) \in \mathcal{B}^{\varepsilon}$ and $H \in \mathcal{A}$, the market participant's controlled process $X^{H ; b, \sigma}=\left(X_{t}^{H ; b, \sigma}\right)_{t \in[0, T]}$ is given by the stochastic integral

$$
\begin{equation*}
X_{t}^{H ; b, \sigma}:=x_{0}+\left(H \cdot S^{b, \sigma}\right)_{t}=x_{0}+\int_{0}^{t} H_{u}^{\top} b_{u} d u+\int_{0}^{t} H_{u}^{\top} \sigma_{u} d W_{u}, \quad t \in[0, T] . \tag{2.4}
\end{equation*}
$$

For shorthand notation, given $H \in \mathcal{A}$, denote by $X^{H ; o}$ the controlled process under the postulated Itô semimartingale $S^{o}$, i.e., $X_{t}^{H ; o}:=x_{0}+\left(H \cdot S^{o}\right)_{t}$.

Assume that the decision maker holds a financial position $h\left(S_{T}^{b, \sigma}\right)$ where $h: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is a given function. Moreover, her cost function (representing her individual preferences) is given by $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$, and she aims to solve the robust optimization problem

$$
V(\varepsilon):=\inf _{H \in \mathcal{A}} \mathcal{V}(H, \varepsilon):=\inf _{H \in \mathcal{A}} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[f\left(X_{T}^{H ; b, \sigma}, h\left(S_{T}^{b, \sigma}\right)\right)\right]
$$

We impose the following conditions on the functions $f$ and $h$.
Assumption 2.7. The function $f: \mathbb{R}^{2} \ni(x, y)^{\top} \rightarrow f(x, y) \in \mathbb{R}$ satisfies the following:
(i) $f \in C^{2}\left(\mathbb{R}^{2}\right)$;
(ii) there is $0<r<\min \left\{\frac{p-2}{2}, p-3\right\}$ and $C_{f}>0$ such that $\left\|D^{2} f(x, y)\right\|_{\mathrm{F}} \leq C_{f}\left(1+\left|(x, y)^{\top}\right|^{r}\right)$ for every $(x, y)^{\top} \in \mathbb{R}^{2}$, where $p>3$ is defined in Assumption 2.1 (i);
(iii) there is $C_{l, 0} \in \mathbb{R}$ such that $f \geq C_{l, 0}$;
(iv) there is $C_{l, 2}>0$ such that $\partial_{x x} f \geq C_{l, 2}$ (i.e. $f$ is uniformly strongly convex in its first argument).

[^1]The function $h: \mathbb{R}^{d} \rightarrow \mathbb{R}$ satisfies the following:
(v) $h \in C^{2}\left(\mathbb{R}^{d}\right)$;
(vi) all first order and second order derivatives of $h$ are Lipschitz continuous and bounded.

Remark 2.8. If Assumption 2.7 is satisfied, a straightforward application of the fundamental theorem of calculus implies that the following growth conditions hold (which we shall use often in what follows).
(i) There is $\widetilde{C}_{f}>0$ such that for every $(x, y)^{\top} \in \mathbb{R}^{2}$,

$$
|f(x, y)| \leq \widetilde{C}_{f}\left(1+|x|^{r+2}+|y|^{r+2}\right) \quad \text { and } \quad|\nabla f(x, y)| \leq \widetilde{C}_{f}\left(1+|x|^{r+1}+|y|^{r+1}\right)
$$

(ii) There is $C_{h}>0$ such that for every $s, \hat{s} \in \mathbb{R}^{d}$,

$$
\begin{aligned}
& |h(s)| \leq C_{h}(1+|s|) \quad \text { and } \quad|\nabla h(s)|+\left\|D^{2} h(s)\right\|_{\mathrm{F}} \leq C_{h}, \\
& |h(s)-h(\hat{s})|+|\nabla h(s)-\nabla h(\hat{s})|+\left\|D^{2} h(s)-D^{2} h(\hat{s})\right\|_{\mathrm{F}} \leq C_{h}|s-\hat{s}| .
\end{aligned}
$$

2.3. Optimization for the baseline parameters and BSDEs. In this section we collect some preliminary results, including the existence of an optimal strategy $H^{*} \in \mathcal{A}$. In particular, it turns out to be convenient to characterize the processes $Y, \mathcal{Y}, Z, \mathcal{Z}$ that appear in the formula for $V^{\prime}(0)$ (see (1.6)) using an auxiliary BSDE formulation.

Proposition 2.9. Suppose that Assumptions 2.1 and 2.7 are satisfied. Then the following hold.
(i) There exists a unique optimizer $H^{*} \in \mathcal{A}$ satisfying

$$
V(0)=\mathcal{V}\left(H^{*}, 0\right)=\mathbb{E}\left[f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\right]
$$

(ii) There exists a unique solution $\left(Y^{*}, Z^{*}, L^{*}\right) \in \mathscr{S}^{2}(\mathbb{R}) \times \mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})$ of

$$
\begin{equation*}
Y_{t}^{*}=\partial_{x} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)-\int_{t}^{T}\left(Z_{u}^{*}\right)^{\top} d W_{u}-\left(L_{T}^{*}-L_{t}^{*}\right), \quad t \in[0, T] \tag{2.5}
\end{equation*}
$$

with $L_{0}^{*}=0$, where $L^{*}$ and $\left(\int_{0}^{t}\left(Z_{s}^{*}\right)^{\top} d W_{s}\right)_{t \in[0, T]} \in \mathscr{M}^{2}(\mathbb{R})$ are strongly orthogonal ${ }^{2}$.
(iii) There exists a unique solution $\left(\mathcal{Y}^{*}, \mathcal{Z}^{*}, \mathcal{L}^{*}\right) \in\left(\mathscr{S}^{2}(\mathbb{R})\right)^{d} \times\left(\mathscr{H}^{2}\left(\mathbb{R}^{d}\right)\right)^{d} \times\left(\mathscr{M}^{2}(\mathbb{R})\right)^{d}$ of

$$
\begin{equation*}
\mathcal{Y}_{t}^{*}=\partial_{y} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right)-\int_{t}^{T} \mathcal{Z}_{u}^{*} d W_{u}-\left(\mathcal{L}_{T}^{*}-\mathcal{L}_{t}^{*}\right), \quad t \in[0, T] \tag{2.6}
\end{equation*}
$$

with $\mathcal{L}_{0}^{*}=0$, where $\mathcal{L}^{*}$ and $\left(\int_{0}^{t} \mathcal{Z}_{s}^{*} d W_{s}\right)_{t \in[0, T]}$ are strongly orthogonal. ${ }^{3}$
Remark 2.10. In the context of solutions to BSDE's, throughout this article we denote the solutions to multi-dimensional BSDE's as in (2.6) by calligraphic letters. In particular, (2.6) is to be understood in the following sense: $\mathcal{Y}^{*}=\left(\mathcal{Y}^{*, 1}, \ldots, \mathcal{Y}^{*, d}\right)^{\top}, \mathcal{Z}^{*}=\left(\mathcal{Z}^{*, 1}, \ldots, \mathcal{Z}^{*, d}\right)^{\top}$, and $\mathcal{L}^{*}=\left(\mathcal{L}^{*, 1}, \ldots, \mathcal{L}^{*, d}\right)^{\top}$ are vectors consisting of processes, and for every $1 \leq i \leq d$,

$$
\mathcal{Y}_{t}^{*, i}=\partial_{y} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right) \partial_{s_{i}} h\left(S_{T}^{o}\right)-\int_{t}^{T}\left(\mathcal{Z}_{u}^{*, i}\right)^{\top} d W_{u}-\left(\mathcal{L}_{T}^{*, i}-\mathcal{L}_{t}^{*, i}\right), \quad t \in[0, T]
$$

[^2]Let us note that the proof of Proposition 2.9 is relatively standard: part (i) follows from a Komlós-type argument and parts (ii) and (iii) follow from employing the Galtchouk-KunitaWatanabe (GKW) decomposition. The complete proof can be found in Section 3.3.

Before we proceed to state our main result, let us briefly comment on some properties of the BSDE-solution in Proposition 2.9.

Lemma 2.11. Suppose that Assumptions 2.1 and 2.7 are satisfied and let $\left(Y^{*}, Z^{*}, L^{*}\right)$ be the unique solution of (2.5). Then, for every $t \in[0, T], \mathbb{P}$-a.s.,

$$
\begin{aligned}
Y_{t}^{*} & =\mathbb{E}\left[\partial_{x} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right) \mid \mathcal{F}_{t}\right] \\
Z_{t}^{*} & =\frac{d}{d t}\left(\left\langle Y^{*}, W^{1}\right\rangle_{t}, \cdots,\left\langle Y^{*}, W^{d}\right\rangle_{t}\right)^{\top}
\end{aligned}
$$

Moreover, if $\left(\mathcal{Y}^{*}, \mathcal{Z}^{*}, \mathcal{L}^{*}\right)$ is the unique solution of (2.6), then for every $t \in[0, T], \mathbb{P}$-a.s.,

$$
\begin{aligned}
\mathcal{Y}_{t}^{*} & =\mathbb{E}\left[\partial_{y} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right) \mid \mathcal{F}_{t}\right] \\
\mathcal{Z}_{t}^{*, i} & =\frac{d}{d t}\left(\left\langle\mathcal{Y}^{*, i}, W^{1}\right\rangle_{t}, \cdots,\left\langle\mathcal{Y}^{*, i}, W^{d}\right\rangle_{t}\right)^{\top}, \quad i=1, \ldots, d .
\end{aligned}
$$

Under certain (rather strong) conditions on the regularity of $Y^{*}$ and $\mathcal{Y}^{*}$, Lemma 2.11 and Itô's formula ensure that $Z^{*}$ and $\mathcal{Z}^{*}$ can be calculated via a Feynman-Kac representation. To formulate the result denote by $\mathcal{C}^{1,2,2}$ the set of all continuous functions from $[0, T] \times \mathbb{R} \times \mathbb{R}^{d}$ to $\mathbb{R}$ which are continuously differentiable on $[0, T)$ and twice continuously differentiable on $\mathbb{R} \times \mathbb{R}^{d}$.

Corollary 2.12. Suppose that Assumptions 2.1 and 2.7 are satisfied. Let $H^{*}$ be the unique optimizer of $V(0)$ defined in Proposition 2.9 (i). Moreover, assume that there are $J \in \mathcal{C}^{1,2,2}$ and $\mathcal{J}=\left(\mathcal{J}^{1}, \ldots, \mathcal{J}^{d}\right)^{\top} \in\left(\mathcal{C}^{1,2,2}\right)^{d}$ such that for every $t \in[0, T], \mathbb{P}$-a.s.

$$
\begin{equation*}
Y_{t}^{*}=J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right), \quad \mathcal{Y}_{t}^{*}=\mathcal{J}\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right) \tag{2.7}
\end{equation*}
$$

Then, for every $t \in[0, T), \mathbb{P}$-a.s.,

$$
\begin{aligned}
Z_{t}^{*} & =\left(\sigma_{t}^{o}\right)^{\top}\left[\partial_{x} J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right) H_{t}^{*}+\nabla_{s} J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right)\right] \\
\mathcal{Z}_{t}^{*, i} & =\left(\sigma_{t}^{o}\right)^{\top}\left[\partial_{x} \mathcal{J}^{i}\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right) H_{t}^{*}+\nabla_{s} \mathcal{J}^{i}\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right)\right], \quad i=1, \ldots, d
\end{aligned}
$$

where $\nabla_{s} \mathcal{J}^{i}$ denotes the gradient of $\mathcal{J}^{i}(t, x, s)$ with respect $s$.
The proofs of Lemma 2.11 and Corollary 2.12 are presented in Section 4, where we also provide some sufficient conditions for the regularity assumption in Corollary 2.12.
2.4. Main results. With all this notation set in place, we present the main result of this article pertaining the characterization of the behavior of

$$
V(\varepsilon)=\inf _{H \in \mathcal{A}} \mathcal{V}(H, \varepsilon)=\inf _{H \in \mathcal{A}} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[f\left(X_{T}^{H ; b, \sigma}, h\left(S_{T}^{b, \sigma}\right)\right)\right]
$$

for small $\varepsilon$. Recall that $\mathcal{B}^{\varepsilon}$ and $\mathcal{A}$ are defined in (2.2) and (2.5), respectively.
Theorem 2.13. Suppose that Assumptions 2.1 and 2.7 are satisfied and set $q:=\frac{p}{p-1}$ to be the conjugate Hölder exponent of $p>3$ (given in Assumption 2.1). Let $H^{*}$ be the unique optimizer of $V(0)$ (see Proposition 2.9 (i)) and let $\left(Y^{*}, Z^{*}\right)$ and $\left(\mathcal{Y}^{*}, \mathcal{Z}^{*}\right)$ be the first two components of the solutions of (2.5) and (2.6), respectively. Then, as $\varepsilon \downarrow 0$,

$$
V(\varepsilon)=V(0)+\varepsilon V^{\prime}(0)+O\left(\varepsilon^{2}\right)
$$

where

$$
\begin{aligned}
V^{\prime}(0) & =\gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H} q} \\
& =\gamma \mathbb{E}\left[\int_{0}^{T}\left|Y_{t}^{*} H_{t}^{*}+\mathcal{Y}_{t}^{*}\right|^{q} d t\right]^{1 / q}+\eta \mathbb{E}\left[\left(\int_{0}^{T}\left\|Z_{t}^{*}\left(H_{t}^{*}\right)^{\top}+\mathcal{Z}_{t}^{*}\right\|_{\mathbb{F}}^{2} d t\right)^{q / 2}\right]^{1 / q} .
\end{aligned}
$$

We emphasize that $Y^{*}, Z^{*}, \mathcal{Y}^{*}, \mathcal{Z}^{*}$ are given explicitly in Lemma 2.11. In particular, the statements made in the introduction on the form of $V^{\prime}(0)$ follow from a combination of Theorem 2.13 and Lemma 2.11.

Next, recall that $H^{*}$ is the unique optimizer for $V(0)$ (given in Proposition 2.9 (i)) and for any $\varepsilon \geq 0$, consider

$$
\begin{equation*}
V^{*}(\varepsilon):=\mathcal{V}\left(H^{*}, \varepsilon\right)=\sup _{(b, \sigma) \in \mathcal{B}^{E}} \mathbb{E}\left[f\left(X_{T}^{H^{*} ; b, \sigma}, h\left(S_{T}^{b, \sigma}\right)\right)\right] \tag{2.8}
\end{equation*}
$$

Therefore, $V^{*}(\varepsilon)$ represents the worst-case value when the market participant sticks to the optimal strategy $H^{*}$ calculated based on $b^{o}$ and $\sigma^{o}$ while the actual parameters lie within $\mathcal{B}^{\varepsilon}$. As it happens, the values of $V^{*}(\varepsilon)$ and $V(\varepsilon)$ are equal up to a second-order correction term.

Theorem 2.14. Suppose that Assumptions 2.1 and 2.7 are satisfied. Then, as $\varepsilon \downarrow 0$,

$$
V^{*}(\varepsilon)=V(\varepsilon)+O\left(\varepsilon^{2}\right) .
$$

In particular, $\left(V^{*}\right)^{\prime}(0)=V^{\prime}(0)$.

## 3. Proof of Proposition 2.9 and Theorem 2.13

We start by highlighting the main ideas used in the proof of Theorem 2.13. For simplicity we focus on the notationally lighter case when $h=0$.

The first step involves a second-order Taylor expansion, showing that for every $H \in \mathcal{A}$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon}$,

$$
\begin{equation*}
\mathbb{E}\left[f\left(X_{T}^{H ; b, \sigma}, 0\right)\right]-\mathbb{E}\left[f\left(X_{T}^{H ; o}, 0\right)\right]=\mathbb{E}\left[\partial_{x} f\left(X_{T}^{H ; o}, 0\right)\left(X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right)\right]+O\left(\varepsilon^{2}\right) . \tag{3.1}
\end{equation*}
$$

Subsequently, a crucial observation is that the expectation appearing in the right-hand side of (3.1) can be expressed as a linear form involving $b-b^{o}$ and $\sigma-\sigma^{o}$ : If $\left(Y^{H}, Z^{H}, L^{H}\right)$ solves the following BSDE

$$
Y_{t}^{H}=\partial_{x} f\left(X_{T}^{H ; o}, 0\right)-\int_{t}^{T}\left(Z_{u}^{H}\right)^{\top} d W_{u}-\left(L_{T}^{H}-L_{t}^{H}\right),
$$

with $L_{0}^{H}=0$, then

$$
\begin{equation*}
\mathbb{E}\left[\partial_{x} f\left(X_{T}^{H ; o}, 0\right)\left(X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right)\right]=\left\langle Y^{H} H, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{H} H^{\top}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathbb{F}} \tag{3.2}
\end{equation*}
$$

Next, setting $q:=\frac{p}{p-1}$ to be the conjugate Hölder conjugate of $p$ and relying on the duality within the pairings $\left\langle\mathbb{L}^{p}\left(\mathbb{R}^{d}\right), \mathbb{L}^{q}\left(\mathbb{R}^{d}\right),\langle\cdot, \cdot\rangle_{\mathbb{P} \otimes d t}\right\rangle$ and $\left\langle\mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right), \mathbb{H}^{q}\left(\mathbb{R}^{d \times d}\right),\langle\cdot, \cdot\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}\right\rangle$, it follows that

$$
\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left(\left\langle Y^{H} H, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{H} H^{\top}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathbb{F}}\right)=\varepsilon\left(\gamma\left\|Y^{H} H\right\|_{\mathbb{L}^{q} q}+\eta\left\|Z^{H} H^{\top}\right\|_{\mathbb{H} q}\right) .
$$

The final ingredient in the proof lies in showing that if $\left(H^{\varepsilon}\right)_{\varepsilon>0}$ are (almost) optimizers for the robust optimization problems $V(\varepsilon)$, then the $H^{\varepsilon}$ s converge to the unique optimizer $H^{*}$ of $V(0)$ as $\varepsilon \downarrow 0$.

In the subsequent sections, we will establish the technical details required to rigorously prove these arguments. Since our results are concerned with the behavior as $\varepsilon \downarrow 0$, we can and do assume without loss of generality that $\gamma, \eta \leq 1$.
3.1. Preliminary estimates and GKW decomposition. Let us provide some simple observations that play an instrumental role in the proof of Proposition 2.9 and Theorem 2.13. In what follows, we often make use of the following elementary inequality: for every $\beta \geq 0$ and $m \in \mathbb{N}$,

$$
\begin{equation*}
\left|\sum_{i=1}^{m} a_{i}\right|^{\beta} \leq m^{\beta} \cdot \sum_{i=1}^{m}\left|a_{i}\right|^{\beta}, \quad \text { for every }\left\{a_{i}\right\}_{i=1}^{m} \subset \mathbb{R} \tag{3.3}
\end{equation*}
$$

We call (3.3) the 'power triangle inequality'.
Let us begin with a priori estimates on $X_{T}^{H ; b, \sigma}$ (given in (2.4)) and $S_{T}^{b, \sigma}$ (given in (2.1)).
Lemma 3.1. Suppose that Assumption 2.1 is satisfied. Then the following hold:
(i) For every $\varepsilon \geq 0$,

$$
\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \sup _{H \in \mathcal{A}}\left\|X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right\|_{L^{p}}^{p} \leq C_{1} \varepsilon^{p}, \quad \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\|S_{T}^{b, \sigma}-S_{T}^{o}\right\|_{L^{p}}^{p} \leq C_{2} \varepsilon^{p}
$$

where $C_{1}:=2^{p} K^{p}\left(T^{p-1}+C_{\mathrm{BDG}, p}\right), C_{2}:=2^{p}\left(T^{p-1}+C_{\mathrm{BDG}, p}\right)$, and $C_{\mathrm{BDG}, p} \geq 1$ is the $B D G$ constant (in Section 2.1) with the exponent $p>3$.
(ii) The constants $C_{3}, C_{4}$ defined by

$$
C_{3}:=\sup _{(b, \sigma) \in \mathcal{B}^{1}} \sup _{H \in \mathcal{A}}\left\|X_{T}^{H ; b, \sigma}\right\|_{L^{p}}^{p}, \quad C_{4}:=\sup _{(b, \sigma) \in \mathcal{B}^{1}}\left\|h\left(S_{T}^{b, \sigma}\right)\right\|_{L^{p}}^{p}
$$

satisfy $C_{3}, C_{4}<\infty$.
Proof. We start by proving (i). An application of the power triangle inequality (see (3.3)) and the BDG inequality shows that for every $\varepsilon \geq 0,(b, \sigma) \in \mathcal{B}^{\varepsilon}$ and $H \in \mathcal{A}$,

$$
\left\|X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right\|_{L^{p}}^{p} \leq 2^{p}\left(\mathbb{E}\left[\left|\int_{0}^{T} H_{t}^{\top}\left(b_{t}-b_{t}^{o}\right) d t\right|^{p}\right]+C_{\mathrm{BDG}, p} \mathbb{E}\left[\left(\int_{0}^{T}\left|H_{t}^{\top}\left(\sigma_{t}-\sigma_{t}^{o}\right)\right|^{2} d t\right)^{\frac{p}{2}}\right]\right)
$$

Moreover, since the function $x \rightarrow|x|^{p}$ with $p>3$ is convex and $\left|H_{t}\right| \leq K \mathbb{P} \otimes d t$-a.e. for every $H \in \mathcal{A}$ (see Definition 2.5), Jensen's inequality ensures that

$$
\mathbb{E}\left[\left|\int_{0}^{T} H_{t}^{\top}\left(b_{t}-b_{t}^{o}\right) d t\right|^{p}\right] \leq T^{p-1} \mathbb{E}\left[\int_{0}^{T}\left|H_{t}^{\top}\left(b_{t}-b_{t}^{o}\right)\right|^{p} d t\right] \leq T^{p-1} K^{p}\left\|b-b^{o}\right\|_{\mathbb{L}^{p}}^{p}
$$

Finally, since $\left|\left(\sigma_{t}-\sigma_{t}^{o}\right)^{\top} H_{t}\right| \leq K\left\|\sigma_{t}-\sigma_{t}^{o}\right\|_{\mathrm{F}} \mathbb{P} \otimes d t$-a.e. for every $H \in \mathcal{A}$, we conclude that

$$
\begin{align*}
& \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \sup _{H \in \mathcal{A}}\left\|X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right\|_{L^{p}}^{p}  \tag{3.4}\\
& \leq 2^{p} K^{p} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left(T^{p-1}\left\|b-b^{o}\right\|_{\mathbb{L}^{p}}^{p}+C_{\mathrm{BDG}, p}\left\|\sigma-\sigma^{o}\right\|_{\mathbb{H}^{p}}^{p}\right) \leq C_{1} \varepsilon^{p} .
\end{align*}
$$

In a similar manner, it follows that

$$
\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\|S_{T}^{b, \sigma}-S_{T}^{o}\right\|_{L^{p}}^{p} \leq 2^{p} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left(T^{p-1}\left\|b-b^{o}\right\|_{\mathbb{L}^{p}}^{p}+C_{\mathrm{BDG}, p}\left\|\sigma-\sigma^{o}\right\|_{\mathbb{H}^{p}}^{p}\right) \leq C_{2} \varepsilon^{p}
$$

Now let us prove (ii). Using Assumption 2.1 (i), the same arguments as presented for the proof of (i) can be used to show that

$$
\sup _{H \in \mathcal{A}}\left\|X_{T}^{H ; o}\right\|_{L^{p}}^{p} \leq 3^{p}\left(\left|x_{0}\right|^{p}+K^{p}\left(T^{p-1}\left\|b^{o}\right\|_{\mathbb{L}^{p}}^{p}+C_{\mathrm{BDG}, p}\left\|\sigma^{o}\right\|_{\mathbb{H}^{p}}^{p}\right)\right)<\infty
$$

Hence, the claim that $C_{3}<\infty$ follows from the triangle inequality and (3.4).
Similarly, since $\sup _{(b, \sigma) \in \mathcal{B}^{1}}\left\|S_{T}^{b, \sigma}\right\|_{L^{p}}^{p}<\infty\left(\right.$ see Remark 2.2) and $|h(s)| \leq C_{h}(1+|s|)$ for every $s \in \mathbb{R}^{d}$ (see Remark 2.8 (ii)), the triangle inequality and the estimate given in (i) ensure that $C_{4}<\infty$.

The following a priori estimate is based on the Galtchouk-Kunita-Watanabe decomposition (see, e.g., [49] or [40, Theorem 4.27, p.126]).

Lemma 3.2. For every real-valued random variable $X \in L^{2}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$, the BSDE given by

$$
\begin{equation*}
Y_{t}=X-\int_{t}^{T}\left(Z_{u}\right)^{\top} d W_{u}-\left(L_{T}-L_{t}\right), \quad t \in[0, T], \quad \text { with } L_{0}=0 \tag{3.5}
\end{equation*}
$$

has a unique solution $(Y, Z, L) \in \mathscr{S}^{2}(\mathbb{R}) \times \mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})$. In particular, $L$ is strongly orthogonal to $\left(\int_{0}^{t}\left(Z_{u}\right)^{\top} d W_{u}\right)_{t \in[0, T]} \in \mathscr{M}^{2}(\mathbb{R})$. Furthermore, setting $C_{\mathrm{ap}}:=20$, the solution satisfies the a priori estimate

$$
\begin{equation*}
\|Y\|_{\mathscr{S}^{2}}^{2}+\|Z\|_{\mathscr{H}^{2}}^{2}+\|L\|_{\mathscr{M}^{2}}^{2} \leq C_{\mathrm{ap}}\|X\|_{L^{2}}^{2} \tag{3.6}
\end{equation*}
$$

Proof. Set $Y_{t}:=\mathbb{E}\left[X \mid \mathcal{F}_{t}\right]$ for $t \in[0, T]$. An application of Doob's inequality shows that

$$
\begin{equation*}
\|Y\|_{\mathscr{S}^{2}}^{2}=\mathbb{E}\left[\sup _{t \in[0, T]} Y_{t}^{2}\right] \leq 2^{2}\|X\|_{L^{2}}^{2}<\infty \tag{3.7}
\end{equation*}
$$

which ensures that $Y=\left(Y_{t}\right)_{t \in[0, T]} \in \mathscr{S}^{2}(\mathbb{R})$. Moreover, since $W^{1}, \ldots, W^{d} \in \mathscr{M}^{2}(\mathbb{R})$, we can apply the Galtchouk-Kunita-Watanabe decomposition in [40, Theorem 4.27, p.126] to represent the càdlàg process $Y$ in terms of the orthogonal decomposition with respect to $W=\left(W^{1}, \ldots, W^{d}\right)^{\top}$, i.e.,

$$
Y_{t}=\mathbb{E}\left[X \mid \mathcal{F}_{t}\right]=Y_{0}+\int_{0}^{t}\left(Z_{s}\right)^{\top} d W_{s}+L_{t}, \quad t \in[0, T]
$$

where $Z=\left(Z_{t}\right)_{t \in[0, T]} \in \mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$ and $L=\left(L_{t}\right)_{t \in[0, T]} \in \mathscr{M}^{2}(\mathbb{R})$ satisfies $L_{0}=0$ and is strongly orthogonal to $W^{i}$ for every $i=1, \ldots, d$. This shows the existence of $(Y, Z, L) \in$ $\mathscr{S}^{2}(\mathbb{R}) \times \mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})$ satisfying the $\operatorname{BSDE}(3.5)$.

Furthermore, by the strong orthogonality between $L$ and $W^{i}, i=1, \ldots, d$, and since $Z \in$ $\mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$ is $\mathbb{F}$-predictable, an application of $[75$, Lemma $2 \&$ Theorem 35, p.149] shows that the two square integrable martingales $L$ and $\left(\int_{0}^{t} Z_{s}^{\top} d W_{s}\right)_{t \in[0, T]}$ are strongly orthogonal, i.e., $\left(L_{t} \int_{0}^{t} Z_{u}^{\top} d W_{u}\right)_{t \in[0, T]}$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale. Therefore, it follows from the Itô-isometry that

$$
\begin{align*}
\|Z\|_{\mathscr{H}^{2}}^{2}+\|L\|_{\mathscr{M}^{2}}^{2} & =\mathbb{E}\left[\int_{0}^{T}\left|Z_{s}\right|^{2} d s+[L, L]_{T}\right]=\mathbb{E}\left[\left|Y_{T}-Y_{0}\right|^{2}\right]  \tag{3.8}\\
& \leq 2^{2} \mathbb{E}\left[\max \left\{\left|Y_{T}\right|^{2},\left|Y_{0}\right|^{2}\right\}\right] \leq 2^{2} \mathbb{E}\left[\sup _{t \in[0, T]}\left|Y_{t}\right|^{2}\right]=2^{2}\|Y\|_{\mathscr{S}^{2}}^{2}
\end{align*}
$$

Hence, the a priori estimate (3.6) follows from combining (3.7) and (3.8). The uniqueness immediately follows from (3.6). Indeed, if there are two solutions to (3.5), then their difference solves (3.5) with $X=0$. This completes the proof.

Proposition 3.3. Suppose that Assumptions 2.1 and 2.7 are satisfied. Let $H \in \mathcal{A}$ and set

$$
\begin{equation*}
A^{H}:=\partial_{x} f\left(X_{T}^{H ; o}, h\left(S_{T}^{o}\right)\right), \quad B^{H}:=\partial_{y} f\left(X_{T}^{H ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right) \tag{3.9}
\end{equation*}
$$

Then the following hold.
(i) $A^{H} \in L^{2}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$ and $B^{H} \in L^{2}\left(\mathcal{F}_{T} ; \mathbb{R}^{d}\right)$.
(ii) There exists a unique solution $\left(Y^{H}, Z^{H}, L^{H}\right) \in \mathscr{S}^{2}(\mathbb{R}) \times \mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})$ of

$$
\begin{equation*}
Y_{t}^{H}=A^{H}-\int_{t}^{T}\left(Z_{s}^{H}\right)^{\top} d W_{s}-\left(L_{T}^{H}-L_{t}^{H}\right), \quad t \in[0, T] \tag{3.10}
\end{equation*}
$$

with $L_{0}^{H}=0$, where $L^{H}$ and $\left(\int_{0}^{t}\left(Z_{s}^{H}\right)^{\top} d W_{s}\right)_{t \in[0, T]} \in \mathscr{M}^{2}(\mathbb{R})$ are strongly orthogonal.
(iii) Similarly, there exists a unique solution $\left(\mathcal{Y}^{H}, \mathcal{Z}^{H}, \mathcal{L}^{H}\right) \in\left(\mathscr{S}^{2}(\mathbb{R})\right)^{d} \times\left(\mathscr{H}^{2}\left(\mathbb{R}^{d}\right)\right)^{d} \times$ $\left(\mathscr{M}^{2}(\mathbb{R})\right)^{d}$ of

$$
\begin{equation*}
\mathcal{Y}_{t}^{H}=B^{H}-\int_{t}^{T} \mathcal{Z}_{u}^{H} d W_{u}-\left(\mathcal{L}_{T}^{H}-\mathcal{L}_{t}^{H}\right), \quad t \in[0, T] \tag{3.11}
\end{equation*}
$$

with $\mathcal{L}_{0}^{H}=0$, where $\mathcal{L}^{H}$ and $\left(\int_{0}^{t} \mathcal{Z}_{s}^{H} d W_{s}\right)_{t \in[0, T]}$ are strongly orthogonal.
Proof. We start by proving (i). Let $0<r<\min \left\{\frac{p-2}{2}, p-3\right\}$ be as in Assumption 2.7 (ii). It follows from Remark 2.8 (i) and the power triangle inequality that

$$
\begin{equation*}
\left|\partial_{x} f(x, y)\right|^{2}+\left|\partial_{y} f(x, y)\right|^{2} \leq\left(3 \widetilde{C}_{f}\right)^{2}\left(1+|x|^{2(r+1)}+|y|^{2(r+1)}\right) \tag{3.12}
\end{equation*}
$$

for every $(x, y)^{\top} \in \mathbb{R}^{2}$. Therefore, by Lemma 3.1 (ii) and Hölder's inequality (with exponent $\frac{p}{2(r+1)}>1$ ), we obtain that

$$
\begin{aligned}
\sup _{H \in \mathcal{A}}\left\|A^{H}\right\|_{L^{2}}^{2} & \leq\left(3 \widetilde{C}_{f}\right)^{2} \sup _{H \in \mathcal{A}} \mathbb{E}\left[1+\left|X_{T}^{H ; o}\right|^{2(r+1)}+\left|h\left(S_{T}^{o}\right)\right|^{2(r+1)}\right] \\
& \leq\left(3 \widetilde{C}_{f}\right)^{2}\left(1+\sup _{H \in \mathcal{A}}\left\|X_{T}^{H ; o}\right\|_{L^{p}}^{2(r+1)}+\left\|h\left(S_{T}^{o}\right)\right\|_{L^{p}}^{2(r+1)}\right)<\infty
\end{aligned}
$$

Next, since $|\nabla h(\cdot)| \leq C_{h}$ (see Remark 2.8 (ii)), the same arguments devoted for showing that $\sup _{H \in \mathcal{A}}\left\|A^{H}\right\|_{L^{2}}<\infty$ ensure that $\sup _{H \in \mathcal{A}}\left\|B^{H}\right\|_{L^{2}}<\infty$, as claimed.

The statements (ii) and (iii) follow directly from Lemma 3.2.
3.2. Stability results. This section is devoted to showing stability results of the forward process $X^{H ; o}$ and the backward triplets $\left(Y^{H}, Z^{H}, L^{H}\right)$ and $\left(\mathcal{Y}^{H}, \mathcal{Z}^{H}, \mathcal{L}^{H}\right)$ (introduced in Proposition 3.3 (ii)) with respect to $H \in \mathcal{A}$, which will play an essential role in the proof of Theorem 2.13. We begin with a priori estimates of the forward process.

Lemma 3.4. Suppose that Assumption 2.1 is satisfied, let $\beta \in(1, p)$, and set $\alpha(\beta)=\frac{p \beta}{p-\beta}>1$. Then, for any $G, H \in \mathcal{A}$,

$$
\left\|X_{T}^{G ; o}-X_{T}^{H ; o}\right\|_{L^{\beta}}^{\beta} \leq C(\beta)\left(\|G-H\|_{\mathbb{L}^{\alpha(\beta)}}^{\beta}\left\|b_{t}^{o}\right\|_{\mathbb{L}^{p}}^{\beta}+\mathbb{E}\left[\left(\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(G_{t}-H_{t}\right)\right|^{2} d t\right)^{\frac{\beta}{2}}\right]\right)
$$

where $C(\beta):=2^{\beta} \max \left\{C_{\mathrm{BDG}, \beta}, T^{\beta-1}\right\}>0$.

Proof. Set $\Delta:=G-H$ and write $\alpha=\alpha(\beta)$. First note that by the power triangle inequality and the BDG inequality,

$$
\left\|X_{T}^{G ; o}-X_{T}^{H ; o}\right\|_{L^{\beta}}^{\beta} \leq 2^{\beta}\left(\mathbb{E}\left[\left|\int_{0}^{T} \Delta_{t}^{\top} b_{t}^{o} d t\right|^{\beta}\right]+C_{\mathrm{BDG}, \beta} \mathbb{E}\left[\left(\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top} \Delta_{t}\right|^{2} d t\right)^{\frac{\beta}{2}}\right]\right)
$$

Moreover, a twofold application of Hölder's inequality (first with exponent $p>3$ followed by exponent $\frac{p}{\beta}>1$ ) shows that

$$
\mathbb{E}\left[\left|\int_{0}^{T} \Delta_{t}^{\top} b_{t}^{o} d t\right|^{\beta}\right] \leq \mathbb{E}\left[\left(\int_{0}^{T}\left|\Delta_{t}\right|^{q} d t\right)^{\frac{\beta}{q}}\left(\int_{0}^{T}\left|b_{t}^{o}\right|^{p} d t\right)^{\frac{\beta}{p}}\right] \leq \mathbb{E}\left[\left(\int_{0}^{T}\left|\Delta_{t}\right|^{q} d t\right)^{\frac{\alpha}{q}}\right]^{\frac{\beta}{\alpha}}\left\|b^{o}\right\|_{\mathbb{L}^{p}}^{\beta}
$$

Finally, since $x \rightarrow|x|^{\frac{\alpha}{q}}$ is convex (noting that $\frac{\alpha}{q}=\frac{\beta p}{q(p-\beta)}>1$ ), Jensen's inequality ensures that $\mathbb{E}\left[\left(\int_{0}^{T}\left|\Delta_{t}\right|^{q} d t\right)^{\frac{\alpha}{q}}\right]^{\frac{\beta}{\alpha}} \leq T^{\beta-1}\|\Delta\|_{\mathbb{L}^{\alpha}}^{\beta}$, as claimed.

Lemma 3.5. Suppose that Assumption 2.1 is satisfied and let $\left(H^{n}\right)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ and $H^{\star} \in \mathcal{A}$ such that $\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right| \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$. Then, for every $\beta \geq 1$,

$$
\left\|H^{n}-H^{\star}\right\|_{\mathbb{L}^{\beta}} \rightarrow 0 \quad \text { as } n \rightarrow 0
$$

Moreover, $\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(H_{t}^{n}-H_{t}^{\star}\right)\right|^{2} d t \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$.
Proof. Recall that the measure $\mathbb{Q} \sim \mathbb{P}$ is defined in (2.3) and that $W^{\mathbb{Q}}=W+\int_{0}^{o}\left(\sigma_{s}^{o}\right)^{-1} b_{s}^{o} d s$ is a $(\mathbb{F}, \mathbb{Q})$-Brownian motion. In particular, $S^{o}$ is an $(\mathbb{F}, \mathbb{Q})$-local martingale since $S^{o}=S_{0}^{o}+$ $\int_{0}^{*} \sigma_{s}^{o} d W_{s}^{\mathbb{Q}}$. Furthermore, for each $n \in \mathbb{N},\left(\left(H^{n} \cdot S^{o}\right)_{t}\right)_{t \in[0, T]}$ is an $(\mathbb{F}, \mathbb{Q})$-local martingale since

$$
\left(H^{n} \cdot S^{o}\right)_{t}=\int_{0}^{t}\left(H_{s}^{n}\right)^{\top} \sigma_{r}^{o} d W_{r}^{\mathbb{Q}}, \quad t \in[0, T]
$$

Next note that by the assumption on $\left(H^{n}\right)_{n \in \mathbb{N}}$,

$$
\begin{equation*}
\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right|^{2} \xrightarrow{\mathbb{Q}} 0 \quad \text { as } n \rightarrow \infty . \tag{3.13}
\end{equation*}
$$

Moreover, as $\left|H_{t}^{\star}\right|,\left|H_{t}^{n}\right| \leq K \mathbb{Q} \otimes d t$-a.e. (by Definition 2.5 and the equivalence $\mathbb{Q} \sim \mathbb{P}$ ), the power triangle inequality, and Hölder's inequality (with exponent $\frac{p}{2 v}>1$ ), it follows for every $1<v<\frac{p}{2}$ that

$$
\begin{aligned}
\sup _{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}}\left[\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right|^{2 v}\right] & =\sup _{n \in \mathbb{N}} \mathbb{E}\left[\mathcal{D}_{T}\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right|^{2 v}\right] \\
& \leq 2^{2 v}\left(\sup _{n \in \mathbb{N}} \mathbb{E}\left[\mathcal{D}_{T}\left|X_{T}^{H^{n} ; o}\right|^{2 v}\right]+\mathbb{E}\left[\mathcal{D}_{T}\left|X_{T}^{H^{\star} ; o}\right|^{2 v}\right]\right) \\
& \leq 2^{2 v}\left\|\mathcal{D}_{T}\right\|_{L^{\frac{p}{p-2 v}}}\left(\sup _{n \in \mathbb{N}}\left\|X_{T}^{H^{n} ; o}\right\|_{L^{p}}^{2 v}+\left\|X_{T}^{H^{\star} ; o}\right\|_{L^{p}}^{2 v}\right) .
\end{aligned}
$$

Hence, using the $L^{\beta}$-boundedness of the exponential martingale $\mathcal{D}_{T}$ (see Assumption 2.1 (ii)), and Lemma 3.1 (ii) with the fact that $\left(H^{n}\right)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ and $H^{\star} \in \mathcal{A}$, it follows that

$$
\sup _{n \in \mathbb{N}} \mathbb{E}^{\mathbb{Q}}\left[\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right|^{2 v}\right]<\infty
$$

In particular, since $v>1$, the de la Vallée Poussin theorem [47, Theorem 6.19] ensures the uniform integrability of $\left(\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right|^{2}\right)_{n \in \mathbb{N}}$ with respect to $\mathbb{Q}$; thus $\mathbb{E}^{\mathbb{Q}}\left[\left|X_{T}^{H^{n ; o}}-X_{T}^{H^{\star} ; o}\right|^{2}\right] \rightarrow 0$
as $n \rightarrow \infty$ by (3.13) and Vitali's convergence theorem [47, Theorem 6.25]. Using that $S^{o}=$ $S_{0}^{o}+\int_{0}^{\prime} \sigma_{s}^{o} d W_{s}^{\mathbb{Q}}$ under $\mathbb{Q}$, the Itô-isometry implies that

$$
\begin{equation*}
0=\lim _{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}}\left[\left|X_{T}^{H^{n ; o}}-X_{T}^{H^{\star} ; o}\right|^{2}\right]=\lim _{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}}\left[\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(H_{t}^{n}-H_{t}^{\star}\right)\right|^{2} d t\right] \tag{3.14}
\end{equation*}
$$

and in particular that $\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(H_{t}^{n}-H_{t}^{\star}\right)\right|^{2} d t \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$ since $\mathbb{Q} \sim \mathbb{P}$.
Moreover, combining (3.14) with the fact that $\sigma^{o}$ is invertible (see Assumption 2.1 (ii)) shows that

$$
\begin{equation*}
H_{t}^{n} \xrightarrow{\mathbb{P} \otimes d t} H_{t}^{\star} \quad \text { as } n \rightarrow \infty \tag{3.15}
\end{equation*}
$$

Finally, since $\left|H_{t}^{\star}\right|,\left|H_{t}^{n}\right| \leq K \mathbb{P} \otimes d t$-a.e., the dominated convergence theorem guarantees that (3.15) implies that for every $\beta \geq 1,\left\|H^{n}-H^{\star}\right\|_{\mathbb{L}^{\beta}} \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof.

Lemma 3.6. Suppose that Assumptions 2.1 and 2.7 are satisfied and let $\left(H^{n}\right)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ and $H^{\star} \in \mathcal{A}$ such that $\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right| \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$.
(i) Denote by $\left(Y^{n}, Z^{n}, L^{n}\right)$ and $\left(Y^{\star}, Z^{\star}, L^{\star}\right)$ the unique solutions of (3.10) (ensured by Proposition 3.3) under the terminal conditions

$$
A^{H^{n}}=\partial_{x} f\left(X_{T}^{H^{n} ; o}, h\left(S_{T}^{o}\right)\right), \quad A^{H^{\star}}=\partial_{x} f\left(X_{T}^{H^{\star} ; o}, h\left(S_{T}^{o}\right)\right),
$$

respectively. Then, as $n \rightarrow \infty$,

$$
\begin{equation*}
\left\|Y^{n}-Y^{\star}\right\|_{\mathscr{S}^{2}}+\left\|Z^{n}-Z^{\star}\right\|_{\mathscr{H}^{2}}+\left\|L^{n}-L^{\star}\right\|_{\mathscr{M}^{2}} \rightarrow 0 \tag{3.16}
\end{equation*}
$$

(ii) Denote by $\left(\mathcal{Y}^{n}, \mathcal{Z}^{n}, \mathcal{L}^{n}\right)$ and $\left(\mathcal{Y}^{\star}, \mathcal{Z}^{\star}, \mathcal{L}^{\star}\right)$ the unique solutions of (3.11) under the terminal conditions

$$
B^{H^{n}}=\partial_{y} f\left(X_{T}^{H^{n} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right), \quad B^{H^{\star}}=\partial_{y} f\left(X_{T}^{H^{\star} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right)
$$

respectively. Then, for every $i=1, \ldots, d$, as $n \rightarrow \infty$,

$$
\begin{equation*}
\left\|\mathcal{Y}^{n, i}-\mathcal{Y}^{\star, i}\right\|_{\mathscr{S}^{2}}+\left\|\mathcal{Z}^{n, i}-\mathcal{Z}^{\star, i}\right\|_{\mathscr{H}^{2}}+\left\|\mathcal{L}^{n, i}-\mathcal{L}^{\star, i}\right\|_{\mathscr{M}^{2}} \rightarrow 0 \tag{3.17}
\end{equation*}
$$

Proof. We start by proving (i). It follows from Proposition 3.3 that ( $\left.Y^{n}-Y^{\star}, Z^{n}-Z^{\star}, L^{n}-L^{\star}\right)$ is the unique solution to the $\operatorname{BSDE}(3.5)$ with the terminal condition $A^{H^{n}}-A^{H^{\star}}$. Therefore, using the a priori estimate given in Lemma 3.2,

$$
\left\|Y^{n}-Y^{\star}\right\|_{\mathscr{S}^{2}}^{2}+\left\|Z^{n}-Z^{\star}\right\|_{\mathscr{H}^{2}}^{2}+\left\|L^{n}-L^{\star}\right\|_{\mathscr{M}^{2}}^{2} \leq C_{\mathrm{ap}} \mathbb{E}\left[\left|A^{H^{n}}-A^{H^{\star}}\right|^{2}\right]
$$

Moreover, by Assumption 2.7 (ii) we have that $\left|\partial_{x x} f(x, y)\right| \leq C_{f}\left(1+\left|(x, y)^{\top}\right|^{r}\right)$ for every $(x, y)^{\top} \in \mathbb{R}^{2}$, hence an application of the fundamental theorem of calculus (and the power triangle inequality) shows that for every $x, x^{*}, y \in \mathbb{R}$,

$$
\left|\partial_{x} f(x, y)-\partial_{x} f\left(x^{*}, y\right)\right|^{2} \leq C_{f}^{2} 2^{r+4}\left(1+|x|^{2 r}+\left|x^{*}\right|^{2 r}+|y|^{2 r}\right)\left|x-x^{*}\right|^{2}
$$

In particular, by the definition of $A^{H^{n}}$ and $A^{H^{\star}}$,

$$
\mathbb{E}\left[\left|A^{H^{n}}-A^{H^{\star}}\right|^{2}\right] \leq C_{f}^{2} 2^{r+4} \mathbb{E}\left[\left(1+\left|X_{T}^{H^{n} ; o}\right|^{2 r}+\left|X_{T}^{H^{\star} ; o}\right|^{2 r}+\left|h\left(S_{T}^{o}\right)\right|^{2 r}\right)\left|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right|^{2}\right]
$$

In order to estimate the last term, set $\widetilde{p}:=\frac{2 p}{p-2 r}$, where we recall that $p>3$ and $0<$ $r<\min \left\{\frac{p-2}{2}, p-3\right\}$ (see Assumptions 2.1 and 2.7 (ii)), and note that $1<\widetilde{p}<p$. It follows
from Hölder's inequality (with exponent $\frac{p}{2 r}>1$ and conjugate exponent $\frac{\widetilde{p}}{2}>1$ ) and the power triangle inequality that

$$
\begin{aligned}
& \mathbb{E}\left[\left|A^{H^{n}}-A^{H^{\star}}\right|^{2}\right] \\
& \leq C_{f}^{2} 2^{r+4} 4^{\frac{p}{2 r}}\left(1+\left\|X_{T}^{H^{n}} ; o\right\|_{L^{p}}^{2 r}+\left\|X_{T}^{H^{*}} ; o\right\|_{L^{p}}^{2 r}+\left\|h\left(S_{T}^{o}\right)\right\|_{L^{p}}^{2 r}\right)\left\|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right\|_{L^{\tilde{p}}}^{2} .
\end{aligned}
$$

Moreover, by Lemma 3.1 (ii),

$$
\begin{equation*}
\sup _{n \in \mathbb{N}}\left(1+\left\|X_{T}^{H^{n} ; o}\right\|_{L^{p}}^{2 r}+\left\|X_{T}^{H^{*} ; o}\right\|_{L^{p}}^{2 r}+\left\|h\left(S_{T}^{o}\right)\right\|_{L^{p}}^{2 r}\right)<\infty \tag{3.18}
\end{equation*}
$$

thus, it remains to show that $\left\|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right\|_{L^{\widetilde{p}}} \rightarrow 0$ as $n \rightarrow \infty$.
To that end, since $1<\widetilde{p}<p$, an application of Lemma 3.4 (with $\beta=\widetilde{p} \in(1, p)$ and $\left.\alpha=\alpha(\widetilde{p})=\frac{\widetilde{p} p}{p-\widetilde{p}}>1\right)$ ensures that

$$
\left\|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right\|_{L^{\widetilde{p}}}^{\widetilde{p}} \leq C(\widetilde{p})\left(\left\|H^{n}-H^{\star}\right\|_{\mathbb{L}^{\alpha}}^{\widetilde{p}}\left\|b_{t}^{o}\right\|_{\mathbb{L}^{p}}^{\widetilde{p}}+\mathbb{E}\left[\left(\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(H_{t}^{n}-H_{t}^{\star}\right)\right|^{2} d t\right)^{\frac{\widetilde{p}}{2}}\right]\right)
$$

where $C(\widetilde{p})=2^{\widetilde{p}} \max \left\{C_{\mathrm{BDG}, \beta}, T^{\widetilde{p}-1}\right\}$. Moreover, since $b^{o} \in \mathbb{L}^{p}\left(\mathbb{R}^{d}\right)$, it follows from Lemma 3.5 that as $n \rightarrow \infty$,

$$
\left\|H^{n}-H^{\star}\right\|_{\mathbb{L}^{\alpha}}^{\widetilde{p}} \cdot\left\|b^{o}\right\|_{\mathbb{L}^{p}}^{\widetilde{p}} \rightarrow 0
$$

Next we note that $\left|H_{t}^{\star}-H_{t}^{n}\right| \leq 2 K \mathbb{P} \otimes d t$-a.e. and $\sigma^{o} \in \mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right)$. Therefore, since $\tilde{p}<p$ and $\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(H_{t}^{n}-H_{t}^{\star}\right)\right|^{2} d t \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$ (by Lemma 3.5), the dominated convergence theorem shows that

$$
\mathbb{E}\left[\left(\int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{\top}\left(H_{t}^{n}-H_{t}^{\star}\right)\right|^{2} d t\right)^{\frac{\tilde{2}}{2}}\right] \rightarrow 0
$$

as $n \rightarrow \infty$. We conclude that indeed $\left\|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right\|_{L^{\tilde{p}}} \rightarrow 0$ as $n \rightarrow \infty$, which completes the proof of part (i).

The proof of part (ii) follows from the same arguments as those used in the proof of (i), and we only sketch it. Note that by Lemma 3.2, for every $i=1, \ldots, d$,

$$
\left\|\mathcal{Y}^{n, i}-\mathcal{Y}^{\star, i}\right\|_{\mathscr{S}^{2}}+\left\|\mathcal{Z}^{n, i}-\mathcal{Z}^{\star, i}\right\|_{\mathscr{H}^{2}}+\left\|\mathcal{L}^{n, i}-\mathcal{L}^{\star, i}\right\|_{\mathscr{M}^{2}} \leq C_{\mathrm{ap}} \mathbb{E}\left[\left|B^{H^{n}, i}-B^{H^{\star}, i}\right|^{2}\right]
$$

and that for every $x, x^{*}, y \in \mathbb{R}$,

$$
\left|\partial_{y} f(x, y)-\partial_{y} f\left(x^{*}, y\right)\right|^{2} \leq C_{f}^{2} 2^{r+4}\left(1+|x|^{2 r}+\left|x^{*}\right|^{2 r}+|y|^{2 r}\right)\left|x-x^{*}\right|^{2}
$$

Moreover, since $\left|\partial_{s_{i}} h(\cdot)\right| \leq C_{h}$ for every $i=1, \ldots, d$, Hölder's inequality shows that

$$
\begin{aligned}
& \mathbb{E}\left[\left|B^{H^{n}, i}-B^{H^{\star}, i}\right|^{2}\right] \\
& \leq C_{h}^{2} C_{f}^{2} 2^{r+4} 4^{\frac{p}{2 r}}\left(1+\left\|X_{T}^{H^{n} ; o}\right\|_{L^{p}}^{2 r}+\left\|X_{T}^{H^{*}} ; o\right\|_{L^{p}}^{2 r}+\left\|h\left(S_{T}^{o}\right)\right\|_{L^{p}}^{2 r}\right)\left\|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right\|_{L^{\tilde{p}}}^{2},
\end{aligned}
$$

where $\widetilde{p}=\frac{2 p}{p-2 r}>1$. The claim follows from (3.18) and since $\lim _{n \rightarrow \infty}\left\|X_{T}^{H^{n} ; o}-X_{T}^{H^{\star} ; o}\right\|_{L^{\widetilde{p}}}=0$, as was shown in part (i).

### 3.3. Proof of Proposition 2.9 \& first order optimality.

Proof of Proposition 2.9. We start by proving the statement (i). Let $\left(H^{n}\right)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ be a sequence such that

$$
V(0)=\lim _{n \rightarrow \infty} \mathbb{E}\left[f\left(X_{T}^{H^{n} ; o}, h\left(S_{T}^{o}\right)\right)\right]
$$

Note that $\mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$ defined in Section 2.1 is a reflexive Banach space. Furthermore, since the sequence $\left(H^{n}\right)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ is $\mathbb{F}$-predictable and $\mathcal{K}$-valued $\mathbb{P} \otimes d t$-a.e. (by Definition 2.5 ), it is bounded in $\mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$. Hence, [19, Theorem 15.1.2, p.320] asserts that there are $H^{*} \in \mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$ and

$$
\begin{equation*}
\widetilde{H}^{n} \in \operatorname{conv}\left(H^{n}, H^{n+1}, \ldots\right), \quad n \in \mathbb{N} \tag{3.19}
\end{equation*}
$$

which satisfy

$$
\begin{equation*}
\left\|\widetilde{H}^{n}-H^{*}\right\|_{\mathscr{H}^{2}} \rightarrow 0 \quad \text { as } n \rightarrow \infty \tag{3.20}
\end{equation*}
$$

Note that as $\left(\widetilde{H}^{n}\right)_{n \in \mathbb{N}}$ is $\mathcal{K}$-valued $\mathbb{P} \otimes d t$-a.e. by (3.19), $H^{*}$ is also $\mathcal{K}$-valued $\mathbb{P} \otimes d t$-a.e. as well, thus $H^{*} \in \mathcal{A}$.

It remains to show that $H^{*}$ is an optimizer. To that end, since $S^{\circ}$ is an Itô $(\mathbb{F}, \mathbb{P})$ semimartingale satisfying $(2.1)$ and $\left(\widetilde{H}^{n}\right)_{n \in \mathbb{M}}$ is $\mathcal{K}$-valued $\mathbb{P} \otimes d t$-a.e. and satisfies $(3.20)$, there is a subsequence $\left(\widetilde{H}^{n}\right)_{n \in \mathbb{N}}$ of the one in (3.19) (for notational simplicity, we do not relabel that sequence) for which

$$
\begin{equation*}
\left(\tilde{H}^{n} \cdot S^{o}\right)_{T} \rightarrow\left(H^{*} \cdot S^{o}\right)_{T} \quad \mathbb{P} \text {-a.s. as } n \rightarrow \infty \tag{3.21}
\end{equation*}
$$

Furthermore, by convexity of $f$ in the first coordinate, $\left(\tilde{H}^{n}\right)_{n \in \mathbb{N}}$ is still a minimizing sequence, i.e., $V(0)=\lim _{n \rightarrow \infty} \mathbb{E}\left[f\left(X_{T}^{\widetilde{H}^{n}} ; o, h\left(S_{T}^{o}\right)\right)\right]$.

Finally, since $f$ is continuous and bounded from below (see Assumption 2.7), an application of Fatou's lemma shows that

$$
\begin{aligned}
\mathbb{E}\left[f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\right] & =\mathbb{E}\left[\lim _{n \rightarrow \infty} f\left(X_{T}^{\widetilde{H}^{n} ; o}, h\left(S_{T}^{o}\right)\right)\right] \\
& \leq \lim _{n \rightarrow \infty} \mathbb{E}\left[f\left(X_{T}^{\widetilde{H}^{n} ; o}, h\left(S_{T}^{o}\right)\right)\right]=V(0)
\end{aligned}
$$

ensuring the optimality of $H^{*} \in \mathcal{A}$.
The uniqueness of an optimizer follows immediately from the strict convexity of $f$ in the first coordinate (see Assumption 2.7 (iv)).

The claims made in part (ii) and (iii) follow immediately from Proposition 3.3.
We wrap up this section by establishing a first order optimality condition for the unique optimizer $H^{*} \in \mathcal{A}$, which is employed in the proof of Theorem 2.13.

Lemma 3.7. Suppose that Assumptions 2.1 and 2.7 are satisfied. Then the unique optimizer $H^{*}$ of $V(0)=\mathcal{V}\left(H^{*}, 0\right)$ (in Proposition 2.9 (i)) satisfies the first-order optimality condition

$$
\mathbb{E}\left[\partial_{x} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\left(X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right)\right] \geq 0, \quad \text { for every } H \in \mathcal{A} .
$$

Proof. Fix $H \in \mathcal{A}$. Clearly $H^{*}+\theta\left(H-H^{*}\right) \in \mathcal{A}$ for any $0<\theta<1$, hence it follows from the optimality of $H^{*}$ that

$$
\begin{equation*}
\mathbb{E}\left[\frac{1}{\theta}\left(f\left(X_{T}^{H^{*} ; o}+\theta\left(X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right), h\left(S_{T}^{o}\right)\right)-f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\right)\right] \geq 0 \tag{3.22}
\end{equation*}
$$

We claim that

$$
\Psi_{H}:=\sup _{\theta \in(0,1)}\left|\frac{1}{\theta}\left(f\left(X_{T}^{H^{*} ; o}+\theta\left(X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right), h\left(S_{T}^{o}\right)\right)-f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\right)\right|
$$

is in $L^{1}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$. If that is the case, then the proof follows from (3.22) and the dominated convergence theorem.

To show that $\mathbb{E}\left[\Psi_{H}\right]<\infty$, first note that by Assumption 2.7 (ii) and Remark 2.8 (i) (and the power triangle inequality),

$$
\left|\partial_{x x} f\left(x+x^{*}, y\right)\right| \leq C_{f} 2^{\frac{3 r}{2}}\left(1+|x|^{r}+\left|x^{*}\right|^{r}+|y|^{r}\right), \quad\left|\partial_{x} f(x, y)\right| \leq \widetilde{C}_{f}\left(1+|x|^{r+1}+|y|^{r+1}\right)
$$

for every $x, x^{*}, y \in \mathbb{R}^{2}$. Hence, a second-order Taylor expansion of $f$ implies that for any $H \in \mathcal{A}$,

$$
\begin{aligned}
\mathbb{E}\left[\Psi_{H}\right] \leq & \widetilde{C}_{f} \mathbb{E}\left[\left(1+\left|X_{T}^{H^{*} ; o}\right|^{r+1}+\left|h\left(S_{T}^{o}\right)\right|^{r+1}\right)\left|X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right|\right] \\
& +C_{f} 2^{\frac{3 r}{2}} \mathbb{E}\left[\left(1+\left|X_{T}^{H ; o}\right|^{r}+\left|X_{T}^{H^{*} ; o}\right|^{r}+\left|h\left(S_{T}^{o}\right)\right|^{r}\right)\left|X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right|^{2}\right] \\
= & : \mathbb{E}\left[\Psi_{H}^{1}\right]+\mathbb{E}\left[\Psi_{H}^{2}\right] .
\end{aligned}
$$

Since $0<r<\min \left\{\frac{p-2}{2}, p-3\right\}$ (see Assumptions 2.1 (i) and 2.7 (ii)), Hölder's inequality (with exponent $\frac{p}{r+1}>1$ and conjugate exponent $\left.\frac{p-(r+1)}{p}=1\right)$ ensures that

$$
\mathbb{E}\left[\Psi_{H}^{1}\right] \leq \widetilde{C}_{f}\left\|1+\left|X_{T}^{H^{*} ; o}\right|^{r+1}+\left|h\left(S_{T}^{o}\right)\right|^{r+1}\right\|_{L^{\frac{p}{r+1}}}\left\|X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right\|_{L^{\frac{p}{p-(r+1)}}}
$$

Furthermore, by Jensen's inequality (noting that $x \rightarrow|x|^{p-(r+1)}$ is convex since $p>r+2$ ), $\left\|X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right\|_{L^{\frac{p}{p-(r+1)}}} \leq\left\|X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right\|_{L^{p}}$ and by the power triangle inequality,

$$
\mathbb{E}\left[\left(1+\left|X_{T}^{H^{*} ; o}\right|^{r+1}+\left|h\left(S_{T}^{o}\right)\right|^{r+1}\right)^{\frac{p}{r+1}}\right] \leq 3^{\frac{p}{r+1}}\left(1+\mathbb{E}\left[\left|X_{T}^{H^{*} ; o}\right|^{p}\right]+\mathbb{E}\left[\left|h\left(S_{T}^{o}\right)\right|^{p}\right]\right)
$$

Hence, the a priori estimates on $X^{H ; b, \sigma}$ and $h\left(S_{T}^{b, \sigma}\right)$ (for any $(b, \sigma) \in \mathcal{B}^{1}$ ) detailed in Lemma 3.1 (ii) ensures that $\mathbb{E}\left[\Psi_{H}^{1}\right]<\infty$.

Similarly, we can deduce that

$$
\mathbb{E}\left[\Psi_{H}^{2}\right] \leq C_{f} 2^{\frac{3 r}{2}} 4\left(1+\mathbb{E}\left[\left|X_{T}^{H ; o}\right|^{p}\right]+\mathbb{E}\left[\left|X_{T}^{H^{*} ; o}\right|^{p}\right]+\mathbb{E}\left[\left|h\left(S_{T}^{o}\right)\right|^{p}\right]\right)^{\frac{r}{p}}\left\|X_{T}^{H ; o}-X_{T}^{H^{*} ; o}\right\|_{L^{p}}^{2}<\infty
$$

thus $\Psi_{H} \in L^{1}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$, as claimed.
3.4. Proof of Theorem 2.13. Following the outline of the proof at the beginning of this section, we first establish the crucial observation that the expectation of the first order derivative can be expressed in a way that is linear in $b-b^{o}$ and $\sigma-\sigma^{o}$, i.e. (3.2). To that end, recall that $\langle X, Y\rangle_{\mathbb{P} \otimes d t}=\mathbb{E}\left[\int_{0}^{T}\left\langle X_{t}, Y_{t}\right\rangle d t\right]$ and similarly for $\langle\cdot, \cdot\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}$, see Section 2.1.

Lemma 3.8. Suppose that Assumptions 2.1 and 2.7 are satisfied. Let $H \in \mathcal{A}$, let $A^{H}$ and $B^{H}$ be given in (3.9), and let $\left(Y^{H}, Z^{H}, L^{H}\right)$ and $\left(\mathcal{Y}^{H}, \mathcal{Z}^{H}, \mathcal{L}^{H}\right)$ be the unique solution of the BSDEs given in (3.10) and (3.11), respectively. Then the following holds: for every $\varepsilon \in[0,1]$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon}$,

$$
\begin{align*}
& \mathbb{E}\left[A^{H}\left(X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right)+\left(B^{H}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right]  \tag{3.23}\\
& =\left\langle Y^{H} H+\mathcal{Y}^{H}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{H} H^{\top}+\mathcal{Z}^{H}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}
\end{align*}
$$

Proof. Step 1. We start by focusing on the first term in (3.23) and show that

$$
\begin{equation*}
\mathbb{E}\left[A^{H}\left(X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right)\right]=\left\langle Y^{H} H, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{H} H^{\top}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}} \tag{3.24}
\end{equation*}
$$

To that end, denote by

$$
\Xi_{b}:=A^{H} \int_{0}^{T} H_{t}^{\top}\left(b_{t}-b_{t}^{o}\right) d t, \quad \Xi_{\sigma}:=A^{H} \int_{0}^{T} H_{t}^{\top}\left(\sigma_{t}-\sigma_{t}^{o}\right) d W_{t}
$$

so that $\mathbb{E}\left[\partial_{x} f\left(X_{T}^{H ; o}, h\left(S_{T}^{o}\right)\right)\left(X_{T}^{H ; b, \sigma}-X_{T}^{H ; o}\right)\right]=\mathbb{E}\left[\Xi_{b}+\Xi_{\sigma}\right]$.
We claim that $\mathbb{E}\left[\Xi_{b}\right]=\left\langle Y^{H} H, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}$, and first note that $\mathbb{E}\left[\left|\Xi_{b}\right|\right]<\infty$. Indeed, by Hölder's inequality (with exponent $p>3$ ) and the fact that $\left|H_{t}\right| \leq K \mathbb{P} \otimes d t$-a.e.,

$$
\begin{aligned}
\mathbb{E}\left[\left|\Xi_{b}\right|\right] & \leq\left\|A^{H}\right\|_{L^{q}}\left\|\int_{0}^{T} H_{t}^{\top}\left(b_{t}-b_{t}^{o}\right) d t\right\|_{L^{p}} \\
& \leq K\left\|A^{H}\right\|_{L^{q}}\left\|\int_{0}^{T}\left|b_{t}-b_{t}^{o}\right| d t\right\|_{L^{p}} \leq K\left\|A^{H}\right\|_{L^{q}} T^{1-\frac{1}{p}}\left\|b-b^{o}\right\|_{\mathbb{L}^{p}}
\end{aligned}
$$

Furthermore, since $\left|\partial_{x} f(x, y)\right| \leq \widetilde{C}_{f}\left(1+|x|^{r+1}+|y|^{r+1}\right)$ for every $x, y \in \mathbb{R}$ (see Remark 2.8 (i)), Hölder's inequality with exponent $\frac{p-1}{(r+1)}>1$ (and the power triangle inequality) ensures that

$$
\begin{aligned}
\mathbb{E}\left[\left|A^{H}\right|^{q}\right] & \leq 3^{q} \widetilde{C}_{f}^{q}\left(1+\mathbb{E}\left[\left|X_{T}^{H ; o}\right|^{q(r+1)}\right]+\mathbb{E}\left[\left|h\left(S_{T}^{o}\right)\right|^{q(r+1)}\right]\right) \\
& \leq 3^{q} \widetilde{C}_{f}^{q}\left(1+\mathbb{E}\left[\left|X_{T}^{H ; o}\right|^{p}\right]^{\frac{r+1}{p-1}}+\mathbb{E}\left[\left|h\left(S_{T}^{o}\right)\right|^{p}\right]^{\frac{r+1}{p-1}}\right)
\end{aligned}
$$

Combining this with Lemma 3.1 (ii) and $\left\|b-b^{o}\right\|_{\mathbb{L}^{p}} \leq \gamma \varepsilon \leq 1$, we conclude that $\mathbb{E}\left[\left|\Xi_{b}\right|\right]<\infty$.
Therefore, by Fubini's theorem and since $Y_{t}^{H}=\mathbb{E}\left[A^{H} \mid \mathcal{F}_{t}\right], t \in[0, T]$ (see Proposition 3.3),

$$
\begin{equation*}
\mathbb{E}\left[\Xi_{b}\right]=\int_{0}^{T} \mathbb{E}\left[\mathbb{E}\left[A^{H} \mid \mathcal{F}_{t}\right] H_{t}^{\top}\left(b_{t}-b_{t}^{o}\right)\right] d t=\left\langle Y^{H} H, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t} \tag{3.25}
\end{equation*}
$$

Next, we claim that $\mathbb{E}\left[\Xi_{\sigma}\right]=\left\langle Z^{H} H^{\top}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}$. Note that by Proposition 3.3 (ii),

$$
\begin{align*}
\mathbb{E}\left[\Xi_{\sigma}\right]=\mathbb{E} & {\left[\int_{0}^{T}\left(Z_{t}^{H}\right)^{\top} d W_{t} \int_{0}^{T} H_{t}^{\top}\left(\sigma_{t}-\sigma_{t}^{o}\right) d W_{t}+\left(L_{T}^{H}-L_{0}^{H}\right) \int_{0}^{T} H_{t}^{\top}\left(\sigma_{t}-\sigma_{t}^{o}\right) d W_{t}\right.}  \tag{3.26}\\
& \left.+Y_{0}^{H} \int_{0}^{T} H_{t}^{\top}\left(\sigma_{t}-\sigma_{t}^{o}\right) d W_{t}\right]=: \mathbb{E}\left[\mathrm{I}_{\sigma}+\mathrm{II}_{\sigma}+\mathrm{III}_{\sigma}\right]
\end{align*}
$$

Since $\left(\sigma-\sigma^{o}\right)^{\top} H \in \mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$ (because $\left|H_{t}\right| \leq K \mathbb{P} \otimes d t$-a.e.) and $\left(Y^{H}, Z^{H}, L^{H}\right) \in \mathscr{S}^{2}(\mathbb{R}) \times$ $\mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})$, it follows that $\mathbb{E}\left[\left|\Xi_{\sigma}\right|\right]<\infty$.

An application of the Itô-isometry shows that

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{I}_{\sigma}\right] & =\mathbb{E}\left[\int_{0}^{T}\left(Z_{t}^{H}\right)^{\top}\left(\sigma_{t}-\sigma_{t}^{o}\right)^{\top} H_{t} d t\right]=\mathbb{E}\left[\int_{0}^{T}\left\langle Z_{t}^{H} H_{t}^{\top}, \sigma_{t}-\sigma_{t}^{o}\right\rangle_{\mathrm{F}} d t\right] \\
& =\left\langle Z^{H} H^{\top}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}
\end{aligned}
$$

Moreover, by [75, Lemma 2 \& Theorem 35, p.149], $L^{H}$ and $\left(\int_{0}^{t} H_{s}^{\top}\left(\sigma_{s}-\sigma_{s}^{o}\right) d W_{s}\right)_{t \in[0, T]}$ are strongly orthogonal, thus $\mathbb{E}\left[\mathrm{II}_{\sigma}\right]=0$. Finally, since $\mathcal{F}_{0}$ is trivial (see Section 2.1 ), $\mathbb{E}\left[\mathrm{III}_{\sigma}\right]=0$.

We conclude that

$$
\begin{equation*}
\mathbb{E}\left[\Xi_{\sigma}\right]=\mathbb{E}\left[\mathrm{I}_{\sigma}\right]=\left\langle Z^{H} H^{\top}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}} \tag{3.27}
\end{equation*}
$$

and combined with (3.25), this shows (3.24).

Step 2. We proceed to analyze the second term in (3.23) and show that

$$
\begin{equation*}
\mathbb{E}\left[\left(B^{H}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right]=\left\langle\mathcal{Y}^{H}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle\mathcal{Z}^{H}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}} \tag{3.28}
\end{equation*}
$$

To that end, we first note that

$$
\left(B^{H}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)=\sum_{i=1}^{d} \partial_{y} f\left(X_{T}^{H ; o}, h\left(S_{T}^{o}\right)\right) \partial_{s_{i}} h\left(S_{T}^{o}\right)\left(\int_{0}^{T}\left(b_{t}^{i}-b_{t}^{o, i}\right) d t+\int_{0}^{T}\left(\sigma_{t}^{i}-\sigma_{t}^{o, i}\right) d W_{t}\right)
$$

where for $i=1, \ldots, d,\left(b_{t}^{i}-b_{t}^{o, i}\right)_{t \in[0, T]}$ denotes the $i$-th component of $b-b^{o}$ and $\left(\sigma_{t}^{i}-\sigma_{t}^{o, i}\right)_{t \in[0, T]}$ denotes the $i$-th row of $\sigma-\sigma^{o}$.

For $i=1, \ldots, d$, set $B^{H, i}:=\partial_{y} f\left(X_{T}^{H ; o}, h\left(S_{T}^{o}\right)\right) \partial_{s_{i}} h\left(S_{T}^{o}\right)$ and denote by

$$
\Xi_{b}^{i}:=B^{H, i} \int_{0}^{T}\left(b_{t}^{i}-b_{t}^{o, i}\right) d t, \quad \Xi_{\sigma}^{i}:=B^{H, i} \int_{0}^{T}\left(\sigma_{t}^{i}-\sigma_{t}^{o, i}\right) d W_{t}
$$

so that

$$
\begin{equation*}
\mathbb{E}\left[\left(B^{H}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right]=\mathbb{E}\left[\sum_{i=1}^{d}\left(\Xi_{b}^{i}+\Xi_{\sigma}^{i}\right)\right] \tag{3.29}
\end{equation*}
$$

First note that for every $\left.i=1, \ldots, d,\left(\mathcal{Y}^{H, i}, \mathcal{Z}^{H, i}, \mathcal{L}^{H, i}\right)\right) \in \mathscr{S}^{2}(\mathbb{R}) \times \mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})$ is the unique solution of BSDE with terminal condition $B^{H, i}$. It follows from the same arguments as given for the proof of (3.25) and the fact that $\left|\partial_{s_{i}} h(\cdot)\right| \leq C_{h}$ (see Remark 2.8 (ii)) that

$$
\begin{equation*}
\mathbb{E}\left[\Xi_{b}^{i}\right]=\mathbb{E}\left[\int_{0}^{T} \mathcal{Y}_{t}^{H, i}\left(b_{t}^{i}-b_{t}^{o, i}\right) d t\right] \tag{3.30}
\end{equation*}
$$

Similarly, it follows from the same argument as given for the proofs of (3.26) and (3.27) (by replacing $Z^{H}$ and $H^{\top}\left(\sigma-\sigma^{o}\right)$ with $\mathcal{Z}^{H, i}$ and $\sigma^{i}-\sigma^{o, i}$, respectively) that

$$
\begin{equation*}
\mathbb{E}\left[\Xi_{\sigma}^{i}\right]=\left\langle\mathcal{Z}^{H, i},\left(\sigma^{i}-\sigma^{o, i}\right)^{\top}\right\rangle_{\mathbb{P} \otimes d t} \tag{3.31}
\end{equation*}
$$

Combining (3.29), (3.30), and (3.31), we obtain that indeed

$$
\begin{aligned}
\mathbb{E}\left[\left(B^{H}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right] & =\sum_{i=1}^{d}\left(\mathbb{E}\left[\int_{0}^{T} \mathcal{Y}_{t}^{H, i}\left(b_{t}^{i}-b_{t}^{o, i}\right) d t\right]+\left\langle\mathcal{Z}^{H, i},\left(\sigma^{i}-\sigma^{o, i}\right)^{\top}\right\rangle_{\mathbb{P} \otimes d t}\right) \\
& =\left\langle\mathcal{Y}^{H}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle\mathcal{Z}^{H}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}
\end{aligned}
$$

The proof of the lemma now follows from (3.24) and (3.28).
Lemma 3.9. Suppose that Assumptions 2.1 and 2.7 are satisfied. Moreover, let $H^{*}$ be the unique optimizer for $V(0)$ (given in Proposition 2.9 (i)) and let $\left(Y^{*}, Z^{*}, L^{*}\right)$ and $\left(\mathcal{Y}^{*}, \mathcal{Z}^{*}, \mathcal{L}^{*}\right)$ be the unique solution of (2.5) and (2.6), respectively. Then there exists a constant $C>0$ such that for any $\varepsilon \in[0,1]$,

$$
0 \leq V(\varepsilon)-V(0) \leq \varepsilon\left(\gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}\right)+C \varepsilon^{2}
$$

In particular, $\lim _{\varepsilon \downarrow 0} V(\varepsilon)=V(0)$.
Proof. Step 1. Set

$$
A^{*}:=A^{H^{*}}=\partial_{x} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right), \quad B^{*}:=B^{H^{*}}=\partial_{y} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right)
$$

A second-order Taylor expansion of $f$ and $h$ around $\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)$ and $S_{T}^{o}$, respectively, implies that for every $\varepsilon \geq 0$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon}$,

$$
\begin{align*}
& f\left(X_{T}^{H^{*} ; b, \sigma}, h\left(S_{T}^{b, \sigma}\right)\right)-f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)  \tag{3.32}\\
& =A^{*}\left(X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}\right)+\left(B^{*}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)+\mathrm{I}^{b, \sigma}+\mathrm{II}^{b, \sigma}
\end{align*}
$$

where $\mathrm{I}^{b, \sigma}$ and $\mathrm{II}^{b, \sigma}$ are given by

$$
\begin{aligned}
& \mathrm{I}^{b, \sigma}:=\partial_{y} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)^{\top} \int_{0}^{1}(1-\theta) D^{2} h\left(S_{T}^{o}+\theta\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right) d \theta\left(S_{T}^{b, \sigma}-S_{T}^{o}\right) \\
& \mathrm{II}^{b, \sigma}:=\left(X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}, h\left(S_{T}^{b, \sigma}\right)-h\left(S_{T}^{o}\right)\right) \int_{0}^{1}(1-\theta) D^{2} f\left(\widetilde{X}^{b, \sigma ; \theta}, \widetilde{h}^{b, \sigma ; \theta}\right) d \theta\binom{X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}}{h\left(S_{T}^{b, \sigma}\right)-h\left(S_{T}^{o}\right)}
\end{aligned}
$$

and, for $\theta \in[0,1], \widetilde{X}^{b, \sigma ; \theta}$ and $\widetilde{h}^{b, \sigma ; \theta}$ are given by

$$
\begin{aligned}
\widetilde{X}^{b, \sigma ; \theta} & :=X_{T}^{H^{*} ; o}+\theta\left(X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}\right), \\
\widetilde{h}^{b, \sigma ; \theta} & :=h\left(S_{T}^{o}\right)+\theta\left(h\left(S_{T}^{b, \sigma}\right)-h\left(S_{T}^{o}\right)\right)
\end{aligned}
$$

Step 2. We claim that there exist $C_{\mathrm{I}}, C_{\mathrm{II}}>0$ such that for every $\varepsilon \in[0,1]$,

$$
\begin{equation*}
\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[\left|\mathrm{I}^{b, \sigma}\right|+\left|\mathrm{II}^{b, \sigma}\right|\right] \leq\left(C_{\mathrm{I}}+C_{\mathrm{II}}\right) \varepsilon^{2} \tag{3.33}
\end{equation*}
$$

Step 2, estimates on $\mathrm{I}^{b, \sigma}$. Since $\left\|D^{2} h(\cdot)\right\|_{\mathrm{F}} \leq C_{h}$ and $\left|\partial_{y} f(x, y)\right| \leq \widetilde{C}_{f}\left(1+|x|^{r+1}+|y|^{r+1}\right)$ for every $x, y \in \mathbb{R}$ (see Remark 2.8), it follows that for every $\varepsilon \in[0,1]$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon}$,

$$
\begin{align*}
\mathbb{E}\left[\left|I^{b, \sigma}\right|\right] & \leq C_{h} \widetilde{C}_{f} \mathbb{E}\left[\left(1+\left|X_{T}^{H^{*} ; o}\right|^{r+1}+\left|h\left(S_{T}^{o}\right)\right|^{r+1}\right)\left|S_{T}^{b, \sigma}-S_{T}^{o}\right|^{2}\right] \\
& \leq C_{h} \widetilde{C}_{f}\left\|1+\left|X_{T}^{H^{*} ; o}\right|^{r+1}+\left|h\left(S_{T}^{o}\right)\right|^{r+1}\right\|_{L^{\frac{p}{p-2}}}\left\|S_{T}^{b, \sigma}-S_{T}^{o}\right\|_{L^{p}}^{2} \tag{3.34}
\end{align*}
$$

where the second inequality follows from Hölder's inequality (with exponent $\frac{p}{2}>1$ ). By Lemma 3.1 (i) we have that for every $\varepsilon \in[0,1]$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon},\left\|S_{T}^{b, \sigma}-S_{T}^{o}\right\|_{L^{p}}^{2} \leq C_{2}^{\frac{2}{p}} \varepsilon^{2}$. Moreover, by the power triangle inequality,

$$
\begin{aligned}
& \mathbb{E}\left[\left(1+\left|X_{T}^{H^{*} ; o}\right|^{r+1}+\left|h\left(S_{T}^{o}\right)\right|^{r+1}\right)^{\frac{p}{p-2}}\right] \\
& \leq 3^{\frac{p}{p-2}}\left(1+\mathbb{E}\left[\left|X_{T}^{H^{*} ; o}\right|^{\frac{(r+1) p}{p-2}}\right]+\mathbb{E}\left[\left|h\left(S_{T}^{o}\right)\right|^{\frac{(r+1) p}{p-2}}\right]\right)<\infty
\end{aligned}
$$

where the last inequality follows from Lemma 3.1 (ii) (noting that $\frac{(r+1) p}{p-2} \leq p$ ).
Combined with (3.34), we conclude that there is some $C_{\mathrm{I}}>0$ such that for every $\varepsilon \in[0,1]$, $\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[\left|I^{b, \sigma}\right|\right] \leq C_{\mathrm{I}} \varepsilon^{2}$.
Step 2, estimates on $\mathrm{II}^{b, \sigma}$. Set

$$
\mathrm{III}^{b, \sigma}:=1+\left|X_{T}^{H^{*} ; b, \sigma}\right|^{r}+\left|X_{T}^{H^{*} ; o}\right|^{r}+\left|h\left(S_{T}^{b, \sigma}\right)\right|^{r}+\left|h\left(S_{T}^{o}\right)\right|^{r}
$$

Then, Hölder's inequality with exponent $\frac{p-2}{r}>1$ (and the power triangle inequality) and the a priori estimates given in Lemma 3.1 (ii) show that for every $\varepsilon \in[0,1]$,

$$
\begin{equation*}
\sup _{\varepsilon \in[0,1]} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\|\operatorname{III}^{b, \sigma}\right\|_{L^{\frac{p}{p-2}}}<\infty \tag{3.35}
\end{equation*}
$$

Moreover, by Assumption 2.7 (ii) we have that for every $x, x^{*}, y, y^{*} \in \mathbb{R}$,

$$
\left\|D^{2} f\left(x+x^{*}, y+y^{*}\right)\right\|_{\mathrm{F}} \leq C_{f} 2^{\frac{3 r}{2}}\left(1+|x|^{r}+\left|x^{*}\right|^{r}+|y|^{r}+\left|y^{*}\right|^{r}\right)
$$

hence it follows that for every $(b, \sigma) \in \mathcal{B}^{\varepsilon}$ and $\varepsilon \in[0,1]$,

$$
\begin{align*}
\mathbb{E}\left[\left|\mathrm{II}^{b, \sigma}\right|\right] & \leq C_{f} 2^{\frac{3}{2} r} \cdot \mathbb{E}\left[\mathrm{III}^{b, \sigma}\left(\left|X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}\right|^{2}+\left|h\left(S_{T}^{b, \sigma}\right)-h\left(S_{T}^{o}\right)\right|^{2}\right)\right]  \tag{3.36}\\
& \leq C_{f} 2^{\frac{3}{2} r} \cdot\left\|\mathrm{III}^{b, \sigma}\right\|_{L^{\frac{p}{p-2}}}\left(\left\|X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}\right\|_{L^{p}}^{2}+\left\|h\left(S_{T}^{b, \sigma}\right)-h\left(S_{T}^{o}\right)\right\|_{L^{p}}^{2}\right)
\end{align*}
$$

where we used Hölder's inequality (with exponent $\frac{p}{2}>1$ ) in the second step.
Note that by Lemma 3.1 (i), $\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\|X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}\right\|_{L^{p}}^{2} \leq C_{1}^{\frac{2}{p}} \varepsilon^{2}$. Furthermore, since $|\nabla h(s)-\nabla h(\hat{s})| \leq C_{h}|s-\hat{s}|$ for every $s, \hat{s} \in \mathbb{R}^{d}$ (see Remark 2.8 (ii)), it follows that for every $\varepsilon \in[0,1]$,

$$
\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\|h\left(S_{T}^{b, \sigma}\right)-h\left(S_{T}^{o}\right)\right\|_{L^{p}}^{2} \leq C_{h}^{2} \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\|S_{T}^{b, \sigma}-S_{T}^{o}\right\|_{L^{p}}^{2} \leq C_{h}^{2} C_{2}^{\frac{2}{p}} \varepsilon^{2}
$$

Combined with (3.35) and (3.36), this ensures that there is $C_{\mathrm{II}}>0$ such that for every $\varepsilon \in[0,1], \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[\left|I I^{b, \sigma}\right| \mid\right] \leq C_{\mathrm{II}} \varepsilon^{2}$.
Step 3. For any $\varepsilon \in[0,1]$, set

$$
\Phi(\varepsilon):=\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[A^{*}\left(X_{T}^{H^{*} ; b, \sigma}-X_{T}^{H^{*} ; o}\right)+\left(B^{*}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right]
$$

Then, by (3.32) and (3.33), for every $\varepsilon \in[0,1]$,

$$
\begin{equation*}
0 \leq V(\varepsilon)-V(0) \leq \Phi(\varepsilon)+\left(C_{\mathrm{I}}+C_{\mathrm{II}}\right) \varepsilon^{2} \tag{3.37}
\end{equation*}
$$

It remains to show that for every $\varepsilon \in[0,1]$,

$$
\Phi(\varepsilon) \leq \varepsilon\left(\gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}\right)
$$

To that end, we note that by Lemma 3.8,

$$
\begin{equation*}
\Phi(\varepsilon)=\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left(\left\langle Y^{*} H^{*}+\mathcal{Y}^{*}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}\right) \tag{3.38}
\end{equation*}
$$

Set $\mathcal{B}_{1}^{\varepsilon}:=\left\{b:(b, \sigma) \in \mathcal{B}^{\varepsilon}\right\}$ and $\mathcal{B}_{2}^{\varepsilon}:=\left\{\sigma:(b, \sigma) \in \mathcal{B}^{\varepsilon}\right\}$ so that $\mathcal{B}^{\varepsilon}=\mathcal{B}_{1}^{\varepsilon} \times \mathcal{B}_{2}^{\varepsilon}$ by the definition of $\mathcal{B}^{\varepsilon}$. It follows from the Cauchy-Schwartz inequality and Hölder's inequality (with exponent $p>3)$ that for every $\varepsilon \in[0,1]$ and $b \in \mathcal{B}_{1}^{\varepsilon}$,

$$
\begin{aligned}
\left\langle Y^{*} H^{*}+\mathcal{Y}^{*}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t} & \leq \mathbb{E}\left[\int_{0}^{T}\left|b_{t}-b_{t}^{o} \| Y_{t}^{*} H_{t}^{*}+\mathcal{Y}_{t}^{*}\right| d t\right] \\
& \leq\left\|b-b^{o}\right\|_{\mathbb{L}^{p}}\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}
\end{aligned}
$$

Hence for every $\varepsilon \in[0,1]$ we have that

$$
\begin{equation*}
\sup _{b \in \mathcal{B}_{1}^{\varepsilon}}\left\langle Y^{*} H^{*}+\mathcal{Y}^{*}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t} \leq \varepsilon \gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}} \tag{3.39}
\end{equation*}
$$

Similarly, using Cauchy-Schwartz inequality for $\|\cdot\|_{F}$ and Hölder's inequality (with exponent $2)$, it follows that for every $\varepsilon \in[0,1]$ and $\sigma \in \mathcal{B}_{2}^{\varepsilon}$,

$$
\begin{aligned}
\left\langle Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}} & \leq \mathbb{E}\left[\int_{0}^{T}\left\|\sigma_{t}-\sigma_{t}^{o}\right\|_{\mathrm{F}}\left\|Z_{t}^{*}\left(H_{t}^{*}\right)^{\top}+\mathcal{Z}_{t}^{*}\right\|_{\mathrm{F}} d t\right] \\
& \leq \mathbb{E}\left[\left(\int_{0}^{T}\left\|\sigma_{t}-\sigma_{t}^{o}\right\|_{\mathrm{F}}^{2} d t\right)^{\frac{1}{2}}\left(\int_{0}^{T}\left\|Z_{t}^{*}\left(H_{t}^{*}\right)^{\top}+\mathcal{Z}_{t}^{*}\right\|_{\mathrm{F}}^{2} d t\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

Therefore, another application of Hölder's inequality shows that for every $\varepsilon \in[0,1]$,

$$
\begin{align*}
\sup _{\sigma \in \mathcal{B}_{2}^{\varepsilon}}\left\langle Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}} & \leq \sup _{\sigma \in \mathcal{B}_{2}^{\varepsilon}}\left\|\sigma-\sigma^{o}\right\|_{\mathbb{H}^{p}}\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}  \tag{3.40}\\
& \leq \varepsilon \eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}
\end{align*}
$$

Combining (3.39) and (3.40) with (3.37) concludes the proof.
Remark 3.10. Recall $V^{*}(\varepsilon)$ defined in (2.8). The proof of Lemma 3.9 actually shows that under the same assumptions as therein, for any $\varepsilon \in[0,1]$,

$$
0 \leq V^{*}(\varepsilon)-V^{*}(0) \leq \varepsilon\left(\gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}\right)+C \varepsilon^{2}
$$

This will be used later in the proof of Theorem 2.14.
From Lemma 3.9, we can deduce the upper bound

$$
\begin{equation*}
V^{\prime}(0) \leq \gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q} q} \tag{3.41}
\end{equation*}
$$

where $H^{*}$ is the unique optimizer for $V(0)$ and $\left(Y^{*}, Z^{*}, \mathcal{Y}^{*}, \mathcal{Z}^{*}\right)$ are defined in Proposition 2.9.
To prove the corresponding lower bound in (3.41), let us consider $\varepsilon^{2}$-optimizers $H^{\varepsilon} \in \mathcal{A}$ of $V(\varepsilon)$, i.e.,

$$
\begin{equation*}
V(\varepsilon)=\inf _{H \in \mathcal{A}} \mathcal{V}(H, \varepsilon)>\mathcal{V}\left(H^{\varepsilon}, \varepsilon\right)-\varepsilon^{2}=\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[f\left(X_{T}^{H^{\varepsilon} ; b, \sigma}, h\left(S_{T}^{b, \sigma}\right)\right)\right]-\varepsilon^{2} \tag{3.42}
\end{equation*}
$$

The lower bound follows from the following lemma, together with an additional result which implies that $H^{\varepsilon}$ converges to $H^{*}$ in a suitable sense.

Lemma 3.11. Suppose that Assumptions 2.1 and 2.7 are satisfied. For any $\varepsilon \in(0,1]$, let $H^{\varepsilon} \in \mathcal{A}$ be an $\varepsilon^{2}$-optimizer of $V(\varepsilon)$. Set

$$
A^{\varepsilon}:=A^{H^{\varepsilon}}=\partial_{x} f\left(X_{T}^{H^{\varepsilon} ; o}, h\left(S_{T}^{o}\right)\right), \quad B^{\varepsilon}:=B^{H^{\varepsilon}}=\partial_{y} f\left(X_{T}^{H^{\varepsilon} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right)
$$

and let

$$
\begin{align*}
& \left(Y^{\varepsilon}, Z^{\varepsilon}, L^{\varepsilon}\right) \in \mathscr{S}^{2}(\mathbb{R}) \times \mathscr{H}^{2}\left(\mathbb{R}^{d}\right) \times \mathscr{M}^{2}(\mathbb{R})  \tag{3.43}\\
& \left(\mathcal{Y}^{\varepsilon}, \mathcal{Z}^{\varepsilon}, \mathcal{L}^{\varepsilon}\right) \in\left(\mathscr{S}^{2}(\mathbb{R})\right)^{d} \times\left(\mathscr{H}^{2}\left(\mathbb{R}^{d}\right)\right)^{d} \times\left(\mathscr{M}^{2}(\mathbb{R})\right)^{d} \tag{3.44}
\end{align*}
$$

be the unique solutions of (3.10) with the terminal condition $A^{\varepsilon}$ and (3.11) with the terminal condition $B^{\varepsilon}$, respectively (ensured by Proposition 3.3). Then,

$$
V(\varepsilon)-V(0) \geq \varepsilon\left(\gamma\left\|Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}\right\|_{\mathbb{H}^{q}}\right)-C_{\mathrm{res}} \varepsilon^{2}
$$

where $C_{\mathrm{res}}:=C_{\mathrm{I}}+C_{\mathrm{II}}+3$ and $C_{\mathrm{I}}, C_{\mathrm{II}}>0$ are given in (3.33).

Proof. It is obvious that

$$
\begin{equation*}
V(\varepsilon)-V(0) \geq \mathcal{V}\left(H^{\varepsilon}, \varepsilon\right)-\mathcal{V}\left(H^{\varepsilon}, 0\right)-\varepsilon^{2} \tag{3.45}
\end{equation*}
$$

Furthermore, from the second-order Taylor expansion of $f$ given in (3.32) and the a priori estimates given in (3.33) with $C_{\mathrm{I}}, C_{\mathrm{II}}>0$ (replacing $H^{*}$ by $H^{\varepsilon}$, see the proof of Lemma 3.9), it follows that for every $\varepsilon \in(0,1]$,

$$
\begin{align*}
\mathcal{V}\left(H^{\varepsilon}, \varepsilon\right)-\mathcal{V}\left(H^{\varepsilon}, 0\right) & =\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[f\left(X_{T}^{H^{\varepsilon} ; b, \sigma}, h\left(S_{T}^{o}\right)\right)-f\left(X_{T}^{H^{\varepsilon} ; o}, h\left(S_{T}^{o}\right)\right)\right] \\
& \geq \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[A^{\varepsilon}\left(X_{T}^{H^{\varepsilon} ; b, \sigma}-X_{T}^{H^{\varepsilon} ; o}\right)+\left(B^{\varepsilon}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right]-\left(C_{\mathrm{I}}+C_{\mathrm{II}}\right) \varepsilon^{2} . \tag{3.46}
\end{align*}
$$

Next, note that for $\left(Y^{\varepsilon}, Z^{\varepsilon}\right)$ and $\left(\mathcal{Y}^{\varepsilon}, \mathcal{Z}^{\varepsilon}\right)$ given in (3.43) and (3.44), Lemma 3.8 implies that

$$
\begin{align*}
& \sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}} \mathbb{E}\left[A^{\varepsilon}\left(X_{T}^{H^{\varepsilon} ; b, \sigma}-X_{T}^{H^{\varepsilon} ; o}\right)+\left(B^{\varepsilon}\right)^{\top}\left(S_{T}^{b, \sigma}-S_{T}^{o}\right)\right]  \tag{3.47}\\
& =\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left(\left\langle Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}, b-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}\right) .
\end{align*}
$$

By [32, Remark 5.3, p.137], the left limit process $Y_{-}^{\varepsilon}:=\left(Y_{t-}^{\varepsilon}\right)_{t \in[0, T]}$ defined by $Y_{t-}^{\varepsilon}:=\lim _{s \uparrow t} Y_{s}^{\varepsilon}$, $t \in(0, T]$ and $Y_{0-}^{\varepsilon}:=Y_{0}^{\varepsilon}$, is the $\mathbb{F}$-predictable projection of $Y^{\varepsilon}$. Moreover, as $Y^{\varepsilon}$ is càdlàg, $Y_{t}^{\varepsilon}=Y_{t-}^{\varepsilon} \mathbb{P} \otimes d t$-a.e.. Using the same notation and arguments, it follows that $\mathcal{Y}_{t}^{\varepsilon}=\mathcal{Y}_{t-}^{\varepsilon} \mathbb{P} \otimes d t-$ a.e.. Therefore, we can invoke the duality between $\mathbb{F}$-predictable spaces $\mathbb{L}^{q}\left(\mathbb{R}^{d}\right)$ and $\mathbb{L}^{p}\left(\mathbb{R}^{d}\right)$ to obtain some $\widetilde{b}^{\varepsilon} \in \mathbb{L}^{p}\left(\mathbb{R}^{d}\right)$ that satisfies $\left\|\widetilde{b}^{\varepsilon}\right\|_{\mathbb{L}^{p}}=1$ and

$$
\begin{align*}
\left\|Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}\right\|_{\mathbb{L}^{q}} & =\left\|Y_{-}^{\varepsilon} H^{\varepsilon}+\mathcal{Y}_{-}^{\varepsilon}\right\|_{\mathbb{L}^{q}} \\
& =\sup _{\|\widetilde{b}\|_{\mathbb{L}^{p}}=1}\left\langle Y_{-}^{\varepsilon} H^{\varepsilon}+\mathcal{Y}_{-}^{\varepsilon}, \widetilde{b}\right\rangle_{\mathbb{P} \otimes d t}  \tag{3.48}\\
& \leq\left\langle Y_{-}^{\varepsilon} H^{\varepsilon}+\mathcal{Y}_{-}^{\varepsilon}, \widetilde{b}^{\varepsilon}\right\rangle_{\mathbb{P} \otimes d t}+\varepsilon=\left\langle Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}, \widetilde{b}^{\varepsilon}\right\rangle_{\mathbb{P} \otimes d t}+\varepsilon
\end{align*}
$$

In a similar manner, since $Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}$ is $\mathbb{F}$-predictable (because $H^{\varepsilon} \in \mathcal{A}, Z^{\varepsilon} \in \mathscr{H}^{2}(\mathbb{R})$ and $\left.\left\{\mathcal{Z}^{\varepsilon, i}\right\}_{i=1, \ldots, d} \subseteq \mathscr{H}^{2}(\mathbb{R})\right)$, we may invoke the duality between $\mathbb{H}^{q}\left(\mathbb{R}^{d \times d}\right)$ and $\mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right)$ (see [39, Theorems 1.3.10\&1.3.21]) to obtain some $\tilde{\sigma}^{\varepsilon} \in \mathbb{H}^{p}\left(\mathbb{R}^{d \times d}\right)$ that satisfies $\left\|\widetilde{\sigma}^{\varepsilon}\right\|_{\mathbb{H}^{p}}=1$ and

$$
\begin{align*}
\left\|Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}\right\|_{\mathbb{H}^{q} q} & =\sup _{\|\widetilde{\sigma}\|_{\mathbb{H} p}=1}\left\langle Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}, \widetilde{\sigma}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}  \tag{3.49}\\
& \leq\left\langle Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}, \widetilde{\sigma}^{\varepsilon}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}+\varepsilon .
\end{align*}
$$

Finally, define $\left(b^{\star, \varepsilon}, \sigma^{\star, \varepsilon}\right) \in \mathcal{B}^{\varepsilon}$ by

$$
b_{t}^{\star, \varepsilon}:=b_{t}^{o}+\varepsilon \gamma \widetilde{b}_{t}^{\varepsilon}, \quad \sigma_{t}^{\star, \varepsilon}:=\sigma_{t}^{o}+\varepsilon \eta \widetilde{\sigma}_{t}^{\varepsilon}, \quad t \in[0, T]
$$

and set $C_{\mathrm{res}, 1}:=C_{\mathrm{I}}+C_{\mathrm{II}}+1$. Then (3.45)-(3.49) imply that

$$
\begin{aligned}
V(\varepsilon)-V(0) & \geq\left(\left\langle Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}, b^{\star, \varepsilon}-b^{o}\right\rangle_{\mathbb{P} \otimes d t}+\left\langle Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}, \sigma^{\star, \varepsilon}-\sigma^{o}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}\right)-C_{\mathrm{res}, 1} \varepsilon^{2} \\
& =\varepsilon\left(\gamma\left\langle Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}, \widetilde{b}^{\varepsilon}\right\rangle_{\mathbb{P} \otimes d t}+\eta\left\langle Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}, \widetilde{\sigma}^{\varepsilon}\right\rangle_{\mathbb{P} \otimes d t, \mathrm{~F}}\right)-C_{\mathrm{res}, 1} \varepsilon^{2} \\
& \geq \varepsilon\left(\gamma\left\|Y^{\varepsilon} H^{\varepsilon}+\mathcal{Y}^{\varepsilon}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{\varepsilon}\left(H^{\varepsilon}\right)^{\top}+\mathcal{Z}^{\varepsilon}\right\|_{\mathbb{H}^{q}}\right)-\left(C_{\mathrm{res}, 1}+\gamma+\eta\right) \varepsilon^{2}
\end{aligned}
$$

This completes the proof.
The final ingredient in the proof of Theorem 2.13 is the following stability result.

Lemma 3.12. Suppose that Assumptions 2.1 and 2.7 are satisfied. Let $\left(\varepsilon_{n}\right)_{n \in \mathbb{N}} \subseteq(0,1]$ with $\lim _{n} \varepsilon_{n}=0$ be such that $\lim _{n} \frac{V\left(\varepsilon_{n}\right)-V(0)}{\varepsilon_{n}}=\liminf _{\varepsilon \downarrow 0} \frac{V(\varepsilon)-V(0)}{\varepsilon}$. Moreover, let $H^{*}$ be the optimizer for $V(0)$ (see Proposition 2.9 (i)). Then for any sequence of $\varepsilon_{n}^{2}$-optimizers $H^{\varepsilon_{n}}$, the following hold:
(i) For every $\beta \geq 1$,

$$
\left\|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right\|_{\mathbb{L}^{\beta}} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

(ii) Denote for each $n \in \mathbb{N}$ by $\left(Y^{\varepsilon_{n}}, Z^{\varepsilon_{n}}, L^{\varepsilon_{n}}\right)$ the unique solution of (3.10) with the terminal condition $A^{H^{\varepsilon_{n}}}=\partial_{x} f\left(X_{T}^{H^{\varepsilon n} ; o}, h\left(S_{T}^{o}\right)\right)$, and by $\left(Y^{*}, Z^{*}, L^{*}\right)$ the unique solution of (2.5). Then, as $n \rightarrow \infty$,

$$
\left\|Y^{\varepsilon_{n}}-Y^{*}\right\|_{\mathscr{S}^{2}}+\left\|Z^{\varepsilon_{n}}-Z^{*}\right\|_{\mathscr{H}^{2}}+\left\|L^{\varepsilon_{n}}-L^{*}\right\|_{\mathscr{M}^{2}} \rightarrow 0
$$

(iii) Denote for each $n \in \mathbb{N}$ by $\left(\mathcal{Y}^{\varepsilon_{n}}, \mathcal{Z}^{\varepsilon_{n}}, \mathcal{L}^{\varepsilon_{n}}\right)$ the unique solution of (3.11) with the terminal condition $B^{H^{\varepsilon n}}=\partial_{y} f\left(X_{T}^{H^{\varepsilon n} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right)$, and by $\left(\mathcal{Y}^{*}, \mathcal{Z}^{*}, \mathcal{L}^{*}\right)$ the unique solution of (2.6). Then, for every $i=1, \ldots, d$, as $n \rightarrow \infty$,

$$
\left\|\mathcal{Y}^{\varepsilon_{n}, i}-\mathcal{Y}^{*, i}\right\|_{\mathscr{S}^{2}}+\left\|\mathcal{Z}^{\varepsilon_{n}, i}-\mathcal{Z}^{*, i}\right\|_{\mathscr{H}^{2}}+\left\|\mathcal{L}^{\varepsilon_{n}, i}-\mathcal{L}^{*, i}\right\|_{\mathscr{M}^{2}} \rightarrow 0
$$

Proof. We start by proving (i). Since $H^{*}$ is optimal for $V(0)$ and $H^{\varepsilon_{n}}$ is $\varepsilon_{n}^{2}$-optimal for $V\left(\varepsilon_{n}\right)$ (see (3.42)), a second-order Taylor expansion of $f$ around $\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)$ shows that

$$
\begin{aligned}
V\left(\varepsilon_{n}\right)-V(0) & \geq \mathbb{E}\left[f\left(X_{T}^{H^{\varepsilon_{n} ; o}}, h\left(S_{T}^{o}\right)\right)\right]-\varepsilon_{n}^{2}-\mathbb{E}\left[f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\right] \\
& =\mathbb{E}\left[\partial_{x} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)\left(X_{T}^{H^{\varepsilon_{n} ; o}}-X_{T}^{H^{*} ; o}\right)+\mathrm{IV}\left|X_{T}^{H^{\varepsilon_{n} ; o}}-X_{T}^{H^{*} ; o}\right|^{2}\right]-\varepsilon_{n}^{2}
\end{aligned}
$$

where by the strong convexity of $f$ (see Assumption 2.7 (iv)),

$$
\mathrm{IV}:=\int_{0}^{1}(1-\theta) \partial_{x x} f\left(\theta X_{T}^{H^{\varepsilon_{n}} ; o}+(1-\theta) X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right) d \theta \geq \frac{1}{2} C_{l, 2}
$$

Hence, using the first-order optimality of $H^{*}$ (see Lemma 3.7),

$$
V\left(\varepsilon_{n}\right)-V(0) \geq \frac{1}{2} C_{l, 2} \mathbb{E}\left[\left|X_{T}^{H^{\varepsilon_{n} ; o}}-X_{T}^{H^{*} ; o}\right|^{2}\right]-\varepsilon_{n}^{2}
$$

Moreover, since $V\left(\varepsilon_{n}\right) \rightarrow V(0)$ as $n \rightarrow \infty$ (by Lemma 3.9), we have that $\mathbb{E}\left[\mid X_{T}^{H^{\varepsilon_{n} ; o}}-\right.$ $\left.\left.X_{T}^{H^{*} ; o}\right|^{2}\right] \rightarrow 0$ as $n \rightarrow \infty$; in particular

$$
\left|X_{T}^{H^{\varepsilon n} ; o}-X_{T}^{H^{*} ; o}\right| \xrightarrow{\mathbb{P}} 0 \quad \text { as } n \rightarrow \infty
$$

Thus, an application of Lemma 3.5 implies (i).
Finally, Lemma 3.6 ensures that (ii) and (iii) hold, which completes the proof.
Proof of Theorem 2.13. First note that Lemma 3.9 immediately implies the upper bound

$$
\limsup _{\varepsilon \downarrow 0} \frac{V(\varepsilon)-V(0)}{\varepsilon} \leq \gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q} q}
$$

Thus, all that is left to do is to prove the corresponding lower bound. That will be achieved in three steps.

Step 1: Let $\left(\varepsilon_{n}\right)_{n \in \mathbb{N}} \subseteq(0,1]$ satisfy that $\lim _{n} \varepsilon_{n}=0$ and $\lim _{n} \frac{V\left(\varepsilon_{n}\right)-V(0)}{\varepsilon_{n}}=\liminf _{\varepsilon \downarrow 0} \frac{V(\varepsilon)-V(0)}{\varepsilon}$. Moreover, for each $n \in \mathbb{N}$, let $H^{\varepsilon_{n}}$ be an $\varepsilon_{n}^{2}$-optimizer of $V\left(\varepsilon_{n}\right)$. Let $\left(Y^{\varepsilon_{n}}, Z^{\varepsilon_{n}}, L^{\varepsilon_{n}}\right)$ and $\left(\mathcal{Y}^{\varepsilon_{n}}, \mathcal{Z}^{\varepsilon_{n}}, \mathcal{L}^{\varepsilon_{n}}\right)$ be the unique solutions of (3.10) and (3.11), respectively, with

$$
A^{H^{\varepsilon_{n}}}=\partial_{x} f\left(X_{T}^{H^{\varepsilon_{n}} ; o}, h\left(S_{T}^{o}\right)\right), \quad B^{H^{\varepsilon_{n}}}=\partial_{y} f\left(X_{T}^{H^{\varepsilon_{n}} ; o}, h\left(S_{T}^{o}\right)\right) \nabla h\left(S_{T}^{o}\right)
$$

Then, by Lemma 3.11,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{V\left(\varepsilon_{n}\right)-V(0)}{\varepsilon_{n}} \geq & \liminf _{n \rightarrow \infty}\left(\gamma\left\|Y^{\varepsilon_{n}} H^{\varepsilon_{n}}+\mathcal{Y}^{\varepsilon_{n}}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}\right)^{\top}+\mathcal{Z}^{\varepsilon_{n}}\right\|_{\mathbb{H}^{q}}\right) \\
\geq & \gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}} \\
& -\limsup _{n \rightarrow \infty}\left(\gamma\left\|Y^{\varepsilon_{n}} H^{\varepsilon_{n}}-Y^{*} H^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}\right)^{\top}-Z^{*}\left(H^{*}\right)^{\top}\right\|_{\mathbb{H}^{q}}\right. \\
& \left.+\gamma\left\|\mathcal{Y}^{\varepsilon_{n}}-\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|\mathcal{Z}^{\varepsilon_{n}}-\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}\right)
\end{aligned}
$$

Therefore, it remains to show that

$$
\begin{align*}
\limsup _{n \rightarrow \infty}\left(\gamma\left\|Y^{\varepsilon_{n}} H^{\varepsilon_{n}}-Y^{*} H^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}\right)^{\top}-Z^{*}\left(H^{*}\right)^{\top}\right\|_{\mathbb{H}^{q}}\right) & =0  \tag{3.50}\\
\limsup _{n \rightarrow \infty}\left(\gamma\left\|\mathcal{Y}^{\varepsilon_{n}}-\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|\mathcal{Z}^{\varepsilon_{n}}-\mathcal{Z}^{*}\right\|_{\mathbb{H}^{q}}\right) & =0 \tag{3.51}
\end{align*}
$$

Step 2: proof of (3.50). Recall that $\left|H_{t}^{*}\right|,\left|H_{t}^{\varepsilon_{n}}\right| \leq K \mathbb{P} \otimes d t$-a.e.. Hence, by the triangle inequality (and using that $\gamma, \eta \leq 1$ ),

$$
\begin{aligned}
& \gamma\left\|Y^{\varepsilon_{n}} H^{\varepsilon_{n}}-Y^{*} H^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}\right)^{\top}-Z^{*}\left(H^{*}\right)^{\top}\right\|_{\mathbb{H}^{q}} \\
& \leq\left\|\left(Y^{\varepsilon_{n}}-Y^{*}\right) H^{*}\right\|_{\mathbb{L}^{q}}+\left\|Y^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}-H^{*}\right)\right\|_{\mathbb{L}^{q}}+\left\|\left(Z^{\varepsilon_{n}}-Z^{*}\right)\left(H^{*}\right)^{\top}\right\|_{\mathbb{H}^{q}}+\left\|Z^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}-H^{*}\right)^{\top}\right\|_{\mathbb{H}^{q}} \\
& \leq K\left(\left\|Y^{\varepsilon_{n}}-Y^{*}\right\|_{\mathscr{S}^{q}}+\left\|Z^{\varepsilon_{n}}-Z^{*}\right\|_{\mathscr{H}^{q}}\right)+\mathbb{E}\left[\sup _{t \in, T]}\left|Y_{t}^{\varepsilon_{n}}\right|^{q} \int_{0}^{T}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{q} d t\right]^{\frac{1}{q}}+ \\
& \quad+\left\|Z^{\varepsilon_{n}}\left(H^{\varepsilon_{n}}-H^{*}\right)^{\top}\right\|_{\mathbb{H}^{q}} \\
&= \mathrm{I}^{n}+\mathrm{II}^{n}+\mathrm{III}^{n} .
\end{aligned}
$$

We will show that $\mathrm{I}^{n}, \mathrm{II}^{n}, \mathrm{III}^{n}$ vanish as $n \rightarrow \infty$.
Step 2, limit of $\mathrm{I}^{n}$ : Note that $1<q=\frac{p}{p-1}<2$ since $p>3$. Hence Lemma 3.12 (ii) implies that $\mathrm{I}^{n} \rightarrow 0$ as $n \rightarrow \infty$, as claimed.
Step 2, limit of $\mathrm{II}^{n}$ : Let $v>1$ satisfy $1<v q<2$. Then by Hölder's inequality (with exponent $v>1$ ) and Jensen's inequality (noting that $x \rightarrow|x|^{\frac{v}{v-1}}$ is convex),

$$
\begin{aligned}
\mathrm{II}^{n} & \leq\left\|Y^{\varepsilon_{n}}\right\|_{\mathscr{S} v q} \mathbb{E}\left[\left(\int_{0}^{T}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{q} d t\right)^{\frac{v}{v-1}}\right]^{\frac{v-1}{q v}} \\
& \leq\left\|Y^{\varepsilon_{n}}\right\|_{\mathscr{S}_{v q}} T^{\frac{1}{q v}}\left\|H^{\varepsilon_{n}}-H^{*}\right\|_{\mathbb{L}^{\frac{q v}{v-1}}}
\end{aligned}
$$

Lemma 3.12 (i) ensures that $\left\|H^{\varepsilon_{n}}-H^{*}\right\|_{\mathbb{L}^{\frac{q v}{v-1}}} \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, by the triangle inequality, for every $n \in \mathbb{N}$,

$$
\left\|Y^{\varepsilon_{n}}\right\|_{\mathscr{S}^{v q}} \leq\left\|Y^{\varepsilon_{n}}-Y^{*}\right\|_{\mathscr{S}^{v q}}+\left\|Y^{*}\right\|_{\mathscr{S}^{v q}}
$$

Note that $\left\|Y^{*}\right\|_{\mathscr{S}^{v q}}<\infty$ since $Y^{*} \in \mathscr{S}^{2}(\mathbb{R})$ and $v q<2$. Moreover, by Lemma 3.12 (ii) and Hölder's inequality (with exponent $\frac{2}{v q}>1$ ), we have that

$$
0 \leq \lim _{n \rightarrow \infty}\left\|Y^{\varepsilon_{n}}-Y^{*}\right\|_{\mathscr{S}^{v q}} \leq \lim _{n \rightarrow \infty}\left\|Y^{\varepsilon_{n}}-Y^{*}\right\|_{\mathscr{S}^{2}}=0
$$

Therefore, we conclude that $\mathrm{II}^{n} \rightarrow 0$ as $n \rightarrow \infty$.
Step 2, limit of $\mathrm{III}^{n}$ : By the triangle inequality and Jensen's inequality (noting that $x \rightarrow|x|^{\frac{2}{q}}$ is convex),

$$
\begin{align*}
\mathrm{III}^{n} & \leq\left\|\left(Z^{\varepsilon_{n}}-Z^{*}\right)\left(H^{\varepsilon_{n}}-H^{*}\right)^{\top}\right\|_{\mathbb{H} q}+\left\|Z^{*}\left(H^{\varepsilon_{n}}-H^{*}\right)^{\top}\right\|_{\mathbb{H} q} \\
& \leq \mathbb{E}\left[\int_{0}^{T}\left|Z_{t}^{\varepsilon_{n}}-Z_{t}^{*}\right|^{2}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{2} d t\right]^{\frac{1}{2}}+\mathbb{E}\left[\int_{0}^{T}\left|Z_{t}^{*}\right|^{2}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{2} d t\right]^{\frac{1}{2}} \tag{3.52}
\end{align*}
$$

Moreover, since $\left|H_{t}^{*}\right|,\left|H_{t}^{\varepsilon_{n}}\right| \leq K \mathbb{P} \otimes d t$-a.e., it follows that

$$
\mathbb{E}\left[\int_{0}^{T}\left|Z_{t}^{\varepsilon_{n}}-Z_{t}^{*}\right|^{2}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{2} d t\right]^{\frac{1}{2}} \leq 2 K\left\|Z^{\varepsilon_{n}}-Z^{*}\right\|_{\mathscr{H}^{2}} \rightarrow 0
$$

as $n \rightarrow \infty$, where we used Lemma 3.12 (ii) in the last step. Therefore, it remains to show that the last term in (3.52) vanishes when $n \rightarrow \infty$. To that end, we note that since $\left\|H^{\varepsilon_{n}}-H^{*}\right\|_{\mathbb{L}^{2}} \rightarrow 0$ as $n \rightarrow \infty$ (by Lemma 3.12 (i)), the continuous mapping theorem implies that

$$
\left|Z_{t}^{*}\right|^{2}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{2} \xrightarrow{\mathbb{P} \otimes d t} 0 \quad \text { as } n \rightarrow \infty
$$

Finally since $Z^{*} \in \mathscr{H}^{2}\left(\mathbb{R}^{d}\right)$ and $\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right| \leq 2 K \mathbb{P} \otimes d t$-a.e., the dominated convergence theorem guarantees that $\mathbb{E}\left[\int_{0}^{T}\left|Z_{t}^{*}\right|^{2}\left|H_{t}^{\varepsilon_{n}}-H_{t}^{*}\right|^{2} d t\right]^{\frac{1}{2}} \rightarrow 0$ as $n \rightarrow \infty$.
Step 3: proof of (3.51). By Hölder's inequality (with exponent $\frac{2}{q}>1$ ),

$$
\begin{aligned}
& \left\|\mathcal{Y}^{\varepsilon_{n}}-\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\left\|\mathcal{Z}^{\varepsilon_{n}}-\mathcal{Z}^{*}\right\|_{\mathbb{H} \boldsymbol{H}} \\
& =\mathbb{E}\left[\int_{0}^{T}\left(\sum_{i=1}^{d}\left(\mathcal{Y}_{t}^{\varepsilon_{n}, i}-\mathcal{Y}_{t}^{*, i}\right)^{2}\right)^{\frac{q}{2}} d t\right]^{\frac{1}{q}}+\mathbb{E}\left[\left(\int_{0}^{T} \sum_{i=1}^{d}\left|\mathcal{Z}_{t}^{\varepsilon_{n}, i}-\mathcal{Z}_{t}^{*, i}\right|^{2} d t\right)^{\frac{q}{2}}\right]^{\frac{1}{q}} \\
& \leq T^{\frac{1}{q}-\frac{1}{2}} \mathbb{E}\left[\sum_{i=1}^{d} \int_{0}^{T}\left(\mathcal{Y}_{t}^{\varepsilon_{n}, i}-\mathcal{Y}_{t}^{*, i}\right)^{2} d t\right]^{\frac{1}{2}}+\mathbb{E}\left[\sum_{i=1}^{d} \int_{0}^{T}\left|\mathcal{Z}_{t}^{\varepsilon_{n}, i}-\mathcal{Z}_{t}^{*, i}\right|^{2} d t\right]^{\frac{1}{2}}=: \mathrm{IV}^{n} .
\end{aligned}
$$

Moreover, it follows from the power triangle inequality that

$$
\begin{aligned}
\mathrm{IV}^{n} & \leq T^{\frac{1}{q}-\frac{1}{2}} d^{\frac{1}{2}} \sum_{i=1}^{d}\left(\mathbb{E}\left[\int_{0}^{T}\left(\mathcal{Y}_{t}^{\varepsilon_{n}, i}-\mathcal{Y}_{t}^{*, i}\right)^{2} d t\right]^{\frac{1}{2}}+\left\|\mathcal{Z}^{\varepsilon_{n}, i}-\mathcal{Z}^{*, i}\right\|_{\mathscr{H}^{2}}\right) \\
& \leq T^{\frac{1}{q}-\frac{1}{2}} d^{\frac{1}{2}} \sum_{i=1}^{d}\left(T^{\frac{1}{2}}\left\|\mathcal{Y}^{\varepsilon_{n}, i}-\mathcal{Y}^{*, i}\right\|_{\mathscr{S}^{2}}+\left\|\mathcal{Z}^{\varepsilon_{n}, i}-\mathcal{Z}^{*, i}\right\|_{\mathscr{H}^{2}}\right)
\end{aligned}
$$

Combined with Lemma 3.12 (iii), this estimate ensures that (3.51) holds. The proof is complete.

## 4. Remaining proofs

Proof of Lemma 2.3. We start by proving (i). Since $\left|b_{t}^{o}\right|+\left\|\sigma_{t}^{o}\right\|_{\mathrm{F}} \leq C_{b, \sigma} \mathbb{P} \otimes d t$-a.e.,

$$
\left\|b^{o}\right\|_{\mathbb{L}^{p}}^{p}=\mathbb{E}\left[\int_{0}^{T}\left|b_{t}^{o}\right|^{p} d t\right] \leq\left(C_{b, \sigma}\right)^{p} T, \quad\left\|\sigma^{o}\right\|_{\mathbb{H}^{p}}^{p}=\mathbb{E}\left[\left(\int_{0}^{T}\left\|\sigma_{t}^{o}\right\|_{\mathrm{F}}^{2} d t\right)^{\frac{p}{2}}\right] \leq\left(C_{b, \sigma}\right)^{p} T^{\frac{p}{2}}
$$

This ensures that Assumption 2.1 (i) holds.
Next, we claim that Assumption 2.1 (ii) holds. Note that from the uniform ellipticity condition on $\left(\sigma^{o}\right)^{\top} \sigma^{o}$, it follows that there exists some constant $C_{c}>0$ such that $y^{\top}\left(\sigma_{t}^{o}\right)^{\top} \sigma_{t}^{o} y \geq$ $\frac{1}{C_{c}}|y|^{2} \mathbb{P} \otimes d t$-a.e. for every $y \in \mathbb{R}^{d}$, hence

$$
\left|\left(\sigma_{t}^{o}\right)^{-1} y\right| \leq \sqrt{C_{c}}|y| \mathbb{P} \otimes d t \text {-a.e., for every } y \in \mathbb{R}^{d}
$$

In particular, using the uniform boundedness of $b^{o}$, it follows that

$$
\begin{equation*}
\frac{1}{2} \int_{0}^{T}\left|\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o}\right|^{2} d u \leq \frac{1}{2} \int_{0}^{T} C_{c}\left|b_{u}^{o}\right|^{2} d u \leq \frac{T}{2} C_{b, \sigma}^{2} C_{c}<\infty \mathbb{P} \text {-a.s.. } \tag{4.1}
\end{equation*}
$$

Thus, $\int_{0}^{\cdot}\left(\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o}\right)^{\top} d W_{u}$ is well-defined and $\mathcal{D}$ satisfies $\mathcal{D}$. $=1-\int_{0}^{*} \mathcal{D}_{u}\left(\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o}\right)^{\top} d W_{u}$ showing that $\mathcal{D}$ is a continuous, $(\mathbb{F}, \mathbb{P})$-local martingale.

Moreover, (4.1) clearly implies that $\mathbb{E}\left[\exp \left(\frac{1}{2} \int_{0}^{T}\left|\left(\sigma_{u}^{o}\right)^{-1} b_{u}^{o}\right|^{2} d u\right)\right]<\infty$. Hence, Novikov's condition in $[45$, Proposition 3.5 .12 , p.198] ensures that $\mathcal{D}$ is a strictly positive $(\mathbb{F}, \mathbb{P})$-martingale.

We proceed to show that $\mathcal{D}_{T} \in L^{\beta}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$ for every $\beta \geq 1$. Indeed, using (4.1), we have that for every $\beta \geq 1$,

$$
\begin{align*}
\mathbb{E}\left[\mathcal{D}_{T}^{\beta}\right] & =\mathbb{E}\left[\mathcal{E}\left(-\beta\left(\left(\sigma^{o}\right)^{-1} b^{o}\right) \cdot W\right)_{T} \exp \left(\left(\frac{\beta^{2}}{2}-\frac{\beta}{2}\right) \int_{0}^{T}\left|\left(\sigma_{t}^{o}\right)^{-1} b_{t}^{o}\right|^{2} d t\right)\right]  \tag{4.2}\\
& \leq \mathbb{E}\left[\mathcal{E}\left(-\beta\left(\left(\sigma^{o}\right)^{-1} b^{o}\right) \cdot W\right)_{T}\right] \exp \left(\left(\frac{\beta^{2}}{2}-\frac{\beta}{2}\right) T C_{b, \sigma}^{2} C_{c}\right)
\end{align*}
$$

Since $\mathcal{E}\left(-\beta\left(\left(\sigma^{o}\right)^{-1} b^{o}\right) \cdot W\right)_{T}$ is a nonnegative continuous $(\mathbb{F}, \mathbb{P})$-local martingale, it is an $(\mathbb{F}, \mathbb{P})$ supermartingale and hence integrable. This together with (4.2) ensures that $\mathcal{D}_{T} \in L^{\beta}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$, for every $\beta \geq 1$.

Now let us prove (ii). Since $b_{t}^{o}=\widetilde{b}^{o}\left(t, S_{t}^{o}\right), \sigma_{t}^{o}=\widetilde{\sigma}^{o}\left(t, S_{t}^{o}\right) \mathbb{P} \otimes d t$-a.e., $S^{o}$ given in (2.1) is driven by the following stochastic differential equation (SDE):

$$
\begin{equation*}
S_{t}^{o}=s_{0}+\int_{0}^{t} \widetilde{b}^{o}\left(u, S_{u}^{o}\right) d u+\int_{0}^{t} \widetilde{\sigma}^{o}\left(u, S_{u}^{o}\right) d W_{u} \quad \mathbb{P} \text {-a.s., } t \in[0, T] . \tag{4.3}
\end{equation*}
$$

Using the Lipschitz and linear growth conditions on ( $\widetilde{b}^{o}, \widetilde{\sigma}^{o}$ ), an application of [54, Theorem 2.3.1] shows that the $\operatorname{SDE}$ (4.3) has a unique solution.

Furthermore, since $S_{0}^{o}=s_{0} \in \mathbb{R}$, an application of [54, Theorem 2.4.1] shows that there is some $C>0$ (depending on $s_{0} \in \mathbb{R}, p>3$ and $T>0$ ) which satisfies

$$
\mathbb{E}\left[\left|S_{t}^{o}\right|^{p}\right] \leq C, \quad \text { for every } t \in[0, T]
$$

Therefore, the linear growth condition on $\widetilde{b}^{o}$ (with the constant $C_{\widetilde{b}, \widetilde{\sigma}}>0$ ), and the power triangle inequality implies

$$
\left\|b^{o}\right\|_{\mathbb{L}^{p}}^{p} \leq\left(C_{\widetilde{b}, \widetilde{\sigma}^{2}} 2\right)^{p} \mathbb{E}\left[\int_{0}^{T}\left(1+\left|S_{t}^{o}\right|^{p}\right) d t\right] \leq\left(C_{\widetilde{b}, \widetilde{\sigma}^{2}} 2\right)^{p} T(1+C)<\infty
$$

Furthermore, using the same arguments as above and Jensen's inequality (noting that $x \rightarrow$ $|x|^{\frac{p}{2}}$ is convex), we have

$$
\begin{aligned}
\left\|\sigma^{o}\right\|_{\mathbb{H} p}^{p} & \leq\left(C_{\widetilde{b}, \widetilde{\sigma}^{2}} 2\right)^{p} \mathbb{E}\left[\left(\int_{0}^{T}\left(1+\left|S_{t}^{o}\right|^{2}\right) d t\right)^{\frac{p}{2}}\right] \\
& \leq\left(C_{\widetilde{b}, \widetilde{\sigma}^{2}} 2\right)^{p} T^{\frac{p}{2}-1} 2^{\frac{p}{2}} \mathbb{E}\left[\int_{0}^{T}\left(1+\left|S_{t}^{o}\right|^{p}\right) d t\right]<\infty
\end{aligned}
$$

Hence Assumption 2.1 (i) holds.
Assumption 2.1 (ii) follows from the Beneš condition [9], i.e., $\left(\sigma_{t}^{o}\right)^{-1} b_{t}^{o}=\theta(t, W) \mathbb{P} \otimes d t$ a.e.. Indeed, by $[45$, Corollary 3.5 .16$], \mathcal{D}$ is a strictly positive $(\mathbb{F}, \mathbb{P})$-martingale. Moreover, the condition that $\mathcal{D}_{T} \in L^{\beta}\left(\mathcal{F}_{T} ; \mathbb{R}\right)$, for every $\beta \geq 1$ follows from [31, Corollary 2]. This completes the proof.

Proof of Lemma 2.11. The expressions for $Y^{*}$ and $\mathcal{Y}^{*}$ follow from the definition of the Galtchouk-Kunita-Watanabe decompositions given in (2.5) and (2.6) by taking conditional expectations.

As for the expressions of $Z^{*}$, recall that $L^{*} \in \mathscr{M}^{2}(\mathbb{R})$ is strongly orthogonal with $W^{i}$ for every $i=1, \ldots, d$, see Proposition 2.9 (ii) and Lemma 3.2. Hence, for every $t \in[0, T], \mathbb{P}$-a.s.,

$$
d\left(\left\langle Y^{*}, W^{1}\right\rangle_{t}, \ldots,\left\langle Y^{*}, W^{d}\right\rangle_{t}\right)^{\top}=Z_{t}^{*} d t+d\left(\left\langle L^{*}, W_{1}\right\rangle_{t}, \ldots,\left\langle L^{*}, W_{d}\right\rangle_{t}\right)^{\top}=Z_{t}^{*} d t
$$

proving the claimed expression of $Z^{*}$.
The proof for the expression of $\mathcal{Z}^{*, i}$ follows from the same arguments (replacing $Z^{*}$ with $\mathcal{Z}^{*, i}$ and $Y^{*}$ with $\left.\mathcal{Y}^{*, i}\right)$.

Proof of Corollary 2.12. By the regularity assumption on $J$, i.e., $J \in \mathcal{C}^{1,2,2}$, an application of Itô's formula ensures that for every $t \in[0, T]$,

$$
\begin{aligned}
J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right)= & \partial_{x} f\left(X_{T}^{H^{*} ; o}, h\left(S_{T}^{o}\right)\right)-\int_{t}^{T}\left(\mathcal{L}_{r}^{1}+\mathcal{L}_{r}^{2}+\mathcal{L}_{r}^{3}\right) d r \\
& -\int_{t}^{T}\left[\partial_{x} J\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)\left(H_{r}^{*}\right)^{\top}+\left(\nabla_{s} J\right)^{\top}\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)\right] \sigma_{r}^{o} d W_{r}
\end{aligned}
$$

where

$$
\begin{aligned}
& \mathcal{L}_{r}^{1}:=\partial_{t} J\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)+\frac{1}{2} \partial_{x x} J\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)\left|\left(\sigma_{r}^{o}\right)^{\top} H_{r}^{*}\right|^{2}+\partial_{x} J\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)\left(H_{r}^{*}\right)^{\top} b_{r}^{o} ; \\
& \mathcal{L}_{r}^{2}:=\sum_{i=1}^{d} \partial_{x, s_{i}} J\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)\left(\left(H_{r}^{*}\right)^{\top} \sigma_{r}^{o}\right)\left(\sigma_{r}^{o, i}\right)^{\top} ; \\
& \mathcal{L}_{r}^{3}:=\frac{1}{2} \operatorname{tr}\left(\left(\sigma_{r}^{o}\right)\left(\sigma_{r}^{o}\right)^{\top} D_{s}^{2} J\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right)\right)+\left(\nabla_{s} J\right)^{\top}\left(r, X_{r}^{H^{*} ; o}, S_{r}^{o}\right) b_{r}^{o} ;
\end{aligned}
$$

with $\left(\sigma_{t}^{o, i}\right)_{t \in[0, T]}, i=1, \ldots, d$, denoting the $i$-th row vector process of $\sigma^{o}, D_{s}^{2} J$ denoting the Hessian of $J$ with respect to $s$.

Hence, since $Y_{t}^{*}=J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right)$ holds for every $t \in[0, T], \mathbb{P}$-a.s. (see (2.7)), by Lemma 2.11 and the regularity on $J$, we have that for every $t \in[0, T), \mathbb{P}$-a.s.,

$$
\begin{aligned}
Z_{t}^{*} & =\frac{d}{d t}\left(\left\langle J\left(\cdot, X^{H^{*} ; o}, S^{o}\right), W^{1}\right\rangle_{t}, \cdots,\left\langle J\left(\cdot, X^{H^{*} ; o}, S^{o}\right), W^{d}\right\rangle_{t}\right)^{\top} \\
& =\left(\sigma_{t}^{o}\right)^{\top}\left[\partial_{x} J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right) H_{t}^{*}+\nabla_{s} J\left(t, X_{t}^{H^{*} ; o}, S_{t}^{o}\right)\right]
\end{aligned}
$$

We can use the same arguments (replacing $Y^{*}=J\left(\cdot, X^{H^{*} ; o}, S^{o}\right)$ with $Y^{*, i}=\mathcal{J}^{i}\left(\cdot, X^{H^{*} ; o}, S^{o}\right)$ and using the regularity assumption on $\mathcal{J}^{i}$ ) to show that the property for $\mathcal{Z}^{*, i}$ holds for every $i=1, \ldots, d$.

Remark 4.1. The regularity assumptions in Corollary 2.12 (see (2.7)) are satisfied if, e.g., the following sufficient conditions (i),(ii),(iii) or (i),(iv) hold (see [65, Theorem 3.2] and [45, Theorem 5.7.6 and Remark 5.7.8]):
(i) $\mathbb{F}$ is the completion of the filtration generated by the Brownian motion $W$;
(ii) $\partial_{x} f(\cdot, h(\cdot))$ and $\partial_{y} f(\cdot, h(\cdot)) \partial_{s_{i}} h(\cdot), i=1, \ldots, d$, are three times continuously differentiable and their derivatives of order less than or equal to 3 grow at most like a polynomial function of the variable at infinity;
(iii) Let $\hat{b}^{o}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\hat{\sigma}^{o}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ be three times continuously differentiable with bounded first, second, and third derivatives. Assume that $H^{*}$ given in Proposition 2.9 (i) is constant, and $b^{o}$ and $\sigma^{o}$ given in (2.1) satisfy $\mathbb{P} \otimes d t$-a.e.,

$$
b_{t}^{o}=\hat{b}^{o}\left(S_{t}^{o}\right), \quad \sigma_{t}^{o}=\hat{\sigma}^{o}\left(S_{t}^{o}\right)
$$

(iv) Let $\bar{b}^{o}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ be bounded, Hölder-continuous. Moreover, let $\bar{\sigma}^{o}:[0, T] \times$ $\mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ be bounded and satisfy that $\bar{\sigma}^{o}\left(\bar{\sigma}^{o}\right)^{\top}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ is Höldercontinuous and satisfies a uniform ellipticity condition, i.e., there is $C_{\sigma}>0$ such that for every $v \in \mathbb{R}^{d}$ and $(t, x) \in[0, T] \times \mathbb{R}^{d}, v^{\top} \bar{\sigma}^{o}\left(\bar{\sigma}^{o}\right)^{\top}(t, x) v \geq C_{\sigma}|v|^{2}$. Assume that $H^{*}$ is deterministic, Hölder-continuous in $[0, T]$, and satisfy $\left|H_{t}^{*}\right|>0$ for every $t \in[0, T]$, and $b^{o}$ and $\sigma^{o}$ satisfy $\mathbb{P} \otimes d t$-a.e.,

$$
b_{t}^{o}=\bar{b}^{o}\left(t, S_{t}^{o}\right), \quad \sigma_{t}^{o}=\bar{\sigma}^{o}\left(t, S_{t}^{o}\right)
$$

Proof of Theorem 2.14. Since $V^{*}(0)=V(0)$ and

$$
V^{\prime}(0)=\gamma\left\|Y^{*} H^{*}+\mathcal{Y}^{*}\right\|_{\mathbb{L}^{q}}+\eta\left\|Z^{*}\left(H^{*}\right)^{\top}+\mathcal{Z}^{*}\right\|_{\mathbb{H} q}
$$

(see Theorem 2.13), the inequality

$$
V^{*}(\varepsilon) \leq V(0)+\varepsilon V^{\prime}(0)+O\left(\varepsilon^{2}\right)=V(\varepsilon)+O\left(\varepsilon^{2}\right)
$$

as $\varepsilon \downarrow 0$, follows directly from Remark 3.10. The reverse inequality follows by using the same arguments as for the proof of Lemma 3.11, but replacing $H^{\varepsilon}$ by $H^{*}$.

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[^1]:    ${ }^{1}$ The following definition is based on [45, Definition 3.5.15]. Given $x \in C\left([0, T] ; \mathbb{R}^{d}\right)$, define $\mathcal{H}_{t}:=\sigma(x(s) ; s \in$ $[0, t])$ for $t \in[0, T]$. A progressively measurable functional on $C\left([0, T] ; \mathbb{R}^{d}\right)$ is a mapping $\theta:[0, T] \times C\left([0, T] ; \mathbb{R}^{d}\right) \rightarrow$ $\mathbb{R}$ which has the property that for each $t \in[0, T],\left.\theta\right|_{[0, t] \times C\left([0, T] ; \mathbb{R}^{d}\right)}$ is $\mathcal{B}([0, t]) \otimes \mathcal{H}_{t} / \mathcal{B}(\mathbb{R})$-measurable.

[^2]:    ${ }^{2}$ We refer to [75, Chapter IV.3, p.148] for the definition of the strong orthogonality and the following equivalences: two martingales $M, N \in \mathscr{M}^{2}(\mathbb{R})$ are strongly orthogonal if and only if $\left(M_{t} N_{t}\right)_{t \in[0, T]}$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale if and only if $\left([M, N]_{t}\right)_{t \in[0, T]}$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale.
    ${ }^{3}$ Two $d$-dimensional martingales $M=\left(M^{1}, \ldots, M^{d}\right)^{\top}$ and $N=\left(N^{1}, \ldots, N^{d}\right)^{\top}$ are strongly orthogonal if $M^{i}, N^{i} \in \mathscr{M}^{2}(\mathbb{R})$ are strongly orthogonal for all $i=1, \ldots, d$.

