# NUMERICAL METHOD FOR NONLINEAR KOLMOGOROV PDES VIA SENSITIVITY ANALYSIS 

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#### Abstract

We examine nonlinear Kolmogorov partial differential equations (PDEs). Here the nonlinear part of the PDE comes from its Hamiltonian where one maximizes over all possible drift and diffusion coefficients which fall within a $\varepsilon$-neighborhood of pre-specified baseline coefficients. Our goal is to quantify and compute how sensitive those PDEs are to such a small nonlinearity, and then use the results to develop an efficient numerical method for their approximation. We show that as $\varepsilon \downarrow 0$, the nonlinear Kolmogorov PDE equals the linear Kolmogorov PDE defined with respect to the corresponding baseline coefficients plus $\varepsilon$ times a correction term which can be also characterized by the solution of another linear Kolmogorov PDE involving the baseline coefficients. As these linear Kolmogorov PDEs can be efficiently solved in high-dimensions by exploiting their Feynman-Kac representation, our derived sensitivity analysis then provides a Monte Carlo based numerical method which can efficiently solve these nonlinear Kolmogorov equations. We provide numerical examples in up to 100 dimensions to empirically demonstrate the applicability of our numerical method.


## 1. Introduction

Kolmogorov partial differential equations (PDEs) are widely used to describe the evolution of underlying diffusion processes over time. These PDEs are applied in various fields, for instance to model dynamics in physics and chemistry (e.g., [56, 78, 99]), to analyze some population growth in biology (e.g., $[59,62])$ to model the evolution of stock prices in finance and economics (e.g., $[2,12,100]$ ), or for climate modeling (e.g., $[42,98]$ ), to name but a few.

Consider the following ${ }^{1}$ Kolmogorov PDE (see, e.g., [15, 17, 20, 28, 40, 46, 61, 77, 90, 91])

$$
\left\{\begin{array}{l}
\partial_{t} v(t, x)+\left\langle b, \nabla_{x} v(t, x)\right\rangle+\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^{\top} D_{x}^{2} v(t, x)\right)=0 \quad \text { on }[0, T) \times \mathbb{R}^{d}  \tag{1.1}\\
v(T, x)=f(x) \quad \text { on } \mathbb{R}^{d}
\end{array}\right.
$$

One of the common modeling challenges arising throughout all fields consists in finding the true drift and volatility parameters $(b, \sigma)$ to describe the underlying evolution process, which is usually unknown. Typically, one would either try to estimate the parameters using historical data or choose them based on experts' opinions. However, it is well-known that model misspecification may lead to wrong outcomes which might be fatal, as e.g., happened during the financial crisis in 2008 when financial derivatives were priced based on solutions of (1.1) but with corresponding parameters which were not consistent with the market behavior during that period.

[^0]To overcome this difficulty of model uncertainty, a common approach is to consider a set $\mathcal{U}$ of parameters $(b, \sigma)$, where each element $(b, \sigma) \in \mathcal{U}$ is considered as a candidate for the true but unknown drift and volatility. Then, one uses this set of candidates $\mathcal{U}$ to describe the evolution of the underlying process robustly with respect to its parameters by considering the following nonlinear Kolmogorov PDE (see, e.g., [14, 21, 30, 58, 79, 83, 85, 101])

$$
\left\{\begin{array}{l}
\partial_{t} v(t, x)+\sup _{(b, \sigma) \in \mathcal{U}}\left\{\left\langle b, \nabla_{x} v(t, x)\right\rangle+\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^{\top} D_{x}^{2} v(t, x)\right)\right\}=0 \quad \text { on }[0, T) \times \mathbb{R}^{d}  \tag{1.2}\\
v(T, x)=f(x) \text { on } \mathbb{R}^{d}
\end{array}\right.
$$

A natural choice for $\mathcal{U}$ we consider throughout this paper is to start with baseline parameters $\left(b^{o}, \sigma^{o}\right)$ that one considers as first best estimates for the true but unknown drift and volatility and then to consider the set

$$
\begin{equation*}
\mathcal{B}^{\varepsilon}:=\left\{(b, \sigma) \in \mathbb{R}^{d} \times \mathbb{R}^{d \times d}:\left|b-b^{o}\right| \leq \gamma \varepsilon,\left\|\sigma-\sigma^{o}\right\|_{\mathrm{F}} \leq \eta \varepsilon\right\} \tag{1.3}
\end{equation*}
$$

of all coefficients that fall within the (weighted by $\gamma, \eta \in[0,1]) \varepsilon$-neighborhood of the baseline coefficients, for some pre-specified $\varepsilon>0$. Typical choices for $\gamma$ and $\eta$ consist of $(\gamma, \eta)=(1,0)$ representing drift uncertainty $[16,19,76,80],(\gamma, \eta)=(0,1)$ representing volatility uncertainty $[23,24,73,81,82]$, as well as $(\gamma, \eta)=(1,1)$ for simultaneous drift and volatility uncertainty [68$70]$. We also refer to $[18,64,94,95]$ for the connection of these nonlinear Kolmogorov PDEs (1.2) with second-order backward stochastic differential equations.

The goal of this paper is to analyze how sensitive Kolmogorov equations are with respect to their parameters $b$ and $\sigma$. More precisely, for small $\varepsilon>0$ let $v^{\varepsilon}(t, x)$ denote the (unique viscosity) solution of the nonlinear Kolmogorov $\operatorname{PDE}(1.2)$ with $\mathcal{U}:=\mathcal{B}^{\varepsilon}$ and let $v^{0}(t, x)$ be the solution of the linear Kolmogorov PDE (1.1) with respect to the baseline parameters $b^{o}$ and $\sigma^{o}$. In this context, we aim to answer the following questions:
. Can we identify and efficiently calculate the sensitivity $\partial_{\varepsilon} v^{0}(t, x):=\lim _{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(v^{\varepsilon}(t, x)-\right.$ $\left.v^{0}(t, x)\right) ?$

- Can we find a numerical method which can efficiently solve high-dimensional nonlinear Kolmogorov PDEs of the form (1.2) with $\mathcal{U}:=\mathcal{B}^{\varepsilon}$ for small $\varepsilon>0$ ?
In Theorem 2.7 we show that if $f$ is sufficiently regular and satisfies some mild growth conditions (see Assumption 2.1) as well as $\sigma^{o}$ is invertible, then the following hold. For every $(t, x) \in[0, T) \times \mathbb{R}^{d}$, as $\varepsilon \downarrow 0$, we obtain that

$$
v^{\varepsilon}(t, x)=v^{0}(t, x)+\varepsilon \cdot \partial_{\varepsilon} v^{0}(t, x)+O\left(\varepsilon^{2}\right)
$$

where $\partial_{\varepsilon} v^{0}(t, x)$ is given by

$$
\partial_{\varepsilon} v^{0}(t, x)=\mathbb{E}\left[\int_{t}^{T} \gamma\left|w\left(s, x+X_{s}^{o}\right)\right|+\eta\left\|\mathrm{J}_{x} w\left(s, x+X_{s}^{o}\right) \sigma^{o}\right\|_{\mathrm{F}} d s \mid X_{t}^{o}=0\right]
$$

Here

- $X_{s}^{o}:=b^{o} s+\sigma^{o} W_{s}, s \in[0, T]$ is a stochastic process driven by a standard $d$-dimensional Brownian motion $\left(W_{s}\right)_{s \in[0, T]}$,
$-\mathbb{E}\left[\cdot \mid X_{t}^{o}=0\right]$ denotes the conditional expectation given $X_{t}^{o}=0$,
$-w=\left(w^{1}, \ldots, w^{d}\right)$ where each $w^{i}$ is the solution of the linear Kolmogorov equation

$$
\left\{\begin{array}{l}
\partial_{s} w^{i}(s, x)+\left\langle b^{o}, \nabla_{x} w^{i}(s, x)\right\rangle+\frac{1}{2} \operatorname{tr}\left(\left(\sigma^{o}\right)\left(\sigma^{o}\right)^{\top} D_{x}^{2} w^{i}(s, x)\right)=0 \quad \text { on }[t, T) \times \mathbb{R}^{d} \\
w^{i}(T, x)=\partial_{x_{i}} f(x) \quad \text { on } \mathbb{R}^{d}
\end{array}\right.
$$

and $\mathrm{J}_{x} w$ stands for the Jacobian of $w$.
We highlight that Theorem 2.7 also provides a methodology to approximate the solution $v^{\varepsilon}$ of the nonlinear Kolmogorov PDE (1.2). Indeed, note that by the Feynman-Kac representation, we have for any $t \leq s \leq T$ and $x \in \mathbb{R}^{d}$ that

$$
\begin{aligned}
v^{0}(t, x) & =\mathbb{E}\left[f\left(x+X_{T}^{o}\right) \mid X_{t}^{o}=0\right] \\
w\left(s, x+X_{s}^{o}\right) & =\mathbb{E}\left[\nabla_{x} f\left(x+X_{s}^{o}+\widetilde{X}_{T}^{o}\right) \mid \widetilde{X}_{s}^{o}=0\right] \\
\left(\mathrm{J}_{x} w\right)\left(s, x+X_{s}^{o}\right) & =\mathbb{E}\left[D_{x}^{2} f\left(x+X_{s}^{o}+\widetilde{X}_{T}^{o}\right) \mid \widetilde{X}_{s}^{o}=0\right]
\end{aligned}
$$

where $\widetilde{X}_{s}^{o}:=b^{o} s+\sigma^{o} \widetilde{W}_{s}, s \in[0, T]$, with $\left(\widetilde{W}_{s}\right)_{s \in[0, T]}$ being another standard $d$-dimensional Brownian motion independent of $\left(W_{s}\right)_{s \in[0, T]}$. Therefore, we can implement the approximation $v^{0}+\varepsilon \cdot \partial_{\varepsilon} v^{0}$ of $v^{\varepsilon}$ by a Monte Carlo based scheme (see Algorithm 1) which is efficient even in high dimensions (see Section 3 for our numerical results in up to $d=100$ dimensions).
Related Literature. Since solutions of Kolmogorov PDEs and parabolic PDEs in general typically cannot be solved explicitly and hence need to be approximately solved, there has been a lot of efforts to develop such numerical approximation methods. We refer e.g. to [93,96, 97] for deterministic approximation methods (e.g., finite difference and finite element methods, spectral Galerkin methods, and sparse grid methods) and to [10, 11, 37-39, 43, 49-51, 51-54, 60, 63, 65, 71,71 ] for stochastic approximation methods including Monte Carlo approximations. Recently, there has been an intensive interest in deep-learning based algorithms that can approximately solve high-dimensional linear/nonlinear parabolic PDEs (e.g., [7-9,25,26,35,41,48,72,86,89, 92]). Moreover, we also refer to [1,36,47] for deep learning algorithms to solve control problems related to (discretized versions of) HJB equations.

Sensitivity analysis of robust optimization problems have been established mostly with respect to 'Wasserstein-type' of uncertainty by considering an (adapted) Wasserstein-ball with radius $\varepsilon$ around an (estimated) baseline probability measure for the underlying process either in a one-period model $[4,13,32-34,66,67,75,84]$ or in a multi-period discrete-time model $[6,55]$. Moreover, in continuous-time, [44,45] provided a sensitivity analysis of a particular robust utility maximization problem under volatility uncertainty, whereas [5] analyzed the sensitivity of general robust optimization problems under both drift and volatility uncertainty.

The contribution of our paper is to provide a sensitivity analysis of nonlinear PDEs of type (1.2) and use this analysis to approximate those PDEs by some suitable linear PDEs as described above, leading to a numerical approximation algorithm which is efficient even in high-dimensions.

## 2. Main Results

Fix $d \in \mathbb{N}$ and endow $\mathbb{R}^{d}$ with the Euclidean inner product $\langle\cdot, \cdot\rangle$, and $\mathbb{R}^{d \times d}$ with the Frobenius inner product $\langle\cdot, \cdot\rangle_{\mathrm{F}}$, respectively. Let $\mathbb{S}^{d}$ be the set of all symmetric $d \times d$ matrices. Then fix a
time horizon $T>0$, and for any $\varepsilon \geq 0$ consider a nonlinear Kolmogorov PDE with the set $\mathcal{B}^{\varepsilon}$ given in (1.3)

$$
\left\{\begin{array}{l}
\partial_{t} v^{\varepsilon}(t, x)+\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\{\left\langle b, \nabla_{x} v^{\varepsilon}(t, x)\right\rangle+\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^{\top} D_{x}^{2} v^{\varepsilon}(t, x)\right)\right\}=0 \quad \text { on }[0, T) \times \mathbb{R}^{d}  \tag{2.1}\\
v^{\varepsilon}(T, x)=f(x) \text { on } \mathbb{R}^{d}
\end{array}\right.
$$

where $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ corresponds to the boundary condition.
We impose certain conditions on the boundary $f$ and baseline coefficient $\sigma^{o}$ given in (1.3).
Assumption 2.1. The function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is continuously differentiable. Moreover, its Hes$\operatorname{sian} D_{x}^{2} f: \mathbb{R}^{d} \rightarrow \mathbb{S}^{d}$ exists in the weak sense ${ }^{2}$ and there are $\alpha \geq 1$ and $C_{f}>0$ such that $\left\|D_{x}^{2} f(x)\right\|_{\mathrm{F}} \leq C_{f}\left(1+|x|^{\alpha}\right)$ for every $x \in \mathbb{R}^{d}$.

Assumption 2.2. The matrix $\sigma^{o}$ is invertible.
Remark 2.3. Assumption 2.2 ensures that $\lambda_{\min }\left(\sigma^{o}\right)$, the smallest singular value of $\sigma^{o}$ is strictly positive; in particular for every $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right)$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon}$, the matrix $\sigma$ is invertible.

We further impose a condition on the solution of the nonlinear Kolmogorov PDE, which relies on the notion of viscosity solutions (see Section 5.3 for the standard definitions of viscosity / strong solutions of PDEs).

Assumption 2.4. For any $\varepsilon \geq 0$, there exists at most one viscosity solution $v^{\varepsilon}$ of (2.1) satisfying that there is $C>0$ such that for all $t \in[0, T]$

$$
\begin{equation*}
\lim _{|x| \rightarrow \infty}\left|v^{\varepsilon}(t, x)\right| e^{-C(\log (|x|))^{2}}=0 \tag{2.2}
\end{equation*}
$$

Remark 2.5. It follows from [69, Proposition 5.5] that Assumption 2.4 is satisfied if, e.g., Assumptions 2.1 and 2.2 are satisfied and $f$ is bounded and Lipschitz continuous. Furthermore, if $\eta=0$, i.e., there is no volatility uncertainty, then Assumptions $2.1 \& 2.2$ directly imply that Assumption 2.4 holds, see [3, Theorem 3.5]. We also refer to [3, Remark 3.6] for a detailed discussion on the growth condition (2.2).

Now we collect some preliminary results in the next proposition on the solution of the nonlinear Kolmogorov PDE (2.1) together with the following linear Kolmogorov PDE defined using the baseline coefficients $b^{o}$ and $\sigma^{o}$ and the boundary condition $\partial_{x_{i}} f, i=1, \ldots, d$,

$$
\left\{\begin{array}{l}
\partial_{s} w^{i}(s, x)+\left\langle b^{o}, \nabla_{x} w^{i}(s, x)\right\rangle+\frac{1}{2} \operatorname{tr}\left(\left(\sigma^{o}\right)\left(\sigma^{o}\right)^{\top} D_{x}^{2} w^{i}(s, x)\right)=0 \quad \text { on }[t, T) \times \mathbb{R}^{d}  \tag{2.3}\\
w^{i}(T, x)=\partial_{x_{i}} f(x) \quad \text { on } \mathbb{R}^{d}
\end{array}\right.
$$

where we note that $f$ is the boundary condition given in (2.1).
The corresponding proofs for the following proposition can be found in Section 5.3.
Proposition 2.6. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied. Then the following hold:
(i) For any $\varepsilon \geq 0$, there exists a unique viscosity solution $v^{\varepsilon}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ of (2.1) satisfying the growth property given in (2.2).

[^1](ii) For any $i=1, \ldots, d$, there exists a unique strong solution $w^{i}:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ of (2.3) with polynomial growth.

We proceed with our main result. To formulate it, denote by $O(\cdot)$ the Landau symbol and if $w^{i}$ is the solution to (2.3) with the boundary condition $\partial_{x_{i}} f$ for every $i=1, \ldots, d$, let us define $w:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\mathrm{J}_{x} w:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ by

$$
w(s, x):=\left(\begin{array}{c}
w^{1}(s, x)  \tag{2.4}\\
\vdots \\
w^{d}(s, x)
\end{array}\right), \quad \mathrm{J}_{x} w(s, x):=\left(\begin{array}{c}
\nabla_{x}^{\top} w^{1}(s, x) \\
\vdots \\
\nabla_{x}^{\top} w^{d}(s, x)
\end{array}\right)
$$

Finally, let $X_{t}^{o}=b^{o} t+\sigma^{o} W_{t}, t \in[0, T]$, be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a fixed $d$-dimensional Brownian motion $W=\left(W_{t}\right)_{t \in[0, T]}$.

Theorem 2.7. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied. For every $\varepsilon \geq 0$, let $v^{\varepsilon}$ be the unique viscosity solution of (2.1) satisfying (2.2), let $w^{i}$ be the unique strong solution of (2.3) with polynomial growth for every $i=1, \ldots, d$, and let $w, \mathrm{~J}_{x} w$ be as in (2.4). Then, for every $(t, x) \in[0, T) \times \mathbb{R}^{d}$ as $\varepsilon \downarrow 0$,

$$
v^{\varepsilon}(t, x)=v^{0}(t, x)+\varepsilon \cdot \partial_{\varepsilon} v^{0}(t, x)+O\left(\varepsilon^{2}\right)
$$

where $\partial_{\varepsilon} v^{0}(t, x)=\lim _{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(v^{\varepsilon}(t, x)-v^{0}(t, x)\right)$ is given by

$$
\partial_{\varepsilon} v^{0}(t, x)=\mathbb{E}\left[\int_{t}^{T}\left(\gamma\left|w\left(s, x+X_{s}^{o}\right)\right|+\eta\left\|\mathrm{J}_{x} w\left(s, x+X_{s}^{o}\right) \sigma^{o}\right\|_{\mathrm{F}}\right) d s \mid X_{t}^{o}=0\right]
$$

with $\mathbb{E}\left[\cdot \mid X_{t}^{o}=0\right]$ denoting the conditional expectation given $X_{t}^{o}=0$.
Remark 2.8. We actually show that the approximation is (locally) uniform in $(t, x)$ : There exists a constant $c$ (that depends on $T, \alpha$ and $C_{f}$ given in Assumption 2.1, and the norms for $b^{o}, \sigma^{o}$ ) such that for every $\varepsilon<\min \left\{1, \lambda_{\min }\left(\sigma^{o}\right)\right\}$ (see Remark 2.3) and every $(t, x) \in[0, T) \times \mathbb{R}^{d}$,

$$
\left|v^{\varepsilon}(t, x)-\left(v^{0}(t, x)+\varepsilon \cdot \partial_{\varepsilon} v^{0}(t, x)\right)\right| \leq c\left(1+|x|^{\alpha}\right) \varepsilon^{2}
$$

Let us mention some basic properties of the sensitivity result given in Theorem 2.7, as well as how it can be used to construct numerical approximations of the $\operatorname{PDE}$ (2.1), as explained in Section 3 below. To that end, recalling the process $X^{o}$ appearing in Theorem 2.7 with the corresponding Brownian motion $W$, let $\widetilde{X}_{t}^{o}:=b^{o} t+\sigma^{o} \widetilde{W}_{t}, t \in[0, T]$, where $\widetilde{W}:=\left(\widetilde{W}_{t}\right)_{t \in[0, T]}$ is another standard $d$-dimensional Brownian motion independent of $W$. We remark the following Feynman-Kac representations: for every $(t, x) \in[0, T] \times \mathbb{R}^{d}$ and $s \in[t, T]$, it holds that

$$
\begin{align*}
v^{0}(t, x) & =\mathbb{E}\left[f\left(x+X_{T}^{o}\right) \mid X_{t}^{o}=0\right]  \tag{2.5}\\
w\left(s, x+X_{s}^{o}\right) & =\mathbb{E}\left[\nabla_{x} f\left(x+X_{s}^{o}+\widetilde{X}_{T}^{o}\right) \mid \widetilde{X}_{s}^{o}=0\right] .
\end{align*}
$$

Furthermore, denote by $\left(\mathrm{J}_{x} w\right)^{k, l}$ for every $k, l \in\{1, \ldots, d\}$ the $(k, l)$-component of $\mathrm{J}_{x} w$ defined in (2.4). If $\nabla_{x} f$ is sufficiently smooth (at least continuously differentiable) and $D_{x}^{2} f$ is at most polynomially growing, then by [61, Theorem 4.32], $\left(\mathrm{J}_{x} w\right)^{k, l}$ also has the following Feynman-Kac representation: for every $(t, x) \in[0, T] \times \mathbb{R}^{d}$ and $s \in[t, T]$,

$$
\begin{equation*}
\left(\mathrm{J}_{x} w\right)^{k, l}\left(s, x+X_{s}^{o}\right)=\mathbb{E}\left[\partial_{x_{k} x_{l}} f\left(x+X_{s}^{o}+\widetilde{X}_{T}^{o}\right) \mid \widetilde{X}_{s}^{o}=0\right] . \tag{2.6}
\end{equation*}
$$

Otherwise, if $\nabla_{x} f$ lacks that kind of regularity, we approximate $\left(\mathrm{J}_{x} w\right)^{k, l}$ via a finite difference quotient as follows: Let $e_{l}$ be a $d$-dimensional vector with value 0 in all the components except for the $l$-th component with value 1 . Then for sufficiently small $h>0$,

$$
\left(\mathrm{J}_{x} w\right)^{k, l}\left(s, x+X_{s}^{o}\right)=\partial_{x_{l}} w^{k}\left(s, x+X_{s}^{o}\right) \approx \frac{1}{h}\left(w^{k}\left(s, x+X_{s}^{o}+h \cdot e_{l}\right)-w^{k}\left(s, x+X_{s}^{o}\right)\right)
$$

In particular, by (2.5) the approximation can be rewritten by

$$
\begin{equation*}
\left(\mathrm{J}_{x} w\right)^{k, l}\left(s, x+X_{s}^{o}\right) \approx \frac{1}{h} \mathbb{E}\left[\partial_{x_{k}} f\left(x+X_{s}^{o}+\widetilde{X}_{T}^{o}+h \cdot e_{l}\right)-\partial_{x_{k}} f\left(x+X_{s}^{o}+\widetilde{X}_{T}^{o}\right) \mid \widetilde{X}_{s}^{o}=0\right] \tag{2.7}
\end{equation*}
$$

Hence, the exact value of $\partial_{\varepsilon} v^{0}$ requires calculations of nested expectations, which will be realized as nested Monte Carlo approximations in the next section.

## 3. Numerical Results

Combining Theorem 2.7 with corresponding probabilistic representations for the functions $v^{0}, w$, and $\mathrm{J}_{x} w$ given in (2.5) and (2.6) (or (2.7)), we derive a Monte Carlo based scheme to implement both the sensitivity $\partial_{\varepsilon} v^{0}$ as well as the approximated solution $v^{0}+\varepsilon \cdot \partial_{\varepsilon} v^{0}$ of the nonlinear PDE (2.1) for every $\varepsilon<\min \left\{1, \lambda_{\min }\left(\sigma^{o}\right)\right\}$ (see Remark 2.8). We provide a pseudocode in Algorithm 1 to show how it can be implemented ${ }^{3}$.

Let us briefly mention the computational complexity of Algorithm 1. For each $N \in \mathbb{N}$ (i.e., the number of steps in the time discretization) and $M_{0}, M_{1} \in \mathbb{N}$ (i.e., the number of samples for each expectation involved in $v^{0}$ and $\partial_{\varepsilon} v^{0}$ ), denote by

$$
\begin{equation*}
\mathfrak{C}\left(d, N, M_{0}, M_{1}\right):=M_{0} d+N M_{1}\left(M_{1}+1\right) d+N M_{1}\left(M_{1}+1+d\right) d^{2} \tag{3.1}
\end{equation*}
$$

the sum of following three components:
(i) $M_{0} d$ is the number of (one-dimensional) samples for the Monte Carlo approximation used in $v^{0}(t, x) \approx \frac{1}{M_{0}} \sum_{j=1}^{M_{0}} f\left(x+\mathcal{X}_{N}(j)\right) ;$
(ii) $N M_{1}\left(M_{1}+1\right) d$ is the sum of the number $N M_{1}^{2} d$ of samples for nested Monte Carlo approximation and the number $N M_{1} d$ of Euclidean norm evaluations used in $\mathbb{E}\left[\int_{t}^{T} \gamma \mid w(s, x+\right.$ $\left.\left.X_{s}^{o}\right) \mid d s\right] \approx \sum_{i=0}^{N-1} \Delta t \frac{1}{M_{1}} \sum_{j=1}^{M_{1}} \gamma|\widehat{w}(i, j)| ;$
(iii) $N M_{1}\left(M_{1}+1+d\right) d^{2}$ is the sum of: the number $N M_{1}^{2} d^{2}$ of samples for the nested MonteCarlo approximation, the number $N M_{1} d^{3}$ of the matrix multiplications, and the number $N M_{1} d^{2}$ of Frobenius norm evaluations used in $\mathbb{E}\left[\int_{t}^{T} \eta\left\|\mathrm{~J}_{x} w\left(s, x+X_{s}^{o}\right) \sigma^{o}\right\|_{\mathrm{F}} d s\right] \approx$ $\sum_{i=0}^{N-1} \Delta t \frac{1}{M_{1}} \sum_{j=1}^{M_{1}} \eta\left\|\widehat{J_{x} w}(i, j) \sigma^{o}\right\|_{\mathrm{F}}$.
Note that while $\mathfrak{C}\left(d, N, M_{0}, M_{1}\right)$ given in (3.1) increases linearly in $M_{0}$, it increases quadratically in $M_{1}$ due to the nested Monte Carlo approximations. To work within constraint of memory in our hardware, we choose $M_{1}$ so that $M_{0} \geq M_{1}$ in all the experiments.

We proceed to calculate the value $v^{0}$ and the sensitivity $\partial_{\varepsilon} v^{0}$ and then compare the approximation $v^{0}+\varepsilon \partial_{\varepsilon} v^{0}$ (given in Theorem 2.7) with $v^{\varepsilon}$. To that end, let us start with the following 1-dimensional example with a specific boundary function: Let $d=1, T=1, b^{o}=1$, $\sigma^{o}=1, f(x)=x^{4},(t, x)=(0,0), N=100, M_{0}=3 \cdot 10^{6}$, and $M_{1}=3 \cdot 10^{4}$, and choose any

[^2]```
Algorithm 1 A Monte Carlo based scheme for \(v^{0}\) and \(\partial_{\varepsilon} v^{0}\).
    Input: \(T>0, b^{o} \in \mathbb{R}^{d}, \sigma^{o} \in \mathbb{R}^{d \times d}, f: \mathbb{R}^{d} \rightarrow \mathbb{R}\) (satisfying Assumptions 2.1, 2.2, and 2.4 ), \((t, x) \in\)
    \([0, T) \times \mathbb{R}^{d}, N \in \mathbb{N}, M_{0}, M_{1} \in \mathbb{N}\) with \(M_{0} \geq M_{1}\), and \(h \geq 0\) (sufficiently small);
    Generate:
        (i) The uniform subdivision \(t_{i}=t+i \Delta t, i \in\{0, \ldots, N\}\) with \(\Delta t=\frac{T-t}{N}\);
        (ii) \(M_{0}\) samples \(\mathcal{W}(j) \sim \mathcal{N}\left(0, \mathrm{Id}_{\mathbb{R}^{d}}\right)\) (i.e., standard \(d\)-dimensional normal distribution), \(j \in\left\{1, \ldots, M_{0}\right\}\);
        (iii) \((N+1) \times M_{0}\) samples \(\mathcal{X}_{i}(j):=b^{o} t_{i}+\sigma^{o} \mathcal{W}(j) \sqrt{t_{i}}, i \in\{0, \ldots, N\}\) and \(j \in\left\{1, \ldots, M_{0}\right\}\);
    Function \(v_{\mathrm{mc}}^{0}\left(t, x ; M_{0}\right)\) :
        Recall \(M_{0}\) realizations \(\mathcal{X}_{N}(j), j \in\left\{1, \ldots, M_{0}\right\} ;\)
    Return \(\frac{1}{M_{0}} \sum_{j=1}^{M_{0}} f\left(x+\mathcal{X}_{N}(j)\right)\)
    Function \(\left(\partial_{\varepsilon} v^{0}\right)_{\mathrm{mc}}\left(t, x ; N, M_{1}\right)\) :
        Recall \(N \times M_{1}\) realizations \(\mathcal{X}_{i}(j), i \in\{0, \ldots, N-1\}\) and \(j \in\left\{1, \ldots, M_{1}\right\} ;\)
        for \(i=0\) to \(N-1\) and \(j=1\) to \(M_{1}\)
            Compute \(\widehat{w}^{k}(i, j):=\frac{1}{M_{1}} \sum_{m=1}^{M_{1}} \partial_{x_{k}} f\left(x+\mathcal{X}_{i}(j)+\mathcal{X}_{N-i}(m)\right) \forall k \in\{1, \ldots, d\} ;\)
        end
        if \(\nabla_{x} f\) is continuously differentiable and has at most polynomial growth
            for \(i=0\) to \(N-1\) and \(j=1\) to \(M_{1}\)
                    Compute \(\left(\widehat{\mathrm{J}_{x} w}\right)^{k, l}(i, j):=\frac{1}{M_{1}} \sum_{m=1}^{M_{1}} \partial_{x_{k}, x_{l}} f\left(x+\mathcal{X}_{i}(j)+\mathcal{X}_{N-i}(m)\right) \forall k, l \in\{1, \ldots, d\} ;\)
            end
        else
            for \(i=0\) to \(N-1\) and \(j=1\) to \(M_{1}\)
                    Compute \(\widehat{w}_{h}^{k, l}(i, j):=\frac{1}{M_{1}} \sum_{m=1}^{M_{1}} \partial_{x_{k}} f\left(x+\mathcal{X}_{i}(j)+h e_{l}+\mathcal{X}_{N-i}(m)\right) \forall k, l \in\{1, \ldots, d\} ;\)
                    Compute \(\left(\widehat{J_{x} w}\right)^{k, l}(i, j):=\frac{1}{h}\left(\widehat{w}_{h}^{k, l}(i, j)-\widehat{w}^{k}(i, j)\right) \forall k, l \in\{1, \ldots, d\}\);
            end
        end
    Return \(\sum_{i=0}^{N-1} \Delta t \frac{1}{M_{1}} \sum_{j=1}^{M_{1}}\left(\gamma|\widehat{w}(i, j)|+\eta\left\|\widehat{\mathrm{J}_{x} w}(i, j) \sigma^{o}\right\|_{\mathrm{F}}\right)\)
```



Figure 1. Comparative analysis between the approximated solution $v^{0}+\varepsilon \cdot \partial_{\varepsilon} v^{0}$ and the actual counterpart $v^{\varepsilon}$ over varying $\varepsilon$.
$\gamma, \eta \leq 1$. Under this case, Assumptions 2.1 and 2.2 are obviously satisfied. Furthermore, since $\lambda_{\min }\left(\sigma^{o}\right)=1$, we can and do choose any $\varepsilon<1$.

As the function $f=x^{4}$ is convex, the probabilistic representation for $v^{\varepsilon}$ given in (4.8) (that will be proven in Lemma 5.6 and Proposition 5.9) ensures that $v^{\varepsilon}(t, x)$ is convex in $x$ and hence the corresponding nonlinear Kolmogorov PDE given in (2.1) can be rewritten by the following


TABLE 1. Implementation of Algorithm 1 for several dimension cases. The average (Avg.) and the standard deviation (Std. Dev.) of the values for $v^{0}$ and $\partial_{\varepsilon} v^{0}$, and the average runtime in seconds are computed over the independent 10 runs of the PYthon code. $\left(b^{o}, \sigma^{o}\right)$ are generated for every $d \in\{1,5,10,20,50,100\}$ but for each $d$, they are fixed during the 10 runs of the code.
quasilinear parabolic equation

$$
\begin{equation*}
\partial_{t} v^{\varepsilon}+\frac{\left(\sigma^{o}+\eta \varepsilon\right)^{2}}{2} \partial_{x x} v^{\varepsilon}+b^{o} \partial_{x} v^{\varepsilon}+\sup _{|\widetilde{b}| \leq \gamma \varepsilon}\left(\widetilde{b} \partial_{x} v^{\varepsilon}\right)=0 \quad \text { on }(t, x) \in[0, T) \times \mathbb{R} \tag{3.2}
\end{equation*}
$$

with $v^{\varepsilon}(T, x)=x^{4}, x \in \mathbb{R}$. In particular, since the $\operatorname{PDE}(3.2)$ is linear in the second derivative and the boundary $f=x^{4}$ is a polynomial, Remark 2.5 guarantees that (3.2) admits a unique viscosity solution satisfying (2.2), which ensures Assumption 2.4 to hold.

Figure 1 shows the comparison between $v^{\varepsilon}$ and $v^{0}+\varepsilon \cdot \partial_{\varepsilon} v^{0}$ with varying $\varepsilon \leq 0.1$, in which we obtain numerical results on $v^{\varepsilon}$ by applying a finite difference approximation on the semilinear PDE (3.2) (we refer to Code_1.m and Code_2.m given in the link provided in Footnote 3; based on [50, Matlab Code 7 in Section 3]) and obtain numerical results on $v^{0}+\varepsilon \cdot \partial_{\varepsilon} v^{0}$ by using our Monte Carlo based scheme given in Algorithm 1 (we refer to Code_3.ipynb given in the mentioned link). As it has been proven in Theorem 2.7, we can observe in all the panels of Figure 1 that the error $\left|v^{0}+\varepsilon \cdot \partial_{\varepsilon} v^{0}-v^{\varepsilon}\right|$ of the approximation increases quadratically in $\varepsilon$.

Now we implement Algorithm 1 for multi-dimensional cases with another boundary function: Let $T=1, f(x)=\sin \left(\sum_{i=1}^{d} x_{i}\right),(t, x)=(0,(0, \cdots, 0)), N=100, M_{0}=3 \cdot 10^{6}$, and $M_{1}=3 \cdot 10^{4}$, and choose any $\gamma, \eta \leq 1$. Furthermore, denote by $\mathrm{U}([a, b])$ for $a, b \in \mathbb{R}$ the uniform distribution with values in $[a, b]$. Then for any $d \in \mathbb{N}$, we generate $b^{o} \in \mathbb{R}^{d}, \sigma^{o} \in \mathbb{R}^{d \times d}$ in the following way:
. $b^{o}:=\widetilde{b} /\left(\sum_{i=1}^{d}\left|\widetilde{b}^{i}\right|\right)$, where $\widetilde{b}$ is a $d$-dimensional random variable such that $\widetilde{b^{i}} \sim \mathrm{U}([0,1])$ for every $i \in\{1, \ldots, d\}$ and satisfy $\sum_{i=1}^{d}\left|\widetilde{b}^{i}\right| \neq 0$;

- $\sigma^{o}:=\tilde{\sigma} /\left(\sum_{l=1}^{d}\left(\sum_{k=1}^{d} \tilde{\sigma}^{k, l}\right)^{2}\right)^{1 / 2}$, where let $\widetilde{\sigma}=\left(\widetilde{\sigma}^{k, l}\right)_{k, l \in\{1, \ldots, d\}}$ is a $d \times d$-valued random variable such that $\widetilde{\sigma}^{k, l} \sim \mathrm{U}([-1,1])$ for every $k, l \in\{1, \ldots, d\}$, satisfy that $\widetilde{\sigma}$ is invertible, and that $\left(\sum_{l=1}^{d}\left(\sum_{k=1}^{d} \tilde{\sigma}^{k, l}\right)^{2}\right)^{1 / 2} \neq 0$.
Under this setup, Assumptions 2.1 and 2.2 are satisfied. Furthermore, since $f=\sin \left(\sum_{i=1}^{d} x_{i}\right)$ is bounded and Lipschitz continuous, by Remark 2.5, Assumption 2.4 is also satisfied. Hence, the corresponding viscosity solution of the nonlinear Kolmogorov PDE is unique. We further
note that unlike the semilinear form of the $\operatorname{PDE}$ (3.2) where the boundary function $f$ is convex, the corresponding Kolmogorov PDE under this setup is fully nonlinear.

Denote by $\Gamma=\left(\Gamma_{t}\right)_{t \in[0, T]}$ an 1-dimensional process satisfying $\Gamma_{t}=t+W_{t}^{1}, t \in[0, T]$, where $W^{1}$ is a standard 1-dimensional Brownian motion independent of $W$. Then, by the nomalization in the parameters $b^{o}$ and $\sigma^{o}$, the following property holds: for every $d \in \mathbb{N}$,

$$
\text { law of } \sum_{i=1}^{d} X^{o, i}=\text { law of } \Gamma
$$

Combined with (2.5) and (2.6) (noting that $\nabla_{x} f=\cos \left(\sum_{i=1}^{d} x_{i}\right) \mathbf{1}_{d}, \mathrm{~J}_{x} f=-\sin \left(\sum_{i=1}^{d} x_{i}\right) \mathbf{1}_{d \times d}$ where here $\mathbf{1}_{d}$ denotes a $d$-dimensional vector with value 1 in all the components and $\mathbf{1}_{d \times d}$ denotes a $d \times d$-matrix with value 1 in all the components), this ensures the following characterizations: for every $d \in \mathbb{N}$,

- $v^{0}(0,0)=\mathbb{E}\left[\sin \left(\sum_{i=1}^{d} X_{T}^{o, i}\right)\right]=\mathbb{E}\left[\sin \left(\Gamma_{T}\right)\right] ;$
- $\partial_{\varepsilon} v^{0}(0,0)$ (when $\left.(\gamma, \eta)=(1,0)\right)$ is characterized by

$$
\begin{aligned}
\mathbb{E}\left[\int_{0}^{T}\left|w\left(t, X_{t}^{o}\right)\right| d t\right] & =\mathbb{E}\left[\int_{0}^{T}\left|\mathbb{E}\left[\cos \left(\sum_{i=1}^{d}\left(X_{t}^{o, i}+\widetilde{X}_{T}^{o, i}\right)\right) \mid \widetilde{X}_{t}^{o}=0\right] \mathbf{1}_{d}\right| d t\right] \\
& =\sqrt{d} \cdot \mathbb{E}\left[\int_{0}^{T}\left|\mathbb{E}\left[\cos \left(\Gamma_{t}+\widetilde{\Gamma}_{T}\right) \mid \widetilde{\Gamma}_{t}=0\right]\right| d t\right]
\end{aligned}
$$

with $\widetilde{X}^{o}$ appearing in (2.5), where we denote by $\widetilde{\Gamma}_{t}:=t+\widetilde{W}_{t}^{1} t \in[0, T]$ a 1-dimensional process with a standard 1-dimensional Brownian motion $\widetilde{W}^{1}$ independent of $W^{1}$;

- $\partial_{\varepsilon} v^{0}(0,0)$ (when $\left.(\gamma, \eta)=(0,1)\right)$ is given by

$$
\begin{aligned}
\mathbb{E}\left[\int_{0}^{T}\left\|\mathrm{~J}_{x} w\left(t, X_{t}^{o}\right) \sigma^{o}\right\|_{\mathrm{F}} d t\right] & =\mathbb{E}\left[\int_{0}^{T}\left\|\mathbb{E}\left[\sin \left(\sum_{i=1}^{d}\left(X_{t}^{o, i}+\widetilde{X}_{T}^{o, i}\right)\right) \mid \widetilde{X}_{t}^{o}=0\right] \mathbf{1}_{d \times d} \sigma^{o}\right\|_{\mathrm{F}} d t\right] \\
& =\sqrt{d} \cdot \mathbb{E}\left[\int_{0}^{T}\left|\mathbb{E}\left[\sin \left(\Gamma_{t}+\widetilde{\Gamma}_{T}\right) \mid \widetilde{\Gamma}_{t}=0\right]\right| d t\right]
\end{aligned}
$$

Table 1 shows the results of several dimension cases based on 10 independent runs of a Python code (Code_4.ipynb) given in the link provided in Footnote 3. As shown in the above characterizations (i)-(iii), the values for $v^{0}$ are invariant as $0.5103( \pm 0.0003)$ over the dimension $d$ (in consideration of the error of the Monte Carlo approximations; see (i)) and the value for $\partial_{\varepsilon} v^{0}$ increases in proportional to $\sqrt{d}$ for all three cases $(\gamma, \eta) \in\{(1,0),(0,1),(1,1)\}$ (see (ii) and (iii)). Furthermore, the average runtime results show that though the complexity $\mathfrak{C}\left(d, N, M_{0}, M_{1}\right)$ is quadratic to the number of samples $M_{1}\left(=3 \cdot 10^{4}\right.$ in this case), the computation for high dimensional cases (e.g., $d=50,100$ ) is still feasible under our Monte Carlo based algorithm.

## 4. Proof of Theorem 2.7

We start by providing some notions. Let $t \in[0, T)$, denote by $C\left([t, T] ; \mathbb{R}^{d}\right)$ the set of all $\mathbb{R}^{d}$-valued continuous functions on $[t, T]$, and set

$$
\Omega^{t}:=\left\{\omega=\left(\omega_{s}\right)_{s \in[t, T]} \in C\left([t, T] ; \mathbb{R}^{d}\right): \omega_{t}=0\right\}
$$

to be the canonical space of continuous paths. Let $W^{t}:=\left(W_{s}^{t}\right)_{s \in[t, T]}$ be the canonical process on $\Omega^{t}$ and $\mathbb{F}^{W^{t}}:=\left(\mathcal{F}_{s}^{W^{t}}\right)_{s \in[t, T]}$ be the raw filtration generated by $W^{t}$. We equip $\Omega^{t}$ with the uniform convergence norm so that the Borel $\sigma$-field $\mathcal{F}^{t}$ on $\Omega^{t}$ coincides with $\mathcal{F}_{T}^{W^{t}}$. Furthermore,
let $\mathbb{P}_{0}^{t}$ be the Wiener measure under which $W^{t}$ is a Brownian motion and write $\mathbb{E}^{\mathbb{P}_{0}^{t}}[\cdot]$ for the expectation under $\mathbb{P}_{0}^{t}$.

On $\left(\Omega^{t}, \mathcal{F}^{t}, \mathbb{F}^{W^{t}}, \mathbb{P}_{0}^{t}\right)$, consider $X^{t, x ; o}:=\left(X_{s}^{t, x ; o}\right)_{s \in[t, T]}$ following the baseline coefficients $b^{o}$ and $\sigma^{o}$ and starting with $x \in \mathbb{R}^{d}$, i.e. for $s \in[t, T]$,

$$
\begin{equation*}
X_{s}^{t, x ; o}=x+b^{o}(s-t)+\sigma^{o} W_{s}^{t} \tag{4.1}
\end{equation*}
$$

Moreover, let $\mathbb{L}^{t, 1}\left(\mathbb{R}^{d}\right)$ and $\mathbb{L}_{\mathrm{F}}^{t, 1}\left(\mathbb{R}^{d \times d}\right)$ be the set of all $\mathbb{F}^{W^{t}}$-predictable processes $L$ defined on $[t, T]$ with values in $\mathbb{R}^{d}$ and $\mathbb{R}^{d \times d}$, respectively. We endow $\mathbb{L}^{t, 1}\left(\mathbb{R}^{d}\right)$ and $\mathbb{L}_{\mathrm{F}}^{t, 1}\left(\mathbb{R}^{d \times d}\right)$ with the norms, respectively,

$$
\|L\|_{\mathbb{L}^{t, 1}}:=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left|L_{s}\right| d s\right], \quad\|L\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}}:=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\|L_{s}\right\|_{\mathrm{F}} d s\right]
$$

In analogy, we define $\mathbb{L}^{t, \infty}\left(\mathbb{R}^{d}\right)$ as the set of all $\mathbb{R}^{d}$-valued, $\mathbb{F}^{W^{t}}$-predictable processes $L$ defined on $[t, T]$ that are bounded $\mathbb{P}_{0}^{t} \otimes d s$-a.e.. Finally, set

$$
\|L\|_{\mathbb{L}^{t, \infty}}:=\inf \left\{C \geq 0:\left|L_{s}\right| \leq C \mathbb{P}_{0}^{t} \otimes d s \text {-a.e. }\right\}<\infty
$$

The space $\mathbb{L}_{\mathrm{F}}^{t, \infty}\left(\mathbb{R}^{d \times d}\right)$ of $\mathbb{R}^{d \times d}$-valued processes is defined analogously to $\mathbb{L}^{t, \infty}\left(\mathbb{R}^{d}\right)$, with $|\cdot|$ replaced by $\|\cdot\|_{\mathrm{F}}$ in the definition of $\|L\|_{\mathbb{L}_{\mathrm{F}}^{t, \infty}}$.

For any $(b, \sigma) \in \mathbb{L}^{t, \infty}\left(\mathbb{R}^{d}\right) \times \mathbb{L}_{\mathrm{F}}^{t, \infty}\left(\mathbb{R}^{d \times d}\right)$, we define an Itô $\left(\mathbb{F}^{W^{t}}, \mathbb{P}_{0}^{t}\right)$-semimartingal $X^{t, x ; b, \sigma}=$ $\left(X_{s}^{t, x ; b, \sigma}\right)_{s \in[t, T]}$ starting with $x \in \mathbb{R}^{d}$ by

$$
\begin{equation*}
X_{s}^{t, x ; b, \sigma}:=x+\int_{t}^{s} b_{u} d u+\int_{t}^{s} \sigma_{u} d W_{u}^{t}, \quad s \in[t, T] \tag{4.2}
\end{equation*}
$$

and note that $X_{s}^{t, x ; o}=X_{s}^{t, x ; b^{o}, \sigma^{o}} s \in[t, T]$; see (4.1). Moreover, for any $\varepsilon \geq 0$ and $t \in[0, T)$, denote by

$$
\begin{equation*}
\mathcal{C}^{\varepsilon}(t):=\left\{(b, \sigma) \in \mathbb{L}^{t, \infty}\left(\mathbb{R}^{d}\right) \times \mathbb{L}_{\mathrm{F}}^{t, \infty}\left(\mathbb{R}^{d \times d}\right) \mid\left(b_{s}, \sigma_{s}\right) \in \mathcal{B}^{\varepsilon} \mathbb{P}_{0}^{t} \otimes d s \text {-a.e. }\right\} \tag{4.3}
\end{equation*}
$$

the set of all $\mathbb{F}^{W^{t}}$-predictable processes taking values within the $\varepsilon$-neighborhood $\mathcal{B}^{\varepsilon}$ of the baseline coefficients $\left(b^{o}, \sigma^{o}\right)$ given in (1.3).

Let us start by providing some a priori estimates for $X_{T}^{t, x ; b, \sigma}$.
Lemma 4.1. For every $p \geq 1$, there is a constant $C_{p}>0$ such that the following holds:
(i) For every $(t, x) \in[0, T) \times \mathbb{R}^{d}$, we have that $\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t, x ; o}\right|^{p}\right] \leq C_{p}\left(1+|x|^{p}\right)$.
(ii) For every $\varepsilon \geq 0$ and $(t, x) \in[0, T) \times \mathbb{R}^{d}$, we have that

$$
\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right|^{p}\right] \leq C_{p} \varepsilon^{p}
$$

Proof. We only prove (ii), as the proof for (i) follows the same line of reasoning.
Fix $\varepsilon \geq 0$ and $(t, x) \in[0, T) \times \mathbb{R}^{d}$, and let $(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)$. Then

$$
X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}=\int_{t}^{T}\left(b_{s}-b^{o}\right) d s+\int_{t}^{T}\left(\sigma_{s}-\sigma^{o}\right) d W_{s}^{t}
$$

We estimate both terms separately. By Jensen's inequality and the definition of $\mathcal{C}^{\varepsilon}(t)$,

$$
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|\int_{t}^{T}\left(b_{s}-b^{o}\right) d s\right|^{p}\right] \leq \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[(T-t)^{p-1} \int_{t}^{T}\left|b_{s}-b^{o}\right|^{p} d s\right] \leq(T-t)^{p} \varepsilon^{p}
$$

Moreover, if $c_{\mathrm{BDG}, p}>0$ denotes the constant appearing in the Burkholder-Davis-Gundy (BDG) inequality (see, e.g, [22, Theorem 92, Chap. VII]), then

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|\int_{t}^{T}\left(\sigma_{s}-\sigma^{o}\right) d W_{s}^{t}\right|^{p}\right] & \leq c_{\mathrm{BDG}, p} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left(\int_{t}^{T}\left\|\sigma_{s}-\sigma^{o}\right\|_{\mathrm{F}}^{2} d s\right)^{p / 2}\right] \\
& \leq c_{\mathrm{BDG}, p}(T-t)^{p / 2} \varepsilon^{p}
\end{aligned}
$$

where the second inequality follows from the definition of $\mathcal{C}^{\varepsilon}(t)$. Thus the proof is completed using the elementary inequality $(a+b)^{p} \leq 2^{p}\left(a^{p}+b^{p}\right)$ for all $a, b \geq 0$.

Remark 4.2. By Assumption 2.1, the (weak) Hessian $D_{x}^{2} f$ has at most polynomial growth of order $\alpha$. In particular, there is a constant $\widetilde{C}_{f}>0$ (that depends on $C_{f}$ in Assumption 2.1) such that for every $(x, y) \in \mathbb{R}^{d}$,

$$
\left|f(y)-f(x)-\nabla_{x}^{\top} f(x)(x-y)\right| \leq \widetilde{C}_{f}\left(1+|x|^{\alpha}+|y|^{\alpha}\right) \cdot|y-x|^{2}
$$

Moreover, $\nabla_{x} f$ and $f$ have at most polynomial growth of of order $\alpha+1$ and $\alpha+2$, respectively.
Next note that if $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is any function with at most polynomial growth, then Lemma 4.1 implies that $g\left(X_{T}^{t, x ; b, \sigma}\right)$ is integrable for every $(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)$. Therefore $f\left(X_{T}^{t, x ; b, \sigma}\right), \partial_{x_{i}} f\left(X_{T}^{t, x ; b, \sigma}\right)$, $\left|\partial_{x_{i}} f\left(X_{T}^{t, x ; b, \sigma}\right)\right|^{2}, \ldots$ are integrable.

Lemma 4.3. Suppose that Assumptions 2.1 and 2.2 are satisfied. For $i=1, \ldots$, d, let $w^{i}:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ be the unique strong solution of (2.3) with polynomial growth (see Proposition 2.6 (ii)). Let $w:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\mathrm{J}_{x} w:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}$ be given in (2.4), let $(t, x) \in[0, T] \times \mathbb{R}^{d}$, and set

$$
\begin{equation*}
Y_{s}^{t, x}:=w\left(s, X_{s}^{t, x ; o}\right), \quad Z_{s}^{t, x}:=\mathrm{J}_{x} w\left(s, X_{s}^{t, x ; o}\right) \sigma^{o}, \quad s \in[t, T] \tag{4.4}
\end{equation*}
$$

Then, for every $i=1, \ldots, d$,

$$
Y_{s}^{t, x, i}=\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right)-\int_{s}^{T}\left(Z_{r}^{t, x, i}\right)^{\top} d W_{r}^{t}, \quad s \in[t, T]
$$

with $Y^{t, x, i}$ and $\left(Z^{t, x, i}\right)^{\top}$ denoting the $i$-th component of $Y^{t, x}$ and the $i$-th row vector of $Z^{t, x}$, respectively. In particular, $Y_{s}^{t, x}=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\nabla_{x} f\left(X_{T}^{t, x, o}\right) \mid \mathcal{F}_{s}^{W^{t}}\right]$ for every $s \in[t, T]$.
Proof of Lemma 4.3. Fix $i \in\{1, \ldots, d\}$. Since $w^{i} \in C^{1,2}\left([t, T) \times \mathbb{R}^{d}\right)$ (see Proposition 2.6 (ii) and Section 5.3), an application of Itô's formula ensures that for every $s \in[t, T]$,

$$
\begin{aligned}
& w^{i}\left(T, X_{T}^{t, x ; o}\right)-w^{i}\left(s, X_{s}^{t, x ; o}\right)=\int_{s}^{T} \nabla_{x}^{\top} w^{i}\left(r, X_{r}^{t, x ; o}\right) \sigma^{o} d W_{r}^{t} \\
& \quad+\int_{s}^{T}\left(\partial_{r} w^{i}\left(r, X_{r}^{t, x ; o}\right)+\left\langle b^{o}, \nabla_{x} w^{i}\left(r, X_{r}^{t, x ; o}\right)\right\rangle+\frac{1}{2} \operatorname{tr}\left(\left(\sigma^{o}\right)\left(\sigma^{o}\right)^{\top} D_{x}^{2} w^{i}\left(r, X_{r}^{t, x ; o}\right)\right)\right) d r
\end{aligned}
$$

The second integral is equal to zero because $w^{i}$ solves the linear Kolmogorov PDE given in (2.3). Therefore, using the boundary condition $w^{i}(T, \cdot)=\partial_{x_{i}} f(\cdot)$ and the definitions of $Y^{t, x}$ and $Z^{t, x}$ given in (4.4), we conclude that for every $s \in[t, T]$,

$$
\begin{equation*}
Y_{s}^{t, x, i}=\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right)-\int_{s}^{T}\left(Z_{u}^{t, x, i}\right)^{\top} d W_{u}^{t} \tag{4.5}
\end{equation*}
$$

as claimed.
The 'in particular' part follows by taking conditional expectations in (4.5). Indeed, by Remark 4.2, $\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right)$ and $Y_{t}^{t, x}=w(t, x)$ are square integrable (because $\partial_{x_{i}} f$ and $w$ have
polynomial growth). Therefore, it follows from (4.5) that $\int_{.}^{T}\left(Z_{s}^{t, x, i}\right)^{\top} d W_{s}^{t}$ is a square integrable martingale. Furthermore, since $Y_{s}^{t, x, i}=w^{i}\left(s, X_{s}^{t, x ; o}\right)$ is $\mathcal{F}_{s}^{W^{t}}$-measurable,

$$
\begin{aligned}
Y_{s}^{t, x, i}=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[Y_{s}^{t, x, i} \mid \mathcal{F}_{s}^{W^{t}}\right] & =\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right)-\int_{s}^{T}\left(Z_{r}^{t, x, i}\right)^{\top} d W_{r}^{t} \mid \mathcal{F}_{s}^{W^{t}}\right] \\
& =\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right) \mid \mathcal{F}_{s}^{W^{t}}\right]
\end{aligned}
$$

as claimed.
For sufficiently integrable $\mathbb{R}^{d}$-valued processes $L=\left(L_{s}\right)_{s \in[t, T]}$ and $M=\left(M_{s}\right)_{s \in[t, T]}$, set

$$
\langle L, M\rangle_{\mathbb{P}_{0}^{t} \otimes d s}:=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\langle L_{s}, M_{s}\right\rangle d s\right]
$$

In a similar manner, we set $\langle L, M\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}}:=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\langle L_{s}, M_{s}\right\rangle_{\mathrm{F}} d s\right]$ for $\mathbb{R}^{d \times d}$-valued processes.
Lemma 4.4. Suppose that Assumptions 2.1 and 2.2 are satisfied and let $Y^{t, x}, Z^{t, x}$ be the processes defined in (4.4). Then, for every $\varepsilon \geq 0$ and $(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)$, we have that

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\nabla_{x}^{\top} f\left(X_{T}^{t, x ; o}\right)\left(X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right)\right]=\left\langle Y^{t, x}, b-b^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}+\left\langle Z^{t, x}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}} \tag{4.6}
\end{equation*}
$$

Proof. For $i=1, \ldots, d$, denote by $\left(b_{s}^{i}-b^{o, i}\right)_{s \in[t, T]}$ and $\left(\sigma_{s}^{i}-\sigma^{o, i}\right)_{s \in[t, T]}$ the $i$-th component of $b-b^{o}$ and $i$-th row vector of $\sigma-\sigma^{o}$, respectively. Using this notation,

$$
\begin{aligned}
& \nabla_{x}^{\top} f\left(X_{T}^{t, x ; o}\right)\left(X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right) \\
& =\sum_{i=1}^{d}\left(\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right) \int_{t}^{T}\left(b_{s}^{i}-b^{o, i}\right) d s+\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right) \int_{t}^{T}\left(\sigma_{s}^{i}-\sigma^{o, i}\right) d W_{s}^{t}\right)=: \sum_{i=1}^{d}\left(\Xi^{b, i}+\Xi^{\sigma, i}\right)
\end{aligned}
$$

It follows from Remark 4.2 that $\Xi^{b, i}, \Xi^{\sigma, i}$ are integrable (noting that $b-b^{0}$ and $\sigma-\sigma^{0}$ are bounded uniformly). In particular,

$$
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\nabla_{x}^{\top} f\left(X_{T}^{t, x ; o}\right)\left(X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right)\right]=\sum_{i=1}^{d}\left(\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{b, i}\right]+\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{\sigma, i}\right]\right)
$$

and it remains to show that

$$
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{b, i}\right]=\left\langle Y^{t, x, i}, b^{i}-b^{o, i}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s} \quad \text { and } \quad \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{\sigma, i}\right]=\left\langle Z^{t, x, i},\left(\sigma^{i}-\sigma^{o, i}\right)^{\top}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}
$$

for every $i=1, \ldots, d$. To that end, fix such $i$.
We first claim that $\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{b, i}\right]=\left\langle Y^{t, x, i}, b^{i}-b^{o, i}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}$. Indeed, an application of Fubini's theorem shows that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{b, i}\right] & =\int_{t}^{T} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right) \mid \mathcal{F}_{s}^{W^{t}}\right]\left(b_{s}^{i}-b^{o, i}\right)\right] d s \\
& =\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T} Y_{s}^{t, x, i}\left(b_{s}^{i}-b^{o, i}\right) d s\right]=\left\langle Y^{t, x, i}, b^{i}-b^{o, i}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s},
\end{aligned}
$$

where the second inequality holds because $Y_{s}^{t, x, i}=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\partial_{x_{i}} f\left(X_{T}^{t, x ; o}\right) \mid \mathcal{F}_{s}^{W^{t}}\right]$, see Lemma 4.3.

Next, we claim that

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{\sigma, i}\right]=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left(Z_{s}^{t, x, i}\right)^{\top}\left(\sigma_{s}^{i}-\sigma^{o, i}\right)^{\top} d s\right]=\left\langle Z^{t, x, i},\left(\sigma^{i}-\sigma^{o, i}\right)^{\top}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s} . \tag{4.7}
\end{equation*}
$$

Note that by Lemma 4.3,

$$
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\Xi^{\sigma, i}\right]=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left(\int_{t}^{T}\left(Z_{s}^{t, x, i}\right)^{\top} d W_{s}^{t}+Y_{t}^{t, x, i}\right) \int_{t}^{T}\left(\sigma_{s}^{i}-\sigma^{o, i}\right) d W_{s}^{t}\right]
$$

and by the Itô-isometry,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left(Z_{s}^{t, x, i}\right)^{\top} d W_{s}^{t} \int_{t}^{T}\left(\sigma_{s}^{i}-\sigma^{o, i}\right) d W_{s}^{t}\right] & =\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left(Z_{s}^{t, x, i}\right)^{\top}\left(\sigma_{s}^{i}-\sigma^{o, i}\right)^{\top} d s\right] \\
& =\left\langle Z^{t, x, i},\left(\sigma^{i}-\sigma^{o, i}\right)^{\top}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s} .
\end{aligned}
$$

Moreover, since $Y_{t}^{t, x, i}=w^{i}(t, x)$ (see (4.4) given in Lemma 4.3),

$$
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[Y_{t}^{t, x, i} \int_{t}^{T}\left(\sigma_{s}^{i}-\sigma^{o, i}\right) d W_{s}^{t}\right]=Y_{t}^{t, x, i} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left(\sigma_{s}^{i}-\sigma^{o, i}\right) d W_{s}^{t} \mid \mathcal{F}_{t}^{W^{t}}\right]=0
$$

and (4.7) follows.
In Section 5.4, we shall show that if Assumptions 2.1, 2.2, and 2.4 are satisfied, then the unique viscosity solution $v^{\varepsilon}$ of (2.1) satisfies the following: For all $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right)$ and $(t, x) \in$ $[0, T) \times \mathbb{R}^{d}$, we have that

$$
\begin{equation*}
v^{\varepsilon}(t, x)=\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[f\left(X_{T}^{t, x ; b, \sigma}\right)\right] \tag{4.8}
\end{equation*}
$$

with $v^{\varepsilon}(T, \cdot)=f(\cdot)$, see Lemma 5.6 and Proposition 5.9. The formula for $v^{\varepsilon}$ given in (4.8) will be crucial in the following proof.

Lemma 4.5. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied and, for every $(t, x) \in$ $[0, T) \times \mathbb{R}^{d}$, let $Y^{t, x}, Z^{t, x}$ be the processes defined in (4.4). Moreover, let $\alpha$ be as in Assumption 2.1. Then, there exists a constant $c$ independent of $t, x, \varepsilon$ such that for every $\varepsilon<$ $\min \left\{1, \lambda_{\min }\left(\sigma^{0}\right)\right\}$ and $(t, x) \in[0, T) \times \mathbb{R}^{d}$, we have that

$$
\begin{aligned}
& \left|v^{\varepsilon}(t, x)-\left(v^{0}(t, x)+\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)}\left(\left\langle Y^{t, x}, b-b^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}+\left\langle Z^{t, x}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}}\right)\right)\right| \\
& \leq c\left(1+|x|^{\alpha}\right) \varepsilon^{2}
\end{aligned}
$$

Proof. Fix $\varepsilon$ as in the lemma and recall the formula for $v^{\varepsilon}$ given in (4.8); in particular

$$
v^{0}(t, x)=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[f\left(X_{T}^{t, x ; o}\right)\right]
$$

Next, using Remark 4.2, for any $(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)$,

$$
\begin{aligned}
& \left|f\left(X_{T}^{t, x ; b, \sigma}\right)-f\left(X_{T}^{t, x ; o}\right)-\nabla_{x}^{\top} f\left(X_{T}^{t, x ; o}\right)\left(X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right)\right| \\
& \leq \widetilde{C}_{f} \cdot\left(1+\left|X_{T}^{t, x ; b, \sigma}\right|^{\alpha}+\left|X_{T}^{t, x ; o}\right|^{\alpha}\right) \cdot\left|X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right|^{2}=: \mathrm{I}^{b, \sigma} .
\end{aligned}
$$

We claim that there is $c>0$ that depends only on $\alpha, \widetilde{C}_{f}$ (see Remark 4.2) such that

$$
\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[I^{b, \sigma}\right] \leq c\left(1+|x|^{\alpha}\right) \varepsilon^{2}
$$

To that end, an application of the Cauchy-Schwartz inequality together with the elementary inequality $(1+a+b)^{2} \leq 3^{2}\left(1+a^{2}+b^{2}\right)$ for all $a, b \geq 0$ shows that

$$
\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\mathrm{I}^{b, \sigma}\right] \leq \widetilde{C}_{f} 3 \cdot \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[1+\left|X_{T}^{t, x ; b, \sigma}\right|^{2 \alpha}+\left|X_{T}^{t, x ; o}\right|^{2 \alpha}\right]^{1 / 2} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right|^{4}\right]^{1 / 2}
$$

Moreover, we have by Lemma 4.1 that

$$
\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right|^{4}\right]^{1 / 2} \leq C_{4}^{1 / 2} \varepsilon^{2}
$$

where $C_{4}$ is the constant appearing in Lemma 4.1. Furthermore, as $\varepsilon<1$, another application of Lemma 4.1 together with the inequality $(a+b)^{2 \alpha} \leq 2^{2 \alpha}\left(a^{2 \alpha}+b^{2 \alpha}\right)$ for all $a, b \geq 0$ implies that

$$
\begin{aligned}
& \sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[1+\left|X_{T}^{t, x ; b, \sigma}\right|^{2 \alpha}+\left|X_{T}^{t, x ; o}\right|^{2 \alpha}\right]^{1 / 2} \\
& \leq \sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)}\left(1+2^{2 \alpha} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right|^{2 \alpha}\right]+\left(2^{2 \alpha}+1\right) \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t, x ; o}\right|^{2 \alpha}\right]\right)^{1 / 2} \\
& \leq\left(1+2^{2 \alpha} C_{2 \alpha}+\left(2^{2 \alpha}+1\right) C_{2 \alpha}\left(1+|x|^{2 \alpha}\right)\right)^{1 / 2} \leq\left(1+2^{2 \alpha+1} C_{2 \alpha}+C_{2 \alpha}\right)^{1 / 2}\left(1+|x|^{\alpha}\right)
\end{aligned}
$$

where $C_{2 \alpha}$ is the constant appearing in Lemma 4.1. Our claim follows by setting $c:=$ $\widetilde{C}_{f} 3 C_{4}^{1 / 2}\left(1+2^{2 \alpha+1} C_{2 \alpha}+C_{2 \alpha}\right)^{1 / 2}$.

Finally, combining all the previous estimates we conclude that

$$
\left|v^{\varepsilon}(t, x)-\left(v^{0}(t, x)+\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}\left[\nabla_{x}^{\top} f\left(X_{T}^{t, x ; o}\right)\left(X_{T}^{t, x ; b, \sigma}-X_{T}^{t, x ; o}\right)\right]\right)\right| \leq c\left(1+|x|^{\alpha}\right) \varepsilon^{2}
$$

Thus the proof is completed by an application of Lemma 4.4.
Lemma 4.6. Suppose that Assumptions 2.1 and 2.2 are satisfied. Then, for every $\varepsilon \geq 0$ and $(t, x) \in[0, T) \times \mathbb{R}^{d}$,

$$
\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)}\left(\left\langle Y^{t, x}, b-b^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}+\left\langle Z^{t, x}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}}\right)=\varepsilon \cdot\left(\gamma\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}}+\eta\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}}\right) .
$$

Proof. Set

$$
\Phi(\varepsilon):=\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)}\left(\left\langle Y^{t, x}, b-b^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}+\left\langle Z^{t, x}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, F}\right)
$$

We first claim that for every $\varepsilon \geq 0$,

$$
\begin{equation*}
\Phi(\varepsilon) \leq \varepsilon \cdot\left(\gamma\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}}+\eta\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}}\right) \tag{4.9}
\end{equation*}
$$

To that end, set

$$
\mathcal{C}_{1}^{\varepsilon}(t):=\left\{b:(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)\right\}, \quad \mathcal{C}_{2}^{\varepsilon}(t):=\left\{\sigma:(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)\right\}
$$

so that $\mathcal{C}^{\varepsilon}(t)=\mathcal{C}_{1}^{\varepsilon}(t) \times \mathcal{C}_{2}^{\varepsilon}(t)$. Using the Cauchy-Schwartz inequality in $\mathbb{R}^{d}$ and Hölder's inequality (with exponents 1 and $\infty$ ),

$$
\sup _{b \in \mathcal{C}_{1}^{\varepsilon}(t)}\left\langle Y^{t, x}, b-b^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s} \leq \sup _{b \in \mathcal{C}_{1}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left|Y_{s}^{t, x}\right|\left|b_{s}-b^{o}\right| d s\right] \leq\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}} \varepsilon \gamma .
$$

In a similarly manner,

$$
\sup _{\sigma \in \mathcal{C}_{2}^{\varepsilon}(t)}\left\langle Z^{t, x}, \sigma-\sigma^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}} \leq \sup _{\sigma \in \mathcal{C}_{2}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\|Z_{s}^{t, x}\right\|_{\mathrm{F}}\left\|\sigma_{s}-\sigma^{o}\right\|_{\mathrm{F}} d s\right] \leq\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}} \varepsilon \eta .
$$

The combination of these two estimates shows (4.9).
Next we claim that for every $\varepsilon \geq 0$,

$$
\begin{equation*}
\Phi(\varepsilon) \geq \varepsilon \cdot\left(\gamma\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}}+\eta\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathbb{F}}^{t, 1}}\right) \tag{4.10}
\end{equation*}
$$

To that end, fix $\varepsilon \geq 0$. Define $\tilde{\sigma}^{*} \in \mathbb{L}_{\mathrm{F}}^{t, \infty}$ by

$$
\tilde{\sigma}_{s}^{*}:= \begin{cases}\frac{Z_{s}^{t, x}}{\left\|Z_{s}^{t, x}\right\|_{\mathrm{F}}} & \text { if }\left\|Z_{s}^{t, x}\right\|_{\mathrm{F}}>0 \\ 0 & \text { else }\end{cases}
$$

which satisfies $\left\|\tilde{\sigma}^{*}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, \infty}} \leq 1$ and $\left\langle Z_{s}^{t, x}, \tilde{\sigma}_{s}^{*}\right\rangle_{\mathrm{F}}=\left\|Z_{s}^{t, x}\right\|_{\mathrm{F}}$. This implies that

$$
\begin{equation*}
\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}}=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\|Z_{s}^{t, x}\right\|_{\mathrm{F}} d s\right]=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\langle Z_{s}^{t, x}, \tilde{\sigma}_{s}^{*}\right\rangle_{\mathrm{F}} d s\right]=\left\langle Z^{t, x}, \tilde{\sigma}^{*}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}} \tag{4.11}
\end{equation*}
$$

In a similar manner, we can construct some $\tilde{b}^{*} \in \mathbb{L}^{t, \infty}\left(\mathbb{R}^{d}\right)$ that satisfies $\left\|\tilde{b}^{*}\right\|_{\mathbb{L}^{t, \infty}} \leq 1$ and

$$
\begin{equation*}
\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}}=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left|Y_{s}^{t, x}\right| d s\right]=\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\int_{t}^{T}\left\langle Y_{s}^{t, x}, \tilde{b}_{s}^{*}\right\rangle d s\right]=\left\langle Y^{t, x}, \tilde{b}^{*}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s} \tag{4.12}
\end{equation*}
$$

Now define

$$
\left(b^{*}, \sigma^{*}\right):=\left(b^{o}+\varepsilon \gamma \tilde{b}^{*}, \sigma^{o}+\varepsilon \eta \tilde{\sigma}^{*}\right) \in \mathcal{C}^{\varepsilon}(t)
$$

Then, by (4.12) and (4.11),

$$
\begin{aligned}
\Phi(\varepsilon) & \geq\left\langle Y^{t, x}, b^{*}-b^{o}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}+\left\langle Z^{t, x}, \sigma^{*}-\sigma^{o}\right\rangle_{\mathbb{P}_{0} \otimes_{t} d s, \mathrm{~F}} \\
& =\varepsilon \cdot\left(\gamma\left\langle Y^{t, x}, \tilde{b}^{*}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s}+\eta\left\langle Z^{t, x}, \tilde{\sigma}^{*}\right\rangle_{\mathbb{P}_{0}^{t} \otimes d s, \mathrm{~F}}\right) \\
& =\varepsilon \cdot\left(\gamma\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}}+\eta\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}}\right) .
\end{aligned}
$$

This shows (4.10), completing the proof.

Proof of Theorem 2.7. Fix $(t, x) \in[0, T) \times \mathbb{R}^{d}$ and let $\left(Y^{t, x}, Z^{t, x}\right)$ be the processes defined in (4.4), that is,

$$
Y_{s}^{t, x}=w\left(s, X_{s}^{t, x ; o}\right), \quad Z_{s}^{t, x}=\mathrm{J}_{x} w\left(s, X_{s}^{t, x ; o}\right) \sigma^{o}, \quad s \in[t, T]
$$

Then, by Lemmas 4.5 and 4.6, we have for every $\varepsilon<\min \left\{1, \lambda_{\min }\left(\sigma^{0}\right)\right\}$ that

$$
\left|v^{\varepsilon}(t, x)-\left(v^{0}(t, x)+\varepsilon \cdot\left(\gamma\left\|Y^{t, x}\right\|_{\mathbb{L}^{t, 1}}+\eta\left\|Z^{t, x}\right\|_{\mathbb{L}_{\mathrm{F}}^{t, 1}}\right)\right)\right| \leq c\left(1+|x|^{\alpha}\right) \varepsilon^{2}
$$

where $c>0$ is the constant (that is independent of $t, x, \varepsilon$ ) appearing in Lemma 4.5. The proof follows from the definitions of the norms on $\mathbb{L}^{t, 1}\left(\mathbb{R}^{d}\right)$ and $\mathbb{L}_{\mathrm{F}}^{t, 1}\left(\mathbb{R}^{d \times d}\right)$, and since the law of $\left(X_{s}^{t, x, o}\right)_{s \in[t, T]}$ under $\mathbb{P}_{0}^{t}$ is equal to the conditional law of $\left(x+X_{s}^{o}\right)_{s \in[t, T]}$ under $\mathbb{P}$ given $X_{t}^{o}=0$, where $\left(X_{t}^{o}\right)_{t \in[0, T]}$ is the process defined in Theorem 2.7.

## 5. Weak and strong formulation of nonlinear Kolmogorov PDE

5.1. Semimartingale measures. In this section we adopt a framework for semimartingale uncertainty introduced by $[68,69]$. For any $(t, x) \in[0, T) \times \mathbb{R}^{d}$, denote by

$$
\Omega^{t, x}:=\left\{\omega=\left(\omega_{s}\right)_{s \in[t, T]} \in C\left([t, T] ; \mathbb{R}^{d}\right): \omega_{t}=x\right\}
$$

under which $X^{t}:=\left(X_{s}^{t}\right)_{s \in[t, T]}$ is the corresponding canonical process starting in $x$. Furthermore, let $\mathbb{F}^{X^{t}}:=\left(\mathcal{F}_{s}^{X^{t}}\right)_{s \in[t, T]}$ be the raw filtration generated by $X^{t}$. We equip $\Omega^{t, x}$ with the uniform norm $\|\omega\|_{t, \infty}:=\max _{t \leq s \leq T}\left|\omega_{s}\right|$ so that the Borel $\sigma$-field on $\Omega^{t, x}$ coincides with $\mathcal{F}_{T}^{X^{t}}$.

We will simplify notations when $t=0$ by setting $X:=X^{0}$ and $\|\omega\|_{\infty}:=\|\omega\|_{0, \infty}$. Then denote by $\mathcal{P}\left(\Omega^{0, x}\right)$ the set of all Borel probability measures on $\Omega^{0, x}$. For each $p \in \mathbb{N}$, set

$$
\begin{equation*}
\mathcal{P}^{p}\left(\Omega^{0, x}\right):=\left\{\mathbb{P} \in \mathcal{P}\left(\Omega^{0, x}\right) \mid \int_{\Omega^{0, x}}\|\omega\|_{\infty}^{p} \mathbb{P}(d \omega)<\infty\right\} \tag{5.1}
\end{equation*}
$$

to be the subset of all Borel probability measures on $\Omega^{0, x}$ with finite $p$-th moment. Furthermore, let $C\left(\Omega^{0, x} ; \mathbb{R}\right)$ be the set of all continuous functions from $\Omega^{0, x}$ to $\mathbb{R}$ and set

$$
\begin{equation*}
C_{p}\left(\Omega^{0, x} ; \mathbb{R}\right):=\left\{\xi \in C\left(\Omega^{0, x} ; \mathbb{R}\right) \mid\|\xi\|_{C_{p}}:=\sup _{\omega \in \Omega^{0, x}} \frac{|\xi(\omega)|}{1+\|\omega\|_{\infty}^{p}}<\infty\right\} \tag{5.2}
\end{equation*}
$$

We equip $\mathcal{P}^{p}\left(\Omega^{0, x}\right)$ with the topology $\tau_{p}$ defined as follows: for any $\mathbb{P} \in \mathcal{P}^{p}\left(\Omega^{0, x}\right)$ and $\left(\mathbb{P}^{n}\right)_{n \in \mathbb{N}} \subseteq$ $\mathcal{P}^{p}\left(\Omega^{0, x}\right)$, we have

$$
\begin{equation*}
\mathbb{P}^{n} \xrightarrow{\tau_{p}} \mathbb{P} \quad \text { as } n \rightarrow \infty \quad \Leftrightarrow \quad \lim _{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}^{n}}[\xi]=\mathbb{E}^{\mathbb{P}}[\xi] \quad \text { for all } \xi \in C_{p}\left(\Omega^{0, x} ; \mathbb{R}\right) \tag{5.3}
\end{equation*}
$$

Recalling the set $\mathbb{S}^{d}$ of all symmetric $d \times d$ matrices, denote by $\mathbb{S}_{+}^{d} \subset \mathbb{S}^{d}$ the subset of all positive semi-definite matrices. Let $\mathcal{P}_{\text {sem }}$ be the set of all $\mathbb{P} \in \mathcal{P}\left(\Omega^{0, x}\right)$ such that $X$ is a semimartingale on $\left(\Omega^{0, x}, \mathcal{F}^{0, x}, \mathbb{F}^{X}, \mathbb{P}\right)$. Moreover, let $\mathcal{P}_{\text {sem }}^{\text {ac }}$ be the subset of all $\mathbb{P} \in \mathcal{P}_{\text {sem }}$ such that $\mathbb{P}$-a.s.

$$
B^{\mathbb{P}} \ll d s, \quad C^{\mathbb{P}} \ll d s
$$

where $B^{\mathbb{P}}$ and $C^{\mathbb{P}}$ denote the finite variation part and quadratic covariation of the local martingale part of $X$ under $\mathbb{P}$ having values in $\mathbb{R}^{d}$ and $\mathbb{S}_{+}^{d}$, respectively (i.e., the first and second characteristics of $X$ ) and are absolutely continuous with respect to $d s$ on $[0, T]$.

Furthermore, we fix a mapping $\mathbb{S}_{+}^{d} \ni A \rightarrow A^{\frac{1}{2}} \in \mathbb{R}^{d \times d}$ so that it is Borel measurable and satisfies $A^{\frac{1}{2}}\left(A^{\frac{1}{2}}\right)^{\top}=A$ for all $A \in \mathbb{S}_{+}^{d}$, (see, e.g., [87, Remarks $\left.1.1 \& 2.1\right]$ ).
5.2. Weak formulation and dynamic programming principle. For any $\varepsilon \geq 0$ and $(t, x) \in$ $[0, T) \times \mathbb{R}^{d}$, define by

$$
\begin{align*}
\mathcal{P}^{\varepsilon}(t, x):=\left\{\mathbb{P} \in \mathcal{P}_{\mathrm{sem}}^{\mathrm{ac}} \mid\right. & \mathbb{P}\left(X_{t \wedge \cdot}=x\right)=1 ; \quad\left(b_{s}^{\mathbb{P}},\left(c_{s}^{\mathbb{P}}\right)^{\frac{1}{2}}\right) \in \mathcal{B}^{\varepsilon}  \tag{5.4}\\
& \text { for } \left.\mathbb{P} \otimes d s \text {-almost every }(\omega, s) \in \Omega^{0, x} \times[t, T]\right\}
\end{align*}
$$

where we recall that $\mathcal{B}^{\varepsilon}$ is given in (1.3).
In particular, under any $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$, the semimartingale $X$ is constant (taking the value $x$ ) up to time $t$ and after that time its differential characterstics $b^{\mathbb{P}}:=\frac{d B^{\mathbb{P}}}{d s}, c^{\mathbb{P}}:=\frac{d C^{\mathbb{P}}}{d s}$ satisfy the value constraint as the set $\mathcal{B}^{\varepsilon}$.

Moreover, recall the function $f$ given in (2.1). For any $\varepsilon \geq 0$, we define the value function $v_{\text {weak }}^{\varepsilon}:[0, T] \times \mathbb{R}^{d} \ni(t, x) \rightarrow v_{\text {weak }}^{\varepsilon}(t, x) \in \mathbb{R}$ by setting for every $(t, x) \in[0, T) \times \mathbb{R}^{d}$,

$$
\begin{equation*}
v_{\text {weak }}^{\varepsilon}(t, x):=\sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right] \tag{5.5}
\end{equation*}
$$

and $v_{\text {weak }}^{\varepsilon}(T, \cdot):=f(\cdot)$ on $\mathbb{R}^{d}$.
The following estimate will be used in next lemma.
Lemma 5.1. For every $p \geq 1$ and $\varepsilon \geq 0$, there is a constant $C_{p, \varepsilon}>0$ such that for every $(t, x) \in[0, T) \times \mathbb{R}^{d}$ and $s \in[t, T]$,

$$
\sup _{\mathbb{P} \in \mathcal{P}^{\mathcal{\varepsilon}}(t, x)} \mathbb{E}^{\mathbb{P}}\left[\sup _{t \leq u \leq s}\left|X_{u}-x\right|^{p}\right] \leq C_{p, \varepsilon}\left((s-t)^{p / 2}+(s-t)^{p}\right)
$$

Proof. Fix $\varepsilon \geq 0,(t, x) \in[0, T) \times \mathbb{R}^{d}$, and $s \in[t, T]$, and let $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$. Then under $\mathbb{P}$, the process $X$ has the canonical representation

$$
X_{s}=x+\int_{t}^{s} b_{r}^{\mathbb{P}} d r+M_{s}^{\mathbb{P}, t}
$$

where $\left(M_{s}^{\mathbb{P}, t}\right)_{s \in[t, T]}$ denotes $\left(\mathbb{F}^{X}, \mathbb{P}\right)$-local martingale part of $\left(X_{s}\right)_{s \in[t, T]}$ satisfying $M_{t}^{\mathbb{P}, t}=0$ with its differential characteristic $c^{\mathbb{P}}$ satisfying the constraint as $\mathcal{B}^{\varepsilon}$ (see (5.4)).

By Jensen's inequality and the definition of $\mathcal{P}^{\varepsilon}(t, x)$,

$$
\mathbb{E}^{\mathbb{P}}\left[\sup _{t \leq u \leq s}\left|\int_{t}^{u} b_{r}^{\mathbb{P}} d r\right|^{p}\right] \leq(s-t)^{p-1} \mathbb{E}^{\mathbb{P}}\left[\int_{t}^{s}\left|b_{r}^{\mathbb{P}}\right|^{p} d r\right] \leq 2^{p}\left(\varepsilon^{p}+\left|b^{o}\right|^{p}\right)(s-t)^{p}
$$

where we use the elementary inequality $(a+b)^{p} \leq 2^{p}\left(a^{p}+b^{p}\right)$ for all $a, b \geq 0$.
Moreover, by the Burkholder-Davis-Gundy inequality and the elementary inequality $\|A B\|_{\mathrm{F}} \leq$ $\|A\|_{\mathrm{F}}\|B\|_{\mathrm{F}}$ for all $A, B \in \mathbb{R}^{d}$,

$$
\begin{align*}
\mathbb{E}^{\mathbb{P}}\left[\sup _{t \leq u \leq s}\left|M_{u}^{\mathbb{P}, t}\right|^{p}\right] & \leq c_{\mathrm{BDG}, p} \mathbb{E}^{\mathbb{P}}\left[\left(\int_{t}^{s}\left\|\left(c_{r}^{\mathbb{P}}\right)^{\frac{1}{2}}\right\|_{\mathrm{F}}^{2} d s\right)^{p / 2}\right]  \tag{5.6}\\
& \leq c_{\mathrm{BDG}, p}\left(2^{2}\left(\varepsilon^{2}+\left\|\sigma^{o}\right\|_{\mathrm{F}}^{2}\right)\right)^{p / 2}(s-t)^{p / 2}
\end{align*}
$$

Our claim follows by using again the inequality $(a+b)^{p} \leq 2^{p}\left(a^{p}+b^{p}\right)$ for all $a, b \geq 0$ and setting $C_{p, \varepsilon}:=2^{p}\left\{2^{p}\left(\varepsilon^{p}+\left|b^{o}\right|^{p}\right)+c_{\mathrm{BDG}, p}\left(2^{2}\left(\varepsilon^{2}+\left\|\sigma^{o}\right\|_{\mathrm{F}}^{2}\right)\right)^{p / 2}\right\}$.

Remark 5.2. Lemma 5.1 implies that $\mathcal{P}^{\varepsilon}(t, x)$ is a subset of $\mathcal{P}^{p}\left(\Omega^{0, x}\right)$ for every $\varepsilon \geq 0$ and $p \geq 1$; see (5.1) and (5.4).

Lemma 5.3. Suppose that Assumption 2.1 is satisfied, let $\varepsilon \geq 0$, and let $v_{\text {weak }}^{\varepsilon}$ be defined in (5.5). Moreover, let $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Then, the following hold:
(i) For any $\mathbb{F}^{X}$-stopping time $\tau$ taking values in $[t, T]$

$$
\begin{equation*}
v_{\text {weak }}^{\varepsilon}(t, x)=\sup _{\mathbb{P}_{\mathcal{P}}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[v_{\text {weak }}^{\varepsilon}\left(\tau, X_{\tau}\right)\right] \tag{5.7}
\end{equation*}
$$

(ii) $v_{\text {weak }}^{\varepsilon}$ is jointly continuous.

Proof. We start by proving the statement (i). We claim that the set

$$
\begin{equation*}
\left\{(\omega, t, \mathbb{P}) \in \Omega^{0, x} \times[0, T] \times \mathcal{P}\left(\Omega^{0, x}\right) \mid \mathbb{P} \in \mathcal{P}^{\varepsilon}\left(t, \omega_{t}\right)\right\} \tag{5.8}
\end{equation*}
$$

is Borel. Indeed, since $\mathcal{B}^{\varepsilon}$ is Borel (see (1.3)) and the map $\mathbb{S}_{+}^{d} \ni A \rightarrow A^{\frac{1}{2}} \in \mathbb{R}^{d \times d}$ is Borelmeasurable (see Section 5.1), the same arguments presented for the proof of [29, Lemma 3.1] using the existence of a Borel-measurable map from $\Omega^{0, x} \times[0, T] \times \mathcal{P}\left(\Omega^{0, x}\right)$ to the differential characteristics of $X$ given in [68, Theorem 2.6] ensure the claim to hold.

Furthermore, from [69, Theorem 2.1], the following stability properties of $\mathcal{P}^{\varepsilon}(t, x)$ also hold: for any $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$ and $\mathbb{F}^{X}$-stopping time $\tau$ having values in $[t, T]$,
(a) There is a set of conditional probability measures $\left(\mathbb{P}_{\omega}\right)_{\omega \in \Omega^{0, x}}$ of $\mathbb{P}$ with respect to $\mathcal{F}_{\tau}^{X}$ such that $\mathbb{P}_{\omega} \in \mathcal{P}^{\varepsilon}\left(\tau(\omega), \omega_{\tau(\omega)}\right)$ for $\mathbb{P}$-almost all $\omega \in \Omega^{0, x}$;
(b) If there is a set of probability measures $\left(\mathbb{Q}_{\omega}\right)_{\omega \in \Omega^{0, x}}$ such that $\mathbb{Q}_{\omega} \in \mathcal{P}^{\varepsilon}\left(\tau(\omega), \omega_{\tau(\omega)}\right)$ for $\mathbb{P}^{-}$ almost all $\omega \in \Omega^{0, x}$, and the map $\omega \rightarrow \mathbb{Q}_{\omega}$ is $\mathcal{F}_{\tau}^{X}$-measurable, then the probability measure

$$
\mathbb{P} \otimes \mathbb{Q}(\cdot):=\int_{\Omega^{0, x}} \mathbb{Q}_{\omega}(\cdot) \mathbb{P}(d \omega)
$$

is an element of $\mathcal{P}^{\varepsilon}(t, x)$.
Therefore, an application of [27, Theorem 2.1] (see also [74, Theorem 2.3]) ensures (5.7) to hold.
Now let us prove (ii). Since $v_{\text {weak }}^{\varepsilon}(T, \cdot)=f(\cdot)$ is continuous (by Assumption 2.1), we can and do consider arbitrary $(t, x) \in[0, T) \times \mathbb{R}^{d}$. The continuity of $v_{\text {weak }}^{\varepsilon}(t, \cdot)$ follows from the definition of $v_{\text {weak }}^{\varepsilon}$ given in (5.5). Indeed, since for every $x, y \in \mathbb{R}^{d}$

$$
v_{\text {weak }}^{\varepsilon}(t, y)=\sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, y)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right]=\sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}+y-x\right)\right]
$$

by Remark 4.2 (with the constants $p \geq 1$ and $c_{1}>0$ ) and the elementary property $(a+b)^{p} \leq$ $2^{p}\left(a^{p}+b^{p}\right)$ for all $a, b \geq 0$, we have that

$$
\begin{aligned}
& \left|v_{\text {weak }}^{\varepsilon}(t, y)-v_{\text {weak }}^{\varepsilon}(t, x)\right| \\
& \leq \sup _{\mathbb{P}_{\mathcal{P}} \varepsilon(t, x)} \mathbb{E}^{\mathbb{P}}\left[\left|f\left(X_{T}+y-x\right)-f\left(X_{T}\right)\right|\right] \\
& \leq \sup _{\mathbb{P}_{\mathcal{P}}(t, x)}\left\{\mathbb{E}^{\mathbb{P}}\left[\left|\nabla_{x}^{\top} f\left(X_{T}\right)\right|\right] \cdot|y-x|+c_{1} \cdot 2^{p}\left(1+|y-x|^{p}+\mathbb{E}^{\mathbb{P}}\left[\left|X_{T}\right|^{p}\right]\right) \cdot|y-x|^{2}\right\} .
\end{aligned}
$$

From Lemma 5.1 together with the polynomial growth property of $\nabla_{x} f$, we hence have that there is a constant $c_{3}>0$ (that depends on $p, \varepsilon, x$, but not on $t$ ) such that

$$
\begin{equation*}
\left|v_{\mathrm{weak}}^{\varepsilon}(t, y)-v_{\mathrm{weak}}^{\varepsilon}(t, x)\right| \leq c_{3}\left(|y-x|+|y-x|^{p+2}\right) \tag{5.9}
\end{equation*}
$$

where we further emphasize that the above estimate holds for every $t \in[0, T)$ and $x, y \in \mathbb{R}^{d}$.
Now we claim that $v_{\text {weak }}^{\varepsilon}(\cdot, x)$ is continuous. To that end, fix any $0 \leq u \leq T-t$. By the dynamic programming principle of $v_{\text {weak }}^{\varepsilon}$ (see Lemma 5.3 (i)), the following holds

$$
v_{\text {weak }}^{\varepsilon}(t, x)=\sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[v_{\text {weak }}^{\varepsilon}\left(t+u, X_{t+u}\right)\right]
$$

Hence, we use again Lemma 5.1 together with the estimates given in (5.9) to have that

$$
\begin{aligned}
\left|v_{\text {weak }}^{\varepsilon}(t, x)-v_{\text {weak }}^{\varepsilon}(t+u, x)\right| & \leq \sup _{\mathbb{P} \in \mathcal{P} \varepsilon(t, x)} \mathbb{P}^{\mathbb{P}}\left[\left|v_{\text {weak }}^{\varepsilon}\left(t+u, X_{t+u}\right)-v_{\text {weak }}^{\varepsilon}(t+u, x)\right|\right] \\
& \leq c_{3} \sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)}\left(\mathbb{E}^{\mathbb{P}}\left[\left|X_{t+u}-x\right|\right]+\mathbb{E}^{\mathbb{P}}\left[\left|X_{t+u}-x\right|^{p+2}\right]\right) \\
& \leq c_{3} \cdot\left(C_{1, \varepsilon}\left(u^{1 / 2}+u\right)+C_{p, \varepsilon}\left(u^{\frac{p+2}{2}}+u^{p+2}\right)\right)
\end{aligned}
$$

where $C_{1, \varepsilon}, C_{p, \varepsilon}$ are the constant (with exponents $1, p$ ) appearing in Lemma 5.1 (and in particular do not depend on $x$ ). Combined with (5.9), this ensures that $v_{\text {weak }}^{\varepsilon}$ is jointly continuous.
5.3. Proof of Proposition 2.6. Let us introduce the notion of viscosity / strong solution of (2.1) and (2.3). To that end, we introduce the following function spaces: for any $t \in[0, T)$

- $C^{1,2}\left([t, T) \times \mathbb{R}^{d} ; \mathbb{R}\right)$ is the set of all real-valued functions on $[t, T) \times \mathbb{R}^{d}$ which are continuously differentiable on $[t, T)$ and twice continuously differentiable on $\mathbb{R}^{d}$;
- $C_{b}^{2,3}\left([t, T) \times \mathbb{R}^{d} ; \mathbb{R}\right)$ is the set of all real-valued functions on $[t, T) \times \mathbb{R}^{d}$ which have bounded continuous derivatives up to the second and third order on $[t, T)$ and $\mathbb{R}^{d}$, respectively.

Definition 5.4 (Viscosity solution (see $[20,30])$ ). Fix any $\varepsilon \geq 0$. We call an upper semicontinuous function $v^{\varepsilon}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ a viscosity subsolution of (2.1) if $v^{\varepsilon}(T, \cdot) \leq f(\cdot)$ on $\mathbb{R}^{d}$ and

$$
-\partial_{t} \varphi(t, x)-\sup _{(b, \sigma) \in \mathcal{B}^{\varepsilon}}\left\{\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^{\top} D_{x x}^{2} \varphi(t, x)\right)+\left\langle b, \nabla_{x} \varphi(t, x)\right\rangle\right\} \leq 0
$$

whenever $\varphi \in C_{b}^{2,3}\left([0, T) \times \mathbb{R}^{d} ; \mathbb{R}\right)$ is such that $\varphi \geq v^{\varepsilon}$ on $[0, T) \times \mathbb{R}^{d}$ and $\varphi(t, x)=v^{\varepsilon}(t, x)$. In a similar manner, the notion of a viscosity supersolution can be defined by reversing the inequalities and replacing upper semicontinuity with lower semicontinuity. Finally, we call a continuous function $v^{\varepsilon}$ from $[0, T] \times \mathbb{R}^{d}$ to $\mathbb{R}$ a viscosity solution if it is both sub- and supersolution of (2.1).
Definition 5.5 (Strong solution). Fix $t \in[0, T)$ and $i=1, \ldots, d$. We call a continuous function $w^{i}:[t, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ a strong solution of (2.3) if it is in $C^{1,2}\left([t, T) \times \mathbb{R}^{d} ; \mathbb{R}\right)$ and satisfies (2.3).

Lemma 5.6. Suppose that Assumptions 2.1 and 2.4 are satisfied and let $\varepsilon \geq 0$. Then $v_{\text {weak }}^{\varepsilon}$ : $[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ defined in (5.5) is a unique viscosity solution of (2.1) satisfying (2.2).

Proof. By Lemma 5.3, $v_{\text {weak }}^{\varepsilon}$ satisfies the dynamic programming principle and is jointly continuous. Hence, the same arguments as presented for the proof of [69, Proposition 5.4] ensure that $v_{\text {weak }}^{\varepsilon}$ is a unique viscosity solution of (2.1). Furthermore, as the function $f$ has at most polynomial growth (see Remark 4.2), Lemma 5.1 ensures that $v_{\text {weak }}^{\varepsilon}$ has polynomial growth with respect to $x \in \mathbb{R}^{d}$ for all $t \in[0, T]$. This implies that $v_{\text {weak }}^{\varepsilon}$ satisfies (2.2) with some $C>0$. Hence by Assumption 2.4, $v_{\text {weak }}^{\varepsilon}$ is the unique viscosity solution satisfying (2.2).

Proof of Proposition 2.6. The statement (i) follows directly from Lemma 5.6. Now let us prove (ii). Note that $\nabla_{x} f$ has at most polynomial growth (see Remark 4.2) and ( $b^{o}, \sigma^{o}$ ) are constant. Furthermore, $\sigma^{o}$ is non-degenerate (see Assumption 2.2). Hence, an application of [57, Theorem 5.7.6 \& Remark 5.7.8] (see also [61, Theorem 4.32]) ensures the existence of a strong solution of (2.3). The uniqueness of the solution with polynomial growth is guaranteed by [31, Corollary 6.4.4].
5.4. Strong formulation and its equivalence. In this section, we construct a set of probability measures corresponding to a strong formulation of the nonlinear Kolmogorov PDE given in (2.1).

Recall the process $X^{t, x ; b, \sigma}$ defined on $[t, T]$ (given in (4.2)) and denote by $\left(x \oplus_{t} X^{t, x ; b, \sigma}\right)$ the constant concatenation of $X^{t, x ; b, \sigma}$ defined on $[0, T]$, i.e.

$$
\begin{equation*}
\left(x \oplus_{t} X^{t, x ; b, \sigma}\right)_{s}:=x \mathbf{1}_{\{s \in[0, t)\}}+X_{s}^{t, x ; b, \sigma} \mathbf{1}_{\{s \in[t, T]\}} . \tag{5.10}
\end{equation*}
$$

Then using the set $\mathcal{C}^{\varepsilon}(t)$ given in (4.3), we define a set of (push-forward) probability measures as follow: for any $\varepsilon \geq 0$ and $(t, x) \in[0, T) \times \mathbb{R}^{d}$

$$
\begin{equation*}
\mathcal{Q}^{\varepsilon}(t, x):=\mathcal{Q}\left(t, x ; \mathcal{C}^{\varepsilon}\right)=\left\{\mathbb{P}_{0}^{t} \circ\left(x \oplus_{t} X^{t, x ; b, \sigma}\right)^{-1} \mid(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)\right\} \subseteq \mathcal{P}\left(\Omega^{0, x}\right) \tag{5.11}
\end{equation*}
$$

Remark 5.7. By the definition of $\left(x \oplus_{t} X^{t, x ; b, \sigma}\right)$ given in (5.10), $\mathcal{Q}^{\varepsilon}(t, x)$ is a subset of $\mathcal{P}^{\varepsilon}(t, x)$ for every $\varepsilon \geq 0$; see (5.4) for the definition.

Proposition 5.8. Suppose that Assumption 2.2 is satisfied. Let $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right)$ (see Remark 2.3) and $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Moreover, let $\mathcal{P}^{\varepsilon}(t, x)$ and $\mathcal{Q}^{\varepsilon}(t, x)$ be defined in (5.4) and (5.11), respectively. Then, there exists $\mathcal{Q}_{\text {sub }}^{\varepsilon}(t, x) \subseteq \mathcal{Q}^{\varepsilon}(t, x)$ such that its convex hull is a dense subset of $\mathcal{P}^{\varepsilon}(t, x)$ with respect to the $\tau_{p}$-topology for all $p \geq 1$.

Recall the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ given in (2.1) and the canonical process $X=\left(X_{s}\right)_{s \in[0, T]}$ defined on $\left(\Omega^{0, x}, \mathcal{F}^{0, x}, \mathbb{F}^{X}\right)$. For any $\varepsilon \geq 0$, we define the value function $v_{\text {strong }}^{\varepsilon}:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ by setting for every $(t, x) \in[0, T) \times \mathbb{R}^{d}$,

$$
\begin{equation*}
v_{\text {strong }}^{\varepsilon}(t, x):=\sup _{\mathbb{P}_{\mathcal{Q}} \mathcal{Q}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right]=\sup _{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[f\left(X_{T}^{t, x ; b, \sigma}\right)\right] \tag{5.12}
\end{equation*}
$$

and $v_{\text {strong }}^{\varepsilon}(T, \cdot):=f(\cdot)$ on $\mathbb{R}^{d}$. We call this the 'strong formulation' of modeling uncertainty of $X$, which will turn out to be equivalent to the weak formulation $v_{\text {weak }}^{\varepsilon}$ in the next proposition.

Proposition 5.9. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied and let $v_{\text {weak }}^{\varepsilon}$ and $v_{\text {strong }}^{\varepsilon}$ be defined in (5.5) and (5.12), respectively. Then the following inequalities hold: for any $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right)$ and $(t, x) \in[0, T] \times \mathbb{R}^{d}$,

$$
v_{\text {strong }}^{\varepsilon}(t, x)=v_{\text {weak }}^{\varepsilon}(t, x)
$$

5.5. Proof of Propositions 5.8 and 5.9. We follow the idea of $[24]$ in order to prove Propositions 5.8 and 5.9. First, we introduce some notions, often employed in this section. Recalling the set $\mathcal{B}^{\varepsilon}$ given in (1.3), we set for any $\varepsilon \geq 0$,

$$
\begin{equation*}
\mathcal{B}^{\varepsilon, 1}:=\left\{b:(b, \sigma) \in \mathcal{B}^{\varepsilon}\right\}, \quad \mathcal{B}^{\varepsilon, 2}:=\left\{\sigma:(b, \sigma) \in \mathcal{B}^{\varepsilon}\right\} \tag{5.13}
\end{equation*}
$$

so that $\mathcal{B}^{\varepsilon}=\mathcal{B}^{\varepsilon, 1} \times \mathcal{B}^{\varepsilon, 2}$ and denote by

$$
\begin{equation*}
\Pi_{\mathcal{B}^{\varepsilon, 1}}: \mathbb{R}^{d} \ni x \rightarrow \Pi_{\mathcal{B}^{\varepsilon, 1}}(x) \in \mathcal{B}^{\varepsilon, 1}, \quad \Pi_{\mathcal{B}^{\varepsilon, 2}}: \mathbb{R}^{d \times d} \ni x \rightarrow \Pi_{\mathcal{B}^{\varepsilon, 2}}(x) \in \mathcal{B}^{\varepsilon, 2} \tag{5.14}
\end{equation*}
$$

the Euclidean projections into the convex, closed sets $\mathcal{B}^{\varepsilon, 1}$ and $\mathcal{B}^{\varepsilon, 2}$ respectively.
For every $n \in \mathbb{N}$ and $t \in[0, T)$, denote by $t_{k}^{n}:=t+\frac{T-t}{n} k$ for $k=0,1, \ldots, n$. Furthermore, for any $\mathbb{P} \in \mathcal{P}_{\text {sem }}^{\text {as }}$ denote by $b^{\mathbb{P}}=\frac{d B^{\mathbb{P}}}{d s}$ the first differential characteristics of $X$ under $\mathbb{P}$, and by

$$
\begin{equation*}
\sigma^{\mathbb{P}}:=\left(c^{\mathbb{P}}\right)^{\frac{1}{2}} \tag{5.15}
\end{equation*}
$$

where $c^{\mathbb{P}}=\frac{d C^{\mathbb{P}}}{d s}$ is the second differential characteristics of $X$ under $\mathbb{P}$. Then we define by $b^{\mathbb{P},(n)}$ and $\sigma^{\mathbb{P},(n)}$ piecewise constant processes defined on $[t, T]$ such that

$$
\begin{align*}
& b_{s}^{\mathbb{P},(n)}:=\mathbf{1}_{\left\{s \in\left[t, t_{1}^{n}\right]\right\}} b^{o}+\sum_{k=1}^{n-1} \mathbf{1}_{\left\{s \in\left(t_{k}^{n}, t_{k+1}^{n}\right]\right\}} \Pi_{\mathcal{B}^{\varepsilon}, 1} \\
& \sigma_{s}^{\mathbb{P},(n)}\left.:=\mathbf{1}_{\left\{s \in\left[t, t_{1}^{n}\right]\right\}} \sigma^{o}+\sum_{k=1}^{n-1} \int_{t_{k-1}^{n}}^{t_{k}^{n}} b_{s}^{\mathbb{P}} d s\right]  \tag{5.16}\\
& \\
&
\end{align*}
$$

Remark 5.10. Fix any $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right)$ and recall $\mathcal{P}^{\varepsilon}(t, x)$ given in (5.4). Then under any $\mathbb{P} \in$ $\mathcal{P}^{\varepsilon}(t, x)$, since $\left\|\sigma_{s}^{\mathbb{P}}-\sigma^{o}\right\|_{\mathrm{F}}<\lambda_{\min }\left(\sigma^{o}\right) \mathbb{P} \otimes d s$-a.e., by Remark 2.3 , there exists the corresponding inverse matrix $\left(\left(\sigma_{s}^{\mathbb{P}}\right)^{-1}\right)_{s \in[t, T]} \mathbb{P} \otimes d s$-almost every $(\omega, s) \in \Omega^{0, x} \times[t, T]$. Therefore, if we denote by $M^{\mathbb{P}, t}:=\left(M_{s}^{\mathbb{P}, t}\right)_{s \in[t, T]}$ the $\left(\mathbb{F}^{X}, \mathbb{P}\right)$-local martingale term of $\left(X_{s}\right)_{s \in[t, T]}$ satisfying $M_{t}^{\mathbb{P}, t}=0$, an application of Lévy's theorem ensures that for $s \in[t, T]$,

$$
\begin{equation*}
W_{s}^{\mathbb{P}, t}:=\int_{t}^{s}\left(\sigma_{u}^{\mathbb{P}}\right)^{-1} d M_{u}^{\mathbb{P}, t} \tag{5.17}
\end{equation*}
$$

is a $d$-dimensional Brownian motion defined on $[t, T]$ under $\mathbb{P}$ satisfying $W_{t}^{\mathbb{P}, t}=0$.
For every $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right)$ and $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$, the piecewise constant processes $b^{\mathbb{P},(n)}$ and $\sigma^{\mathbb{P},(n)}$, $n \in \mathbb{N}$, given in (5.16) and the Brownian motion $W^{\mathbb{P}, t}=\left(W_{s}^{\mathbb{P}, t}\right)_{s \in[t, T]}$ given in (5.17) enable to define $X^{\mathbb{P},(n)}$ (that is defined on $[0, T]$ ) for every $n \in \mathbb{N}$ by letting

$$
\begin{equation*}
X^{\mathbb{P},(n)}:=x \oplus_{t}\left(x+\int_{t} b_{s}^{\mathbb{P},(n)} d s+\int_{t} \sigma_{s}^{\mathbb{P},(n)} d W_{s}^{\mathbb{P}, t}\right) \tag{5.18}
\end{equation*}
$$

Finally, for any $\mathbb{P} \in \mathcal{P}\left(\Omega^{0, x}\right)$ denote $^{4}$ by $\mathcal{H}^{2}\left(\Omega^{0, x}, \mathcal{F}^{0, x}, \mathbb{F}^{X}, \mathbb{P}\right)$ the space of all semimartingales $S$ defined on $[0, T]$ such that

$$
\begin{equation*}
\|S\|_{\mathcal{H}_{\mathbb{P}}^{2}}:=\mathbb{E}^{\mathbb{P}}\left[\langle N, N\rangle_{T}\right]^{1 / 2}+\mathbb{E}^{\mathbb{P}}\left[\left(\int_{0}^{T}\left|d A_{t}\right|\right)^{2}\right]^{1 / 2}<\infty \tag{5.19}
\end{equation*}
$$

where $N=\left(N_{t}\right)_{t \in[0, T]}$ and $A=\left(A_{t}\right)_{t \in[0, T]}$ denote the $\left(\mathbb{F}^{X}, \mathbb{P}\right)$-local martingale and $\mathbb{F}^{X}$ predictable finite variation process of $S$, respectively (i.e., the canocial decomposition).

Lemma 5.11. Suppose that Assumption 2.2 is satisfied. Let $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right),(t, x) \in[0, T) \times \mathbb{R}^{d}$, and $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$. Let $\left(X^{\mathbb{P},(n)}\right)_{n \in \mathbb{N}}$ be the sequence defined in (5.18). Then $X^{\mathbb{P},(n)}$ converges to $X$ in $\mathcal{H}^{2}\left(\Omega^{0, x}, \mathcal{F}^{0, x}, \mathbb{F}^{X}, \mathbb{P}\right)$, i.e. as $n \rightarrow \infty$

$$
\left\|X^{\mathbb{P},(n)}-X\right\|_{\mathcal{H}_{\mathbb{P}}^{2}} \rightarrow 0
$$

Proof. Let $b^{\mathbb{P}}$ be the first differential characteristic of $X$ and $\sigma^{\mathbb{P}}$ be given in (5.15). Using the Brownian motion $W^{\mathbb{P}, t}$ defined in (5.17), we have that $\mathbb{P}$-a.s.

$$
\begin{equation*}
X=x \oplus_{t}\left(x+\int_{t} b_{s}^{\mathbb{P}} d s+\int_{t} \sigma_{s}^{\mathbb{P}} d W_{s}^{\mathbb{P}, t}\right) \tag{5.20}
\end{equation*}
$$

That is, the canonical process $X$ can be represented by an Itô $\left(\mathbb{F}^{X}, \mathbb{P}\right)$-semimartingale with constant $x$-path up to time $t$.

Recall $\|\cdot\|_{\mathcal{H}_{\mathbb{P}}^{2}}$ defined in (5.19) and the piecewise constant processes $\left(b^{\mathbb{P},(n)}, \sigma^{\mathbb{P},(n)}\right)_{n \in \mathbb{N}}$ defined in (5.16). By (5.20), Hölder's inequality (with exponent 2) ensures that for every $n \in \mathbb{N}$,

$$
\begin{equation*}
\left\|X^{\mathbb{P},(n)}-X\right\|_{\mathcal{H}_{\mathbb{P}}^{2}} \leq \mathbb{E}^{\mathbb{P}}\left[\int_{t}^{T}\left\|\sigma_{s}^{\mathbb{P},(n)}-\sigma_{s}^{\mathbb{P}}\right\|_{\mathrm{F}}^{2} d s\right]^{\frac{1}{2}}+(T-t) \mathbb{E}^{\mathbb{P}}\left[\int_{t}^{T}\left|b_{s}^{\mathbb{P},(n)}-b_{s}^{\mathbb{P}}\right|^{2} d s\right]^{\frac{1}{2}} \tag{5.21}
\end{equation*}
$$

In particular, by the definition of $\sigma^{\mathbb{P},(n)}$ in (5.16), $\int_{t}^{T}\left\|\sigma_{s}^{\mathbb{P},(n)}-\sigma_{s}^{\mathbb{P}}\right\|_{\mathrm{F}} d s \rightarrow 0$ as $n \rightarrow \infty$ for every $\omega \in \Omega^{0, x}$. Furthermore, since $\sigma^{\mathbb{P},(n)}, \sigma^{\mathbb{P}}$ are uniformly bounded, the dominated convergence theorem implies that the first term of the right hand side of (5.21) vanishes as $n \rightarrow \infty$. The same arguments ensure that the second term vanishes. This completes the proof.

[^3]Lemma 5.12. Suppose that Assumption 2.2 is satisfied. Let $\varepsilon<\lambda_{\min }\left(\sigma^{o}\right),(t, x) \in[0, T) \times \mathbb{R}^{d}$, and $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$. Let $W^{\mathbb{P}, t}$ and $\left(X^{\mathbb{P},(n)}\right)_{n \in \mathbb{N}}$ be given in (5.17) and (5.18). Then for each $n \in \mathbb{N}$ and $p \geq 1$, the law of $X^{\mathbb{P},(n)}$ is contained in the $\tau_{p}$-closure of the convex hull of the laws of

$$
\begin{equation*}
\left\{x \oplus_{t}\left(x+\int_{t} \mu^{u}\left(s, W^{\mathbb{P}, t}\right) d s+\int_{t}^{\cdot} \Sigma^{v}\left(s, W^{\mathbb{P}, t}\right) d W_{s}^{\mathbb{P}, t}\right) \mid(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}\right\} \tag{5.22}
\end{equation*}
$$

where for every $(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}, \mu^{u}:[t, T] \times \Omega^{t} \rightarrow \mathcal{B}^{\varepsilon, 1}$ and $\Sigma^{v}:[t, T] \times \Omega^{t} \rightarrow \mathcal{B}^{\varepsilon, 2}$ are adapted ${ }^{5}$ Borel functionals on $\Omega^{t}$.

Proof. Fix $n \in \mathbb{N}$ and denote by $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}:=\left(\widehat{\mathcal{F}}_{t}\right)_{t \in[0, T]}, \widehat{\mathbb{P}}\right)$ another filtered probability space which carries a $d$-dimensional Brownian motion $\widehat{W}^{t}$ defined on $[t, T]$ satisfying $\widehat{W}_{t}^{t}=0$ and a sequence $\left\{\left(U^{k}, V^{k}\right) \mid 1 \leq k \leq n\right\}$ of $\left(\mathbb{R}^{d}, \mathbb{R}^{d \times d}\right)$-valued random variables such that the components $\left\{\left(U_{i}^{k}, V_{j, l}^{k}\right) \mid 1 \leq k \leq n ; 1 \leq i, j, l \leq d\right\}$ are i.i.d. uniformly distributed on $(0,1)$ and independent of $\widehat{W}^{t}$. For notational simplicity, set

$$
U:=\left(U^{1}, \ldots, U^{n}\right), \quad V:=\left(V^{1}, \ldots, V^{n}\right)
$$

Recall $\left(b^{\mathbb{P},(n)}, \sigma^{\mathbb{P},(n)}\right)$ and $X^{\mathbb{P},(n)}$ given in (5.16) and (5.18). For each $k=1, \ldots, n$, denote by $C\left(\left[t, t_{k}^{n}\right] ; \mathbb{R}^{d}\right)$ the set of all $\mathbb{R}^{d}$-valued, continuous functions on $\left[t, t_{k}^{n}\right]$ (recalling that $t_{k}^{n}=t+$ $\frac{k(T-t)}{n}$ with $\left.k=0,1, \ldots, n\right)$. Then the existence of regular conditional probability distributions guarantees that there exist measurable functions for every $k=1, \ldots, n$

$$
\Theta_{k}^{1}: C\left(\left[t, t_{k}^{n}\right] ; \mathbb{R}^{d}\right) \times(0,1)^{k d} \rightarrow \mathcal{B}^{\varepsilon, 1}, \quad \Theta_{k}^{2}: C\left(\left[t, t_{k}^{n}\right] ; \mathbb{R}^{d}\right) \times(0,1)^{k d^{2}} \rightarrow \mathcal{B}^{\varepsilon, 2}
$$

such that the random variables defined by

$$
\begin{equation*}
\widehat{b}^{(n)}(k):=\Theta_{k}^{1}\left(\left.\widehat{W}^{t}\right|_{\left[t, t_{k}^{n}\right]}, U^{1}, \ldots, U^{k}\right), \quad \widehat{\sigma}^{(n)}(k):=\Theta_{k}^{2}\left(\left.\widehat{W}^{t}\right|_{\left[t, t_{k}^{n}\right]}, V^{1}, \ldots, V^{k}\right) \tag{5.23}
\end{equation*}
$$

satisfy

$$
\begin{align*}
& \text { law of }\left\{\widehat{W}^{t},\left(\widehat{b}^{(n)}(1), \widehat{\sigma}^{(n)}(1)\right), \ldots,\left(\widehat{b}^{(n)}(n), \widehat{\sigma}^{(n)}(n)\right)\right\} \text { under } \widehat{\mathbb{P}}  \tag{5.24}\\
& =\text { law of }\left\{W^{\mathbb{P}, t},\left(b_{t_{1}^{n}}^{\mathbb{P}^{\prime}(n)}, \sigma_{t_{1}^{n}}^{\mathbb{P},(n)}\right), \ldots,\left(b_{t_{n}^{n}}^{\mathbb{P}^{\prime}(n)}, \sigma_{t_{n}^{n}}^{\mathbb{P},(n)}\right)\right\} \text { under } \mathbb{P} .
\end{align*}
$$

Now for each $(u, v):=\left(u^{1}, \ldots, u^{n}\right) \times\left(v^{1}, \ldots, v^{n}\right) \in(0,1)^{n d} \times(0,1)^{n d^{2}}$ and $k=1, \ldots, n$, set

$$
\begin{equation*}
\widehat{b}^{(n)}(k ; u):=\Theta_{k}^{1}\left(\left.\widehat{W}^{t}\right|_{\left[t, t_{k}^{n}\right]}, u^{1}, \ldots, u^{k}\right), \quad \widehat{\sigma}^{(n)}(k ; v):=\Theta_{k}^{2}\left(\left.\widehat{W}^{t}\right|_{\left[t, t_{k}^{n}\right]}, v^{1}, \ldots, v^{k}\right) \tag{5.25}
\end{equation*}
$$

and denote by $\widehat{b}^{(n) ; u}$ and $\widehat{\sigma}^{(n) ; v}$ other piecewise constant processes defined on $[t, T]$ such that

$$
\begin{aligned}
& \widehat{b}_{s}^{(n) ; u}:=\mathbf{1}_{\left\{s \in\left[t, t_{1}^{n}\right]\right\}} b^{o}+\sum_{k=1}^{n-1} \mathbf{1}_{\left\{s \in\left(t_{k}^{n}, t_{k+1}^{n}\right]\right\}} \widehat{b}^{(n)}(k ; u), \\
& \widehat{\sigma}_{s}^{(n) ; v}:=\mathbf{1}_{\left\{s \in\left[t, t_{1}^{n}\right]\right\}} \sigma^{o}+\sum_{k=1}^{n} \mathbf{1}_{\left\{s \in\left(t_{k}^{n}, t_{k+1}^{n}\right]\right\}} \widehat{\sigma}^{(n)}(k ; v) .
\end{aligned}
$$

[^4]With the notations in place, we define for each $(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}$

$$
\begin{equation*}
\widehat{X}^{(n) ; u, v}=x \oplus_{t}\left(x+\int_{t} \widehat{b}_{s}^{(n) ; u} d s+\int_{t} \widehat{\sigma}_{s}^{(n) ; v} d \widehat{W}_{s}^{t}\right) . \tag{5.26}
\end{equation*}
$$

Note that for every $k=1, \ldots, n$ (see (5.23) and (5.25))

$$
\widehat{b}^{(n)}(k ; U)=\widehat{b}^{(n)}(k), \quad \widehat{\sigma}^{(n)}(k ; V)=\widehat{\sigma}^{(n)}(k) .
$$

Then by the definitions of $X^{\mathbb{P},(n)}$ and $\widehat{X}^{(n) ; u, v}$ (see (5.18) and (5.26)) and the property given in (5.24),

$$
\text { law of } \widehat{X}^{(n) ; U, V} \text { under } \widehat{\mathbb{P}}=\text { law of } X^{\mathbb{P},(n)} \text { under } \mathbb{P} \text {. }
$$

Furthermore, since $\left(\widehat{b}^{(n)}(k), \widehat{\sigma}^{(n)}(k)\right)$ and $\left(\widehat{b}^{(n)}(k ; u), \widehat{\sigma}^{(n)}(k ; v)\right)$ are uniformly bounded for every $k=1, \ldots, n$ and $(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}$, by using the same arguments given in Lemma 5.1, the following holds for every $p \geq 1$,

$$
\begin{equation*}
\sup _{(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}} \mathbb{E}^{\widehat{\mathbb{P}}}\left[\sup _{0 \leq t \leq T}\left|\widehat{X}_{t}^{(n) ; u, v}\right|^{p}\right]+\mathbb{E}^{\widehat{\mathbb{P}}}\left[\sup _{0 \leq t \leq T}\left|\widehat{X}_{t}^{(n) ; U, V}\right|^{p}\right]<\infty \tag{5.27}
\end{equation*}
$$

Therefore, an application of Fubini theorem and (5.27) ensure that for every $p \geq 1$ and $\xi \in C_{p}\left(\Omega^{0, x} ; \mathbb{R}\right)$ (see (5.2) for the definition)

$$
\begin{equation*}
\mathbb{E}^{\mathbb{P}}\left[\xi\left(X^{\mathbb{P},(n)}\right)\right]=\mathbb{E}^{\widehat{\mathbb{P}}}\left[\xi\left(\widehat{X}^{(n) ; U, V}\right)\right] \leq \sup _{(u, v) \in(0,1)^{n d} \times(0,1)^{n d} d^{2}} \mathbb{E}^{\widehat{\mathbb{P}}}\left[\xi\left(\widehat{X}^{(n) ; u, v}\right)\right]<\infty \tag{5.28}
\end{equation*}
$$

Furthermore, from (5.27), it follows that the laws of $\widehat{X}_{t}^{(n) ; U, V}$ and $\left(\widehat{X}_{t}^{(n) ; u, v}\right)_{(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}}$ belong to $\mathcal{P}^{p}\left(\Omega^{0, x}\right)$ for every $p \geq 1$ (that is equipped with the topology $\tau_{p}$; see (5.1) and (5.3)).

Therefore an application of Hahn-Banach theorem guarantees that for each $n \in \mathbb{N}$ and $p \geq 1$, the law of $X^{\mathbb{P},(n)}$ is contained in the $\tau_{p}$-closure of the convex hull of the laws of

$$
\begin{equation*}
\left\{x \oplus_{t}\left(x+\int_{t} \mu^{u}\left(s, \widehat{W}^{t}\right) d s+\int_{t} \Sigma^{v}\left(s, \widehat{W}^{t}\right) d \widehat{W}_{s}^{t}\right) \mid(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}\right\} \tag{5.29}
\end{equation*}
$$

where for every $(u, v) \in(0,1)^{n d} \times(0,1)^{n d^{2}}, \mu^{u}:[t, T] \times \Omega^{t} \rightarrow \mathcal{B}^{\varepsilon, 1}$ and $\Sigma^{v}:[t, T] \times \Omega^{t} \rightarrow \mathcal{B}^{\varepsilon, 2}$ are adapted Borel functionals on $\Omega^{t}$.

Replacing $\widehat{W}^{t}$ with $W^{\mathbb{P}, t}$ in the set (5.29) ensures the claim to hold.
Define by $\mathcal{G}\left([t, T] \times \Omega^{t} ; \mathcal{B}^{\varepsilon, 1}\right)$ the set of all adapted, $\mathcal{B}^{\varepsilon, 1}$-valued, Borel functionals on $\Omega^{t}$. Define $\mathcal{G}\left([t, T] \times \Omega^{t} ; \mathcal{B}^{\varepsilon, 2}\right)$ analogously, with $\mathcal{B}^{\varepsilon, 1}$ replaced by $\mathcal{B}^{\varepsilon, 2}$, and set

$$
\begin{equation*}
\mathcal{G}^{\varepsilon}(t):=\left\{(\mu, \Sigma) \in \mathcal{G}\left([t, T] \times \Omega^{t} ; \mathcal{B}^{\varepsilon, 1}\right) \times \mathcal{G}\left([t, T] \times \Omega^{t} ; \mathcal{B}^{\varepsilon, 2}\right)\right\} \tag{5.30}
\end{equation*}
$$

Proof of Proposition 5.8. Recall the Wiener measure $\mathbb{P}_{0}^{t}$ defined on $\left(\Omega^{t}, \mathcal{F}^{t}, \mathbb{F}^{W^{t}}\right)$ under which the canonical process $W^{t}=\left(W_{s}^{t}\right)_{s \in[t, T]}$ is a Brownian motion satisfying $W_{t}^{t}=0$ (see Section 4). Moreover, recalling the set $\mathcal{G}^{\varepsilon}(t)$ given in (5.30), we denote by

$$
\begin{equation*}
\mathcal{D}^{\varepsilon}(t):=\left\{(b, \sigma) \mid\left(b_{s}, \sigma_{s}\right):=\left(\mu\left(s, W^{t}\right), \Sigma\left(s, W^{t}\right)\right) \text { for } s \in[t, T],(\mu, \Sigma) \in \mathcal{G}^{\varepsilon}(t)\right\} \tag{5.31}
\end{equation*}
$$

and we define

$$
\begin{equation*}
\mathcal{Q}_{\mathrm{sub}}^{\varepsilon}(t, x):=\mathcal{Q}\left(t, x ; \mathcal{D}^{\varepsilon}\right):=\left\{\mathbb{P}_{0}^{t} \circ\left(x \oplus_{t} X^{t, x ; b, \sigma}\right)^{-1} \mid(b, \sigma) \in \mathcal{D}^{\varepsilon}(t)\right\} \tag{5.32}
\end{equation*}
$$

where $X^{t, x ; b, \sigma}$ is defined in (4.2).

From the definition of $\mathcal{G}^{\varepsilon}(t)$, it follows that $\mathcal{D}^{\varepsilon}(t) \subseteq \mathcal{C}^{\varepsilon}(t)$ (see (4.3)). Furthermore, since $\mathcal{Q}^{\varepsilon}(t, x)=\mathcal{Q}\left(t, x ; \mathcal{C}^{\varepsilon}\right)($ see $(5.11))$, by Remark 5.7 we have

$$
\mathcal{Q}_{\mathrm{sub}}^{\varepsilon}(t, x) \subseteq \mathcal{Q}^{\varepsilon}(t, x) \subseteq \mathcal{P}^{\varepsilon}(t, x)
$$

Now let $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$. Lemma 5.11 ensures that $X^{\mathbb{P},(n)}, n \in \mathbb{N}$, given in (5.18), converges to the canonical process $X$ in $\mathcal{H}^{2}\left(\Omega^{0, x}, \mathcal{F}^{0, x}, \mathbb{F}^{X}, \mathbb{P}\right)$. Thus Lemma 5.12 together with (5.31) and (5.32) ensures that $\mathbb{P}$ is contained in the $\tau_{p}$-closure of the convex hull of $\mathcal{Q}_{\text {sub }}^{\varepsilon}(t, x)$ for every $p \geq 1$.

Proof of Proposition 5.9. Recall the sets $\mathcal{P}^{\varepsilon}(t, x)$ and $\mathcal{Q}^{\varepsilon}(t, x)$ defined in (5.4) and (5.11). Denote by $\mathcal{Q}_{\text {sub }}^{\varepsilon}(t, x)$ the subset of $\mathcal{Q}^{\varepsilon}(t, x)$ such that its' convex hull is a dense subset of $\mathcal{P}^{\varepsilon}(t, x)$ with respect to the $\tau_{p}$-topology for all $p \geq 1$ (see Proposition 5.8). Furthermore, $\mathcal{P}^{\varepsilon}(t, x)$ is a subset of $\mathcal{P}^{p}\left(\Omega^{0, x}\right)$ for every $p \geq 1$ (see Remark 5.2 ). Since for every $\xi \in C_{p}\left(\Omega^{0, x} ; \mathbb{R}\right)$ the map $\mathcal{P}^{p}\left(\Omega^{0, x}\right) \ni \mathbb{P} \rightarrow \mathbb{E}^{\mathbb{P}}[\xi]$ is continuous and linear, it follows that

$$
\sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}[\xi]=\sup _{\mathbb{P} \in \mathcal{Q}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}[\xi]
$$

Therefore, as the function $f$ has at most polynomial growth (see Remark 4.2),

$$
v_{\text {weak }}^{\varepsilon}(t, x)=\sup _{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right]=\sup _{\mathbb{P} \in \mathcal{Q}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right]=v_{\text {strong }}^{\varepsilon}(t, x)
$$

This completes the proof.

## References

[1] A. Bachouch, C. Huré, N. Langrené, and H. Pham. Deep neural networks algorithms for stochastic control problems on finite horizon: numerical applications. Methodol. Comput. Appl. Probab., 24(1):143-178, 2022.
[2] G. Barles. Convergence of numerical schemes for degenerate parabolic equations arising in finance theory. In Rogers L. C. G., Talay D. eds. Numerical Methods in Finance.
[3] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial differential equations. Stochastics, 60(1-2):57-83, 1997.
[4] D. Bartl, S. Drapeau, J. Obłój, and J. Wiesel. Sensitivity analysis of Wasserstein distributionally robust optimization problems. Proc. R. Soc. A, 477(2256):20210176, 2021.
[5] D. Bartl, A. Neufeld, and K. Park. Sensitivity of robust optimization problems under drift and volatility uncertainty. arXiv preprint arXiv:2311.11248, 2023.
[6] D. Bartl and J. Wiesel. Sensitivity of multiperiod optimization problems with respect to the adapted Wasserstein distance. SIAM J. Financial Math., 14(2):704-720, 2023.
[7] C. Beck, S. Becker, P. Cheridito, A. Jentzen, and A. Neufeld. Deep splitting method for parabolic PDEs. SIAM J. Sci. Comput, 43(5):A3135-A3154, 2021.
[8] C. Beck, S. Becker, P. Grohs, N. Jaafari, and A. Jentzen. Solving the Kolmogorov PDE by means of deep learning. J. Sci. Comput., 88:1-28, 2021.
[9] C. Beck, W. E, and A. Jentzen. Machine learning approximation algorithms for high-dimensional fully nonlinear partial differential equations and second-order backward stochastic differential equations. $J$. Nonlinear Sci., 29:1563-1619, 2019.
[10] C. Beck, F. Hornung, M. Hutzenthaler, A. Jentzen, and T. Kruse. Overcoming the curse of dimensionality in the numerical approximation of Allen-Cahn partial differential equations via truncated full-history recursive multilevel Picard approximations. J. Numer. Math., 28(4):197-222, 2020.
[11] S. Becker, R. Braunwarth, M. Hutzenthaler, A. Jentzen, and P. von Wurstemberger. Numerical simulations for full history recursive multilevel Picard approximations for systems of high-dimensional partial differential equations. arXiv preprint arXiv:2005.10206, 2020.
[12] F. Black and M. Scholes. The pricing of options and corporate liabilities. J. Polit. Econ., 81(3):637-654, 1973.
[13] J. Blanchet and K. Murthy. Quantifying distributional model risk via optimal transport. Math. Oper. Res., 44(2):565-600, 2019.
[14] J. Blessing, R. Denk, M. Kupper, and M. Nendel. Convex monotone semigroups and their generators with respect to $\Gamma$-convergence. arXiv preprint arXiv:2202.08653, 2022.
[15] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov. Fokker-Planck-Kolmogorov Equations, volume 207 of Mathematical Surveys and Monographs. American Mathematical Society, 2022.
[16] P. Briand, F. Coquet, Y. Hu, J. Mémin, and S. Peng. A converse comparison theorem for BSDEs and related properties of $g$-expectation. Elect. Comm. in Probab., 5:101-117, 2000.
[17] S. Cerrai. Second Order PDE's in Finite and Infinite Dimension: A Probabilistic Approach. Lecture Notes in Math. Springer, Berlin, 2001.
[18] P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. Comm. Pure Appl. Math., 60(7):1081-1110, 2007.
[19] F. Coquet, Y. Hu, J. Mémin, and S. Peng. Filtration-consistent nonlinear expectations and related $g$ expectations. Probab. Theory Relat. Fields, 123(1):1-27, 2002.
[20] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc., 27(1):1-67, 1992.
[21] D. Criens and L. Niemann. Markov selections and Feller properties of nonlinear diffusions. arXiv preprint arXiv:2205.15200, 2022.
[22] C. Dellacherie and P. Meyer. Probabilities and Potential, B: Theory of Martingales.
[23] L. Denis, M. Hu, and S. Peng. Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion paths. Potential Anal., 34:139-161, 2011.
[24] Y. Dolinsky, M. Nutz, and H. M. Soner. Weak approximation of G-expectations. Stoch. Process. Appl., 122(2):664-675, 2012.
[25] Q. Du, Y. Gu, H. Yang, and C. Zhou. The discovery of dynamics via linear multistep methods and deep learning: error estimation. SIAM J. Numer. Anal., 60(4):2014-2045, 2022.
[26] W. E, J. Han, and A. Jentzen. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. Commun. Math. Stat, 5(4):349380, 2017.
[27] N. El Karoui and X. Tan. Capacities, measurable selection and dynamic programming part ii: application in stochastic control problems. arXiv preprint arXiv:1310.3364, 2013.
[28] L. C. Evans. Partial Differential Equations, volume 19. American Mathematical Society, 2010.
[29] T. Fadina, A. Neufeld, and T. Schmidt. Affine processes under parameter uncertainty. Probab. Uncertain. Quant. Risk, 4:1-35, 2019.
[30] W. H. Fleming and H. M. Soner. Controlled Markov processes and viscosity solutions, volume 25. Springer Science \& Business Media, 2006.
[31] A. Friedman. Stochastic differential equations and applications. In Stochastic differential equations, pages 75-148. Springer, 1975.
[32] S. Fuhrmann, M. Kupper, and M. Nendel. Wasserstein perturbations of Markovian transition semigroups. Ann. Inst. Henri Poincaré Probab. Stat., 59(2):904-932, 2023.
[33] R. Gao, X. Chen, and A. J. Kleywegt. Wasserstein distributionally robust optimization and variation regularization. Oper. Res., 2022.
[34] R. Gao and A. Kleywegt. Distributionally robust stochastic optimization with wasserstein distance. Math. Oper. Res., 48(2):603-655, 2023.
[35] M. Germain, H. Pham, and X. Warin. Approximation error analysis of some deep backward schemes for nonlinear PDEs. SIAM J. Sci. Comput., 44(1):A28-A56, 2022.
[36] M. Germain, H. Pham, and X. Warin. Neural Networks-Based Algorithms for Stochastic Control and PDEs in Finance. In Machine Learning and Data Sciences for Financial Markets: A Guide to Contemporary Practices, pages 426-452. Cambridge University Press, 2023.
[37] M. B. Giles. Multilevel Monte Carlo path simulation. Oper. Res., 56(3):607-617, 2008.
[38] M. B. Giles, A. Jentzen, and T. Welti. Generalised multilevel Picard approximations. arXiv preprint arXiv:1911.03188, 2019.
[39] C. Graham and D. Talay. Stochastic simulation and Monte Carlo methods: mathematical foundations of stochastic simulation, volume 68. Springer Science \& Business Media, 2013.
[40] M. Hairer, M. Hutzenthaler, and A. Jentzen. Loss of regularity for Kolmogorov equations. Ann. Probab., 43(2):468-527, 2015.
[41] J. Han, A. Jentzen, and W. E. Solving high-dimensional partial differential equations using deep learning. Proc. Natl. Acad. Sci., 115(34):8505-8510, 2018.
[42] K. Hasselmann. Stochastic climate models part I. Theory. Tellus, 28(6):473-485, 1976.
[43] S. Heinrich. Multilevel Monte Carlo methods. In Large-Scale Scientific Computing: Third International Conference, LSSC 2001 Sozopol, Bulgaria, June 6-10, 2001 Revised Papers 3, pages 58-67. Springer, 2001.
[44] S. Herrmann and J. Muhle-Karbe. Model uncertainty, recalibration, and the emergence of delta-vega hedging. Finance Stoch., 21:873-930, 2017.
[45] S. Herrmann, J. Muhle-Karbe, and F. T. Seifried. Hedging with small uncertainty aversion. Finance Stoch., 21:1-64, 2017.
[46] L. Hörmander. Hypoelliptic second order differential equations. Acta Math., 119:147-171, 1967.
[47] C. Huré, H. Pham, A. Bachouch, and N. Langrené. Deep neural networks algorithms for stochastic control problems on finite horizon: convergence analysis. SIAM J. Numer. Anal., 59(1):525-557, 2021.
[48] C. Huré, H. Pham, and X. Warin. Deep backward schemes for high-dimensional nonlinear PDEs. Math. Comp., 89(324):1547-1579, 2020.
[49] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. Ann. Appl. Probab., 22(4):1611, 2012.
[50] M. Hutzenthaler, A. Jentzen, and T. Kruse. On multilevel Picard numerical approximations for highdimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. J. Sci. Comput., 79(3):1534-1571, 2019.
[51] M. Hutzenthaler, A. Jentzen, T. Kruse, T. Anh Nguyen, and P. von Wurstemberger. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. Proc. R. Soc. A, 476(2244):20190630, 2020.
[52] M. Hutzenthaler, A. Jentzen, T. Kruse, et al. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. Partial Differ. Equ. Appl, 2(6):1-31, 2021.
[53] M. Hutzenthaler and T. Kruse. Multilevel picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. SIAM J. Numer. Anal., 58(2):929-961, 2020.
[54] M. Hutzenthaler and T. A. Nguyen. Multilevel Picard approximations of high-dimensional semilinear partial differential equations with locally monotone coefficient functions. Appl. Numer. Math., 181:151175, 2022.
[55] Y. Jiang. Duality of causal distributionally robust optimization: the discrete-time case. arXiv preprint arXiv:2401.16556, 2024.
[56] N. G. V. Kampen. Stochastic processes in physics and chemistry, volume 1. North-Holland, Oxford, 1981.
[57] I. Karatzas and S. Shreve. Brownian motion and stochastic calculus, volume 113. Springer, 1991.
[58] I. Karatzas, S. E. Shreve, I. Karatzas, and S. E. Shreve. Methods of Mathematical Finance, volume 39. Springer, 1998.
[59] M. Kimura. Some problems of stochastic processes in genetics. Ann. Math. Stat., 28:882-901, 1957.
[60] P. E. Kloeden and E. Platen. Numerical Solution of Stochastic Differential Equations, volume 23. Springer.
[61] N. V. Krylov. On Kolmogorov's equations for finite-dimensional diffusions. In Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions (Cetraro, 1998). Lecture Notes in Math. 1715, 1-63. Springer, Berlin, 1999.
[62] J. D. Logan and W. Wolesensky. Mathematical methods in biology, volume 96. John Wiley \& Sons, 2009.
[63] G. Maruyama. Continuous Markov processes and stochastic equations. Rendiconti del Circolo Matematico di Palermo, 4:48-90, 1955.
[64] A. Matoussi, D. Possamaï, and C. Zhou. Robust utility maximization in nondominated models with 2BSDE: the uncertain volatility model. Math. Finance, 25(2):258-287, 2015.
[65] N. Metropolis and S. Ulam. The Monte Carlo method. J. Amer. Statist. Assoc., 44(247):335-341, 1949.
[66] P. Mohajerin Esfahani and D. Kuhn. Data-driven distributionally robust optimization using the Wasserstein metric: performance guarantees and tractable reformulations. Math. Programming, 171(1-2):115-166, 2018.
[67] M. Nendel and A. Sgarabottolo. A parametric approach to the estimation of convex risk functionals based on Wasserstein distance. arXiv preprint arXiv:2210.14340, 2022.
[68] A. Neufeld and M. Nutz. Measurability of semimartingale characteristics with respect to the probability law. Stoch. Process. Appl., 124(11):3819-3845, 2014.
[69] A. Neufeld and M. Nutz. Nonlinear Lévy processes and their characteristics. Trans. Am. Math. Soc., 369(1):69-95, 2017.
[70] A. Neufeld and M. Nutz. Robust utility maximization with Lévy processes. Math. Finance, 28(1):82-105, 2018.
[71] A. Neufeld and S. Wu. Multilevel Picard algorithm for general semilinear parabolic PDEs with gradientdependent nonlinearities. arXiv preprint arXiv:2310.12545, 2023.
[72] J. Y. Nguwi, G. Penent, and N. Privault. A deep branching solver for fully nonlinear partial differential equations. J. Comput. Phys., 499:112712, 2024.
[73] M. Nutz. Random G-expectations. Ann. Appl. Probab., 23(5):1755-1777, 2013.
[74] M. Nutz and R. van Handel. Constructing sublinear expectations on path space. Stoch. Process. Appl., 123(8):3100-3121, 2013.
[75] J. Obłój and J. Wiesel. Distributionally robust portfolio maximization and marginal utility pricing in one period financial markets. Math. Finance, 31(4):1454-1493, 2021.
[76] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. Systems Control Lett., 14(1):55-61, 1990.
[77] E. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In Stochastic Partial Differential Equations and Their Applications: Proceedings of IFIP WG 7/1 International Conference University of North Carolina at Charlotte, NC June 6-8, 1991, pages 200-217. Springer, 2005.
[78] A. Pascucci. Kolmogorov equations in physics and in finance. Elliptic and parabolic problems. Prog. Nonlinear Differ. Equ. Appl., 63:353-364, 2005.
[79] S. Peng. Backward stochastic differential equations and applications to optimal control. Appl. Math. Optim., 27(2):125-144, 1993.
[80] S. Peng. BSDE and related g-expectations. In Backward Stochastic Differential Equations, ed. N. El Karoui and L. Mazliak. pages 141-159. Longman, Harlow, 1997.
[81] S. Peng. $G$-expectation, $G$-Brownian motion and related stochastic calculus of Itô type. In Stochastic Analysis and Applications, Abel Symp., pages 541-567. Springer, Berlin, Heidelberg, 2007.
[82] S. Peng. Multi-dimensional $G$-Brownian motion and related stochastic calculus under $G$-expectation. Stochastic Process. Appl., 118(12):2223-2253., 2008.
[83] S. Peng. Nonlinear expectations and stochastic calculus under uncertainty. arXiv preprint arXiv:1002.4546, 24, 2010.
[84] G. Pflug and D. Wozabal. Ambiguity in portfolio selection. Quant. Finance, 7(4):435-442, 2007.
[85] H. Pham. Continuous-time stochastic control and optimization with financial applications, volume 61. Springer Science \& Business Media, 2009.
[86] H. Pham, X. Warin, and M. Germain. Neural networks-based backward scheme for fully nonlinear PDEs. Partial Differ. Equ. Appl., 2(1):16, 2021.
[87] D. Possamaï, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. Ann. Probab., 46(1):551-603, 2018.
[88] P. E. Protter. Stochastic Integration and Differential Equations, volume 21. Springer Science \& Business Media, 2005.
[89] M. Raissi, P. Perdikaris, and G. E. Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. $J$. Comput. Phys., 378:686-707, 2019.
[90] M. Röckner. $L^{p}$-analysis of finite and infinite dimensional diffusion operators. In Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998) Lecture Notes in Math., volume 1715, pages 65-116, 2006.
[91] M. Röckner and Z. Sobol. Kolmogorov equations in infinite dimensions: Well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations. Ann. Probab., pages 663-727, 2006.
[92] J. Sirignano and K. Spiliopoulos. DGM: A deep learning algorithm for solving partial differential equations. J. Comput. Phys., 375:1339-1364, 2018.
[93] S. A. Smolyak. Quadrature and interpolation formulas for tensor products of certain classes of functions. In Doklady Akademii Nauk, volume 148, pages 1042-1045. Russian Academy of Sciences, 1963.
[94] H. M. Soner, N. Touzi, and J. Zhang. Wellposedness of second order backward SDEs. Probab. Theory Related Fields, 153(1-2):149-190, 2012.
[95] H. M. Soner, N. Touzi, and J. Zhang. Dual formulation of second order target problems. Ann. Appl. Probab., 23(1):308-347, 2013.
[96] E. Tadmor. A review of numerical methods for nonlinear partial differential equations. Bull. Amer. Math. Soc., 49(4):507-554, 2012.
[97] V. Thomée. Galerkin finite element methods for parabolic problems, volume 25. Springer Science \& Business Media, 2007.
[98] J. Thuburn. Climate sensitivities via a Fokker-Planck adjoint approach. Q. J. R. Meteorol. Soc., 131(605):73-92, 2005.
[99] D. V. Widder. The heat equation. Academic Press, New York, 1976.
[100] P. Wilmott, J. Dewynne, and S. Howison. Option Pricing: Mathematical Models and Computation. Oxford Financial Press, 1993.
[101] J. Yong and X. Y. Zhou. Stochastic controls: Hamiltonian systems and HJB equations, volume 43. Springer Science \& Business Media, 2012.

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    ${ }^{1}$ The choice of the boundary $v(T, x)=f(x)$ to be defined at terminal time is arbitrary and could be placed at initial time using the time change $t \mapsto T-t$.

[^1]:    ${ }^{2}$ We refer to, e.g., [28, Section 5.2] for the definition of weak derivatives.

[^2]:    ${ }^{3}$ All the numerical experiments have been performed with the following hardware configurations: a Macbook Pro with Apple M2 Max chip, 32 GBytes of memory, and Mac OS 13.2.1. While we implement the Matlab code only on the CPU, we take advantage of the GPU acceleration (Metal Performance Shaders (MPS) backend) for implementing the Python codes. All the codes are provided in the following link: https://github.com/ kyunghyunpark1/Sensitivity_nonlinear_Kolmogorov

[^3]:    ${ }^{4}$ See, e.g. [88, Chapter IV.2, p.124] for the definition.

[^4]:    ${ }^{5}$ An adapted functional on $\Omega^{t}$ is a mapping $\theta:[t, T] \times \Omega^{t} \rightarrow \mathbb{R}$ such that $\theta(s, \cdot)$ is $\mathcal{F}_{s}^{W^{t}}$-measurable for every $s \in[t, T]$ (noting that $\mathbb{F}^{W^{t}}$ is the raw filtration of the canonical process $W^{t}$ defined on $[t, T]$; see Section 4). Similarly, an ( $\mathbb{R}^{d}$-valued) adapted functional on $\Omega^{t}$ is a mapping $\Theta:=\left(\theta_{1}, \ldots, \theta_{d}\right)^{\top}:[t, T] \times \Omega^{t} \rightarrow \mathbb{R}^{d}$ such that each $\theta_{i}, i=1, \ldots, d$, is an adapted functional on $\Omega^{t}$.

