NUMERICAL METHOD FOR NONLINEAR KOLMOGOROV PDES VIA SENSITIVITY ANALYSIS

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ABSTRACT. We examine nonlinear Kolmogorov partial differential equations (PDEs). Here the nonlinear part of the PDE comes from its Hamiltonian where one maximizes over all possible drift and diffusion coefficients which fall within a ε -neighborhood of pre-specified baseline coefficients. Our goal is to quantify and compute how sensitive those PDEs are to such a small nonlinearity, and then use the results to develop an efficient numerical method for their approximation. We show that as $\varepsilon \downarrow 0$, the nonlinear Kolmogorov PDE equals the linear Kolmogorov PDE defined with respect to the corresponding baseline coefficients plus ε times a correction term which can be also characterized by the solution of another linear Kolmogorov PDE involving the baseline coefficients. As these linear Kolmogorov PDEs can be efficiently solved in high-dimensions by exploiting their Feynman-Kac representation, our derived sensitivity analysis then provides a Monte Carlo based numerical method which can efficiently solve these nonlinear Kolmogorov equations. We provide numerical examples in up to 100 dimensions to empirically demonstrate the applicability of our numerical method.

1. Introduction

Kolmogorov partial differential equations (PDEs) are widely used to describe the evolution of underlying diffusion processes over time. These PDEs are applied in various fields, for instance to model dynamics in physics and chemistry (e.g., [56, 78, 99]), to analyze some population growth in biology (e.g., [59,62]) to model the evolution of stock prices in finance and economics (e.g., [2,12,100]), or for climate modeling (e.g., [42,98]), to name but a few.

Consider the following Kolmogorov PDE (see, e.g., [15, 17, 20, 28, 40, 46, 61, 77, 90, 91])

(1.1)
$$\begin{cases} \partial_t v(t,x) + \langle b, \nabla_x v(t,x) \rangle + \frac{1}{2} \operatorname{tr}(\sigma \sigma^\top D_x^2 v(t,x)) = 0 & \text{on } [0,T) \times \mathbb{R}^d; \\ v(T,x) = f(x) & \text{on } \mathbb{R}^d. \end{cases}$$

One of the common modeling challenges arising throughout all fields consists in finding the true drift and volatility parameters (b, σ) to describe the underlying evolution process, which is usually unknown. Typically, one would either try to estimate the parameters using historical data or choose them based on experts' opinions. However, it is well-known that model misspecification may lead to wrong outcomes which might be fatal, as e.g., happened during the financial crisis in 2008 when financial derivatives were priced based on solutions of (1.1) but with corresponding parameters which were not consistent with the market behavior during that period.

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¹The choice of the boundary v(T, x) = f(x) to be defined at terminal time is arbitrary and could be placed at initial time using the time change $t \mapsto T - t$.

To overcome this difficulty of model uncertainty, a common approach is to consider a set \mathcal{U} of parameters (b, σ) , where each element $(b, \sigma) \in \mathcal{U}$ is considered as a candidate for the true but unknown drift and volatility. Then, one uses this set of candidates \mathcal{U} to describe the evolution of the underlying process robustly with respect to its parameters by considering the following nonlinear Kolmogorov PDE (see, e.g., [14, 21, 30, 58, 79, 83, 85, 101])

$$(1.2) \qquad \begin{cases} \partial_t v(t,x) + \sup_{(b,\sigma) \in \mathcal{U}} \left\{ \langle b, \nabla_x v(t,x) \rangle + \frac{1}{2} \operatorname{tr}(\sigma \sigma^\top D_x^2 v(t,x)) \right\} = 0 & \text{ on } [0,T) \times \mathbb{R}^d; \\ v(T,x) = f(x) & \text{ on } \mathbb{R}^d. \end{cases}$$

A natural choice for \mathcal{U} we consider throughout this paper is to start with baseline parameters (b^o, σ^o) that one considers as first best estimates for the true but unknown drift and volatility and then to consider the set

(1.3)
$$\mathcal{B}^{\varepsilon} := \left\{ (b, \sigma) \in \mathbb{R}^d \times \mathbb{R}^{d \times d} : |b - b^o| \le \gamma \varepsilon, \|\sigma - \sigma^o\|_{\mathcal{F}} \le \eta \varepsilon \right\}$$

of all coefficients that fall within the (weighted by γ , $\eta \in [0,1]$) ε -neighborhood of the baseline coefficients, for some pre-specified $\varepsilon > 0$. Typical choices for γ and η consist of $(\gamma, \eta) = (1,0)$ representing drift uncertainty [16, 19, 76, 80], $(\gamma, \eta) = (0,1)$ representing volatility uncertainty [23,24,73,81,82], as well as $(\gamma, \eta) = (1,1)$ for simultaneous drift and volatility uncertainty [68–70]. We also refer to [18,64,94,95] for the connection of these nonlinear Kolmogorov PDEs (1.2) with second-order backward stochastic differential equations.

The goal of this paper is to analyze how sensitive Kolmogorov equations are with respect to their parameters b and σ . More precisely, for small $\varepsilon > 0$ let $v^{\varepsilon}(t,x)$ denote the (unique viscosity) solution of the nonlinear Kolmogorov PDE (1.2) with $\mathcal{U} := \mathcal{B}^{\varepsilon}$ and let $v^{0}(t,x)$ be the solution of the linear Kolmogorov PDE (1.1) with respect to the baseline parameters b^{o} and σ^{o} . In this context, we aim to answer the following questions:

- · Can we identify and efficiently calculate the sensitivity $\partial_{\varepsilon}v^{0}(t,x) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(v^{\varepsilon}(t,x) v^{0}(t,x))$?
- · Can we find a numerical method which can efficiently solve high-dimensional nonlinear Kolmogorov PDEs of the form (1.2) with $\mathcal{U} := \mathcal{B}^{\varepsilon}$ for small $\varepsilon > 0$?

In Theorem 2.7 we show that if f is sufficiently regular and satisfies some mild growth conditions (see Assumption 2.1) as well as σ^o is invertible, then the following hold. For every $(t,x) \in [0,T) \times \mathbb{R}^d$, as $\varepsilon \downarrow 0$, we obtain that

$$v^{\varepsilon}(t,x) = v^{0}(t,x) + \varepsilon \cdot \partial_{\varepsilon} v^{0}(t,x) + O(\varepsilon^{2}),$$

where $\partial_{\varepsilon} v^0(t,x)$ is given by

$$\partial_{\varepsilon} v^{0}(t,x) = \mathbb{E}\left[\int_{t}^{T} \gamma \left|w\left(s,x+X_{s}^{o}\right)\right| + \eta \left\|\mathbf{J}_{x} w\left(s,x+X_{s}^{o}\right) \sigma^{o}\right\|_{F} ds \,\middle|\, X_{t}^{o} = 0\right].$$

Here

- $-X_s^o := b^o s + \sigma^o W_s$, $s \in [0,T]$ is a stochastic process driven by a standard d-dimensional Brownian motion $(W_s)_{s \in [0,T]}$,
- $-\mathbb{E}[\cdot|X_t^o=0]$ denotes the conditional expectation given $X_t^o=0$,

 $-w = (w^1, \dots, w^d)$ where each w^i is the solution of the linear Kolmogorov equation

$$\begin{cases} \partial_s w^i(s,x) + \langle b^o, \nabla_x w^i(s,x) \rangle + \frac{1}{2} \operatorname{tr} \left((\sigma^o)(\sigma^o)^\top D_x^2 w^i(s,x) \right) = 0 & \text{on } [t,T) \times \mathbb{R}^d; \\ w^i(T,x) = \partial_{x_i} f(x) & \text{on } \mathbb{R}^d, \end{cases}$$

and $J_x w$ stands for the Jacobian of w.

We highlight that Theorem 2.7 also provides a methodology to approximate the solution v^{ε} of the nonlinear Kolmogorov PDE (1.2). Indeed, note that by the Feynman-Kac representation, we have for any $t \leq s \leq T$ and $x \in \mathbb{R}^d$ that

$$\begin{split} v^0(t,x) &= \mathbb{E}\Big[f(x+X_T^o)\Big|X_t^o = 0\Big],\\ w(s,x+X_s^o) &= \mathbb{E}\Big[\nabla_x f(x+X_s^o + \widetilde{X}_T^o)\Big|\widetilde{X}_s^o = 0\Big],\\ (\mathbf{J}_x w)(s,x+X_s^o) &= \mathbb{E}\left[D_x^2 f(x+X_s^o + \widetilde{X}_T^o)\Big|\widetilde{X}_s^o = 0\right], \end{split}$$

where $\widetilde{X}_s^o := b^o s + \sigma^o \widetilde{W}_s$, $s \in [0,T]$, with $(\widetilde{W}_s)_{s \in [0,T]}$ being another standard d-dimensional Brownian motion independent of $(W_s)_{s \in [0,T]}$. Therefore, we can implement the approximation $v^0 + \varepsilon \cdot \partial_{\varepsilon} v^0$ of v^{ε} by a Monte Carlo based scheme (see Algorithm 1) which is efficient even in high dimensions (see Section 3 for our numerical results in up to d = 100 dimensions).

Related Literature. Since solutions of Kolmogorov PDEs and parabolic PDEs in general typically cannot be solved explicitly and hence need to be approximately solved, there has been a lot of efforts to develop such numerical approximation methods. We refer e.g. to [93,96,97] for deterministic approximation methods (e.g., finite difference and finite element methods, spectral Galerkin methods, and sparse grid methods) and to [10,11,37–39,43,49–51,51–54,60,63,65,71,71] for stochastic approximation methods including Monte Carlo approximations. Recently, there has been an intensive interest in deep-learning based algorithms that can approximately solve high-dimensional linear/nonlinear parabolic PDEs (e.g., [7–9,25,26,35,41,48,72,86,89,92]). Moreover, we also refer to [1,36,47] for deep learning algorithms to solve control problems related to (discretized versions of) HJB equations.

Sensitivity analysis of robust optimization problems have been established mostly with respect to 'Wasserstein-type' of uncertainty by considering an (adapted) Wasserstein-ball with radius ε around an (estimated) baseline probability measure for the underlying process either in a one-period model [4,13,32–34,66,67,75,84] or in a multi-period discrete-time model [6,55]. Moreover, in continuous-time, [44,45] provided a sensitivity analysis of a particular robust utility maximization problem under volatility uncertainty, whereas [5] analyzed the sensitivity of general robust optimization problems under both drift and volatility uncertainty.

The contribution of our paper is to provide a sensitivity analysis of nonlinear PDEs of type (1.2) and use this analysis to approximate those PDEs by some suitable linear PDEs as described above, leading to a numerical approximation algorithm which is efficient even in high-dimensions.

2. Main results

Fix $d \in \mathbb{N}$ and endow \mathbb{R}^d with the Euclidean inner product $\langle \cdot, \cdot \rangle$, and $\mathbb{R}^{d \times d}$ with the Frobenius inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$, respectively. Let \mathbb{S}^d be the set of all symmetric $d \times d$ matrices. Then fix a

time horizon T > 0, and for any $\varepsilon \ge 0$ consider a nonlinear Kolmogorov PDE with the set $\mathcal{B}^{\varepsilon}$ given in (1.3)

$$(2.1) \quad \left\{ \begin{aligned} \partial_t v^\varepsilon(t,x) + \sup_{(b,\sigma) \in \mathcal{B}^\varepsilon} \left\{ \langle b, \nabla_x v^\varepsilon(t,x) \rangle + \frac{1}{2} \operatorname{tr} \left(\sigma \sigma^\top D_x^2 v^\varepsilon(t,x) \right) \right\} &= 0 \quad \text{on } [0,T) \times \mathbb{R}^d; \\ v^\varepsilon(T,x) = f(x) \quad \text{on } \mathbb{R}^d, \end{aligned} \right.$$

where $f: \mathbb{R}^d \to \mathbb{R}$ corresponds to the boundary condition.

We impose certain conditions on the boundary f and baseline coefficient σ^o given in (1.3).

Assumption 2.1. The function $f: \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable. Moreover, its Hessian $D_x^2 f: \mathbb{R}^d \to \mathbb{S}^d$ exists in the weak sense² and there are $\alpha \geq 1$ and $C_f > 0$ such that $\|D_x^2 f(x)\|_{\mathcal{F}} \leq C_f (1+|x|^{\alpha})$ for every $x \in \mathbb{R}^d$.

Assumption 2.2. The matrix σ^o is invertible.

Remark 2.3. Assumption 2.2 ensures that $\lambda_{\min}(\sigma^o)$, the smallest singular value of σ^o is strictly positive; in particular for every $\varepsilon < \lambda_{\min}(\sigma^o)$ and $(b, \sigma) \in \mathcal{B}^{\varepsilon}$, the matrix σ is invertible.

We further impose a condition on the solution of the nonlinear Kolmogorov PDE, which relies on the notion of viscosity solutions (see Section 5.3 for the standard definitions of viscosity / strong solutions of PDEs).

Assumption 2.4. For any $\varepsilon \geq 0$, there exists at most one viscosity solution v^{ε} of (2.1) satisfying that there is C > 0 such that for all $t \in [0, T]$

(2.2)
$$\lim_{|x| \to \infty} |v^{\varepsilon}(t, x)| e^{-C(\log(|x|))^2} = 0.$$

Remark 2.5. It follows from [69, Proposition 5.5] that Assumption 2.4 is satisfied if, e.g., Assumptions 2.1 and 2.2 are satisfied and f is bounded and Lipschitz continuous. Furthermore, if $\eta = 0$, i.e., there is no volatility uncertainty, then Assumptions 2.1 & 2.2 directly imply that Assumption 2.4 holds, see [3, Theorem 3.5]. We also refer to [3, Remark 3.6] for a detailed discussion on the growth condition (2.2).

Now we collect some preliminary results in the next proposition on the solution of the nonlinear Kolmogorov PDE (2.1) together with the following linear Kolmogorov PDE defined using the baseline coefficients b^o and σ^o and the boundary condition $\partial_{x_i} f$, $i = 1, \ldots, d$,

(2.3)
$$\begin{cases} \partial_s w^i(s,x) + \langle b^o, \nabla_x w^i(s,x) \rangle + \frac{1}{2} \operatorname{tr} \left((\sigma^o)(\sigma^o)^\top D_x^2 w^i(s,x) \right) = 0 & \text{on } [t,T) \times \mathbb{R}^d; \\ w^i(T,x) = \partial_{x_i} f(x) & \text{on } \mathbb{R}^d, \end{cases}$$

where we note that f is the boundary condition given in (2.1).

The corresponding proofs for the following proposition can be found in Section 5.3.

Proposition 2.6. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied. Then the following hold:

(i) For any $\varepsilon \geq 0$, there exists a unique viscosity solution $v^{\varepsilon} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ of (2.1) satisfying the growth property given in (2.2).

²We refer to, e.g., [28, Section 5.2] for the definition of weak derivatives.

(ii) For any i = 1, ..., d, there exists a unique strong solution $w^i : [t, T] \times \mathbb{R}^d \to \mathbb{R}$ of (2.3) with polynomial growth.

We proceed with our main result. To formulate it, denote by $O(\cdot)$ the Landau symbol and if w^i is the solution to (2.3) with the boundary condition $\partial_{x_i} f$ for every $i = 1, \ldots, d$, let us define $w : [t, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $J_x w : [t, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ by

(2.4)
$$w(s,x) := \begin{pmatrix} w^1(s,x) \\ \vdots \\ w^d(s,x) \end{pmatrix}, \quad J_x w(s,x) := \begin{pmatrix} \nabla_x^\top w^1(s,x) \\ \vdots \\ \nabla_x^\top w^d(s,x) \end{pmatrix}.$$

Finally, let $X_t^o = b^o t + \sigma^o W_t$, $t \in [0, T]$, be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a fixed d-dimensional Brownian motion $W = (W_t)_{t \in [0, T]}$.

Theorem 2.7. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied. For every $\varepsilon \geq 0$, let v^{ε} be the unique viscosity solution of (2.1) satisfying (2.2), let w^{i} be the unique strong solution of (2.3) with polynomial growth for every $i = 1, \ldots, d$, and let w, $J_{x}w$ be as in (2.4). Then, for every $(t, x) \in [0, T) \times \mathbb{R}^{d}$ as $\varepsilon \downarrow 0$,

$$v^{\varepsilon}(t,x) = v^{0}(t,x) + \varepsilon \cdot \partial_{\varepsilon} v^{0}(t,x) + O(\varepsilon^{2}),$$

where $\partial_{\varepsilon}v^{0}(t,x) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}(v^{\varepsilon}(t,x) - v^{0}(t,x))$ is given by

$$\partial_{\varepsilon} v^{0}(t,x) = \mathbb{E}\left[\int_{t}^{T} \left(\gamma \left|w\left(s,x+X_{s}^{o}\right)\right| + \eta \left\|J_{x}w\left(s,x+X_{s}^{o}\right)\sigma^{o}\right\|_{F}\right) ds \,\middle|\, X_{t}^{o} = 0\right],$$

with $\mathbb{E}[\cdot|X_t^o=0]$ denoting the conditional expectation given $X_t^o=0$.

Remark 2.8. We actually show that the approximation is (locally) uniform in (t, x): There exists a constant c (that depends on T, α and C_f given in Assumption 2.1, and the norms for b^o , σ^o) such that for every $\varepsilon < \min\{1, \lambda_{\min}(\sigma^o)\}$ (see Remark 2.3) and every $(t, x) \in [0, T) \times \mathbb{R}^d$,

$$|v^{\varepsilon}(t,x) - (v^{0}(t,x) + \varepsilon \cdot \partial_{\varepsilon}v^{0}(t,x))| \le c(1+|x|^{\alpha})\varepsilon^{2}.$$

Let us mention some basic properties of the sensitivity result given in Theorem 2.7, as well as how it can be used to construct numerical approximations of the PDE (2.1), as explained in Section 3 below. To that end, recalling the process X^o appearing in Theorem 2.7 with the corresponding Brownian motion W, let $\widetilde{X}_t^o := b^o t + \sigma^o \widetilde{W}_t$, $t \in [0,T]$, where $\widetilde{W} := (\widetilde{W}_t)_{t \in [0,T]}$ is another standard d-dimensional Brownian motion independent of W. We remark the following Feynman-Kac representations: for every $(t,x) \in [0,T] \times \mathbb{R}^d$ and $s \in [t,T]$, it holds that

(2.5)
$$v^{0}(t,x) = \mathbb{E}\left[f(x+X_{T}^{o})\middle|X_{t}^{o}=0\right],$$

$$w(s,x+X_{s}^{o}) = \mathbb{E}\left[\nabla_{x}f(x+X_{s}^{o}+\widetilde{X}_{T}^{o})\middle|\widetilde{X}_{s}^{o}=0\right].$$

Furthermore, denote by $(J_x w)^{k,l}$ for every $k, l \in \{1, \ldots, d\}$ the (k, l)-component of $J_x w$ defined in (2.4). If $\nabla_x f$ is sufficiently smooth (at least continuously differentiable) and $D_x^2 f$ is at most polynomially growing, then by [61, Theorem 4.32], $(J_x w)^{k,l}$ also has the following Feynman-Kac representation: for every $(t, x) \in [0, T] \times \mathbb{R}^d$ and $s \in [t, T]$,

$$(2.6) (J_x w)^{k,l}(s, x + X_s^o) = \mathbb{E} \left[\partial_{x_k x_l} f(x + X_s^o + \widetilde{X}_T^o) \middle| \widetilde{X}_s^o = 0 \right].$$

Otherwise, if $\nabla_x f$ lacks that kind of regularity, we approximate $(J_x w)^{k,l}$ via a finite difference quotient as follows: Let e_l be a d-dimensional vector with value 0 in all the components except for the l-th component with value 1. Then for sufficiently small h > 0,

$$(J_x w)^{k,l}(s, x + X_s^o) = \partial_{x_l} w^k(s, x + X_s^o) \approx \frac{1}{h} (w^k(s, x + X_s^o + h \cdot e_l) - w^k(s, x + X_s^o)).$$

In particular, by (2.5) the approximation can be rewritten by

$$(2.7) \left(\mathbf{J}_x w \right)^{k,l} (s, x + X_s^o) \approx \frac{1}{h} \mathbb{E} \left[\partial_{x_k} f(x + X_s^o + \widetilde{X}_T^o + h \cdot e_l) - \partial_{x_k} f(x + X_s^o + \widetilde{X}_T^o) \middle| \widetilde{X}_s^o = 0 \right].$$

Hence, the exact value of $\partial_{\varepsilon}v^0$ requires calculations of nested expectations, which will be realized as nested Monte Carlo approximations in the next section.

3. Numerical results

Combining Theorem 2.7 with corresponding probabilistic representations for the functions v^0 , w, and $J_x w$ given in (2.5) and (2.6) (or (2.7)), we derive a Monte Carlo based scheme to implement both the sensitivity $\partial_{\varepsilon} v^0$ as well as the approximated solution $v^0 + \varepsilon \cdot \partial_{\varepsilon} v^0$ of the nonlinear PDE (2.1) for every $\varepsilon < \min\{1, \lambda_{\min}(\sigma^o)\}$ (see Remark 2.8). We provide a pseudocode in Algorithm 1 to show how it can be implemented³.

Let us briefly mention the computational complexity of Algorithm 1. For each $N \in \mathbb{N}$ (i.e., the number of steps in the time discretization) and $M_0, M_1 \in \mathbb{N}$ (i.e., the number of samples for each expectation involved in v^0 and $\partial_{\varepsilon} v^0$), denote by

(3.1)
$$\mathfrak{C}(d, N, M_0, M_1) := M_0 d + N M_1 (M_1 + 1) d + N M_1 (M_1 + 1 + d) d^2$$

the sum of following three components:

- (i) M₀d is the number of (one-dimensional) samples for the Monte Carlo approximation used in v⁰(t, x) ≈ 1/M₀ ∑_{j=1}^{M₀} f(x + X_N(j));
 (ii) NM₁(M₁ + 1)d is the sum of the number NM₁²d of samples for nested Monte Carlo ap-
- (ii) $NM_1(M_1+1)d$ is the sum of the number NM_1^2d of samples for nested Monte Carlo approximation and the number NM_1d of Euclidean norm evaluations used in $\mathbb{E}[\int_t^T \gamma |w(s,x+X_s^o)|ds] \approx \sum_{i=0}^{N-1} \Delta t \frac{1}{M_1} \sum_{j=1}^{M_1} \gamma |\widehat{w}(i,j)|;$ (iii) $NM_1(M_1+1+d)d^2$ is the sum of: the number $NM_1^2d^2$ of samples for the nested Monte-
- (iii) $NM_1(M_1+1+d)d^2$ is the sum of: the number $NM_1^2d^2$ of samples for the nested Monte-Carlo approximation, the number NM_1d^3 of the matrix multiplications, and the number NM_1d^2 of Frobenius norm evaluations used in $\mathbb{E}[\int_t^T \eta \|J_x w(s, x + X_s^o) \sigma^o\|_F ds] \approx \sum_{i=0}^{N-1} \Delta t \frac{1}{M_1} \sum_{j=1}^{M_1} \eta \|\widehat{J_x w}(i,j)\sigma^o\|_F$.

Note that while $\mathfrak{C}(d, N, M_0, M_1)$ given in (3.1) increases linearly in M_0 , it increases quadratically in M_1 due to the nested Monte Carlo approximations. To work within constraint of memory in our hardware, we choose M_1 so that $M_0 \geq M_1$ in all the experiments.

We proceed to calculate the value v^0 and the sensitivity $\partial_{\varepsilon}v^0$ and then compare the approximation $v^0 + \varepsilon \partial_{\varepsilon}v^0$ (given in Theorem 2.7) with v^{ε} . To that end, let us start with the following 1-dimensional example with a specific boundary function: Let d=1, T=1, $b^o=1$, $\sigma^o=1$, $f(x)=x^4$, (t,x)=(0,0), N=100, $M_0=3\cdot 10^6$, and $M_1=3\cdot 10^4$, and choose any

³All the numerical experiments have been performed with the following hardware configurations: a Macbook Pro with Apple M2 Max chip, 32 GBytes of memory, and Mac OS 13.2.1. While we implement the Matlab code only on the CPU, we take advantage of the GPU acceleration (Metal Performance Shaders (MPS) backend) for implementing the Python codes. All the codes are provided in the following link: https://github.com/kyunghyunpark1/Sensitivity_nonlinear_Kolmogorov

Algorithm 1 A Monte Carlo based scheme for v^0 and $\partial_{\varepsilon} v^0$.

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1: Input: T > 0, b^o \in \mathbb{R}^d, \sigma^o \in \mathbb{R}^{d \times d}, f : \mathbb{R}^d \to \mathbb{R} (satisfying Assumptions 2.1, 2.2, and 2.4), (t, x) \in \mathbb{R}^d
      [0,T)\times\mathbb{R}^d, N\in\mathbb{N}, M_0,M_1\in\mathbb{N} with M_0\geq M_1, and h\geq 0 (sufficiently small);
2: Generate:
             (i) The uniform subdivision t_i = t + i\Delta t, i \in \{0, \dots, N\} with \Delta t = \frac{T - t}{N};
             (ii) M_0 samples W(j) \sim \mathcal{N}(0, \mathrm{Id}_{\mathbb{R}^d}) (i.e., standard d-dimensional normal distribution), j \in \{1, \ldots, M_0\};
             (iii) (N+1) \times M_0 samples \mathcal{X}_i(j) := b^o t_i + \sigma^o \mathcal{W}(j) \sqrt{t_i}, i \in \{0, \dots, N\} \text{ and } j \in \{1, \dots, M_0\};
3: Function v_{\mathrm{mc}}^0(t, x; M_0):
     Recall M_0 realizations \mathcal{X}_N(j), j \in \{1, \dots, M_0\};
Return \frac{1}{M_0} \sum_{j=1}^{M_0} f(x + \mathcal{X}_N(j))
4: Function (\partial_{\varepsilon} v^0)_{mc}(t, x; N, M_1):
             Recall N \times M_1 realizations \mathcal{X}_i(j), i \in \{0, \dots, N-1\} and j \in \{1, \dots, M_1\};
            for i=0 to N-1 and j=1 to M_1
Compute \widehat{w}^k(i,j) := \frac{1}{M_1} \sum_{m=1}^{M_1} \partial_{x_k} f(x+\mathcal{X}_i(j)+\mathcal{X}_{N-i}(m)) \ \forall k \in \{1,\dots,d\};
            if \nabla_x f is continuously differentiable and has at most polynomial growth
                   \begin{array}{l} \textbf{for } i=0 \textbf{ to } N-1 \textbf{ and } j=1 \textbf{ to } M_1 \\ \text{ Compute } (\widehat{\mathbf{J}_x w})^{k,l}(i,j) := \frac{1}{M_1} \sum_{m=1}^{M_1} \partial_{x_k,x_l} f(x+\mathcal{X}_i(j)+\mathcal{X}_{N-i}(m)) \ \forall k,l \in \{1,\dots,d\}; \end{array}
                   end
            else
                   for i = 0 to N - 1 and j = 1 to M_1

Compute \widehat{w}_h^{k,l}(i,j) := \frac{1}{M_1} \sum_{m=1}^{M_1} \partial_{x_k} f(x + \mathcal{X}_i(j) + he_l + \mathcal{X}_{N-i}(m)) \ \forall k,l \in \{1,\ldots,d\};

Compute (\widehat{J_x w})^{k,l}(i,j) := \frac{1}{h} (\widehat{w}_h^{k,l}(i,j) - \widehat{w}^k(i,j)) \ \forall k,l \in \{1,\ldots,d\};
                   end
     Return \sum_{i=0}^{N-1} \Delta t \frac{1}{M_1} \sum_{j=1}^{M_1} (\gamma \left| \widehat{w}(i,j) \right| + \eta \| \widehat{\mathbf{J}_x w}(i,j) \sigma^o \|_{\mathbf{F}})
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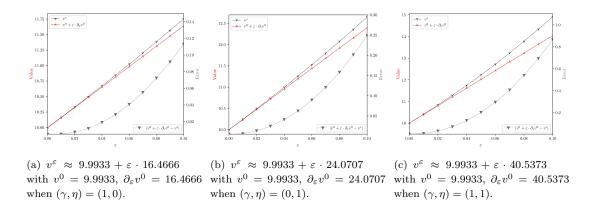


FIGURE 1. Comparative analysis between the approximated solution $v^0 + \varepsilon \cdot \partial_{\varepsilon} v^0$ and the actual counterpart v^{ε} over varying ε .

 $\gamma, \eta \leq 1$. Under this case, Assumptions 2.1 and 2.2 are obviously satisfied. Furthermore, since $\lambda_{\min}(\sigma^o) = 1$, we can and do choose any $\varepsilon < 1$.

As the function $f = x^4$ is convex, the probabilistic representation for v^{ε} given in (4.8) (that will be proven in Lemma 5.6 and Proposition 5.9) ensures that $v^{\varepsilon}(t,x)$ is convex in x and hence the corresponding nonlinear Kolmogorov PDE given in (2.1) can be rewritten by the following

Dimension $(=d)$			1	5	10	20	50	100
	v^0	Avg. Std. Dev.	0.51033 0.00038	0.51028 0.00025	0.51030 0.00020	0.51062 0.00023	0.51018 0.00032	0.51052 0.00026
$\partial_{arepsilon}v^{0}$	$(\gamma,\eta)=(1,0)$	Avg. Std. Dev.	0.45018 0.00267	1.00558 0.00451	$1.42421 \\ 0.00977$	2.01173 0.00595	3.18120 0.00869	4.49794 0.01729
	$(\gamma,\eta)=(0,1)$	Avg. Std. Dev.	0.55718 0.00360	$\begin{array}{c} 1.24767 \\ 0.00506 \end{array}$	1.76609 0.00940	2.49299 0.00822	3.94644 0.01327	5.58518 0.02033
	$(\gamma,\eta)=(1,1)$	Avg. Std. Dev.	1.00736 0.00217	2.25325 0.00536	3.19029 0.00679	4.50473 0.00806	7.12765 0.01324	10.08312 0.02257
$\lambda_{\min}(\sigma^o)$			1.00	0.21807	0.06382	0.01078	0.00235	0.00021
Runtime in Sec. (Avg.)			14.338	49.329	89.643	168.205	393.547	807.328

Table 1. Implementation of Algorithm 1 for several dimension cases. The average (Avg.) and the standard deviation (Std. Dev.) of the values for v^0 and $\partial_{\varepsilon}v^0$, and the average runtime in seconds are computed over the independent 10 runs of the Python code. $(b^{\circ}, \sigma^{\circ})$ are generated for every $d \in \{1, 5, 10, 20, 50, 100\}$ but for each d, they are fixed during the 10 runs of the code.

quasilinear parabolic equation

(3.2)
$$\partial_t v^{\varepsilon} + \frac{(\sigma^o + \eta \varepsilon)^2}{2} \partial_{xx} v^{\varepsilon} + b^o \partial_x v^{\varepsilon} + \sup_{|\widetilde{b}| \le \gamma \varepsilon} \left(\widetilde{b} \partial_x v^{\varepsilon} \right) = 0 \quad \text{on } (t, x) \in [0, T) \times \mathbb{R};$$

with $v^{\varepsilon}(T,x)=x^4$, $x\in\mathbb{R}$. In particular, since the PDE (3.2) is linear in the second derivative and the boundary $f = x^4$ is a polynomial, Remark 2.5 guarantees that (3.2) admits a unique viscosity solution satisfying (2.2), which ensures Assumption 2.4 to hold.

Figure 1 shows the comparison between v^{ε} and $v^0 + \varepsilon \cdot \partial_{\varepsilon} v^0$ with varying $\varepsilon < 0.1$, in which we obtain numerical results on v^{ε} by applying a finite difference approximation on the semilinear PDE (3.2) (we refer to Code_1.m and Code_2.m given in the link provided in Footnote 3; based on [50, MATLAB Code 7 in Section 3]) and obtain numerical results on $v^0 + \varepsilon \cdot \partial_{\varepsilon} v^0$ by using our Monte Carlo based scheme given in Algorithm 1 (we refer to Code_3.ipynb given in the mentioned link). As it has been proven in Theorem 2.7, we can observe in all the panels of Figure 1 that the error $|v^0 + \varepsilon \cdot \partial_{\varepsilon} v^0 - v^{\varepsilon}|$ of the approximation increases quadratically in ε .

Now we implement Algorithm ${\bf 1}$ for multi-dimensional cases with another boundary function: Let T = 1, $f(x) = \sin(\sum_{i=1}^{d} x_i)$, $(t, x) = (0, (0, \dots, 0))$, N = 100, $M_0 = 3 \cdot 10^6$, and $M_1 = 3 \cdot 10^4$, and choose any $\gamma, \eta \leq 1$. Furthermore, denote by $\mathrm{U}([a,b])$ for $a,b \in \mathbb{R}$ the uniform distribution with values in [a, b]. Then for any $d \in \mathbb{N}$, we generate $b^o \in \mathbb{R}^d$, $\sigma^o \in \mathbb{R}^{d \times d}$ in the following way:

- $b^o:=\widetilde{b}/(\sum_{i=1}^d|\widetilde{b}^i|),$ where \widetilde{b} is a d-dimensional random variable such that $\widetilde{b}^i\sim \mathrm{U}([0,1])$ for
- every $i \in \{1, ..., d\}$ and satisfy $\sum_{i=1}^{d} |\widetilde{b}^{i}| \neq 0$; $\cdot \sigma^{o} := \widetilde{\sigma}/(\sum_{l=1}^{d} (\sum_{k=1}^{d} \widetilde{\sigma}^{k,l})^{2})^{1/2}$, where let $\widetilde{\sigma} = (\widetilde{\sigma}^{k,l})_{k,l \in \{1,...,d\}}$ is a $d \times d$ -valued random variable such that $\widetilde{\sigma}^{k,l} \sim \mathrm{U}([-1,1])$ for every $k,l \in \{1,...,d\}$, satisfy that $\widetilde{\sigma}$ is invertible, and that $(\sum_{l=1}^{d} (\sum_{k=1}^{d} \tilde{\sigma}^{k,l})^2)^{1/2} \neq 0$.

Under this setup, Assumptions 2.1 and 2.2 are satisfied. Furthermore, since $f = \sin(\sum_{i=1}^{d} x_i)$ is bounded and Lipschitz continuous, by Remark 2.5, Assumption 2.4 is also satisfied. Hence, the corresponding viscosity solution of the nonlinear Kolmogorov PDE is unique. We further note that unlike the semilinear form of the PDE (3.2) where the boundary function f is convex, the corresponding Kolmogorov PDE under this setup is fully nonlinear.

Denote by $\Gamma = (\Gamma_t)_{t \in [0,T]}$ an 1-dimensional process satisfying $\Gamma_t = t + W_t^1$, $t \in [0,T]$, where W^1 is a standard 1-dimensional Brownian motion independent of W. Then, by the nomalization in the parameters b^o and σ^o , the following property holds: for every $d \in \mathbb{N}$,

law of
$$\sum_{i=1}^{d} X^{o,i} = \text{law of } \Gamma$$
.

Combined with (2.5) and (2.6) (noting that $\nabla_x f = \cos(\sum_{i=1}^d x_i) \mathbf{1}_d$, $J_x f = -\sin(\sum_{i=1}^d x_i) \mathbf{1}_{d \times d}$ where here $\mathbf{1}_d$ denotes a d-dimensional vector with value 1 in all the components and $\mathbf{1}_{d\times d}$ denotes a $d \times d$ -matrix with value 1 in all the components), this ensures the following characterizations: for every $d \in \mathbb{N}$,

- $v^0(0,0) = \mathbb{E}[\sin(\sum_{i=1}^d X_T^{o,i})] = \mathbb{E}[\sin(\Gamma_T)];$ $\partial_{\varepsilon} v^0(0,0)$ (when $(\gamma,\eta) = (1,0)$) is characterized by

$$\begin{split} \mathbb{E}\left[\int_{0}^{T}|w(t,X_{t}^{o})|dt\right] &= \mathbb{E}\left[\int_{0}^{T}\left|\mathbb{E}\left[\cos\left(\sum_{i=1}^{d}\left(X_{t}^{o,i}+\widetilde{X}_{T}^{o,i}\right)\right)\middle|\widetilde{X}_{t}^{o}=0\right]\mathbf{1}_{d}\left|dt\right]\right] \\ &= \sqrt{d}\cdot\mathbb{E}\left[\int_{0}^{T}\left|\mathbb{E}\left[\cos(\Gamma_{t}+\widetilde{\Gamma}_{T})\middle|\widetilde{\Gamma}_{t}=0\right]\middle|dt\right], \end{split}$$

with \widetilde{X}^o appearing in (2.5), where we denote by $\widetilde{\Gamma}_t := t + \widetilde{W}_t^1$ $t \in [0,T]$ a 1-dimensional process with a standard 1-dimensional Brownian motion \widetilde{W}^1 independent of W^1 ;

 $\partial_{\varepsilon} v^{0}(0,0)$ (when $(\gamma,\eta)=(0,1)$) is given by

$$\mathbb{E}\left[\int_{0}^{T} \|\mathbf{J}_{x}w(t, X_{t}^{o})\sigma^{o}\|_{\mathbf{F}}dt\right] = \mathbb{E}\left[\int_{0}^{T} \left\|\mathbb{E}\left[\sin\left(\sum_{i=1}^{d} \left(X_{t}^{o,i} + \widetilde{X}_{T}^{o,i}\right)\right) \middle| \widetilde{X}_{t}^{o} = 0\right] \mathbf{1}_{d \times d} \sigma^{o} \right\|_{\mathbf{F}}dt\right]$$
$$= \sqrt{d} \cdot \mathbb{E}\left[\int_{0}^{T} \left|\mathbb{E}\left[\sin(\Gamma_{t} + \widetilde{\Gamma}_{T}) \middle| \widetilde{\Gamma}_{t} = 0\right] \middle| dt\right].$$

Table 1 shows the results of several dimension cases based on 10 independent runs of a PYTHON code (Code_4.ipynb) given in the link provided in Footnote 3. As shown in the above characterizations (i)-(iii), the values for v^0 are invariant as 0.5103 (± 0.0003) over the dimension d (in consideration of the error of the Monte Carlo approximations; see (i)) and the value for $\partial_{\varepsilon}v^0$ increases in proportional to \sqrt{d} for all three cases $(\gamma, \eta) \in \{(1, 0), (0, 1), (1, 1)\}$ (see (ii) and (iii)). Furthermore, the average runtime results show that though the complexity $\mathfrak{C}(d, N, M_0, M_1)$ is quadratic to the number of samples M_1 (= $3 \cdot 10^4$ in this case), the computation for high dimensional cases (e.g., d = 50, 100) is still feasible under our Monte Carlo based algorithm.

4. Proof of Theorem 2.7

We start by providing some notions. Let $t \in [0,T)$, denote by $C([t,T];\mathbb{R}^d)$ the set of all \mathbb{R}^d -valued continuous functions on [t, T], and set

$$\Omega^t := \{ \omega = (\omega_s)_{s \in [t,T]} \in C([t,T]; \mathbb{R}^d) : \omega_t = 0 \}$$

to be the canonical space of continuous paths. Let $W^t := (W^t_s)_{s \in [t,T]}$ be the canonical process on Ω^t and $\mathbb{F}^{W^t} := (\mathcal{F}^{W^t}_s)_{s \in [t,T]}$ be the raw filtration generated by W^t . We equip Ω^t with the uniform convergence norm so that the Borel σ -field \mathcal{F}^t on Ω^t coincides with $\mathcal{F}_T^{W^t}$. Furthermore,

let \mathbb{P}_0^t be the Wiener measure under which W^t is a Brownian motion and write $\mathbb{E}^{\mathbb{P}_0^t}[\cdot]$ for the expectation under \mathbb{P}_0^t .

On $(\Omega^t, \mathcal{F}^t, \mathbb{F}^{W^t}, \mathbb{P}^t_0)$, consider $X^{t,x;o} := (X^{t,x;o}_s)_{s \in [t,T]}$ following the baseline coefficients b^o and σ^o and starting with $x \in \mathbb{R}^d$, i.e. for $s \in [t,T]$,

$$(4.1) X_{\circ}^{t,x;o} = x + b^{o}(s-t) + \sigma^{o}W_{\circ}^{t}.$$

Moreover, let $\mathbb{L}^{t,1}(\mathbb{R}^d)$ and $\mathbb{L}^{t,1}_{\mathrm{F}}(\mathbb{R}^{d\times d})$ be the set of all \mathbb{F}^{W^t} -predictable processes L defined on [t,T] with values in \mathbb{R}^d and $\mathbb{R}^{d\times d}$, respectively. We endow $\mathbb{L}^{t,1}(\mathbb{R}^d)$ and $\mathbb{L}^{t,1}_{\mathrm{F}}(\mathbb{R}^{d\times d})$ with the norms, respectively,

$$\|L\|_{\mathbb{L}^{t,1}}:=\mathbb{E}^{\mathbb{P}^t_0}\left[\int_t^T|L_s|ds\right],\qquad \|L\|_{\mathbb{L}^{t,1}_{\mathrm{F}}}:=\mathbb{E}^{\mathbb{P}^t_0}\left[\int_t^T\|L_s\|_{\mathrm{F}}ds\right].$$

In analogy, we define $\mathbb{L}^{t,\infty}(\mathbb{R}^d)$ as the set of all \mathbb{R}^d -valued, \mathbb{F}^{W^t} -predictable processes L defined on [t,T] that are bounded $\mathbb{P}^t_0 \otimes ds$ -a.e.. Finally, set

$$||L||_{\mathbb{L}^{t,\infty}} := \inf \{ C \ge 0 : |L_s| \le C \mathbb{P}_0^t \otimes ds$$
-a.e. $\} < \infty$.

The space $\mathbb{L}_{\mathrm{F}}^{t,\infty}(\mathbb{R}^{d\times d})$ of $\mathbb{R}^{d\times d}$ -valued processes is defined analogously to $\mathbb{L}^{t,\infty}(\mathbb{R}^d)$, with $|\cdot|$ replaced by $\|\cdot\|_{\mathrm{F}}$ in the definition of $\|L\|_{\mathbb{L}_{\mathrm{F}}^{t,\infty}}$.

For any $(b, \sigma) \in \mathbb{L}^{t, \infty}(\mathbb{R}^d) \times \mathbb{L}^{t, \infty}_{F}(\mathbb{R}^{d \times d})$, we define an Itô $(\mathbb{F}^{W^t}, \mathbb{P}^t_0)$ -semimartingal $X^{t, x; b, \sigma} = (X^{t, x; b, \sigma}_s)_{s \in [t, T]}$ starting with $x \in \mathbb{R}^d$ by

$$(4.2) X_s^{t,x;b,\sigma} := x + \int_t^s b_u du + \int_t^s \sigma_u dW_u^t, \quad s \in [t,T],$$

and note that $X_s^{t,x;o} = X_s^{t,x;b^o,\sigma^o}$ $s \in [t,T]$; see (4.1). Moreover, for any $\varepsilon \geq 0$ and $t \in [0,T)$, denote by

$$(4.3) \mathcal{C}^{\varepsilon}(t) := \left\{ (b, \sigma) \in \mathbb{L}^{t, \infty}(\mathbb{R}^d) \times \mathbb{L}^{t, \infty}_{\mathbb{F}}(\mathbb{R}^{d \times d}) \mid (b_s, \sigma_s) \in \mathcal{B}^{\varepsilon} \ \mathbb{P}^t_0 \otimes ds \text{-a.e.} \right\}$$

the set of all \mathbb{F}^{W^t} -predictable processes taking values within the ε -neighborhood $\mathcal{B}^{\varepsilon}$ of the baseline coefficients (b^o, σ^o) given in (1.3).

Let us start by providing some a priori estimates for $X_T^{t,x;b,\sigma}$.

Lemma 4.1. For every $p \ge 1$, there is a constant $C_p > 0$ such that the following holds:

- (i) For every $(t,x) \in [0,T) \times \mathbb{R}^d$, we have that $\mathbb{E}^{\mathbb{P}_0^t}[|X_T^{t,x;o}|^p] \leq C_p(1+|x|^p)$.
- (ii) For every $\varepsilon \geq 0$ and $(t,x) \in [0,T) \times \mathbb{R}^d$, we have that

$$\sup_{(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}^t_0} \left[\left| X_T^{t,x;b,\sigma} - X_T^{t,x;o} \right|^p \right] \leq C_p \varepsilon^p.$$

Proof. We only prove (ii), as the proof for (i) follows the same line of reasoning.

Fix $\varepsilon \geq 0$ and $(t,x) \in [0,T) \times \mathbb{R}^d$, and let $(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)$. Then

$$X_T^{t,x;b,\sigma} - X_T^{t,x;o} = \int_t^T (b_s - b^o) ds + \int_t^T (\sigma_s - \sigma^o) dW_s^t.$$

We estimate both terms separately. By Jensen's inequality and the definition of $\mathcal{C}^{\varepsilon}(t)$,

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\left| \int_t^T (b_s - b^o) ds \right|^p \right] \le \mathbb{E}^{\mathbb{P}_0^t} \left[(T - t)^{p-1} \int_t^T |b_s - b^o|^p ds \right] \le (T - t)^p \varepsilon^p.$$

Moreover, if $c_{\text{BDG},p} > 0$ denotes the constant appearing in the Burkholder-Davis-Gundy (BDG) inequality (see, e.g, [22, Theorem 92, Chap. VII]), then

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\left| \int_t^T (\sigma_s - \sigma^o) dW_s^t \right|^p \right] \le c_{\mathrm{BDG},p} \mathbb{E}^{\mathbb{P}_0^t} \left[\left(\int_t^T \|\sigma_s - \sigma^o\|_{\mathrm{F}}^2 ds \right)^{p/2} \right]$$

$$\le c_{\mathrm{BDG},p} (T - t)^{p/2} \varepsilon^p,$$

where the second inequality follows from the definition of $C^{\varepsilon}(t)$. Thus the proof is completed using the elementary inequality $(a+b)^p \leq 2^p(a^p+b^p)$ for all $a,b \geq 0$.

Remark 4.2. By Assumption 2.1, the (weak) Hessian $D_x^2 f$ has at most polynomial growth of order α . In particular, there is a constant $\widetilde{C}_f > 0$ (that depends on C_f in Assumption 2.1) such that for every $(x,y) \in \mathbb{R}^d$,

$$|f(y) - f(x) - \nabla_x^{\top} f(x)(x - y)| \le \widetilde{C}_f (1 + |x|^{\alpha} + |y|^{\alpha}) \cdot |y - x|^2$$

Moreover, $\nabla_x f$ and f have at most polynomial growth of of order $\alpha+1$ and $\alpha+2$, respectively. Next note that if $g: \mathbb{R}^d \to \mathbb{R}$ is any function with at most polynomial growth, then Lemma 4.1 implies that $g(X_T^{t,x;b,\sigma})$ is integrable for every $(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)$. Therefore $f(X_T^{t,x;b,\sigma})$, $\partial_{x_i} f(X_T^{t,x;b,\sigma})$, $|\partial_{x_i} f(X_T^{t,x;b,\sigma})|^2$, ... are integrable.

Lemma 4.3. Suppose that Assumptions 2.1 and 2.2 are satisfied. For i = 1, ..., d, let $w^i : [t, T] \times \mathbb{R}^d \to \mathbb{R}$ be the unique strong solution of (2.3) with polynomial growth (see Proposition 2.6 (ii)). Let $w : [t, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $J_x w : [t, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be given in (2.4), let $(t, x) \in [0, T] \times \mathbb{R}^d$, and set

$$(4.4) Y_s^{t,x} := w\left(s, X_s^{t,x;o}\right), Z_s^{t,x} := J_x w\left(s, X_s^{t,x;o}\right) \sigma^o, \quad s \in [t, T].$$

Then, for every $i = 1, \ldots, d$,

$$Y_s^{t,x,i} = \partial_{x_i} f(X_T^{t,x,o}) - \int_s^T (Z_r^{t,x,i})^\top dW_r^t, \qquad s \in [t,T],$$

with $Y^{t,x,i}$ and $(Z^{t,x,i})^{\top}$ denoting the *i*-th component of $Y^{t,x}$ and the *i*-th row vector of $Z^{t,x}$, respectively. In particular, $Y_s^{t,x} = \mathbb{E}^{\mathbb{P}_0^t}[\nabla_x f(X_T^{t,x,o})|\mathcal{F}_s^{W^t}]$ for every $s \in [t,T]$.

Proof of Lemma 4.3. Fix $i \in \{1, ..., d\}$. Since $w^i \in C^{1,2}([t,T) \times \mathbb{R}^d)$ (see Proposition 2.6 (ii) and Section 5.3), an application of Itô's formula ensures that for every $s \in [t,T]$,

$$\begin{split} w^i(T, X_T^{t,x;o}) - w^i(s, X_s^{t,x;o}) &= \int_s^T \nabla_x^\top w^i(r, X_r^{t,x;o}) \; \sigma^o dW_r^t \\ &+ \int_s^T \left(\partial_r w^i(r, X_r^{t,x;o}) + \langle b^o, \nabla_x w^i(r, X_r^{t,x;o}) \rangle + \frac{1}{2} \operatorname{tr} \left((\sigma^o)(\sigma^o)^\top D_x^2 w^i(r, X_r^{t,x;o}) \right) \right) dr. \end{split}$$

The second integral is equal to zero because w^i solves the linear Kolmogorov PDE given in (2.3). Therefore, using the boundary condition $w^i(T,\cdot) = \partial_{x_i} f(\cdot)$ and the definitions of $Y^{t,x}$ and $Z^{t,x}$ given in (4.4), we conclude that for every $s \in [t,T]$,

$$(4.5) Y_s^{t,x,i} = \partial_{x_i} f(X_T^{t,x,o}) - \int_s^T (Z_u^{t,x,i})^\top dW_u^t,$$

as claimed.

The 'in particular' part follows by taking conditional expectations in (4.5). Indeed, by Remark 4.2, $\partial_{x_i} f(X_T^{t,x;o})$ and $Y_t^{t,x} = w(t,x)$ are square integrable (because $\partial_{x_i} f$ and w have

polynomial growth). Therefore, it follows from (4.5) that $\int_{\cdot}^{T} (Z_{s}^{t,x,i})^{\top} dW_{s}^{t}$ is a square integrable martingale. Furthermore, since $Y_{s}^{t,x,i} = w^{i}(s, X_{s}^{t,x,o})$ is $\mathcal{F}_{s}^{w^{t}}$ -measurable,

$$\begin{split} Y_s^{t,x,i} &= \mathbb{E}^{\mathbb{P}_0^t} \left[Y_s^{t,x,i} \middle| \mathcal{F}_s^{W^t} \right] = \mathbb{E}^{\mathbb{P}_0^t} \left[\partial_{x_i} f(X_T^{t,x;o}) - \int_s^T (Z_r^{t,x,i})^\top dW_r^t \middle| \mathcal{F}_s^{W^t} \right] \\ &= \mathbb{E}^{\mathbb{P}_0^t} \left[\partial_{x_i} f(X_T^{t,x;o}) \middle| \mathcal{F}_s^{W^t} \right], \end{split}$$

as claimed. \Box

For sufficiently integrable \mathbb{R}^d -valued processes $L=(L_s)_{s\in[t,T]}$ and $M=(M_s)_{s\in[t,T]}$, set

$$\langle L, M \rangle_{\mathbb{P}^t_0 \otimes ds} := \mathbb{E}^{\mathbb{P}^t_0} \left[\int_t^T \langle L_s, M_s \rangle ds \right].$$

In a similar manner, we set $\langle L, M \rangle_{\mathbb{P}_0^t \otimes ds, F} := \mathbb{E}^{\mathbb{P}_0^t} [\int_t^T \langle L_s, M_s \rangle_F ds]$ for $\mathbb{R}^{d \times d}$ -valued processes.

Lemma 4.4. Suppose that Assumptions 2.1 and 2.2 are satisfied and let $Y^{t,x}, Z^{t,x}$ be the processes defined in (4.4). Then, for every $\varepsilon \geq 0$ and $(b, \sigma) \in C^{\varepsilon}(t)$, we have that

$$(4.6) \qquad \mathbb{E}^{\mathbb{P}_0^t} \left[\nabla_x^\top f(X_T^{t,x;o}) \left(X_T^{t,x;b,\sigma} - X_T^{t,x;o} \right) \right] = \langle Y^{t,x}, b - b^o \rangle_{\mathbb{P}_0^t \otimes ds} + \langle Z^{t,x}, \sigma - \sigma^o \rangle_{\mathbb{P}_0^t \otimes ds, F}.$$

Proof. For $i=1,\ldots,d$, denote by $(b^i_s-b^{o,i})_{s\in[t,T]}$ and $(\sigma^i_s-\sigma^{o,i})_{s\in[t,T]}$ the *i*-th component of $b-b^o$ and *i*-th row vector of $\sigma-\sigma^o$, respectively. Using this notation,

$$\nabla_x^{\top} f(X_T^{t,x;o}) \left(X_T^{t,x;b,\sigma} - X_T^{t,x;o} \right)$$

$$= \sum_{i=1}^{d} \left(\partial_{x_i} f(X_T^{t,x;o}) \int_t^T (b_s^i - b^{o,i}) ds + \partial_{x_i} f(X_T^{t,x;o}) \int_t^T (\sigma_s^i - \sigma^{o,i}) dW_s^t \right) =: \sum_{i=1}^{d} \left(\Xi^{b,i} + \Xi^{\sigma,i} \right) ds$$

It follows from Remark 4.2 that $\Xi^{b,i}, \Xi^{\sigma,i}$ are integrable (noting that $b-b^0$ and $\sigma-\sigma^0$ are bounded uniformly). In particular,

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\nabla_x^\top f(X_T^{t,x;o}) (X_T^{t,x;b,\sigma} - X_T^{t,x;o}) \right] = \sum_{i=1}^d \left(\mathbb{E}^{\mathbb{P}_0^t} \left[\Xi^{b,i} \right] + \mathbb{E}^{\mathbb{P}_0^t} \left[\Xi^{\sigma,i} \right] \right)$$

and it remains to show that

$$\mathbb{E}^{\mathbb{P}^t_0}[\Xi^{b,i}] = \langle Y^{t,x,i}, b^i - b^{o,i} \rangle_{\mathbb{P}^t_0 \otimes ds} \quad \text{and} \quad \mathbb{E}^{\mathbb{P}^t_0}[\Xi^{\sigma,i}] = \langle Z^{t,x,i}, (\sigma^i - \sigma^{o,i})^\top \rangle_{\mathbb{P}^t_0 \otimes ds}$$

for every i = 1, ..., d. To that end, fix such i.

We first claim that $\mathbb{E}^{\mathbb{P}_0^t}[\Xi^{b,i}] = \langle Y^{t,x,i}, b^i - b^{o,i} \rangle_{\mathbb{P}_0^t \otimes ds}$. Indeed, an application of Fubini's theorem shows that

$$\begin{split} \mathbb{E}^{\mathbb{P}_0^t} \left[\Xi^{b,i} \right] &= \int_t^T \mathbb{E}^{\mathbb{P}_0^t} \left[\mathbb{E}^{\mathbb{P}_0^t} \left[\partial_{x_i} f(X_T^{t,x;o}) | \mathcal{F}_s^{W^t} \right] (b_s^i - b^{o,i}) \right] ds \\ &= \mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^T Y_s^{t,x,i} (b_s^i - b^{o,i}) ds \right] = \langle Y^{t,x,i}, b^i - b^{o,i} \rangle_{\mathbb{P}_0^t \otimes ds}, \end{split}$$

where the second inequality holds because $Y_s^{t,x,i} = \mathbb{E}^{\mathbb{P}_0^t}[\partial_{x_i} f(X_T^{t,x;o}) | \mathcal{F}_s^{W^t}]$, see Lemma 4.3.

Next, we claim that

$$(4.7) \qquad \mathbb{E}^{\mathbb{P}_0^t}[\Xi^{\sigma,i}] = \mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^T (Z_s^{t,x,i})^\top (\sigma_s^i - \sigma^{o,i})^\top ds \right] = \langle Z^{t,x,i}, (\sigma^i - \sigma^{o,i})^\top \rangle_{\mathbb{P}_0^t \otimes ds}.$$

Note that by Lemma 4.3,

$$\mathbb{E}^{\mathbb{P}^t_0}\left[\Xi^{\sigma,i}\right] = \mathbb{E}^{\mathbb{P}^t_0}\left[\left(\int_t^T (Z^{t,x,i}_s)^\top dW^t_s + Y^{t,x,i}_t\right) \int_t^T (\sigma^i_s - \sigma^{o,i}) dW^t_s\right]$$

and by the Itô-isometry,

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^T (Z_s^{t,x,i})^\top dW_s^t \int_t^T (\sigma_s^i - \sigma^{o,i}) dW_s^t \right] = \mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^T (Z_s^{t,x,i})^\top (\sigma_s^i - \sigma^{o,i})^\top ds \right]$$
$$= \langle Z^{t,x,i}, (\sigma^i - \sigma^{o,i})^\top \rangle_{\mathbb{P}_0^t \otimes ds}.$$

Moreover, since $Y_t^{t,x,i} = w^i(t,x)$ (see (4.4) given in Lemma 4.3).

$$\mathbb{E}^{\mathbb{P}^t_0}\left[Y^{t,x,i}_t\int_t^T(\sigma^i_s-\sigma^{o,i})dW^t_s\right]=Y^{t,x,i}_t\,\mathbb{E}^{\mathbb{P}^t_0}\left[\int_t^T(\sigma^i_s-\sigma^{o,i})dW^t_s\Big|\mathcal{F}^{W^t}_t\right]=0$$

and (4.7) follows.

In Section 5.4, we shall show that if Assumptions 2.1, 2.2, and 2.4 are satisfied, then the unique viscosity solution v^{ε} of (2.1) satisfies the following: For all $\varepsilon < \lambda_{\min}(\sigma^{o})$ and $(t, x) \in [0, T) \times \mathbb{R}^{d}$, we have that

(4.8)
$$v^{\varepsilon}(t,x) = \sup_{(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_0^t} \left[f\left(X_T^{t,x;b,\sigma}\right) \right],$$

with $v^{\varepsilon}(T,\cdot) = f(\cdot)$, see Lemma 5.6 and Proposition 5.9. The formula for v^{ε} given in (4.8) will be crucial in the following proof.

Lemma 4.5. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied and, for every $(t,x) \in [0,T) \times \mathbb{R}^d$, let $Y^{t,x}, Z^{t,x}$ be the processes defined in (4.4). Moreover, let α be as in Assumption 2.1. Then, there exists a constant c independent of t,x,ε such that for every $\varepsilon < \min\{1, \lambda_{\min}(\sigma^0)\}$ and $(t,x) \in [0,T) \times \mathbb{R}^d$, we have that

$$\left| v^{\varepsilon}(t,x) - \left(v^{0}(t,x) + \sup_{(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)} \left(\langle Y^{t,x}, b - b^{o} \rangle_{\mathbb{P}_{0}^{t} \otimes ds} + \langle Z^{t,x}, \sigma - \sigma^{o} \rangle_{\mathbb{P}_{0}^{t} \otimes ds, \mathbf{F}} \right) \right) \right|$$

$$\leq c(1 + |x|^{\alpha}) \varepsilon^{2}.$$

Proof. Fix ε as in the lemma and recall the formula for v^{ε} given in (4.8); in particular

$$v^{0}(t,x) = \mathbb{E}^{\mathbb{P}_{0}^{t}} \left[f\left(X_{T}^{t,x;o}\right) \right].$$

Next, using Remark 4.2, for any $(b, \sigma) \in C^{\varepsilon}(t)$,

$$\left| f(X_T^{t,x;b,\sigma}) - f(X_T^{t,x;o}) - \nabla_x^\top f(X_T^{t,x;o}) \left(X_T^{t,x;b,\sigma} - X_T^{t,x;o} \right) \right|$$

$$\leq \widetilde{C}_f \cdot \left(1 + \left| X_T^{t,x;b,\sigma} \right|^{\alpha} + \left| X_T^{t,x;o} \right|^{\alpha} \right) \cdot \left| X_T^{t,x;b,\sigma} - X_T^{t,x;o} \right|^2 =: \mathbf{I}^{b,\sigma}.$$

We claim that there is c>0 that depends only on α , \widetilde{C}_f (see Remark 4.2) such that

$$\sup_{(b,\sigma)\in\mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_0^t}[\mathbf{I}^{b,\sigma}] \le c(1+|x|^{\alpha})\varepsilon^2.$$

To that end, an application of the Cauchy-Schwartz inequality together with the elementary inequality $(1+a+b)^2 \le 3^2(1+a^2+b^2)$ for all $a,b \ge 0$ shows that

$$\mathbb{E}^{\mathbb{P}^t_0}\left[\mathbf{I}^{b,\sigma}\right] \leq \widetilde{C}_f 3 \cdot \mathbb{E}^{\mathbb{P}^t_0}\left[1 + \left|X_T^{t,x;b,\sigma}\right|^{2\alpha} + \left|X_T^{t,x;o}\right|^{2\alpha}\right]^{1/2} \mathbb{E}^{\mathbb{P}^t_0}\left[\left|X_T^{t,x;b,\sigma} - X_T^{t,x;o}\right|^4\right]^{1/2}.$$

Moreover, we have by Lemma 4.1 that

$$\sup_{(b,\sigma)\in\mathcal{C}^{\varepsilon}(t)}\mathbb{E}^{\mathbb{P}_{0}^{t}}\left[\left|X_{T}^{t,x;b,\sigma}-X_{T}^{t,x;o}\right|^{4}\right]^{1/2}\leq C_{4}^{1/2}\varepsilon^{2},$$

where C_4 is the constant appearing in Lemma 4.1. Furthermore, as $\varepsilon < 1$, another application of Lemma 4.1 together with the inequality $(a+b)^{2\alpha} \le 2^{2\alpha}(a^{2\alpha}+b^{2\alpha})$ for all $a,b \ge 0$ implies that

$$\sup_{(b,\sigma)\in\mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}} \left[1 + \left| X_{T}^{t,x;b,\sigma} \right|^{2\alpha} + \left| X_{T}^{t,x;o} \right|^{2\alpha} \right]^{1/2} \\
\leq \sup_{(b,\sigma)\in\mathcal{C}^{\varepsilon}(t)} \left(1 + 2^{2\alpha} \mathbb{E}^{\mathbb{P}_{0}^{t}} \left[\left| X_{T}^{t,x;b,\sigma} - X_{T}^{t,x;o} \right|^{2\alpha} \right] + (2^{2\alpha} + 1) \mathbb{E}^{\mathbb{P}_{0}^{t}} \left[\left| X_{T}^{t,x;o} \right|^{2\alpha} \right] \right)^{1/2} \\
\leq \left(1 + 2^{2\alpha} C_{2\alpha} + (2^{2\alpha} + 1) C_{2\alpha} (1 + |x|^{2\alpha}) \right)^{1/2} \leq \left(1 + 2^{2\alpha + 1} C_{2\alpha} + C_{2\alpha} \right)^{1/2} (1 + |x|^{\alpha}),$$

where $C_{2\alpha}$ is the constant appearing in Lemma 4.1. Our claim follows by setting $c:=\widetilde{C}_f 3C_4^{1/2}(1+2^{2\alpha+1}C_{2\alpha}+C_{2\alpha})^{1/2}$.

Finally, combining all the previous estimates we conclude that

$$\left| v^{\varepsilon}(t,x) - \left(v^{0}(t,x) + \sup_{(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}\left[\nabla_{x}^{\top} f\left(X_{T}^{t,x;o}\right) \left(X_{T}^{t,x;b,\sigma} - X_{T}^{t,x;o}\right) \right] \right) \right| \leq c(1 + |x|^{\alpha}) \varepsilon^{2}.$$

Thus the proof is completed by an application of Lemma 4.4.

Lemma 4.6. Suppose that Assumptions 2.1 and 2.2 are satisfied. Then, for every $\varepsilon \geq 0$ and $(t,x) \in [0,T) \times \mathbb{R}^d$,

$$\sup_{(b,\sigma)\in\mathcal{C}^{\varepsilon}(t)} \left(\langle Y^{t,x}, b-b^{o} \rangle_{\mathbb{P}_{0}^{t}\otimes ds} + \langle Z^{t,x}, \sigma-\sigma^{o} \rangle_{\mathbb{P}_{0}^{t}\otimes ds,\mathcal{F}} \right) = \varepsilon \cdot \left(\gamma \|Y^{t,x}\|_{\mathbb{L}^{t,1}} + \eta \|Z^{t,x}\|_{\mathbb{L}^{t,1}_{\mathcal{F}}} \right).$$

Proof. Set

$$\Phi(\varepsilon) := \sup_{(b,\sigma) \in \mathcal{C}^{\varepsilon}(t)} \left(\langle Y^{t,x}, b - b^o \rangle_{\mathbb{P}^t_0 \otimes ds} + \langle Z^{t,x}, \sigma - \sigma^o \rangle_{\mathbb{P}^t_0 \otimes ds, \mathcal{F}} \right).$$

We first claim that for every $\varepsilon > 0$,

(4.9)
$$\Phi(\varepsilon) \leq \varepsilon \cdot \left(\gamma \| Y^{t,x} \|_{\mathbb{L}^{t,1}} + \eta \| Z^{t,x} \|_{\mathbb{L}^{t,1}_{F}} \right).$$

To that end, set

$$C_1^{\varepsilon}(t) := \{b : (b, \sigma) \in C^{\varepsilon}(t)\}, \qquad C_2^{\varepsilon}(t) := \{\sigma : (b, \sigma) \in C^{\varepsilon}(t)\}$$

so that $C^{\varepsilon}(t) = C_1^{\varepsilon}(t) \times C_2^{\varepsilon}(t)$. Using the Cauchy-Schwartz inequality in \mathbb{R}^d and Hölder's inequality (with exponents 1 and ∞),

$$\sup_{b \in \mathcal{C}_1^\varepsilon(t)} \langle Y^{t,x}, b - b^o \rangle_{\mathbb{P}_0^t \otimes ds} \leq \sup_{b \in \mathcal{C}_1^\varepsilon(t)} \mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^T |Y_s^{t,x}| |b_s - b^o| ds \right] \leq \|Y^{t,x}\|_{\mathbb{L}^{t,1}} \ \varepsilon \gamma.$$

In a similarly manner,

$$\sup_{\sigma \in \mathcal{C}^{\varepsilon}_{2}(t)} \langle Z^{t,x}, \sigma - \sigma^{o} \rangle_{\mathbb{P}^{t}_{0} \otimes ds, \mathcal{F}} \leq \sup_{\sigma \in \mathcal{C}^{\varepsilon}_{2}(t)} \mathbb{E}^{\mathbb{P}^{t}_{0}} \left[\int_{t}^{T} \|Z^{t,x}_{s}\|_{\mathcal{F}} \|\sigma_{s} - \sigma^{o}\|_{\mathcal{F}} ds \right] \leq \|Z^{t,x}\|_{\mathbb{L}^{t,1}_{\mathcal{F}}} \; \varepsilon \eta.$$

The combination of these two estimates shows (4.9).

Next we claim that for every $\varepsilon \geq 0$,

(4.10)
$$\Phi(\varepsilon) \ge \varepsilon \cdot \left(\gamma \| Y^{t,x} \|_{\mathbb{L}^{t,1}} + \eta \| Z^{t,x} \|_{\mathbb{L}^{t,1}_{F}} \right).$$

To that end, fix $\varepsilon \geq 0$. Define $\tilde{\sigma}^* \in \mathbb{L}_F^{t,\infty}$ by

$$\tilde{\sigma}_{s}^{*} := \begin{cases} \frac{Z_{s}^{t,x}}{\|Z_{s}^{t,x}\|_{F}} & \text{if } \|Z_{s}^{t,x}\|_{F} > 0; \\ 0 & \text{else,} \end{cases}$$

which satisfies $\|\tilde{\sigma}^*\|_{\mathbb{L}^{t,\infty}_F} \leq 1$ and $\langle Z_s^{t,x}, \tilde{\sigma}_s^* \rangle_F = \|Z_s^{t,x}\|_F$. This implies that

$$(4.11) \qquad \|Z^{t,x}\|_{\mathbb{L}^{t,1}_{\mathrm{F}}} = \mathbb{E}^{\mathbb{P}^t_0} \left[\int_t^T \|Z^{t,x}_s\|_{\mathrm{F}} ds \right] = \mathbb{E}^{\mathbb{P}^t_0} \left[\int_t^T \langle Z^{t,x}_s, \tilde{\sigma}^*_s \rangle_{\mathrm{F}} ds \right] = \langle Z^{t,x}, \tilde{\sigma}^* \rangle_{\mathbb{P}^t_0 \otimes ds, \mathrm{F}}.$$

In a similar manner, we can construct some $\tilde{b}^* \in \mathbb{L}^{t,\infty}(\mathbb{R}^d)$ that satisfies $\|\tilde{b}^*\|_{\mathbb{L}^{t,\infty}} \leq 1$ and

Now define

$$(b^*, \sigma^*) := \left(b^o + \varepsilon \gamma \tilde{b}^*, \sigma^o + \varepsilon \eta \tilde{\sigma}^*\right) \in \mathcal{C}^{\varepsilon}(t).$$

Then, by (4.12) and (4.11),

$$\begin{split} \Phi(\varepsilon) &\geq \langle Y^{t,x}, b^* - b^o \rangle_{\mathbb{P}^t_0 \otimes ds} + \langle Z^{t,x}, \sigma^* - \sigma^o \rangle_{\mathbb{P}_0 \otimes_t ds, \mathbf{F}} \\ &= \varepsilon \cdot \left(\gamma \langle Y^{t,x}, \tilde{b}^* \rangle_{\mathbb{P}^t_0 \otimes ds} + \eta \langle Z^{t,x}, \tilde{\sigma}^* \rangle_{\mathbb{P}^t_0 \otimes ds, \mathbf{F}} \right) \\ &= \varepsilon \cdot \left(\gamma \|Y^{t,x}\|_{\mathbb{L}^{t,1}} + \eta \|Z^{t,x}\|_{\mathbb{L}^{t,1}_{\mathbf{F}}} \right). \end{split}$$

This shows (4.10), completing the proof.

Proof of Theorem 2.7. Fix $(t,x) \in [0,T) \times \mathbb{R}^d$ and let $(Y^{t,x},Z^{t,x})$ be the processes defined in (4.4), that is,

$$Y_s^{t,x} = w\left(s, X_s^{t,x;o}\right), \qquad Z_s^{t,x} = \mathbf{J}_x w\left(s, X_s^{t,x;o}\right) \sigma^o, \quad s \in [t,T].$$

Then, by Lemmas 4.5 and 4.6, we have for every $\varepsilon < \min\{1, \lambda_{\min}(\sigma^0)\}$ that

$$\left| v^{\varepsilon}(t,x) - \left(v^{0}(t,x) + \varepsilon \cdot \left(\gamma \| Y^{t,x} \|_{\mathbb{L}^{t,1}} + \eta \| Z^{t,x} \|_{\mathbb{L}^{t,1}_{\mathbf{F}}} \right) \right) \right| \leq c(1 + |x|^{\alpha}) \varepsilon^{2},$$

where c>0 is the constant (that is independent of t,x,ε) appearing in Lemma 4.5. The proof follows from the definitions of the norms on $\mathbb{L}^{t,1}(\mathbb{R}^d)$ and $\mathbb{L}^{t,1}_{\mathrm{F}}(\mathbb{R}^{d\times d})$, and since the law of $(X^{t,x,o}_s)_{s\in[t,T]}$ under \mathbb{P}^t_0 is equal to the conditional law of $(x+X^o_s)_{s\in[t,T]}$ under \mathbb{P} given $X^o_t=0$, where $(X^o_t)_{t\in[0,T]}$ is the process defined in Theorem 2.7.

5. Weak and strong formulation of nonlinear Kolmogorov PDE

5.1. Semimartingale measures. In this section we adopt a framework for semimartingale uncertainty introduced by [68,69]. For any $(t,x) \in [0,T) \times \mathbb{R}^d$, denote by

$$\Omega^{t,x} := \left\{ \omega = (\omega_s)_{s \in [t,T]} \in C([t,T]; \mathbb{R}^d) : \omega_t = x \right\}$$

under which $X^t := (X_s^t)_{s \in [t,T]}$ is the corresponding canonical process starting in x. Furthermore, let $\mathbb{F}^{X^t} := (\mathcal{F}_s^{X^t})_{s \in [t,T]}$ be the raw filtration generated by X^t . We equip $\Omega^{t,x}$ with the uniform norm $\|\omega\|_{t,\infty} := \max_{t \leq s \leq T} |\omega_s|$ so that the Borel σ -field on $\Omega^{t,x}$ coincides with $\mathcal{F}_T^{X^t}$.

We will simplify notations when t=0 by setting $X:=X^0$ and $\|\omega\|_{\infty}:=\|\omega\|_{0,\infty}$. Then denote by $\mathcal{P}(\Omega^{0,x})$ the set of all Borel probability measures on $\Omega^{0,x}$. For each $p\in\mathbb{N}$, set

(5.1)
$$\mathcal{P}^p(\Omega^{0,x}) := \left\{ \mathbb{P} \in \mathcal{P}(\Omega^{0,x}) \mid \int_{\Omega^{0,x}} \|\omega\|_{\infty}^p \mathbb{P}(d\omega) < \infty \right\}$$

to be the subset of all Borel probability measures on $\Omega^{0,x}$ with finite p-th moment. Furthermore, let $C(\Omega^{0,x};\mathbb{R})$ be the set of all continuous functions from $\Omega^{0,x}$ to \mathbb{R} and set

(5.2)
$$C_p(\Omega^{0,x}; \mathbb{R}) := \left\{ \xi \in C(\Omega^{0,x}; \mathbb{R}) \mid \|\xi\|_{C_p} := \sup_{\omega \in \Omega^{0,x}} \frac{|\xi(\omega)|}{1 + \|\omega\|_{\infty}^p} < \infty \right\}.$$

We equip $\mathcal{P}^p(\Omega^{0,x})$ with the topology τ_p defined as follows: for any $\mathbb{P} \in \mathcal{P}^p(\Omega^{0,x})$ and $(\mathbb{P}^n)_{n \in \mathbb{N}} \subseteq \mathcal{P}^p(\Omega^{0,x})$, we have

$$(5.3) \qquad \mathbb{P}^n \xrightarrow{\tau_p} \mathbb{P} \quad \text{as} \quad n \to \infty \quad \Leftrightarrow \quad \lim_{n \to \infty} \mathbb{E}^{\mathbb{P}^n} \left[\xi \right] = \mathbb{E}^{\mathbb{P}} \left[\xi \right] \quad \text{for all} \quad \xi \in C_p(\Omega^{0,x}; \mathbb{R}).$$

Recalling the set \mathbb{S}^d of all symmetric $d \times d$ matrices, denote by $\mathbb{S}^d_+ \subset \mathbb{S}^d$ the subset of all positive semi-definite matrices. Let \mathcal{P}_{sem} be the set of all $\mathbb{P} \in \mathcal{P}(\Omega^{0,x})$ such that X is a semimartingale on $(\Omega^{0,x}, \mathcal{F}^{0,x}, \mathbb{F}^X, \mathbb{P})$. Moreover, let $\mathcal{P}_{\text{sem}}^{\text{ac}}$ be the subset of all $\mathbb{P} \in \mathcal{P}_{\text{sem}}$ such that \mathbb{P} -a.s.

$$B^{\mathbb{P}} \ll ds$$
, $C^{\mathbb{P}} \ll ds$.

where $B^{\mathbb{P}}$ and $C^{\mathbb{P}}$ denote the finite variation part and quadratic covariation of the local martingale part of X under \mathbb{P} having values in \mathbb{R}^d and \mathbb{S}^d_+ , respectively (i.e., the first and second characteristics of X) and are absolutely continuous with respect to ds on [0,T].

Furthermore, we fix a mapping $\mathbb{S}^d_+ \ni A \to A^{\frac{1}{2}} \in \mathbb{R}^{d \times d}$ so that it is Borel measurable and satisfies $A^{\frac{1}{2}}(A^{\frac{1}{2}})^{\top} = A$ for all $A \in \mathbb{S}^d_+$, (see, e.g., [87, Remarks 1.1 & 2.1]).

5.2. Weak formulation and dynamic programming principle. For any $\varepsilon \geq 0$ and $(t, x) \in [0, T) \times \mathbb{R}^d$, define by

$$\mathcal{P}^{\varepsilon}(t,x) := \Big\{ \mathbb{P} \in \mathcal{P}^{\mathrm{ac}}_{\mathrm{sem}} \ \Big| \ \mathbb{P}(X_{t \wedge \cdot} = x) = 1; \ (b_{s}^{\mathbb{P}}, (c_{s}^{\mathbb{P}})^{\frac{1}{2}}) \in \mathcal{B}^{\varepsilon}$$

$$\text{for } \mathbb{P} \otimes ds\text{-almost every } (\omega, s) \in \Omega^{0,x} \times [t, T] \Big\},$$

where we recall that $\mathcal{B}^{\varepsilon}$ is given in (1.3).

In particular, under any $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)$, the semimartingale X is constant (taking the value x) up to time t and after that time its differential characteristics $b^{\mathbb{P}} := \frac{dB^{\mathbb{P}}}{ds}$, $c^{\mathbb{P}} := \frac{dC^{\mathbb{P}}}{ds}$ satisfy the value constraint as the set $\mathcal{B}^{\varepsilon}$.

Moreover, recall the function f given in (2.1). For any $\varepsilon \geq 0$, we define the value function $v_{\text{weak}}^{\varepsilon} : [0, T] \times \mathbb{R}^{d} \ni (t, x) \to v_{\text{weak}}^{\varepsilon}(t, x) \in \mathbb{R}$ by setting for every $(t, x) \in [0, T) \times \mathbb{R}^{d}$,

$$v_{\text{weak}}^{\varepsilon}(t, x) := \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right]$$

and $v_{\text{weak}}^{\varepsilon}(T,\cdot) := f(\cdot)$ on \mathbb{R}^d .

The following estimate will be used in next lemma.

Lemma 5.1. For every $p \geq 1$ and $\varepsilon \geq 0$, there is a constant $C_{p,\varepsilon} > 0$ such that for every $(t,x) \in [0,T) \times \mathbb{R}^d$ and $s \in [t,T]$,

$$\sup_{\mathbb{P}\in\mathcal{P}^{\varepsilon}(t,x)} \mathbb{E}^{\mathbb{P}} \left[\sup_{t\leq u\leq s} |X_u - x|^p \right] \leq C_{p,\varepsilon} \left((s-t)^{p/2} + (s-t)^p \right).$$

Proof. Fix $\varepsilon \geq 0$, $(t,x) \in [0,T) \times \mathbb{R}^d$, and $s \in [t,T]$, and let $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)$. Then under \mathbb{P} , the process X has the canonical representation

$$X_s = x + \int_t^s b_r^{\mathbb{P}} dr + M_s^{\mathbb{P},t},$$

where $(M_s^{\mathbb{P},t})_{s\in[t,T]}$ denotes $(\mathbb{F}^X,\mathbb{P})$ -local martingale part of $(X_s)_{s\in[t,T]}$ satisfying $M_t^{\mathbb{P},t}=0$ with its differential characteristic $c^{\mathbb{P}}$ satisfying the constraint as $\mathcal{B}^{\varepsilon}$ (see (5.4)).

By Jensen's inequality and the definition of $\mathcal{P}^{\varepsilon}(t,x)$,

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t < u < s} \left| \int_t^u b_r^{\mathbb{P}} dr \right|^p \right] \leq (s-t)^{p-1} \mathbb{E}^{\mathbb{P}}\left[\int_t^s |b_r^{\mathbb{P}}|^p dr \right] \leq 2^p (\varepsilon^p + |b^o|^p) (s-t)^p,$$

where we use the elementary inequality $(a+b)^p \leq 2^p(a^p+b^p)$ for all $a,b\geq 0$.

Moreover, by the Burkholder-Davis-Gundy inequality and the elementary inequality $||AB||_F \le ||A||_F ||B||_F$ for all $A, B \in \mathbb{R}^d$,

(5.6)
$$\mathbb{E}^{\mathbb{P}}\left[\sup_{t\leq u\leq s}\left|M_{u}^{\mathbb{P},t}\right|^{p}\right]\leq c_{\mathrm{BDG},p}\mathbb{E}^{\mathbb{P}}\left[\left(\int_{t}^{s}\|(c_{r}^{\mathbb{P}})^{\frac{1}{2}}\|_{\mathrm{F}}^{2}ds\right)^{p/2}\right]$$
$$\leq c_{\mathrm{BDG},p}\left(2^{2}(\varepsilon^{2}+\|\sigma^{o}\|_{\mathrm{F}}^{2})\right)^{p/2}(s-t)^{p/2}.$$

Our claim follows by using again the inequality $(a+b)^p \leq 2^p(a^p+b^p)$ for all $a,b \geq 0$ and setting $C_{p,\varepsilon} := 2^p \{2^p(\varepsilon^p + |b^o|^p) + c_{\mathrm{BDG},p}(2^2(\varepsilon^2 + \|\sigma^o\|_{\mathrm{F}}^2))^{p/2}\}.$

Remark 5.2. Lemma 5.1 implies that $\mathcal{P}^{\varepsilon}(t,x)$ is a subset of $\mathcal{P}^{p}(\Omega^{0,x})$ for every $\varepsilon \geq 0$ and $p \geq 1$; see (5.1) and (5.4).

Lemma 5.3. Suppose that Assumption 2.1 is satisfied, let $\varepsilon \geq 0$, and let $v_{\text{weak}}^{\varepsilon}$ be defined in (5.5). Moreover, let $(t, x) \in [0, T) \times \mathbb{R}^d$. Then, the following hold:

(i) For any \mathbb{F}^X -stopping time τ taking values in [t,T]

(5.7)
$$v_{\text{weak}}^{\varepsilon}(t, x) = \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}} \left[v_{\text{weak}}^{\varepsilon}(\tau, X_{\tau}) \right].$$

(ii) $v_{\text{weak}}^{\varepsilon}$ is jointly continuous.

Proof. We start by proving the statement (i). We claim that the set

(5.8)
$$\left\{ (\omega, t, \mathbb{P}) \in \Omega^{0,x} \times [0, T] \times \mathcal{P}(\Omega^{0,x}) \mid \mathbb{P} \in \mathcal{P}^{\varepsilon}(t, \omega_t) \right\}$$

is Borel. Indeed, since $\mathcal{B}^{\varepsilon}$ is Borel (see (1.3)) and the map $\mathbb{S}^d_+ \ni A \to A^{\frac{1}{2}} \in \mathbb{R}^{d \times d}$ is Borel-measurable (see Section 5.1), the same arguments presented for the proof of [29, Lemma 3.1] using the existence of a Borel-measurable map from $\Omega^{0,x} \times [0,T] \times \mathcal{P}(\Omega^{0,x})$ to the differential characteristics of X given in [68, Theorem 2.6] ensure the claim to hold.

Furthermore, from [69, Theorem 2.1], the following stability properties of $\mathcal{P}^{\varepsilon}(t,x)$ also hold: for any $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)$ and \mathbb{F}^{X} -stopping time τ having values in [t,T],

- (a) There is a set of conditional probability measures $(\mathbb{P}_{\omega})_{\omega \in \Omega^{0,x}}$ of \mathbb{P} with respect to \mathcal{F}_{τ}^{X} such that $\mathbb{P}_{\omega} \in \mathcal{P}^{\varepsilon}(\tau(\omega), \omega_{\tau(\omega)})$ for \mathbb{P} -almost all $\omega \in \Omega^{0,x}$;
- (b) If there is a set of probability measures $(\mathbb{Q}_{\omega})_{\omega \in \Omega^{0,x}}$ such that $\mathbb{Q}_{\omega} \in \mathcal{P}^{\varepsilon}(\tau(\omega), \omega_{\tau(\omega)})$ for \mathbb{P} almost all $\omega \in \Omega^{0,x}$, and the map $\omega \to \mathbb{Q}_{\omega}$ is \mathcal{F}_{τ}^{X} -measurable, then the probability measure

$$\mathbb{P} \otimes \mathbb{Q}(\cdot) := \int_{\Omega^{0,x}} \mathbb{Q}_{\omega}(\cdot) \mathbb{P}(d\omega)$$

is an element of $\mathcal{P}^{\varepsilon}(t,x)$.

Therefore, an application of [27, Theorem 2.1] (see also [74, Theorem 2.3]) ensures (5.7) to hold.

Now let us prove (ii). Since $v_{\text{weak}}^{\varepsilon}(T,\cdot)=f(\cdot)$ is continuous (by Assumption 2.1), we can and do consider arbitrary $(t,x)\in[0,T)\times\mathbb{R}^d$. The continuity of $v_{\text{weak}}^{\varepsilon}(t,\cdot)$ follows from the definition of $v_{\text{weak}}^{\varepsilon}$ given in (5.5). Indeed, since for every $x,y\in\mathbb{R}^d$

$$v_{\text{weak}}^{\varepsilon}(t,y) = \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,y)} \mathbb{E}^{\mathbb{P}} \Big[f\left(X_{T}\right) \Big] = \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)} \mathbb{E}^{\mathbb{P}} \Big[f\left(X_{T} + y - x\right) \Big],$$

by Remark 4.2 (with the constants $p \ge 1$ and $c_1 > 0$) and the elementary property $(a + b)^p \le 2^p (a^p + b^p)$ for all a, b > 0, we have that

$$\begin{aligned} &|v_{\text{weak}}^{\varepsilon}(t,y) - v_{\text{weak}}^{\varepsilon}(t,x)| \\ &\leq \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)} \mathbb{E}^{\mathbb{P}} \Big[|f(X_T + y - x) - f(X_T)| \Big] \\ &\leq \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)} \Big\{ \mathbb{E}^{\mathbb{P}} \Big[|\nabla_x^{\top} f(X_T)| \Big] \cdot |y - x| + c_1 \cdot 2^p \Big(1 + |y - x|^p + \mathbb{E}^{\mathbb{P}} \Big[|X_T|^p \Big] \Big) \cdot |y - x|^2 \Big\} \,. \end{aligned}$$

From Lemma 5.1 together with the polynomial growth property of $\nabla_x f$, we hence have that there is a constant $c_3 > 0$ (that depends on p, ε, x , but not on t) such that

$$(5.9) |v_{\text{weak}}^{\varepsilon}(t,y) - v_{\text{weak}}^{\varepsilon}(t,x)| \le c_3 \left(|y - x| + |y - x|^{p+2} \right),$$

where we further emphasize that the above estimate holds for every $t \in [0,T)$ and $x,y \in \mathbb{R}^d$.

Now we claim that $v_{\text{weak}}^{\varepsilon}(\cdot, x)$ is continuous. To that end, fix any $0 \le u \le T - t$. By the dynamic programming principle of $v_{\text{weak}}^{\varepsilon}$ (see Lemma 5.3 (i)), the following holds

$$v_{\text{weak}}^{\varepsilon}(t, x) = \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}} \Big[v_{\text{weak}}^{\varepsilon}(t + u, X_{t+u}) \Big].$$

Hence, we use again Lemma 5.1 together with the estimates given in (5.9) to have that

$$\begin{aligned} |v_{\text{weak}}^{\varepsilon}(t,x) - v_{\text{weak}}^{\varepsilon}(t+u,x)| &\leq \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)} \mathbb{E}^{\mathbb{P}} \left[\left| v_{\text{weak}}^{\varepsilon}(t+u,X_{t+u}) - v_{\text{weak}}^{\varepsilon}(t+u,x) \right| \right] \\ &\leq c_{3} \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)} \left(\mathbb{E}^{\mathbb{P}} \left[|X_{t+u} - x| \right] + \mathbb{E}^{\mathbb{P}} \left[|X_{t+u} - x|^{p+2} \right] \right) \\ &\leq c_{3} \cdot \left(C_{1,\varepsilon} \left(u^{1/2} + u \right) + C_{p,\varepsilon} \left(u^{\frac{p+2}{2}} + u^{p+2} \right) \right), \end{aligned}$$

where $C_{1,\varepsilon}, C_{p,\varepsilon}$ are the constant (with exponents 1, p) appearing in Lemma 5.1 (and in particular do not depend on x). Combined with (5.9), this ensures that $v_{\text{weak}}^{\varepsilon}$ is jointly continuous.

- 5.3. **Proof of Proposition 2.6.** Let us introduce the notion of viscosity / strong solution of (2.1) and (2.3). To that end, we introduce the following function spaces: for any $t \in [0, T)$
 - · $C^{1,2}([t,T)\times\mathbb{R}^d;\mathbb{R})$ is the set of all real-valued functions on $[t,T)\times\mathbb{R}^d$ which are continuously differentiable on [t,T) and twice continuously differentiable on \mathbb{R}^d ;
 - · $C_b^{2,3}([t,T)\times\mathbb{R}^d;\mathbb{R})$ is the set of all real-valued functions on $[t,T)\times\mathbb{R}^d$ which have bounded continuous derivatives up to the second and third order on [t,T) and \mathbb{R}^d , respectively.

Definition 5.4 (Viscosity solution (see [20,30])). Fix any $\varepsilon \geq 0$. We call an upper semicontinuous function $v^{\varepsilon}: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ a viscosity subsolution of (2.1) if $v^{\varepsilon}(T,\cdot) \leq f(\cdot)$ on \mathbb{R}^d and

$$-\partial_t \varphi(t,x) - \sup_{(b,\sigma) \in \mathcal{B}^{\varepsilon}} \left\{ \frac{1}{2} \operatorname{tr} \left(\sigma \sigma^{\top} D_{xx}^2 \varphi(t,x) \right) + \langle b, \nabla_x \varphi(t,x) \rangle \right\} \le 0$$

whenever $\varphi \in C_b^{2,3}([0,T) \times \mathbb{R}^d;\mathbb{R})$ is such that $\varphi \geq v^{\varepsilon}$ on $[0,T) \times \mathbb{R}^d$ and $\varphi(t,x) = v^{\varepsilon}(t,x)$. In a similar manner, the notion of a viscosity supersolution can be defined by reversing the inequalities and replacing upper semicontinuity with lower semicontinuity. Finally, we call a continuous function v^{ε} from $[0,T] \times \mathbb{R}^d$ to \mathbb{R} a viscosity solution if it is both sub- and supersolution of (2.1).

Definition 5.5 (Strong solution). Fix $t \in [0, T)$ and i = 1, ..., d. We call a continuous function $w^i : [t, T] \times \mathbb{R}^d \to \mathbb{R}$ a strong solution of (2.3) if it is in $C^{1,2}([t, T) \times \mathbb{R}^d; \mathbb{R})$ and satisfies (2.3).

Lemma 5.6. Suppose that Assumptions 2.1 and 2.4 are satisfied and let $\varepsilon \geq 0$. Then $v_{\text{weak}}^{\varepsilon}$: $[0,T] \times \mathbb{R}^d \to \mathbb{R}$ defined in (5.5) is a unique viscosity solution of (2.1) satisfying (2.2).

Proof. By Lemma 5.3, $v_{\text{weak}}^{\varepsilon}$ satisfies the dynamic programming principle and is jointly continuous. Hence, the same arguments as presented for the proof of [69, Proposition 5.4] ensure that $v_{\text{weak}}^{\varepsilon}$ is a unique viscosity solution of (2.1). Furthermore, as the function f has at most polynomial growth (see Remark 4.2), Lemma 5.1 ensures that $v_{\text{weak}}^{\varepsilon}$ has polynomial growth with respect to $x \in \mathbb{R}^d$ for all $t \in [0, T]$. This implies that $v_{\text{weak}}^{\varepsilon}$ satisfies (2.2) with some C > 0. Hence by Assumption 2.4, $v_{\text{weak}}^{\varepsilon}$ is the unique viscosity solution satisfying (2.2).

Proof of Proposition 2.6. The statement (i) follows directly from Lemma 5.6. Now let us prove (ii). Note that $\nabla_x f$ has at most polynomial growth (see Remark 4.2) and (b^o, σ^o) are constant. Furthermore, σ^o is non-degenerate (see Assumption 2.2). Hence, an application of [57, Theorem 5.7.6 & Remark 5.7.8] (see also [61, Theorem 4.32]) ensures the existence of a strong solution of (2.3). The uniqueness of the solution with polynomial growth is guaranteed by [31, Corollary 6.4.4].

5.4. Strong formulation and its equivalence. In this section, we construct a set of probability measures corresponding to a strong formulation of the nonlinear Kolmogorov PDE given in (2.1).

Recall the process $X^{t,x;b,\sigma}$ defined on [t,T] (given in (4.2)) and denote by $(x \oplus_t X^{t,x;b,\sigma})$ the constant concatenation of $X^{t,x;b,\sigma}$ defined on [0,T], i.e.

$$(5.10) (x \oplus_t X^{t,x;b,\sigma})_s := x \mathbf{1}_{\{s \in [0,t)\}} + X_s^{t,x;b,\sigma} \mathbf{1}_{\{s \in [t,T]\}}.$$

Then using the set $C^{\varepsilon}(t)$ given in (4.3), we define a set of (push-forward) probability measures as follow: for any $\varepsilon \geq 0$ and $(t, x) \in [0, T) \times \mathbb{R}^d$

$$(5.11) \mathcal{Q}^{\varepsilon}(t,x) := \mathcal{Q}(t,x;\mathcal{C}^{\varepsilon}) = \left\{ \mathbb{P}_{0}^{t} \circ \left(x \oplus_{t} X^{t,x;b,\sigma} \right)^{-1} \mid (b,\sigma) \in \mathcal{C}^{\varepsilon}(t) \right\} \subseteq \mathcal{P}(\Omega^{0,x}).$$

Remark 5.7. By the definition of $(x \oplus_t X^{t,x;b,\sigma})$ given in (5.10), $\mathcal{Q}^{\varepsilon}(t,x)$ is a subset of $\mathcal{P}^{\varepsilon}(t,x)$ for every $\varepsilon \geq 0$; see (5.4) for the definition.

Proposition 5.8. Suppose that Assumption 2.2 is satisfied. Let $\varepsilon < \lambda_{\min}(\sigma^o)$ (see Remark 2.3) and $(t,x) \in [0,T) \times \mathbb{R}^d$. Moreover, let $\mathcal{P}^{\varepsilon}(t,x)$ and $\mathcal{Q}^{\varepsilon}(t,x)$ be defined in (5.4) and (5.11), respectively. Then, there exists $\mathcal{Q}^{\varepsilon}_{\text{sub}}(t,x) \subseteq \mathcal{Q}^{\varepsilon}(t,x)$ such that its convex hull is a dense subset of $\mathcal{P}^{\varepsilon}(t,x)$ with respect to the τ_p -topology for all $p \geq 1$.

Recall the function $f: \mathbb{R}^d \to \mathbb{R}$ given in (2.1) and the canonical process $X = (X_s)_{s \in [0,T]}$ defined on $(\Omega^{0,x}, \mathcal{F}^{0,x}, \mathbb{F}^X)$. For any $\varepsilon \geq 0$, we define the value function $v_{\text{strong}}^{\varepsilon} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ by setting for every $(t,x) \in [0,T) \times \mathbb{R}^d$,

$$(5.12) v_{\text{strong}}^{\varepsilon}(t, x) := \sup_{\mathbb{P} \in \mathcal{Q}^{\varepsilon}(t, x)} \mathbb{E}^{\mathbb{P}}\left[f\left(X_{T}\right)\right] = \sup_{(b, \sigma) \in \mathcal{C}^{\varepsilon}(t)} \mathbb{E}^{\mathbb{P}_{0}^{t}}\left[f\left(X_{T}^{t, x; b, \sigma}\right)\right]$$

and $v_{\text{strong}}^{\varepsilon}(T,\cdot) := f(\cdot)$ on \mathbb{R}^d . We call this the 'strong formulation' of modeling uncertainty of X, which will turn out to be equivalent to the weak formulation $v_{\text{weak}}^{\varepsilon}$ in the next proposition.

Proposition 5.9. Suppose that Assumptions 2.1, 2.2, and 2.4 are satisfied and let $v_{\text{weak}}^{\varepsilon}$ and $v_{\text{strong}}^{\varepsilon}$ be defined in (5.5) and (5.12), respectively. Then the following inequalities hold: for any $\varepsilon < \lambda_{\min}(\sigma^o)$ and $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$v_{\text{strong}}^{\varepsilon}(t,x) = v_{\text{weak}}^{\varepsilon}(t,x).$$

5.5. **Proof of Propositions 5.8 and 5.9.** We follow the idea of [24] in order to prove Propositions 5.8 and 5.9. First, we introduce some notions, often employed in this section. Recalling the set $\mathcal{B}^{\varepsilon}$ given in (1.3), we set for any $\varepsilon \geq 0$,

(5.13)
$$\mathcal{B}^{\varepsilon,1} := \{ b : (b,\sigma) \in \mathcal{B}^{\varepsilon} \}, \qquad \mathcal{B}^{\varepsilon,2} := \{ \sigma : (b,\sigma) \in \mathcal{B}^{\varepsilon} \}$$

so that $\mathcal{B}^{\varepsilon} = \mathcal{B}^{\varepsilon,1} \times \mathcal{B}^{\varepsilon,2}$ and denote by

$$(5.14) \Pi_{\mathcal{B}^{\varepsilon,1}}: \mathbb{R}^d \ni x \to \Pi_{\mathcal{B}^{\varepsilon,1}}(x) \in \mathcal{B}^{\varepsilon,1}, \Pi_{\mathcal{B}^{\varepsilon,2}}: \mathbb{R}^{d\times d} \ni x \to \Pi_{\mathcal{B}^{\varepsilon,2}}(x) \in \mathcal{B}^{\varepsilon,2}$$

the Euclidean projections into the convex, closed sets $\mathcal{B}^{\varepsilon,1}$ and $\mathcal{B}^{\varepsilon,2}$ respectively.

For every $n \in \mathbb{N}$ and $t \in [0, T)$, denote by $t_k^n := t + \frac{T - t}{n} k$ for $k = 0, 1, \dots, n$. Furthermore, for any $\mathbb{P} \in \mathcal{P}_{\text{sem}}^{\text{as}}$ denote by $b^{\mathbb{P}} = \frac{dB^{\mathbb{P}}}{ds}$ the first differential characteristics of X under \mathbb{P} , and by

(5.15)
$$\sigma^{\mathbb{P}} := (c^{\mathbb{P}})^{\frac{1}{2}}$$

where $c^{\mathbb{P}} = \frac{dC^{\mathbb{P}}}{ds}$ is the second differential characteristics of X under \mathbb{P} . Then we define by $b^{\mathbb{P},(n)}$ and $\sigma^{\mathbb{P},(n)}$ piecewise constant processes defined on [t,T] such that

$$(5.16) b_s^{\mathbb{P},(n)} := \mathbf{1}_{\{s \in [t,t_1^n]\}} b^o + \sum_{k=1}^{n-1} \mathbf{1}_{\{s \in (t_k^n,t_{k+1}^n]\}} \Pi_{\mathcal{B}^{\varepsilon,1}} \left[\frac{n}{T-t} \int_{t_{k-1}^n}^{t_k^n} b_s^{\mathbb{P}} ds \right],$$

$$\sigma_s^{\mathbb{P},(n)} := \mathbf{1}_{\{s \in [t,t_1^n]\}} \sigma^o + \sum_{k=1}^{n-1} \mathbf{1}_{\{s \in (t_k^n,t_{k+1}^n]\}} \Pi_{\mathcal{B}^{\varepsilon,2}} \left[\frac{n}{T-t} \int_{t_{k-1}^n}^{t_k^n} \sigma_s^{\mathbb{P}} ds \right].$$

Remark 5.10. Fix any $\varepsilon < \lambda_{\min}(\sigma^o)$ and recall $\mathcal{P}^{\varepsilon}(t,x)$ given in (5.4). Then under any $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)$, since $\|\sigma_s^{\mathbb{P}} - \sigma^o\|_{\mathbb{F}} < \lambda_{\min}(\sigma^o) \mathbb{P} \otimes ds$ -a.e., by Remark 2.3, there exists the corresponding inverse matrix $((\sigma_s^{\mathbb{P}})^{-1})_{s \in [t,T]} \mathbb{P} \otimes ds$ -almost every $(\omega,s) \in \Omega^{0,x} \times [t,T]$. Therefore, if we denote by $M^{\mathbb{P},t} := (M_s^{\mathbb{P},t})_{s \in [t,T]}$ the $(\mathbb{F}^X,\mathbb{P})$ -local martingale term of $(X_s)_{s \in [t,T]}$ satisfying $M_t^{\mathbb{P},t} = 0$, an application of Lévy's theorem ensures that for $s \in [t,T]$,

$$(5.17) W_s^{\mathbb{P},t} := \int_t^s (\sigma_u^{\mathbb{P}})^{-1} dM_u^{\mathbb{P},t}$$

is a d-dimensional Brownian motion defined on [t,T] under \mathbb{P} satisfying $W_t^{\mathbb{P},t}=0$.

For every $\varepsilon < \lambda_{\min}(\sigma^o)$ and $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)$, the piecewise constant processes $b^{\mathbb{P},(n)}$ and $\sigma^{\mathbb{P},(n)}$, $n \in \mathbb{N}$, given in (5.16) and the Brownian motion $W^{\mathbb{P},t} = (W_s^{\mathbb{P},t})_{s \in [t,T]}$ given in (5.17) enable to define $X^{\mathbb{P},(n)}$ (that is defined on [0,T]) for every $n \in \mathbb{N}$ by letting

(5.18)
$$X^{\mathbb{P},(n)} := x \oplus_t \left(x + \int_t^{\cdot} b_s^{\mathbb{P},(n)} ds + \int_t^{\cdot} \sigma_s^{\mathbb{P},(n)} dW_s^{\mathbb{P},t} \right).$$

Finally, for any $\mathbb{P} \in \mathcal{P}(\Omega^{0,x})$ denote⁴ by $\mathcal{H}^2(\Omega^{0,x}, \mathcal{F}^{0,x}, \mathbb{F}^X, \mathbb{P})$ the space of all semimartingales S defined on [0,T] such that

(5.19)
$$||S||_{\mathcal{H}_{\mathbb{P}}^2} := \mathbb{E}^{\mathbb{P}} \left[\langle N, N \rangle_T \right]^{1/2} + \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |dA_t| \right)^2 \right]^{1/2} < \infty,$$

where $N = (N_t)_{t \in [0,T]}$ and $A = (A_t)_{t \in [0,T]}$ denote the $(\mathbb{F}^X, \mathbb{P})$ -local martingale and \mathbb{F}^X -predictable finite variation process of S, respectively (i.e., the canocial decomposition).

Lemma 5.11. Suppose that Assumption 2.2 is satisfied. Let $\varepsilon < \lambda_{\min}(\sigma^o)$, $(t, x) \in [0, T) \times \mathbb{R}^d$, and $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$. Let $(X^{\mathbb{P},(n)})_{n \in \mathbb{N}}$ be the sequence defined in (5.18). Then $X^{\mathbb{P},(n)}$ converges to X in $\mathcal{H}^2(\Omega^{0,x}, \mathcal{F}^{0,x}, \mathbb{F}^X, \mathbb{P})$, i.e. as $n \to \infty$

$$||X^{\mathbb{P},(n)} - X||_{\mathcal{H}^2_{\mathbb{P}}} \to 0.$$

Proof. Let $b^{\mathbb{P}}$ be the first differential characteristic of X and $\sigma^{\mathbb{P}}$ be given in (5.15). Using the Brownian motion $W^{\mathbb{P},t}$ defined in (5.17), we have that \mathbb{P} -a.s.

(5.20)
$$X = x \oplus_t \left(x + \int_t^{\cdot} b_s^{\mathbb{P}} ds + \int_t^{\cdot} \sigma_s^{\mathbb{P}} dW_s^{\mathbb{P}, t} \right).$$

That is, the canonical process X can be represented by an Itô $(\mathbb{F}^X, \mathbb{P})$ -semimartingale with constant x-path up to time t.

Recall $\|\cdot\|_{\mathcal{H}^2_{\mathbb{P}}}$ defined in (5.19) and the piecewise constant processes $(b^{\mathbb{P},(n)},\sigma^{\mathbb{P},(n)})_{n\in\mathbb{N}}$ defined in (5.16). By (5.20), Hölder's inequality (with exponent 2) ensures that for every $n\in\mathbb{N}$,

$$(5.21) \quad \left\| X^{\mathbb{P},(n)} - X \right\|_{\mathcal{H}^{2}_{\mathbb{P}}} \leq \mathbb{E}^{\mathbb{P}} \left[\int_{t}^{T} \| \sigma_{s}^{\mathbb{P},(n)} - \sigma_{s}^{\mathbb{P}} \|_{\mathcal{F}}^{2} ds \right]^{\frac{1}{2}} + (T - t) \, \mathbb{E}^{\mathbb{P}} \left[\int_{t}^{T} |b_{s}^{\mathbb{P},(n)} - b_{s}^{\mathbb{P}}|^{2} ds \right]^{\frac{1}{2}}.$$

In particular, by the definition of $\sigma^{\mathbb{P},(n)}$ in (5.16), $\int_t^T \|\sigma_s^{\mathbb{P},(n)} - \sigma_s^{\mathbb{P}}\|_{\mathrm{F}} ds \to 0$ as $n \to \infty$ for every $\omega \in \Omega^{0,x}$. Furthermore, since $\sigma^{\mathbb{P},(n)}$, $\sigma^{\mathbb{P}}$ are uniformly bounded, the dominated convergence theorem implies that the first term of the right hand side of (5.21) vanishes as $n \to \infty$. The same arguments ensure that the second term vanishes. This completes the proof.

⁴See, e.g. [88, Chapter IV.2, p.124] for the definition.

Lemma 5.12. Suppose that Assumption 2.2 is satisfied. Let $\varepsilon < \lambda_{\min}(\sigma^o)$, $(t, x) \in [0, T) \times \mathbb{R}^d$, and $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t, x)$. Let $W^{\mathbb{P}, t}$ and $(X^{\mathbb{P}, (n)})_{n \in \mathbb{N}}$ be given in (5.17) and (5.18). Then for each $n \in \mathbb{N}$ and $p \geq 1$, the law of $X^{\mathbb{P}, (n)}$ is contained in the τ_p -closure of the convex hull of the laws of

$$(5.22) \left\{ x \oplus_t \left(x + \int_t^{\cdot} \mu^u(s, W^{\mathbb{P}, t}) ds + \int_t^{\cdot} \Sigma^v(s, W^{\mathbb{P}, t}) dW_s^{\mathbb{P}, t} \right) \mid (u, v) \in (0, 1)^{nd} \times (0, 1)^{nd^2} \right\},$$

where for every $(u, v) \in (0, 1)^{nd} \times (0, 1)^{nd^2}$, $\mu^u : [t, T] \times \Omega^t \to \mathcal{B}^{\varepsilon, 1}$ and $\Sigma^v : [t, T] \times \Omega^t \to \mathcal{B}^{\varepsilon, 2}$ are adapted Borel functionals on Ω^t .

Proof. Fix $n \in \mathbb{N}$ and denote by $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{t \in [0,T]}, \widehat{\mathbb{P}})$ another filtered probability space which carries a d-dimensional Brownian motion \widehat{W}^t defined on [t,T] satisfying $\widehat{W}^t_t = 0$ and a sequence $\{(U^k, V^k) \mid 1 \leq k \leq n\}$ of $(\mathbb{R}^d, \mathbb{R}^{d \times d})$ -valued random variables such that the components $\{(U^k_i, V^k_{j,l}) \mid 1 \leq k \leq n; 1 \leq i, j, l \leq d\}$ are i.i.d. uniformly distributed on (0,1) and independent of \widehat{W}^t . For notational simplicity, set

$$U := (U^1, \dots, U^n), \qquad V := (V^1, \dots, V^n).$$

Recall $(b^{\mathbb{P},(n)}, \sigma^{\mathbb{P},(n)})$ and $X^{\mathbb{P},(n)}$ given in (5.16) and (5.18). For each $k=1,\ldots,n$, denote by $C([t,t_k^n];\mathbb{R}^d)$ the set of all \mathbb{R}^d -valued, continuous functions on $[t,t_k^n]$ (recalling that $t_k^n=t+\frac{k(T-t)}{n}$ with $k=0,1,\ldots,n$). Then the existence of regular conditional probability distributions guarantees that there exist measurable functions for every $k=1,\ldots,n$

$$\Theta_k^1: C([t, t_k^n]; \mathbb{R}^d) \times (0, 1)^{kd} \to \mathcal{B}^{\varepsilon, 1}, \qquad \Theta_k^2: C([t, t_k^n]; \mathbb{R}^d) \times (0, 1)^{kd^2} \to \mathcal{B}^{\varepsilon, 2},$$

such that the random variables defined by

(5.23)
$$\widehat{b}^{(n)}(k) := \Theta_k^1 \left(\widehat{W}^t |_{[t,t_k^n]}, U^1, \dots, U^k \right), \quad \widehat{\sigma}^{(n)}(k) := \Theta_k^2 \left(\widehat{W}^t |_{[t,t_k^n]}, V^1, \dots, V^k \right)$$
 satisfy

$$(5.24) \quad \text{law of } \left\{ \widehat{W}^t, (\widehat{b}^{(n)}(1), \widehat{\sigma}^{(n)}(1)), \dots, (\widehat{b}^{(n)}(n), \widehat{\sigma}^{(n)}(n)) \right\} \text{ under } \widehat{\mathbb{P}}$$

$$= \text{law of } \left\{ W^{\mathbb{P},t}, (b^{\mathbb{P},(n)}_{t^n}, \sigma^{\mathbb{P},(n)}_{t^n}), \dots, (b^{\mathbb{P},(n)}_{t^n}, \sigma^{\mathbb{P},(n)}_{t^n}) \right\} \text{ under } \mathbb{P}.$$

Now for each $(u, v) := (u^1, \dots, u^n) \times (v^1, \dots, v^n) \in (0, 1)^{nd} \times (0, 1)^{nd^2}$ and $k = 1, \dots, n$, set

$$(5.25) \qquad \widehat{b}^{(n)}(k;u) := \Theta_k^1 \left(\widehat{W}^t|_{[t,t_k^n]}, u^1, \dots, u^k \right), \quad \widehat{\sigma}^{(n)}(k;v) := \Theta_k^2 \left(\widehat{W}^t|_{[t,t_k^n]}, v^1, \dots, v^k \right),$$

and denote by $\hat{b}^{(n);u}$ and $\hat{\sigma}^{(n);v}$ other piecewise constant processes defined on [t,T] such that

$$\widehat{b}_s^{(n);u} := \mathbf{1}_{\{s \in [t,t_1^n]\}} \ b^o + \sum_{l=1}^{n-1} \mathbf{1}_{\{s \in (t_k^n,t_{k+1}^n]\}} \ \widehat{b}^{(n)}(k;u),$$

$$\widehat{\sigma}_s^{(n);v} := \mathbf{1}_{\{s \in [t,t_1^n]\}} \ \sigma^o + \sum_{k=1}^n \mathbf{1}_{\{s \in (t_k^n,t_{k+1}^n]\}} \ \widehat{\sigma}^{(n)}(k;v).$$

⁵An adapted functional on Ω^t is a mapping $\theta:[t,T]\times\Omega^t\to\mathbb{R}$ such that $\theta(s,\cdot)$ is $\mathcal{F}^{W^t}_s$ -measurable for every $s\in[t,T]$ (noting that \mathbb{F}^{W^t} is the raw filtration of the canonical process W^t defined on [t,T]; see Section 4). Similarly, an (\mathbb{R}^d -valued) adapted functional on Ω^t is a mapping $\Theta:=(\theta_1,\ldots,\theta_d)^\top:[t,T]\times\Omega^t\to\mathbb{R}^d$ such that each $\theta_i,\ i=1,\ldots,d$, is an adapted functional on Ω^t .

With the notations in place, we define for each $(u, v) \in (0, 1)^{nd} \times (0, 1)^{nd^2}$

(5.26)
$$\widehat{X}^{(n);u,v} = x \oplus_t \left(x + \int_t^{\cdot} \widehat{b}_s^{(n);u} ds + \int_t^{\cdot} \widehat{\sigma}_s^{(n);v} d\widehat{W}_s^t \right).$$

Note that for every k = 1, ..., n (see (5.23) and (5.25))

$$\widehat{b}^{(n)}(k;U) = \widehat{b}^{(n)}(k), \qquad \widehat{\sigma}^{(n)}(k;V) = \widehat{\sigma}^{(n)}(k).$$

Then by the definitions of $X^{\mathbb{P},(n)}$ and $\widehat{X}^{(n);u,v}$ (see (5.18) and (5.26)) and the property given in (5.24),

$$\text{law of } \widehat{X}^{(n);U,V} \text{ under } \widehat{\mathbb{P}} = \text{law of } X^{\mathbb{P},(n)} \text{ under } \mathbb{P}.$$

Furthermore, since $(\widehat{b}^{(n)}(k), \widehat{\sigma}^{(n)}(k))$ and $(\widehat{b}^{(n)}(k; u), \widehat{\sigma}^{(n)}(k; v))$ are uniformly bounded for every k = 1, ..., n and $(u, v) \in (0, 1)^{nd} \times (0, 1)^{nd^2}$, by using the same arguments given in Lemma 5.1, the following holds for every $p \ge 1$,

$$(5.27) \qquad \sup_{(u,v)\in(0,1)^{nd}\times(0,1)^{nd^2}}\mathbb{E}^{\widehat{\mathbb{P}}}\left[\sup_{0\leq t\leq T}\left|\widehat{X}_t^{(n);u,v}\right|^p\right]+\mathbb{E}^{\widehat{\mathbb{P}}}\left[\sup_{0\leq t\leq T}\left|\widehat{X}_t^{(n);U,V}\right|^p\right]<\infty.$$

Therefore, an application of Fubini theorem and (5.27) ensure that for every $p \ge 1$ and $\xi \in C_p(\Omega^{0,x};\mathbb{R})$ (see (5.2) for the definition)

$$(5.28) \quad \mathbb{E}^{\mathbb{P}}\left[\xi\left(X^{\mathbb{P},(n)}\right)\right] = \mathbb{E}^{\widehat{\mathbb{P}}}\left[\xi\left(\widehat{X}^{(n);U,V}\right)\right] \leq \sup_{(u,v)\in(0.1)^{nd}\times(0.1)^{nd^2}}\mathbb{E}^{\widehat{\mathbb{P}}}\left[\xi\left(\widehat{X}^{(n);u,v}\right)\right] < \infty.$$

Furthermore, from (5.27), it follows that the laws of $\widehat{X}_t^{(n);U,V}$ and $(\widehat{X}_t^{(n);u,v})_{(u,v)\in(0,1)^{nd}\times(0,1)^{nd^2}}$ belong to $\mathcal{P}^p(\Omega^{0,x})$ for every $p\geq 1$ (that is equipped with the topology τ_p ; see (5.1) and (5.3)).

Therefore an application of Hahn-Banach theorem guarantees that for each $n \in \mathbb{N}$ and $p \ge 1$, the law of $X^{\mathbb{P},(n)}$ is contained in the τ_p -closure of the convex hull of the laws of

$$(5.29) \qquad \left\{ x \oplus_t \left(x + \int_t \dot{u}^u(s, \widehat{W}^t) ds + \int_t \dot{\Sigma}^v(s, \widehat{W}^t) d\widehat{W}_s^t \right) \mid (u, v) \in (0, 1)^{nd} \times (0, 1)^{nd^2} \right\},$$

where for every $(u, v) \in (0, 1)^{nd} \times (0, 1)^{nd^2}$, $\mu^u : [t, T] \times \Omega^t \to \mathcal{B}^{\varepsilon, 1}$ and $\Sigma^v : [t, T] \times \Omega^t \to \mathcal{B}^{\varepsilon, 2}$ are adapted Borel functionals on Ω^t .

Replacing \widehat{W}^t with $W^{\mathbb{P},t}$ in the set (5.29) ensures the claim to hold.

Define by $\mathcal{G}([t,T] \times \Omega^t; \mathcal{B}^{\varepsilon,1})$ the set of all adapted, $\mathcal{B}^{\varepsilon,1}$ -valued, Borel functionals on Ω^t . Define $\mathcal{G}([t,T] \times \Omega^t; \mathcal{B}^{\varepsilon,2})$ analogously, with $\mathcal{B}^{\varepsilon,1}$ replaced by $\mathcal{B}^{\varepsilon,2}$, and set

(5.30)
$$\mathcal{G}^{\varepsilon}(t) := \left\{ (\mu, \Sigma) \in \mathcal{G}([t, T] \times \Omega^{t}; \mathcal{B}^{\varepsilon, 1}) \times \mathcal{G}([t, T] \times \Omega^{t}; \mathcal{B}^{\varepsilon, 2}) \right\}.$$

Proof of Proposition 5.8. Recall the Wiener measure \mathbb{P}_0^t defined on $(\Omega^t, \mathcal{F}^t, \mathbb{F}^{W^t})$ under which the canonical process $W^t = (W_s^t)_{s \in [t,T]}$ is a Brownian motion satisfying $W_t^t = 0$ (see Section 4). Moreover, recalling the set $\mathcal{G}^{\varepsilon}(t)$ given in (5.30), we denote by

$$(5.31) \qquad \mathcal{D}^{\varepsilon}(t) := \left\{ (b, \sigma) \mid (b_s, \sigma_s) := (\mu(s, W^t), \Sigma(s, W^t)) \text{ for } s \in [t, T], \ (\mu, \Sigma) \in \mathcal{G}^{\varepsilon}(t) \right\},$$

and we define

$$(5.32) \qquad \mathcal{Q}_{\mathrm{sub}}^{\varepsilon}(t,x) := \mathcal{Q}\left(t,x;\mathcal{D}^{\varepsilon}\right) := \left\{\mathbb{P}_{0}^{t} \circ \left(x \oplus_{t} X^{t,x;b,\sigma}\right)^{-1} \;\middle|\; (b,\sigma) \in \mathcal{D}^{\varepsilon}(t)\right\},$$

where $X^{t,x;b,\sigma}$ is defined in (4.2).

From the definition of $\mathcal{G}^{\varepsilon}(t)$, it follows that $\mathcal{D}^{\varepsilon}(t) \subseteq \mathcal{C}^{\varepsilon}(t)$ (see (4.3)). Furthermore, since $\mathcal{Q}^{\varepsilon}(t,x) = \mathcal{Q}(t,x;\mathcal{C}^{\varepsilon})$ (see (5.11)), by Remark 5.7 we have

$$Q_{\mathrm{sub}}^{\varepsilon}(t,x) \subseteq Q^{\varepsilon}(t,x) \subseteq \mathcal{P}^{\varepsilon}(t,x).$$

Now let $\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)$. Lemma 5.11 ensures that $X^{\mathbb{P},(n)}$, $n \in \mathbb{N}$, given in (5.18), converges to the canonical process X in $\mathcal{H}^2(\Omega^{0,x},\mathcal{F}^{0,x},\mathbb{F}^X,\mathbb{P})$. Thus Lemma 5.12 together with (5.31) and (5.32) ensures that \mathbb{P} is contained in the τ_p -closure of the convex hull of $\mathcal{Q}^{\varepsilon}_{\mathrm{sub}}(t,x)$ for every $p \geq 1$.

Proof of Proposition 5.9. Recall the sets $\mathcal{P}^{\varepsilon}(t,x)$ and $\mathcal{Q}^{\varepsilon}(t,x)$ defined in (5.4) and (5.11). Denote by $\mathcal{Q}^{\varepsilon}_{\mathrm{sub}}(t,x)$ the subset of $\mathcal{Q}^{\varepsilon}(t,x)$ such that its' convex hull is a dense subset of $\mathcal{P}^{\varepsilon}(t,x)$ with respect to the τ_p -topology for all $p \geq 1$ (see Proposition 5.8). Furthermore, $\mathcal{P}^{\varepsilon}(t,x)$ is a subset of $\mathcal{P}^p(\Omega^{0,x})$ for every $p \geq 1$ (see Remark 5.2). Since for every $\xi \in C_p(\Omega^{0,x};\mathbb{R})$ the map $\mathcal{P}^p(\Omega^{0,x}) \ni \mathbb{P} \to \mathbb{E}^{\mathbb{P}}[\xi]$ is continuous and linear, it follows that

$$\sup_{\mathbb{P}\in\mathcal{P}^{\varepsilon}(t,x)}\mathbb{E}^{\mathbb{P}}\left[\xi\right]=\sup_{\mathbb{P}\in\mathcal{Q}^{\varepsilon}(t,x)}\mathbb{E}^{\mathbb{P}}\left[\xi\right].$$

Therefore, as the function f has at most polynomial growth (see Remark 4.2),

$$v_{\text{weak}}^{\varepsilon}(t,x) = \sup_{\mathbb{P} \in \mathcal{P}^{\varepsilon}(t,x)} \mathbb{E}^{\mathbb{P}}\left[f(X_T)\right] = \sup_{\mathbb{P} \in \mathcal{Q}^{\varepsilon}(t,x)} \mathbb{E}^{\mathbb{P}}\left[f(X_T)\right] = v_{\text{strong}}^{\varepsilon}(t,x).$$

This completes the proof.

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