Mathematical Statistics

MAS 713

Tutorial about Chapter 2
SOLUTIONS

Exercise 1

Example

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What is the probability that it must be cast more than five time?

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Solution:

$$\mathbb{P}(x = 6) = 1/6, \qquad \mathbb{P}(x \neq 6) = 5/6$$

- We would have to throw the die more than 5 times if the 6 never showed up in those 5 throws: So, by independence

$$\mathbb{P}(x_1 \neq 6, x_2 \neq 6, x_3 \neq 6, x_4 \neq 6, x_5 \neq 6) = \prod_{i=1}^{5} \mathbb{P}(x_i \neq 6) = (5/6)^5 \approx 0.4$$

Exercise 2

Example

Among the students taking the engineering program, there are:

1. Four boys enrolled in the civil engineering program,
2. Six girls enrolled in the civil engineering program,
3. Six boys enrolled in the electrical engineering program.

How many girls must be enrolled in the electrical engineering program if gender and engineering program are to be independent when a student is selected at random ?

Recall

Definition

Two events E_1 and E_2 are said to be independent **if and only if**

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1) \times \mathbb{P}(E_2)$$

Important property of independence

If E_1, E_2 are independent, then also:

- E_1^c, E_2 are independent
- E_1, E_2^c are independent
- E_1^c, E_2^c are independent

Solution:

- Let x be the unknown number of girls enrolled in the electrical engineering program.
Then the total number of students is $4+6+6+x=16+x$ students.
- Let $C=$ "the selected student is in the civil engineering program"
Let $E=$"the selected student is in the electrical engineering program"
Let $B=$ "the selected student is a boy"
Let $G=$ "the selected student is a girl"
- Notice the sample space $S=\{(B,C),(G,C),(B,E),(G,E)\}$
- For gender and engineering program to be independent when a student is selected at random, we require that:

$$\mathbb{P}(B\cap C)=\mathbb{P}(B)\times\mathbb{P}(C),\qquad \mathbb{P}(B\cap E)=\mathbb{P}(B)\times\mathbb{P}(E),$$
$$\mathbb{P}(G\cap C)=\mathbb{P}(G)\times\mathbb{P}(C),\qquad \mathbb{P}(G\cap E)=\mathbb{P}(G)\times\mathbb{P}(E).$$

- Note that $G=B^c$ and $E=C^c$
Therefore, it suffices to show that $\mathbb{P}(B\cap C)=\mathbb{P}(B)\times\mathbb{P}(C)$

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Therefore, it suffices to show that $\mathbb{P}(B\cap C)=\mathbb{P}(B)\times\mathbb{P}(C)$

$$\mathbb{P}(C) = \frac{10}{16 + x}$$

(10 students in the civil engineering program out of $16 + x$ students)

$$\mathbb{P}(B) = \frac{10}{16 + x}$$

(10 boys out of $16 + x$ students).

$$\mathbb{P}(B \cap C) = \frac{4}{16 + x}$$

(4 boys in the civil engineering program out of $16 + x$ students)

- Events B and C will be independent $\iff \mathbb{P}(B \cap C) = \mathbb{P}(B) \times \mathbb{P}(C)$, that is

$$\frac{4}{16 + x} = \frac{10}{16 + x} \times \frac{10}{16 + x}$$

Solving for x yields $x = 9$ girls enrolled in electrical engineering.

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Exercise 3

You are imprisoned in a dungeon together with two fellow prisoners. You are informed by the jailer that one of you has been chosen at random to be hanged, and the other two are to be freed.
You ask the jailer to tell you privately which of your fellow prisoners will be set free, claiming that there would be no harm in sharing this information, since you already know that at least one of them will go free.

a) The jailer refuses to answer the question, pointing out that if you knew which of your fellows were to be set free, then your own probability of being executed would rise from $1/3$ to $1/2$, since you would then be one of two prisoners. Show that the jailer is wrong.
b) You convince the jailer, and he tells you which of the other two will be set free. You say this information to your fellow prisoners, and while the spared guy is jumping for joy, the other one asks you to switch your identities. Would you accept?

Recall: Law of Total Probability

Law of Total Probability

Given a partition $\{E_1, E_2, \ldots, E_n\}$ of S such that $\mathbb{P}(E_i) > 0$ for all i, the probability of any event A can be written

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|E_i) \times \mathbb{P}(E_i)$$

- So, for any event A and any event E such that $0 < \mathbb{P}(E) < 1$, we have

$$\mathbb{P}(A) = \mathbb{P}(A|E)\mathbb{P}(E) + \mathbb{P}(A|E^c)(1 - \mathbb{P}(E))$$

Recall: Bayes' second rule

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Given a partition $\{E_1, E_2, \ldots, E_n\}$ of S such that $\mathbb{P}(E_i) > 0$ for all i, we have, for any event A such that $\mathbb{P}(A) > 0$,

$$\mathbb{P}(E_i|A) = \frac{\mathbb{P}(A|E_i)\mathbb{P}(E_i)}{\sum_{j=1}^{n} \mathbb{P}(A|E_j)\mathbb{P}(E_j)}$$

In particular :

$$\mathbb{P}(E|A) = \frac{\mathbb{P}(A|E)\mathbb{P}(E)}{\mathbb{P}(A|E)\mathbb{P}(E) + \mathbb{P}(A|E^c)(1 - \mathbb{P}(E))}$$

Solution: a)

- There are three prisoners, you, A and B.

Define events $Y/A/B$ as "you/A/B gonna be hanged". So $\mathbb{P}(Y) = \mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{3}$

- Denote $J_Y/J_A/J_B$ the events "the jailer says that you/A/B will be set free".

Want: $\mathbb{P}(Y) = \mathbb{P}(Y|J_A)$, and $\mathbb{P}(Y) = \mathbb{P}(Y|J_B)$

- By Bayes' first rule,

$$\mathbb{P}(Y|J_A) = \mathbb{P}(J_A|Y)\frac{\mathbb{P}(Y)}{\mathbb{P}(J_A)}$$

- If you are chosen to be hanged, it is reasonable to suppose that the answer of the jailer is equally likely to be A or B, that is,

$$\mathbb{P}(J_A|Y) = \mathbb{P}(J_B|Y) = \frac{1}{2}$$

- By **law of total probability**, with partition $\{A, B, Y\}$ of the sample space, we have

$$\begin{aligned}\mathbb{P}(J_A) &= \mathbb{P}(J_A|A)\mathbb{P}(A) + \mathbb{P}(J_A|B)\mathbb{P}(B) + \mathbb{P}(J_A|Y)\mathbb{P}(Y) \\ &= 0 + 1 \times 1/3 + 1/2 \times 1/3 = \frac{1}{2}\end{aligned}$$

Hence,

$$\mathbb{P}(Y|J_A) = 1/2 \times \frac{1/3}{1/2} = \frac{1}{3},$$

so that indeed $\mathbb{P}(Y|J_A) = \mathbb{P}(Y)$. Similarly, we could find $\mathbb{P}(Y|J_B) = \mathbb{P}(Y)$.
$\Longrightarrow$ you'll be hanged or not independently of the answer of the jailer.

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Solution: **b)**

- Without loss of generality assume that A gets free.

Therefore, we need to compare $\mathbb{P}(Y|J_A)$ and $\mathbb{P}(B|J_A)$.

- We know from a) that $\mathbb{P}(Y|J_A) = 1/3$

$$\mathbb{P}(B|J_A) = \mathbb{P}(J_A|B)\frac{\mathbb{P}(B)}{\mathbb{P}(J_A)} = 1 \times \frac{1/3}{1/2} = \frac{2}{3}.$$

$\rightsquigarrow$ you'd better not take his identity.

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Exercise 4

Example

Suppose that 5% of men and 0.25% of women are color-blind. A person is chosen at random and that person is color-blind.

What is the probability that the person is male?

Solution:

This is an application of Bayes' Theorem.

- Let A be the event the "person is male".
  Let U be the event that the "person is color-blind".
- Bayes' Theorem gives us

$$\mathbb{P}(A|U) = \frac{\mathbb{P}(U|A)\mathbb{P}(A)}{\mathbb{P}(U|A)\mathbb{P}(A) + \mathbb{P}(U|A^C)\mathbb{P}(A^C)}$$
$$= \frac{0.05\ 0.5}{0.05\ 0.5 + 0.0025\ 0.5} = 0.95$$

So there is about a 99.5% chance that the person is male if we know the subject is color-blind.

Exercise 5

Example

A box contains 3 white balls and 2 red balls. We remove at random and without replacement two balls in succession.

What is the probability that the first removed ball is white and the second is red?

Recall: Multiplicative Law of Probability:

For any events $E_1, \dots, E_n$

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \mathbb{P}(E_1) \times \mathbb{P}(E_2|E_1) \times \mathbb{P}(E_3|E_1 \cap E_2) \times \ldots \times \mathbb{P}\left(E_n \middle| \bigcap_{i=1}^{n-1} E_i\right)$$

Solution:

- A box contains 3 white balls and 2 red balls.

$$\begin{aligned}&\mathbb{P}(\text{first ball is white} \cap \text{second ball is red})\\ &= \mathbb{P}(\text{second ball is red}|\text{first ball is white})\,\mathbb{P}(\text{first ball is white})\end{aligned}$$

- Notice $\quad \mathbb{P}(\text{first ball is white}) = 3/5$
- Moreover, $\quad \mathbb{P}(\text{second ball is red}|\text{first ball is white}) = 1/2$

Therefore,

$$\begin{aligned}&\mathbb{P}(\text{first ball is white} \cap \text{second ball is red})\\ &= \mathbb{P}(\text{second ball is red}|\text{first ball is white})\,\mathbb{P}(\text{first ball is white})\\ &= 1/2 \times 3/5\\ &= 3/10\end{aligned}$$

Exercise 6

Example

- Machines A and B produce 10% and 90% respectively of the production of a component intended for the motor industry.
- From experience, it is known that the probability that machine A produces a defective component is 0.01 while
the probability that machine B produces a defective component is 0.05.

If a component is selected at random and is found to be defective, find the probability that it was made by machine A?

Solution:

- Let A = item from machine A,
  Let B = item from machine B,
  Let D = item is defective.
- We know: $\mathbb{P}(A) = 0.1,\ \mathbb{P}(B) = 0.9,\ \mathbb{P}(D|A) = 0.01,\ \mathbb{P}(D|B) = 0.05.$

Therefore, by Bayes

$$\begin{aligned}\mathbb{P}(A|D) &= \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D)}\\ &= \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D|A)\mathbb{P}(A) + \mathbb{P}(D|B)\mathbb{P}(D|B)}\\ &= \frac{0.01 \times 0.1}{0.01 \times 0.1 + 0.05 \times 0.9}\\ &= 0.02\end{aligned}$$