Let $X_1, \dots, X_n, X$ be random variables

- We say that $(X_n)_{n\in\mathbb{N}}$ converges in probability to X if:

$$\forall \varepsilon > 0: \quad \lim_{n\to\infty} P[|X_n - X| > \varepsilon] = 0.$$

- We say that $(X_n)_{n\in\mathbb{N}}$ converges almost surely (a.s.) to X if

$$P[\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1.$$

Exercise: 1) Almost sure convergence implies convergence in probability

2) convergence in probability does not imply almost sure convergence

3) convergence in probability implies the existence of subsequence $(X_{n_k})_{k\in\mathbb{N}}$ which converges almost surely

Hints: For 2) Consider sequence of independent $(X_n)_{n\in\mathbb{N}}$ with $P[X_n = 1] = \frac{1}{n}$, $P[X_n = 0] = 1 - \frac{1}{n}$ and use 2. statement of Borel-Cantelli lemma

For 3) Use 1. statement of Borel-Cantelli lemma.

- We say that $X_1, \dots, X_n$ converges in L^p to X if

$$\lim_{n\to\infty} E[|X_n - X|^p] = 0.$$

Note: If $(X_n) \to X$ in L^1 then you can interchange limit with Expectation...

In general this FAILS:

Example: Let $U \sim \mathrm{Unif}[0,1]$, let $(X_n)_{n\in\mathbb{N}}$ be defined by $X_n := n \cdot \mathbb{1}_{[0,\frac{1}{n}]}(U)$

$$= \begin{cases} n & \text{if } U \in [0,\frac{1}{n}] \\ 0 & \text{else} \end{cases}$$

Exercise $\Rightarrow$ X_n converges to 0 in probability

But X_n does NOT converge in L^1 to 0.

Remark: By Markov inequality convergence in L^1 implies convergence in probability

Goal: What do we need in addition to convergence in probability to attain convergence in L^1

Def: A family of integrable random variables $(X_n)_{n\in\mathbb{N}}$ is called Uniformly Integrable (U.I.) if

$$\lim_{c\to\infty} \sup_n E\left[|X_n| \mathbb{1}_{\{|X_n|>c\}}\right] = 0 .$$

Note: Hard to verify a priori

Theorem of de la Vallee-Poussin: If there exists a function $g: [0,\infty) \to [0,\infty)$ which satisfies

1) g is monotonically increasing
2) g is convex
3) $\lim_{x\to\infty} \frac{g(x)}{x} = \infty$

for which we have $\sup_n E[g(|X_n|)] < \infty$

$\Rightarrow$ $(X_n)_{n\in\mathbb{N}}$ is (U.I.)

Example: $g: [0,\infty) \to [0,\infty)$ defined by $g(x) = x^p$ for $p > 1$ satisfies 1) - 3).

Theorem: (Extension of Lebesgue's Thm) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence converging to X in probability and $(X_n)_{n\in\mathbb{N}}$ is (U.I.)

$\Rightarrow$ $(X_n)_{n\in\mathbb{N}}$ converges in L^1 to X

Remark 1 If $(X_n)_{n\in\mathbb{N}}$ converges in probability to X
and $(X_n^2)_{n\in\mathbb{N}}$ is Uniformly Integrable
$\Rightarrow$ $X_n \to X$ in L^2

Remark 2 By de-la-Vallee-Poussin Theorem (applied to $g(x) = x^{1+\delta/2}$):
it is sufficient to have $\sup_n E[|X_n|^{2+\delta}] < \infty$ for some $\delta > 0$
to see that $(X_n^2)_{n\in\mathbb{N}}$ is (U.I.)