Uniform Integrability Notes

1 Modes of convergence

Definition 1 (Convergence in probability)**.** *Let $X_1, X_2, \dots$ and X be random variables on the same probability space. We say $X_n \to X$* in probability *if for every $\varepsilon > 0$,*

$$\mathbb{P}\big(|X_n - X| > \varepsilon\big) \xrightarrow[n\to\infty]{} 0.$$

Definition 2 (Almost sure convergence)**.** *We say $X_n \to X$* almost surely *(a.s.) if*

$$\mathbb{P}\big(\{\omega : X_n(\omega) \to X(\omega)\}\big) = 1.$$

Definition 3 (L^p convergence)**.** *Let $p \geq 1$. We say $X_n \to X$* in L^p *if $\mathbb{E}[|X_n|^p] < \infty$ and*

$$\mathbb{E}\big[|X_n - X|^p\big] \xrightarrow[n\to\infty]{} 0.$$

Remarks

Remark 1. *Almost sure convergence implies convergence in probability:*

$$X_n \to X \text{ a.s.} \implies X_n \to X \text{ in probability.}$$

Remark 2. *Convergence in probability does* not *imply almost sure convergence in general.*

Remark 3. *If $X_n \to X$ in probability, then there exists a subsequence (X_{n_k}) such that $X_{n_k} \to X$ almost surely.*

L^p convergence versus convergence in probability

Remark 4. *If $p \geq 1$ and $X_n \to X$ in L^p, then $X_n \to X$ in probability.*

Remark 5. *The converse generally fails: convergence in probability does* not *imply convergence in L^p (even for $p = 1$).*

Example 1 (A standard counterexample)**.** *Let $U \sim \mathrm{Unif}[0,1]$ and define*

$$X_n := n\,\mathbf{1}_{\{U \leq 1/n\}}.$$

Then $X_n \to 0$ in probability, since for any $\varepsilon > 0$ and all $n > \varepsilon$,

$$\mathbb{P}(|X_n| > \varepsilon) = \mathbb{P}(X_n = n) = \mathbb{P}(U \leq 1/n) = \frac{1}{n} \to 0.$$

However, $X_n \not\to 0$ in L^1 because

$$\mathbb{E}[|X_n|] = n \cdot \frac{1}{n} = 1 \quad \text{for all } n.$$

Remark 6. *If $X_n \to X$ a.s. or in probability, when can we interchange limit with expectation? Observe that if $X_n \to X$ also in L^1, then we can interchange limit and expectation.*
Question: *What do we need in addition to convergence in probability to obtain convergence in L^1?*

2 Uniform integrability

Definition 4 (Uniform integrability)**.** *A family of integrable random variables $\mathcal{F}$ is uniformly integrable (UI) if*

$$\lim_{K\to\infty} \sup_{X\in\mathcal{F}} \mathbb{E}\big[|X|\mathbf{1}_{\{|X|>K\}}\big] = 0.$$

A sequence (X_n) is uniformly integrable if $\{X_n : n \geq 1\}$ is UI.

3 Key theorems

Theorem 1 (de la Vallée–Poussin)**.** *Let (X_n) be integrable random variables. Suppose there exists $g : [0,\infty) \to [0,\infty)$ such that*

1. *g is increasing,*
2. *g is convex,*
3. $\displaystyle\lim_{x\to\infty} \frac{g(x)}{x} = \infty$,

and

$$\sup_n \mathbb{E}[g(|X_n|)] < \infty.$$

Then (X_n) is uniformly integrable.

Theorem 2 (Vitali convergence theorem)**.** *Let (X_n) be integrable random variables and X integrable. If $X_n \to X$ in probability and (X_n) is uniformly integrable, then*

$$\mathbb{E}[|X_n - X|] \to 0,$$

i.e. $X_n \to X$ in L^1.