An efficient Monte Carlo scheme for Zakai equations

Christian Beck[1,2], Sebastian Becker[1], Patrick Cheridito[1],
Arnulf Jentzen[2,3], and Ariel Neufeld[4]

[1] Department of Mathematics, ETH Zurich, Switzerland

[2] Faculty of Mathematics and Computer Science, University of Münster, Germany

[3] School of Data Science and Shenzhen Research Institute of Big Data,
The Chinese University of Hong Kong, Shenzhen, China

[4] Division of Mathematical Sciences, School of Physical and Mathematical Sciences,
Nanyang Technological University, Singapore

Abstract

In this paper we develop a numerical method for efficiently approximating solutions of certain Zakai equations in high dimensions. The key idea is to transform a given Zakai SPDE into a PDE with random coefficients. We show that under suitable regularity assumptions on the coefficients of the Zakai equation the corresponding random PDE admits a solution random field which, conditionally on the random coefficients, can be written as a classical solution of a second order linear parabolic PDE. This makes it possible to apply the Feynman–Kac formula to obtain an efficient Monte Carlo scheme for computing approximate solutions of Zakai equations. The approach achieves good results in up to 100 dimensions with fast run times.

Keywords: Stochastic partial differential equations, nonlinear filtering problems, Zakai equation, Feynman–Kac representation, Doss–Sussmann transformation

1 Introduction

The goal of stochastic filtering is to estimate the conditional distribution of a not directly observable stochastic process blurred by measurement noise. The process of interest is usually called *signal process*, *state process* or *system process*, while the observed process is typically referred to as *observation process*. Whereas the signal process follows a hidden dynamics, probing the system only reveals the observation process, which, in general, might depend nonlinearly on the signal process and, in addition, is blurred by measurement noise. Stochastic filtering problems were first studied in connection with tracking and signal processing (see the seminal works by Kalman [26] and Kalman & Bucy [27]) but soon turned out to also be relevant in a variety of other applications in finance, the natural sciences and engineering. Among others, nonlinear filtering problems naturally arise in e.g., financial engineering ([3, 10, 13, 18, 20, 21]), weather forecasting ([8, 9, 11, 17, 19, 33]) or chemical engineering ([7, 12, 34, 35, 36, 38]). For further applications of nonlinear filtering, we refer to the survey paper [31]. Stochastic filtering problems are naturally related to stochastic partial differential equations (SPDEs) since in continuous time, the (unnormalized) density of the unobserved signal process given the observations is described by a suitable SPDE, such as the Zakai equation [39] or Kushner equation [30]. The SPDEs arising in this context can typically not be solved explicitly but instead, have to be computed numerically. Moreover, they often are high-dimensional as the number of dimensions corresponds to the state space dimension of the filtering problem.

In this paper, we focus on Zakai equations with coefficients that satisfy certain regularity conditions. Let us assume the signal and observation processes follow d-dimensional dynamics of the form

$$Y_t = Y_0 + \int_0^t \mu(Y_s)\, ds + \sigma W_t \quad \text{and} \quad Z_t = \int_0^t h(Y_s) ds + V_t, \tag{1}$$

respectively, for a d-dimensional random vector Y_0 with density $\varphi\colon \mathbb{R}^d \to [0,\infty)$, sufficiently regular functions $\mu, h\colon \mathbb{R}^d \to \mathbb{R}^d$, a constant $d \times d$-matrix σ, and independent d-dimensional Brownian motions $(W_t)_{t\in[0,T]}$ and $(V_t)_{t\in[0,T]}$ that are independent of Y_0. Then the solution of the corresponding Zakai equation

$$X_t(x) = \varphi(x) + \int_0^t \big[\tfrac{1}{2} \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess} X_s)(x)\big) - (\operatorname{div}(\mu X_s))(x)\big]\, ds + \int_0^t X_s(x)\langle h(x), dZ_s\rangle_{\mathbb{R}^d} \tag{2}$$

describes the evolution of an unnormalized density of the conditional distribution of Y_t given observations of Z_s, $s \le t$; that is,

$$\mathbb{P}[Y_t \in A \,|\, \mathfrak{S}(Z_s; s \le t)] = \frac{\int_A X_t(x)\, dx}{\int_{\mathbb{R}^d} X_t(x)\, dx} \quad \text{for every Borel subset } A \subseteq \mathbb{R}^d. \tag{3}$$

Our numerical method is based on a transformation which transforms a Zakai SPDE of the form (2) into a PDE with random coefficients. We show that under suitable conditions on the coefficients of the Zakai SPDE, the solution of the resulting random PDE is ω-wise a classical solution of a linear second order parabolic PDE. This makes it possible to apply the Feynman–Kac formula to obtain an efficient Monte Carlo scheme for the numerical approximation of solutions of high-dimensional Zakai equations. The following is this paper's main theoretical result.

Theorem 1.1. *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and $\sigma \in \mathbb{R}^{d\times d}$. Consider functions $\varphi \in C^2(\mathbb{R}^d, [0,\infty))$, $\mu \in C^3(\mathbb{R}^d, \mathbb{R}^d)$ and $h \in C^4(\mathbb{R}^d, \mathbb{R}^d)$ such that φ has at most polynomially growing derivatives up to the second order, μ has bounded derivatives up to the third order, and h has bounded derivatives up to the fourth order. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions[1]. Let $W, V, U\colon [0,T] \times \Omega \to \mathbb{R}^d$ be standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motions with continuous sample paths such that W and V are independent. Let $Y, Z\colon [0,T] \times \Omega \to \mathbb{R}^d$ be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes such that $\mathbb{P}(Y_0 \in A) = \int_A \varphi(x)\, dx$ for every Borel subset $A \subseteq \mathbb{R}^d$ and*

$$Y_t = Y_0 + \int_0^t \mu(Y_s)\, ds + \sigma W_t, \quad Z_t = \int_0^t h(Y_s)\, ds + V_t \quad \textit{for all } t \in [0,T]. \tag{4}$$

[1] We say that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ satisfies the *usual conditions* if for all $t \in [0,T)$, one has $\bigcup_{A\in\mathcal{F}, \mathbb{P}(A)=0}\{B \subseteq \Omega\colon B \subseteq A\} \subseteq \mathcal{F}_t = \bigcap_{s\in(t,T]} \mathcal{F}_s$.

For every $v \in C([0,T],\mathbb{R}^d)$, $t \in [0,T]$, and $x \in \mathbb{R}^d$, let $R^{v,t,x}\colon [0,t] \times \Omega \to \mathbb{R}^d$ be an $(\mathcal{F}_s)_{s\in[0,t]}$-adapted stochastic processes satisfying

$$R^{v,t,x}_s = x + \int_0^s \big[\sigma\sigma^*[h'(R^{v,t,x}_r)]^* v(t-r) - \mu(R^{v,t,x}_r)\big]\, dr + \sigma U_s \quad \text{for all } s \in [0,t]. \tag{5}$$

Let the functions $B_{v,y}, u_{v,y}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $v, y \in C([0,T],\mathbb{R}^d)$ be given by[2]

$$\begin{aligned} B_{v,y}(t,x) &= \tfrac{1}{2}\left\langle \sigma\sigma^*[h'(x)]^* v(t), [h'(x)]^* v(t)\right\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}(\langle h, v(t)\rangle_{\mathbb{R}^d}))(x)\big) \\ &\quad + \big[\langle h(x), h(y(t))\rangle_{\mathbb{R}^d} - \tfrac{1}{2}\|h(x)\|^2_{\mathbb{R}^d} - \langle \mu(x), [h'(x)]^* v(t)\rangle_{\mathbb{R}^d} - (\operatorname{div}\mu)(x)\big] \end{aligned} \tag{6}$$

and

$$u_{v,y}(t,x) = \mathbb{E}\Big[\varphi(R^{v,t,x}_t)\exp\Big(\int_0^t B_{v,y}(t-s, R^{v,t,x}_s)\, ds\Big)\Big] \tag{7}$$

for $v, y \in C([0,T],\mathbb{R}^d)$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. Then, the random field $X\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by

$$X_t(x,\omega) = u_{V(\omega),Y(\omega)}(t,x)\exp\big(\langle h(x), V_t(\omega)\rangle_{\mathbb{R}^d}\big), \quad t \in [0,T],\ x \in \mathbb{R}^d,\ \omega \in \Omega, \tag{8}$$

satisfies the following:

(i) *for all $t \in [0,T]$, $x \in \mathbb{R}^d$, the mapping $X_t(x)\colon \Omega \to \mathbb{R}$ is $\mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable,*

(ii) *for all $\omega \in \Omega$, the mapping $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto X_t(x,\omega) \in \mathbb{R})$ is in $C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$,*

(iii)
$$X_t(x) = \varphi(x) + \int_0^t \Big[\tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess} X_s)(x)\big) - \operatorname{div}\big(\mu X_s\big)(x)\Big]\, ds + \int_0^t X_s(x)\,\langle h(x), dZ_s\rangle_{\mathbb{R}^d} \tag{9}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

Representation (8) makes it possible to approximate the solution $X_t(x,\omega)$ of the Zakai equation (9) for given $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$ by averaging over different Monte Carlo simulations of the process $R^{V(\omega),t,x}$ given in (5). Note that expression (8) formally depends on ω through the realizations of the processes Y and V, while in a typical stochastic filtering application, the Zakai equation has to be solved along a given path of the observation process Z. Y and V determine Z through (4), but the path realizations of Y and V cannot be recovered from a realized path of Z. In fact, the whole purpose of stochastic filtering is to estimate Y_t from observations of Z_s, $s \le t$. However, it follows from (9) that $X_t(x,\omega)$ only depends on the realization of $(Z_s)_{s\in[0,t]}$. In other words, two different realizations of $(Y_s, V_s)_{s\in[0,t]}$ resulting in the same path of $(Z_s)_{s\in[0,t]}$ lead to the same value of $X_t(x,\omega)$. This means that for numerical purposes, a given realization $z = (Z_s(\omega))_{s\in[0,t]}$ can artificially be decomposed by first fixing a likely path y of $(Y_s)_{s\in[0,t]}$. In view of (4), a natural choice of y is the solution of the ODE

$$y(s) = \overline{\varphi} + \int_0^s \mu(y(r))dr, \quad s \in [0,t], \quad \text{for the mean } \overline{\varphi} = \int_{\mathbb{R}^d} x\varphi(x)dx \text{ of } \varphi. \tag{10}$$

[2] For $d \in \mathbb{N}$, we denote by $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the standard scalar product given by $\langle x, y\rangle_{\mathbb{R}^d} = \sum_{i=1}^d x_i y_i$ and by $\|.\|_{\mathbb{R}^d}\colon \mathbb{R}^d \to [0,\infty)$ the corresponding norm given by $\|x\|_{\mathbb{R}^d} = \sqrt{\langle x, x\rangle}$.

The corresponding realization v of V can then be chosen so that $\int_0^s h(y(r))dr + v(s) = z(s)$ for all $s \in [0,t]$. Theorem 1.1 ensures that formula (8) with input (y,v) yields the correct value for $X_t(x,\omega)$ along the path $z = (Z_s(\omega))_{s\in[0,t]}$.

The idea of transforming a stochastic differential equation into an ordinary differential equation with random coefficients goes back to Doss [16] and Sussmann [37]. An extension to SPDEs has been used by Buckdahn and Ma [4, 5] to introduce a notion of stochastic viscosity solution for SPDEs and show existence and uniqueness results as well as connections to backward doubly stochastic differential equations. The same approach has been employed by Buckdahn and Ma [6] and Boufoussi et al. [2] to study stochastic viscosity solutions of stochastic Hamilton–Jacobi–Bellman (HJB) equations. In this paper, we analyze the regularity properties of such transformations and use them to develop a Monte Carlo method for approximating solutions of Zakai equations. For different numerical approximation methods for Zakai equations, see e.g., [1, 14, 15, 22, 23].

The rest of the paper is organized as follows. In Section 2 we provide preliminary regularity and stability results needed for the proof of the Feynman–Kac representation of Proposition 3.2 in Section 3.1, which is the main ingredient in the proof of Theorem 1.1, given in Section 3.2. In Section 4 we provide numerical results for a Zakai equation of the form (2) for dimensions $d \in \{1,2,5,10,20,50,75,100\}$.

2 Preliminary regularity and stability results

In this section we present essentially well-known regularity and stability results needed to show the Feynman–Kac representation in Proposition 3.2, which is used in the proof of Theorem 1.1.

2.1 Approximation and mollification results for at most polynomially growing functions

Lemma 2.1. *Let $c,p \in [0,\infty)$, $d \in \mathbb{N}$, $\alpha, T \in (0,\infty)$, and consider at most polynomially growing functions $G \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ and $H \in C(\mathbb{R}^d,\mathbb{R})$. Moreover, assume that*

$$|G(t,x) - G(s,x)| \le c\,(1+\|x\|_{\mathbb{R}^d})^p\,|t-s|^\alpha \quad \text{for all } t,s\in[0,T] \text{ and } x \in \mathbb{R}^d, \tag{11}$$

and let $G_n \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ and $H_n \in C(\mathbb{R}^d,\mathbb{R})$ for all $n\in\mathbb{N}$, $t\in[0,T]$ and $x\in\mathbb{R}^d$ be given by

$$G_n(t,x) = \big(\tfrac{n}{2\pi}\big)^{1/2} \int_{-\infty}^{\infty} G(\min\{T,\max\{s,0\}\},x)\exp\big(-\tfrac{n}{2}(t-s)^2\big)\,ds \tag{12}$$

and

$$H_n(x) = \big(\tfrac{n}{2\pi}\big)^{d/2} \int_{\mathbb{R}^d} H(y)\exp\big(-\tfrac{n}{2}\|x-y\|^2_{\mathbb{R}^d}\big)\,dy. \tag{13}$$

Then

(i) $\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\frac{|G_n(t,x)-G(t,x)|}{(1+\|x\|_{\mathbb{R}^d})^p} = 0$ *and*

(ii) $\limsup_{n\to\infty}\sup_{x\in[-q,q]^d}|H_n(x)-H(x)| = 0$ *for all $q\in(0,\infty)$.*

Proof. From (11) and the fact that $|\min\{T,\max\{s,0\}\}-t|\leq|s-t|$ for all $s\in\mathbb{R}$ and $t\in[0,T]$, one obtains

$$\begin{aligned}\frac{|G_n(t,x)-G(t,x)|}{(1+\|x\|_{\mathbb{R}^d})^p}&\leq\left(\tfrac{n}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}\frac{|G(\min\{T,\max\{s,0\}\},x)-G(t,x)|}{(1+\|x\|_{\mathbb{R}^d})^p}\exp(-\tfrac{n(t-s)^2}{2})\,ds\\&\leq\left(\tfrac{n}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}c\,|\min\{T,\max\{s,0\}\}-t|^{\alpha}\exp(-\tfrac{n(t-s)^2}{2})\,ds\\&\leq\left(\tfrac{n}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}c\,|s-t|^{\alpha}\exp(-\tfrac{n(t-s)^2}{2})\,ds=c\left(\tfrac{1}{2\pi n^{\alpha}}\right)^{1/2}\int_{-\infty}^{\infty}|z|^{\alpha}\exp(-\tfrac{|z|^2}{2})\,dz\end{aligned}\tag{14}$$

for all $t\in[0,T]$ and $x\in\mathbb{R}^d$. In particular, $\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\left(\frac{|G_n(t,x)-G(t,x)|}{(1+\|x\|_{\mathbb{R}^d})^p}\right)=0$, which shows (i). Next, note that one has

$$\begin{aligned}|H_n(x)-H(x)|&\leq\left(\tfrac{n}{2\pi}\right)^{d/2}\int_{\mathbb{R}^d}|H(y)-H(x)|\exp(-\tfrac{n}{2}\|x-y\|^2_{\mathbb{R}^d})\,dy\\&=\left(\tfrac{1}{2\pi}\right)^{d/2}\int_{\mathbb{R}^d}\left|H\big(x+\tfrac{z}{\sqrt{n}}\big)-H\big(x\big)\right|\exp(-\tfrac{1}{2}\|z\|^2_{\mathbb{R}^d})\,dz\quad\text{for all } n\in\mathbb{N}\text{ and } x\in\mathbb{R}^d,\end{aligned}\tag{15}$$

and the assumption that $H\in C(\mathbb{R}^d,\mathbb{R})$ is at most polynomially growing implies that

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d}\sup_{x\in[-q,q]^d}\left|H\big(x+\tfrac{z}{\sqrt{n}}\big)-H\big(x\big)\right|^q\exp(-\tfrac{1}{2}\|z\|^2_{\mathbb{R}^d})\,dz<\infty\quad\text{for all } q\in(0,\infty).\tag{16}$$

Combining (15), (16), the assumption that $H\in C(\mathbb{R}^d,\mathbb{R})$, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) yields that $\limsup_{n\to\infty}\sup_{x\in[-q,q]^d}|H_n(x)-H(x)|=0$ for all $q\in(0,\infty)$. This establishes (ii) and completes the proof of the lemma. □

Lemma 2.2. *Let $d,m\in\mathbb{N}$, $T\in(0,\infty)$, and consider a family of functions $f^{n,t,x}\in C([0,t],\mathbb{R}^m)$, $n\in\mathbb{N}_0$, $t\in[0,T]$, $x\in\mathbb{R}^d$, such that for every $p\in(0,\infty)$,*

$$\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|f^{0,t,x}(s)\|_{\mathbb{R}^m}<\infty\tag{17}$$

and

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|f^{n,t,x}(s)-f^{0,t,x}(s)\|_{\mathbb{R}^m}=0.\tag{18}$$

Let $g_n\in C(\mathbb{R}^m,\mathbb{R})$, $n\in\mathbb{N}_0$, such that for every $p\in(0,\infty)$,

$$\limsup_{n\to\infty}\sup_{y\in[-p,p]^m}|g_n(y)-g_0(y)|=0.\tag{19}$$

Then, one has for all $p\in(0,\infty)$,

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}|g_n(f^{n,t,x}(s))-g_0(f^{0,t,x}(x))|=0.\tag{20}$$

Proof. Observe that (18) ensures that there exist $n_p, N_p \in \mathbb{N}$, $p \in (0,\infty)$ such that

$$\sup_{n\in\{0\}\cup(\mathbb{N}\cap[n_p,\infty))} \sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} \|f^{n,t,x}(s)\|_{\mathbb{R}^m} \leq N_p < \infty \quad \text{for all } p \in (0,\infty). \tag{21}$$

The assumption that $g_0 \in C(\mathbb{R}^m, \mathbb{R})$ and (18) hence imply that

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} |g_0(f^{n,t,x}(s)) - g_0(f^{0,t,x}(s))| = 0 \quad \text{for all } p \in (0,\infty). \tag{22}$$

Combining this with (19), (21), and the triangle inequality shows that

$$\begin{aligned}
&\limsup_{n\to\infty} \left[\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} |g_n(f^{n,t,x}(s)) - g_0(f^{0,t,x}(s))|\right] \\
&\leq \limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} |g_n(f^{n,t,x}(s)) - g_0(f^{n,t,x}(s))| \\
&+ \limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} |g_0(f^{n,t,x}(s)) - g_0(Y^{0,t,x}(s))| \\
&\leq \limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{y\in[-N_p,N_p]^m} |g_n(y) - g_0(y)| = 0
\end{aligned} \tag{23}$$

for all $p \in (0,\infty)$, which completes the proof of the lemma. □

2.2 Exponential integrability of Brownian motions

The following is a consequence of [32, Corollary 3.2]. For more details; see e.g., [25, Lemma 6.3].

Lemma 2.3. *Let $T, c \in [0,\infty)$, $\alpha \in [0,2)$ and $d \in \mathbb{N}$. Let $W\colon [0,T]\times\Omega \to \mathbb{R}^d$ be a standard Brownian motion with continuous sample paths defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then*

(i) *the mapping $\Omega \ni \omega \mapsto \sup_{t\in[0,T]} \|W_t(\omega)\|_{\mathbb{R}^d}^\alpha \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, and*

(ii) $\mathbb{E}\left[\exp(c \sup_{t\in[0,T]} \|W_t\|_{\mathbb{R}^d}^\alpha)\right] < \infty.$

2.3 A priori estimates and regularity properties for solutions of ODEs with additive noise

Lemma 2.4. *Let $c, T \in (0,\infty)$, $d \in \mathbb{N}$, and $\sigma \in \mathbb{R}^{d\times d}$. Let $b\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a Borel measurable function satisfying $\|b(t,x)\|_{\mathbb{R}^d} \leq c(1 + \|x\|_{\mathbb{R}^d})$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U\colon [0,T]\times\Omega \to \mathbb{R}^d$ with continuous sample paths. Consider stochastic processes $R^{t,x} = (R^{t,x}_s)_{s\in[0,t]}\colon [0,t]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, $t \in [0,T]$ satisfying*

$$R^{t,x}_s = x + \int_0^s b(t-r, R^{t,x}_r)\,dr + \sigma U_r \quad \text{for all } t \in [0,T],\ s \in [0,t] \text{ and } x \in \mathbb{R}^d. \tag{24}$$

Then

(i)
$$\begin{aligned}
&\mathbb{E}\left[\exp\Big(p \sup_{t\in[0,T]} \sup_{s\in[0,t]} \|R^{t,x}_s\|_{\mathbb{R}^d}\Big)\right] \\
&\leq \exp\big(pe^{cT}\|x\|_{\mathbb{R}^d}\big) \exp\big(pcTe^{cT}\big)\, \mathbb{E}\left[\exp\big(pe^{cT} \sup_{t\in[0,T]} \|\sigma U_t\|_{\mathbb{R}^d}\big)\right] < \infty
\end{aligned} \tag{25}$$

for all $p \in (0,\infty)$ and $x \in \mathbb{R}^d$, and

(ii)
$$\mathbb{E}\Big[\exp\Big(p\big[\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|R_s^{t,x}\|_{\mathbb{R}^d}\big]\Big)\Big] < \infty \quad \textit{for all } p\in(0,\infty). \tag{26}$$

Proof. The triangle inequality, the assumption that $\|b(t,x)\|_{\mathbb{R}^d} \leq c(1+\|x\|_{\mathbb{R}^d})$ for all $t\in[0,T]$ and $x\in\mathbb{R}^d$, and (24) assure that

$$\begin{aligned}\|R_s^{t,x}\|_{\mathbb{R}^d} &\leq \|x\|_{\mathbb{R}^d} + \int_0^s c\big(1+\|R_r^{t,x}\|_{\mathbb{R}^d}\big)\,dr + \sup_{r\in[0,T]}\|\sigma U_r\|_{\mathbb{R}^d} \\ &\leq \|x\|_{\mathbb{R}^d} + cT + \sup_{r\in[0,T]}\|\sigma U_r\|_{\mathbb{R}^d} + c\int_0^s \|R_r^{t,x}\|_{\mathbb{R}^d}\,dr\end{aligned} \tag{27}$$

for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and $s\in[0,t]$. Hence, one obtains from Gronwall's integral inequality (cf., e.g., [24, Lemma 2.11]) that

$$\|R_s^{t,x}\|_{\mathbb{R}^d} \leq \Big(\|x\|_{\mathbb{R}^d} + cT + \sup_{r\in[0,T]}\|\sigma U_r\|_{\mathbb{R}^d}\Big)\,e^{cT} \tag{28}$$

for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and $s\in[0,t]$. So it follows from Lemma 2.3 that

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(p\sup_{t\in[0,T]}\sup_{s\in[0,t]}\|R_s^{t,x}\|_{\mathbb{R}^d}\Big)\Big] \\ &\leq \exp\big(pe^{cT}\|x\|_{\mathbb{R}^d}\big)\exp\big(pcTe^{cT}\big)\,\mathbb{E}\Big[\exp\Big(pe^{cT}\sup_{r\in[0,T]}\|\sigma U_r\|_{\mathbb{R}^d}\Big)\Big] < \infty\end{aligned} \tag{29}$$

for all $p\in(0,\infty)$ and $x\in\mathbb{R}^d$, as well as

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(p\big[\sup_{t\in[0,T]}\sup_{x\in[-p,p]^d}\sup_{s\in[0,t]}\|R_s^{t,x}\|_{\mathbb{R}^d}\big]\Big)\Big] \\ &\leq \exp\big(pe^{cT}\sqrt{d}p\big)\exp\big(pcTe^{cT}\big)\,\mathbb{E}\Big[\exp\Big(pe^{cT}\sup_{r\in[0,T]}\|\sigma U_r\|_{\mathbb{R}^d}\Big)\Big] < \infty\end{aligned} \tag{30}$$

for all $p\in(0,\infty)$, which proves the lemma. □

Lemma 2.5. *Let $T\in(0,\infty)$, $d\in\mathbb{N}$, and $\sigma\in\mathbb{R}^{d\times d}$. Denote by e_i, $i\in\{1,2,\dots,d\}$, the standard unit vectors in $\mathbb{R}^d$. Let $b\in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ have bounded partial derivatives of first and second order with respect to the x-variables. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting a standard Brownian motion $U\colon[0,T]\times\Omega\to\mathbb{R}^d$ with continuous sample paths, and let $R^{t,x}=(R_s^{t,x})_{s\in[0,t]}\colon[0,t]\times\Omega\to\mathbb{R}^d$, $t\in[0,T]$, $x\in\mathbb{R}^d$, be stochastic processes satisfying*

$$R_s^{t,x} = x + \int_0^s b(t-r,R_r^{t,x})\,dr + \sigma U_s \quad \textit{for all } t\in[0,T],\ s\in[0,t] \textit{ and } x\in\mathbb{R}^d. \tag{31}$$

Then

(i) *for all $t\in[0,T]$ and $\omega\in\Omega$, the mapping $[0,t]\times\mathbb{R}^d\ni(s,x)\mapsto R_s^{t,x}(\omega)\in\mathbb{R}^d$ is in $C^{0,2}([0,t]\times\mathbb{R}^d,\mathbb{R}^d)$,*

(ii)
$$\tfrac{\partial}{\partial x_i}R_s^{t,x} = e_i + \int_0^s \big[(\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_r^{t,x}\big)\,dr \tag{32}$$

for all $i\in\{1,2,\dots,d\}$, $t\in[0,T]$, $s\in[0,t]$ and $x\in\mathbb{R}^d$,

(iii) $\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}^d} \|\frac{\partial}{\partial x_i} R_s^{t,x}\|_{\mathbb{R}^d} \leq \exp\Big(T \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\frac{\partial}{\partial x} b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) < \infty$ (33)

for all[3] $i \in \{1, 2, \ldots, d\}$,

(iv)
$$\frac{\partial^2}{\partial x_i \partial x_j} R_s^{t,x} = \int_0^s \big[(\tfrac{\partial^2}{\partial x^2} b)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i} R_r^{t,x}, \tfrac{\partial}{\partial x_j} R_r^{t,x}\big)\, dr + \int_0^s \big[(\tfrac{\partial}{\partial x} b)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j} R_r^{t,x}\big)\, dr \tag{34}$$

for all $i, j \in \{1, \ldots, d\}$, $t \in [0,T]$, $s \in [0,t]$ *and* $x \in \mathbb{R}^d$,

(v)
$$\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}^d} \|\tfrac{\partial^2}{\partial x_i \partial x_j} R_s^{t,x}\|_{\mathbb{R}^d} \leq T\Big[\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial^2}{\partial x^2} b)(t,x)\big\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}\Big] \times \exp\Big(3T \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) < \infty \tag{35}$$

for all $i, j \in \{1, 2, \ldots, d\}$.

Proof. Note that (31), the assumption that $b \in C^{0,2}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives with respect to the x-variables, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) ensure that for all $i \in \{1, 2, \ldots, d\}$, $t \in [0,T]$, $s \in [0,t]$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$, the mapping $[0,t]\times\mathbb{R}^d \ni (s,x) \mapsto R_s^{t,x}(\omega) \in \mathbb{R}^d$ is in $C^{0,1}([0,t]\times\mathbb{R}^d, \mathbb{R}^d)$ and

$$\tfrac{\partial}{\partial x_i} R_s^{t,x} = e_i + \int_0^s \big[(\tfrac{\partial}{\partial x} b)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i} R_r^{t,x}\big)\, dr \tag{36}$$

(cf. also, e.g., [29, Theorem 4.6.5]). This shows (ii).

Moreover, (ii) and the assumption that $b \in C^{0,2}([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives with respect to the x-variables imply that for all $i \in \{1, 2, \ldots, d\}$, $x \in \mathbb{R}^d$, $t \in [0,T]$ and $s \in [0,t]$, one has

$$\|\tfrac{\partial}{\partial x_i} R_s^{t,x}\|_{\mathbb{R}^d} \leq 1 + \int_0^s \sup_{(w,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(w,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \|\tfrac{\partial}{\partial x_i} R_r^{t,x}\|_{\mathbb{R}^d}\, dr. \tag{37}$$

Gronwall's integral inequality (cf., e.g., [24, Lemma 2.11]) therefore ensures that for every $i \in \{1, 2, \ldots, d\}$,

$$\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}^d} \|\tfrac{\partial}{\partial x_i} R_s^{t,x}\|_{\mathbb{R}^d} \leq \exp\Big(T \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) < \infty, \tag{38}$$

which establishes (iii).

[3]For $d, k, n \in \mathbb{N}$ we denote by $L^{(n)}(\mathbb{R}^d, \mathbb{R}^k)$ the set of all continuous n-linear functions from $(\mathbb{R}^d)^n$ to $\mathbb{R}^k$. By $\|\cdot\|_{L^{(n)}(\mathbb{R}^d,\mathbb{R}^k)}$ we denote the operator norm on $L^{(n)}(\mathbb{R}^d, \mathbb{R}^k)$ given by $\|f\|_{L^{(n)}(\mathbb{R}^d,\mathbb{R}^k)} = \sup_{v_1,v_2,\ldots,v_n\in\mathbb{R}^d\setminus\{0\}} \frac{\|f(v_1,v_2,\ldots,v_n)\|_{\mathbb{R}^k}}{\|v_1\|_{\mathbb{R}^d}\|v_2\|_{\mathbb{R}^d}\cdots\|v_n\|_{\mathbb{R}^d}}$. For simplicity, we set $L(\mathbb{R}^d, \mathbb{R}^k) = L^{(1)}(\mathbb{R}^d, \mathbb{R}^k)$ and denote the norm $\|\cdot\|_{L^{(1)}(\mathbb{R}^d,\mathbb{R}^k)}$ by $\|\cdot\|_{L(\mathbb{R}^d,\mathbb{R}^k)}$.

In the next step we observe that (31), (ii), the assumption that $b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives of second order with respect to the x-variables, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) ensure that for every $i, j \in \{1, 2, \ldots, d\}$, $t \in [0,T]$, $s \in [0,t]$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$, the mapping $[0,t] \times \mathbb{R}^d \ni (s,x) \mapsto R_s^{t,x}(\omega) \in \mathbb{R}^d$ is in $C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and

$$\tfrac{\partial^2}{\partial x_i \partial x_j} R_s^{t,x} = \int_0^s \big[(\tfrac{\partial^2}{\partial x^2} b)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i} R_r^{t,x}, \tfrac{\partial}{\partial x_j} R_r^{t,x}\big)\, dr + \int_0^s \big[(\tfrac{\partial}{\partial x} b)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j} R_r^{t,x}\big)\, dr \tag{39}$$

(cf. also, e.g., [29, Theorem 4.6.5]). This proves (i) and (iv).

Moreover, (iv), (iii) and the assumption that $b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives of second order with respect to the x-variables imply that for all $i, j \in \{1, 2, \ldots, d\}$, $x \in \mathbb{R}^d$ $t \in [0,T]$ and $s \in [0,t]$, one has

$$\begin{aligned} \|\tfrac{\partial^2}{\partial x_i \partial x_j} R_s^{t,x}\|_{\mathbb{R}^d} \leq T\Big[&\sup\nolimits_{(w,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial^2}{\partial x^2} b)(w,y)\big\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}\Big] \\ &\times \exp\Big(2T \sup\nolimits_{(w,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(w,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\ &+ \int_0^s \Big[\sup\nolimits_{(w,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(w,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big] \|\tfrac{\partial^2}{\partial x_i \partial x_j} R_r^{t,x}\|_{\mathbb{R}^d}\, dr. \end{aligned} \tag{40}$$

Gronwall's integral inequality (cf., e.g., [24, Lemma 2.11]) therefore ensures that for every $i, j \in \{1, 2, \ldots, d\}$,

$$\begin{aligned} &\sup\nolimits_{t\in[0,T]} \sup\nolimits_{s\in[0,t]} \sup\nolimits_{x\in\mathbb{R}^d} \|\tfrac{\partial^2}{\partial x_i \partial x_j} R_s^{t,x}\|_{\mathbb{R}^d} \\ &\leq T\Big[\sup\nolimits_{(t,x)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial^2}{\partial x^2} b)(t,x)\big\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}\Big] \exp\Big(3T \sup\nolimits_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) < \infty, \end{aligned} \tag{41}$$

which establishes (v) and completes the proof of the lemma. □

2.4 Stability properties of solutions of ODEs with additive noise

Lemma 2.6. *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and $\sigma \in \mathbb{R}^{d\times d}$. Let $b, \mathfrak{b} \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ have bounded partial derivatives of first and second order with respect to the x-variables. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U \colon [0,T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths, and consider stochastic processes $R^{t,x} = (R_s^{t,x})_{s\in[0,t]} \colon [0,t] \times \Omega \to \mathbb{R}^d$ and $\mathcal{R}^{t,x} = (\mathcal{R}_s^{t,x})_{s\in[0,t]} \colon [0,t] \times \Omega \to \mathbb{R}^d$, $t \in [0,T]$, $x \in \mathbb{R}^d$, satisfying*

$$R_s^{t,x} = x + \int_0^s b(t-r, R_r^{t,x})\, dr + \sigma U_s \quad \text{and} \quad \mathcal{R}_s^{t,x} = x + \int_0^s \mathfrak{b}(t-r, \mathcal{R}_r^{t,x})\, dr + \sigma U_s \tag{42}$$

for all $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Then

(i)

$$\sup_{s\in[0,t]} \|\mathcal{R}^{t,x}_s - R^{t,x}_s\|_{\mathbb{R}^d} \\ \leq T \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \|\mathfrak{b}(r,y) - b(r,y)\|_{\mathbb{R}^d} \exp\Big(T \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial}{\partial x} b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big), \tag{43}$$

for all $t \in [0,T]$ *and* $x \in \mathbb{R}^d$,

(ii)
$$\sup_{s\in[0,t]} \|\tfrac{\partial}{\partial x_i}\mathcal{R}^{t,x}_s - \tfrac{\partial}{\partial x_i}R^{t,x}_s\|_{\mathbb{R}^d} \\ \leq T \sup_{s\in[0,t]} \big\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-s,R^{t,x}_s) - (\tfrac{\partial}{\partial x}b)(t-s,\mathcal{R}^{t,x}_s)\big\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \\ \times \exp\Big(T\Big[\sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} + \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial}{\partial x}b)(r,y)\big\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big]\Big), \tag{44}$$

for all $i \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, *and*

(iii)
$$\sup_{s\in[0,t]}\|\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{R}^{t,x}_s - \tfrac{\partial^2}{\partial x_i \partial x_j}R^{t,x}_s\|_{\mathbb{R}^d} \\ \leq \exp\Big(T\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\ \times \Big[2T^2 \sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(r,y)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\ \times \exp\Big(3T\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\max\Big\{\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}, \|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big\}\Big) \\ \times \sup_{s\in[0,t]}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-s,\mathcal{R}^{t,x}_s) - (\tfrac{\partial}{\partial x}b)(t-s,R^{t,x}_s)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \\ + T\exp\Big(2T\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\ \times \sup_{s\in[0,t]}\|(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(t-s,\mathcal{R}^{t,x}_s) - (\tfrac{\partial^2}{\partial x^2}b)(t-s,R^{t,x}_s)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\ + T^2\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial^2}{\partial x^2}b)(r,y)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\ \times \exp\Big(3T\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\ \times \sup_{s\in[0,t]}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-s,\mathcal{R}^{t,x}_s) - (\tfrac{\partial}{\partial x}b)(t-s,R^{t,x}_s)\|_{\mathbb{R}^d}\Big]. \tag{45}$$

for all $i,j \in \{1,2,\ldots,d\}$, $t \in [0,T]$ *and* $x \in \mathbb{R}^d$.

Proof. Throughout this proof we assume without loss of generality that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|\mathfrak{b}(t,x) - b(t,x)\|_{\mathbb{R}^d} < \infty. \tag{46}$$

From (42) and the triangle inequality one obtains

$$\|\mathcal{R}^{t,x}_s - R^{t,x}_s\|_{\mathbb{R}^d} \leq \int_0^s \|\mathfrak{b}(t-r,\mathcal{R}^{t,x}_r) - b(t-r,R^{t,x}_r)\|_{\mathbb{R}^d}\,dr \\ \leq \int_0^s \Big[\|\mathfrak{b}(t-r,\mathcal{R}^{t,x}_r) - b(t-r,\mathcal{R}^{t,x}_r)\|_{\mathbb{R}^d} + \|\mathfrak{b}(t-r,\mathcal{R}^{t,x}_r) - \mathfrak{b}(t-r,R^{t,x}_r)\|_{\mathbb{R}^d}\Big]\,dr \tag{47}$$

for all $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Combining this with (46) and the assumption that b has bounded partial derivatives with respect to the x-variables yields

$$\begin{aligned} \|\mathcal{R}_s^{t,x} - R_s^{t,x}\|_{\mathbb{R}^d} &\leq T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|\mathfrak{b}(r,y) - b(r,y)\|_{\mathbb{R}^d} \\ &\quad + \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial}{\partial x} b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \int_0^s \|\mathcal{R}_r^{t,x} - R_r^{t,x}\|_{\mathbb{R}^d}\, dr \end{aligned} \tag{48}$$

for all $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Gronwall's integral inequality (cf., e.g., [24, Lemma 2.11]) therefore gives

$$\begin{aligned} &\sup\nolimits_{s\in[0,t]} \|\mathcal{R}_s^{t,x} - R_s^{t,x}\|_{\mathbb{R}^d} \\ &\leq T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|\mathfrak{b}(r,y) - b(r,y)\|_{\mathbb{R}^d} \exp\Big(T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial}{\partial x} b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \end{aligned} \tag{49}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, which establishes (i).

Next, observe that (42), Lemma 2.5.(ii) and the triangle inequality imply that

$$\begin{aligned} \|\tfrac{\partial}{\partial x_i}\mathcal{R}_s^{t,x} - \tfrac{\partial}{\partial x_i}R_s^{t,x}\|_{\mathbb{R}^d} &\leq \int_0^s \big\|\big[(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x})\big](\tfrac{\partial}{\partial x_i}\mathcal{R}_r^{t,x}) - \big[(\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\big](\tfrac{\partial}{\partial x_i}R_r^{t,x})\big\|_{\mathbb{R}^d}\, dr \\ &\leq \int_0^s \|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \|\tfrac{\partial}{\partial x_i}\mathcal{R}_r^{t,x} - \tfrac{\partial}{\partial x_i}R_r^{t,x}\|_{\mathbb{R}^d}\, dr \\ &\quad + \int_0^s \|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \|\tfrac{\partial}{\partial x_i}R_r^{t,x}\|_{\mathbb{R}^d}\, dr \end{aligned} \tag{50}$$

for all $i \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Therefore, Lemma 2.5.(iii) and the assumption that $\mathfrak{c}$ has bounded partial derivatives with respect to the x-variables yield

$$\begin{aligned} \|\tfrac{\partial}{\partial x_i}\mathcal{R}_s^{t,x} - \tfrac{\partial}{\partial x_i}R_s^{t,x}\|_{\mathbb{R}^d} &\leq \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\big\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \int_0^s \|\tfrac{\partial}{\partial x_i}\mathcal{R}_r^{t,x} - \tfrac{\partial}{\partial x_i}R_r^{t,x}\|_{\mathbb{R}^d}\, dr \\ &\quad + T \sup\nolimits_{r\in[0,t]} \big\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \\ &\quad \times \exp\Big(T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \end{aligned} \tag{51}$$

for all $i \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Gronwall's inequality (cf., e.g., [24, Lemma 2.11]) hence ensures that

$$\begin{aligned} &\|\tfrac{\partial}{\partial x_i}\mathcal{R}_s^{t,x} - \tfrac{\partial}{\partial x_i}R_s^{t,x}\|_{\mathbb{R}^d} \leq T \sup\nolimits_{r\in[0,t]} \big\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\big\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \\ &\times \exp\Big(T\Big[\sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} + \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \big\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\big\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big]\Big) \end{aligned} \tag{52}$$

for all $i \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. This shows (ii).

Now, note that it follows from (42) and Lemma 2.5.(iv) that

$$\begin{aligned}\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{R}_s^{t,x} - \tfrac{\partial^2}{\partial x_i \partial x_j}R_s^{t,x} &= \int_0^s \big[(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}\mathcal{R}_r^{t,x}, \tfrac{\partial}{\partial x_j}\mathcal{R}_r^{t,x} - \tfrac{\partial}{\partial x_j}R_r^{t,x}\big)\,dr \\ &+ \int_0^s \big[(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}\mathcal{R}_r^{t,x} - \tfrac{\partial}{\partial x_i}R_r^{t,x}, \tfrac{\partial}{\partial x_j}R_r^{t,x}\big)\,dr \\ &+ \int_0^s \big[(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial^2}{\partial x^2}b)(t-r,R_r^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_r^{t,x}, \tfrac{\partial}{\partial x_j}R_r^{t,x}\big)\,dr \\ &+ \int_0^s \big[(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{R}_r^{t,x} - \tfrac{\partial^2}{\partial x_i \partial x_j}R_r^{t,x}\big)\,dr \\ &+ \int_0^s \big[(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j}R_r^{t,x}\big)\,dr\end{aligned} \tag{53}$$

for all $i,j \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $s \in [0,s]$ and $x \in \mathbb{R}^d$. (ii) together with (iii) and (v) of Lemma 2.5 therefore ensure that

$$\begin{aligned}&\|\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{R}_s^{t,x} - \tfrac{\partial^2}{\partial x_i \partial x_j}R_s^{t,x}\|_{\mathbb{R}^d} \\ &\le 2T^2 \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(r,y)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\ &\quad \times \exp\Big(3T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \max\Big\{\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}, \|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big\}\Big) \\ &\quad \times \sup\nolimits_{r\in[0,t]}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \\ &+ T\exp\Big(2T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\ &\quad \times \sup\nolimits_{r\in[0,t]}\|(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial^2}{\partial x^2}b)(t-r,R_r^{t,x})\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\ &+ T^2 \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial^2}{\partial x^2}b)(r,y)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\ &\quad \times \exp\Big(3T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\ &\quad \times \sup\nolimits_{r\in[0,t]}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-u,R_u^{t,x})\|_{\mathbb{R}^d} \\ &+ \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \int_0^s \|\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{R}_r^{t,x} - \tfrac{\partial^2}{\partial x_i \partial x_j}R_r^{t,x}\|_{\mathbb{R}^d}\,dr\end{aligned} \tag{54}$$

for all $i,j \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $s \in [0,t]$ $x \in \mathbb{R}^d$. So it follows from Gronwall's integral inequality (cf., e.g., [24, Lemma 2.11]) that

$$
\begin{aligned}
&\|\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{R}_s^{t,x} - \tfrac{\partial^2}{\partial x_i \partial x_j} R_s^{t,x}\|_{\mathbb{R}^d} \\
&\leq \exp\Big(T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\
&\times \Bigg[2T^2 \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(r,y)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\
&\qquad \times \exp\Big(3T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \max\Big\{ \|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}, \|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \Big\}\Big) \\
&\qquad \times \sup\nolimits_{s\in[0,t]} \|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-s,\mathcal{R}_s^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-s,R_s^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \\
&\quad + T \exp\Big(2T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\
&\qquad \times \sup\nolimits_{r\in[0,t]} \|(\tfrac{\partial^2}{\partial x^2}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial^2}{\partial x^2}b)(t-r,R_r^{t,x})\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\
&\quad + T^2 \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial^2}{\partial x^2}b)(r,y)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \\
&\qquad \times \exp\Big(3T \sup\nolimits_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x}\mathfrak{b})(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) \\
&\qquad \times \sup\nolimits_{r\in[0,t]} \|(\tfrac{\partial}{\partial x}\mathfrak{b})(t-r,\mathcal{R}_r^{t,x}) - (\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\|_{\mathbb{R}^d} \Bigg]
\end{aligned} \tag{55}
$$

for all $i,j \in \{1,2,\ldots,d\}$, $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. This shows (iii) and completes the proof of the lemma. □

2.5 Differentiability properties of certain random fields defined in terms of ODEs with additive noise

Lemma 2.7. *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and $\sigma \in \mathbb{R}^{d\times d}$. Let $b \in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ have bounded partial derivatives of first and second order with respect to the x-variables. Let $\varphi \in C^2(\mathbb{R}^d,[0,\infty))$ and $B \in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty$ and the first and second order partial derivatives of B with respect to the x-variables are at most polynomially growing. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting a standard Brownian motion $U\colon [0,T]\times\Omega\to\mathbb{R}^d$ with continuous sample paths. Let $R^{t,x} = (R_s^{t,x})_{s\in[0,t]}\colon [0,t]\times\Omega\to\mathbb{R}^d$, $t\in[0,T]$, $x\in\mathbb{R}^d$, be stochastic processes satisfying*

$$R_s^{t,x} = x + \int_0^s b(t-r,R_r^{t,x})\,dr + \sigma U_r \tag{56}$$

for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and $s\in[0,t]$. Let the function $\mathcal{U}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ be given by

$$\mathcal{U}(t,x) = \varphi(R_t^{t,x})\exp\Big(\int_0^t B(t-s,R_s^{t,x})\,ds\Big) \tag{57}$$

for all $t\in[0,T]$ and $x\in\mathbb{R}^d$. Then

(i) *for all $\omega\in\Omega$, the mapping $[0,T]\times\mathbb{R}^d \ni (t,x)\mapsto \mathcal{U}(t,x,\omega)\in\mathbb{R}$ is in $C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$,*

(ii)
$$(\tfrac{\partial}{\partial x_i}\mathcal{U})(t,x) = \exp\Big(\int\limits_0^t B(t-s,R_s^{t,x})\,ds\Big)$$
$$\times\Big[\big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_t^{t,x}\big) + \varphi(R_t^{t,x})\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}\big)\,ds\Big] \quad (58)$$

for all $i \in \{1,2,\dots,d\}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, *and*

(iii)
$$\begin{aligned}(\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{U})(t,x) = \exp\Big(\int\limits_0^t B(t-s,R_s^{t,x})\,ds\Big)\Big[&\big[(\tfrac{\partial^2}{\partial x^2}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_t^{t,x}, \tfrac{\partial}{\partial x_j}R_t^{t,x}\big)\\
&+ \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j}R_t^{t,x}\big)\\
&+ \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_j}R_t^{t,x}\big)\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}\big)\,ds\\
&+ \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_t^{t,x}\big)\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_j}R_s^{t,x}\big)\,ds\\
&+ \varphi(R_t^{t,x})\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_j}R_s^{t,x}\big)\,ds\\
&\quad\times\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}\big)\,ds\\
&+ \varphi(R_t^{t,x})\int_0^t \big[(\tfrac{\partial^2}{\partial x^2}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}, \tfrac{\partial}{\partial x_j}R_s^{t,x}\big)\,ds\\
&+ \varphi(R_t^{t,x})\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j}R_s^{t,x}\big)\Big]\,ds\Big].\end{aligned} \quad (59)$$

for all $i,j \in \{1,2,\dots,d\}$, $t \in [0,T]$ *and* $x \in \mathbb{R}^d$.

Proof. It follows from Lemma 2.5.(i), the assumptions on b, φ, B, the chain rule, the fundamental theorem of calculus, (56), and (57) that for all $t \in [0,T]$ and $\omega \in \Omega$, the mapping $\mathbb{R}^d \ni x \mapsto \mathcal{U}(t,x,\omega) \in \mathbb{R}$ belongs to $C^1(\mathbb{R}^d,\mathbb{R})$ and

$$\begin{aligned}(\tfrac{\partial}{\partial x_i}\mathcal{U})(t,x) &= \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_t^{t,x}\big)\exp\Big(\int\limits_0^t B(t-s,R_s^{t,x})\,ds\Big)\\
&\quad+ \varphi(R_t^{t,x})\exp\Big(\int\limits_0^t B(t-s,R_s^{t,x})\,ds\Big)\int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}\big)\,ds\end{aligned} \quad (60)$$

for all $i \in \{1,2,\dots,d\}$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. This shows (ii).

Similarly, it follows from Lemma 2.5.(i), (ii), and the assumptions that for all $t \in [0,T]$ and

$\omega \in \Omega$, the mapping $\mathbb{R}^d \ni x \mapsto \mathcal{U}(t,x,\omega) \in \mathbb{R}$ is in $C^2(\mathbb{R}^d,\mathbb{R})$ and

$$\begin{aligned}
&\exp\Big(-\int_0^t B(t-s,R_s^{t,x})\,ds\Big)\big(\tfrac{\partial^2}{\partial x_i \partial x_j}\mathcal{U}\big)(t,x) \\
&= \big[(\tfrac{\partial^2}{\partial x^2}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_t^{t,x}, \tfrac{\partial}{\partial x_j}R_t^{t,x}\big) + \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j}R_t^{t,x}\big) \\
&+ \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_j}R_t^{t,x}\big) \int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}\big)\,ds \\
&+ \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_t^{t,x}\big) \int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_j}R_s^{t,x}\big)\,ds \\
&+ \varphi(R_t^{t,x}) \int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_j}R_s^{t,x}\big)\,ds \int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}\big)\,ds \\
&+ \varphi(R_t^{t,x}) \int_0^t \big[(\tfrac{\partial^2}{\partial x^2}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial}{\partial x_i}R_s^{t,x}, \tfrac{\partial}{\partial x_j}R_s^{t,x}\big)\,ds \\
&+ \varphi(R_t^{t,x}) \int_0^t \big[(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\big]\big(\tfrac{\partial^2}{\partial x_i \partial x_j}R_s^{t,x}\big)\,ds
\end{aligned} \tag{61}$$

for all $i,j \in \{1,2,\dots,d\}$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. This shows (i) and (iii) and completes the proof of the lemma. □

Lemma 2.8. *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and $\sigma \in \mathbb{R}^{d\times d}$. Let $b \in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ have bounded partial derivatives of first and second order with respect to the x-variables. Let $\varphi \in C^2(\mathbb{R}^d,[0,\infty))$ and $B \in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ have at most polynomially growing partial derivatives of first and second order with respect to the x-variables, and assume that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting a standard Brownian motion $U\colon [0,T]\times\Omega \to \mathbb{R}^d$ with continuous sample paths. Let $R^{t,x} = (R_s^{t,x})_{s\in[0,t]}\colon [0,t]\times\Omega \to \mathbb{R}^d$, $t\in[0,T]$, $x \in \mathbb{R}^d$, be stochastic processes satisfying*

$$R_s^{t,x} = x + \int_0^s b(t-r,R_r^{t,x})\,dr + \sigma U_s \quad \text{for all } t\in[0,T],\ s\in[0,t] \text{ and } x\in\mathbb{R}^d, \tag{62}$$

let $\mathcal{U}\colon [0,T]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$ and $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be given by

$$\mathcal{U}(t,x) = \varphi(R_t^{t,x}) \exp\Big(\int_0^t B(t-s,R_s^{t,x})\,ds\Big) \quad \text{and} \quad u(t,x) = \mathbb{E}\big[\mathcal{U}(t,x)\big] \tag{63}$$

for $x \in \mathbb{R}^d$ and $t\in[0,T]$. Then

(i) *$u \in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$,*

(ii) *$(\frac{\partial}{\partial x}u)(t,x) = \mathbb{E}\big[(\frac{\partial}{\partial x}\mathcal{U})(t,x)\big]$ for all $t\in[0,T]$, $x\in\mathbb{R}^d$, and*

(iii) *$(\frac{\partial^2}{\partial x^2}u)(t,x) = \mathbb{E}\big[(\frac{\partial^2}{\partial x^2}\mathcal{U})(t,x)\big]$ for all $t\in[0,T]$ and $x\in\mathbb{R}^d$.*

Proof. Lemma 2.4 and the assumption that $\varphi \in C^2(\mathbb{R}^d, [0,\infty))$ has at most polynomially growing partial derivatives of first and second order ensure that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} \big(|\varphi(R_s^{t,x})|^p + \|(\tfrac{\partial}{\partial x}\varphi)(R_s^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R})}^p + \|(\tfrac{\partial^2}{\partial x^2}\varphi)(R_s^{t,x})\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}^p\big)\Big] < \infty \quad (64)$$

for all $p \in (0,\infty)$. Similarly, Lemma 2.4 and the assumption that $B \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ has at most polynomially growing partial derivatives of first and second order with respect to the x-variables guarantee that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} \big(|B(t-s, R_s^{t,x})|^p + \|(\tfrac{\partial}{\partial x}B)(t-s, R_s^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R})}^p + \|(\tfrac{\partial^2}{\partial x^2}B)(t-s, R_s^{t,x})\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}^p\big)\Big] < \infty \quad (65)$$

for all $p \in (0,\infty)$. By Lemma 2.4 and the assumption that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty$, one also has

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in[-p,p]^d} \big|\exp\big(\int_0^s B(t-r, R_r^{t,x})\,dr\big)\big|^p\Big] < \infty \quad (66)$$

for all $p \in (0,\infty)$. Combining this, (64), and (65) with (63) and Hölder's inequality shows that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} |\mathcal{U}(t,x)|^p\Big] < \infty \quad \text{for all } p \in (0,\infty),$$

which together with the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]), and Lemma 2.7.(i) implies that $u \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$. Next, we note that (64)–(66), Lemma 2.7.(ii), Lemma 2.5.(iii), and Hölder's inequality imply that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \sup_{x\in[-p,p]^d} \|(\tfrac{\partial}{\partial x}\mathcal{U})(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})}^p\Big] < \infty \quad (67)$$

for all $p \in (0,\infty)$. Therefore, one obtains from Lemma 2.7.(i), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]), and the fundamental theorem of calculus that

(a) for all $t \in [0,T]$, the mapping $\mathbb{R}^d \ni x \mapsto u(t,x) \in \mathbb{R}$ is in $C^1(\mathbb{R}^d, \mathbb{R})$,

(b) for all $i \in \{1, 2, \ldots, d\}$, the mapping $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \big(\tfrac{\partial}{\partial x_i}u)(t,x) \in \mathbb{R}\big)$ is in $C([0,T] \times \mathbb{R}^d, \mathbb{R})$, and

(c) $(\tfrac{\partial}{\partial x}u)(t,x) = \mathbb{E}[(\tfrac{\partial}{\partial x}\mathcal{U})(t,x)]$ for all $x \in \mathbb{R}^d$ and $t \in [0,T]$.

This establishes (ii).

Next, note that it follows from (64)–(66), Lemma 2.7.(iii), items (iii) and (v) of Lemma 2.5, and Hölder's inequality that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \sup_{x\in[-p,p]} \|(\tfrac{\partial^2}{\partial x^2}\mathcal{U})(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}^p\Big] < \infty \quad (68)$$

for all $p \in (0,\infty)$. Hence, one obtains from Lemma 2.7.(i), (a)–(b) above, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]), and the fundamental theorem of calculus that

(A) for all $t \in [0,T]$, the mapping $\mathbb{R}^d \ni x \mapsto u(t,x) \in \mathbb{R}$ is in $C^2(\mathbb{R}^d,\mathbb{R})$,

(B) for all $i,j \in \{1,2,\dots,d\}$, the mapping $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto (\frac{\partial^2}{\partial x_i \partial x_j}u)(t,x) \in \mathbb{R}$ is in $C([0,T]\times\mathbb{R}^d,\mathbb{R})$,

(C) $(\frac{\partial^2}{\partial x^2}u)(t,x) = \mathbb{E}[(\frac{\partial^2}{\partial x^2}\mathcal{U})(t,x)]$ for all $x \in \mathbb{R}^d$ and $t \in [0,T]$.

(C) directly establishes (iii). Moreover, (a)–(c), (A)–(C), and the fact that $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ imply (i), which completes the proof of the lemma. □

3 Feynman–Kac formulas and proof of Theorem 1.1

In this section we derive the Feynman–Kac representation of Proposition 3.2 below from the results of Section 2 and use it to prove Theorem 1.1.

3.1 Feynman–Kac representations for linear parabolic partial differential equations

The following lemma is a stepping stone towards the proof of the Feynman–Kac representation of Proposition 3.2. It makes stronger regularity assumptions on the coefficients. Proposition 3.2 can be derived from it by mollifying the coefficients.

Lemma 3.1. *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and $\sigma \in \mathbb{R}^{d\times d}$. Let $b \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ have a bounded partial derivative with respect to t and bounded partial derivatives of first and second order with respect to the x-variables. Let $\varphi \in C^3(\mathbb{R}^d,[0,\infty))$ have at most polynomially growing partial derivatives of first, second and third order, let $B \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ have an at most polynomially growing partial derivative with respect to t and at most polynomially growing partial derivatives of first and second order with respect to the x variables. In addition, assume that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty$, and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space supporting a standard Brownian motion $U\colon [0,T]\times\Omega \to \mathbb{R}^d$ with continuous sample paths. Let $R^{t,x}\colon [0,t]\times\Omega \to \mathbb{R}^d$, $t \in [0,T]$, $x \in \mathbb{R}^d$, be stochastic processes satisfying*

$$R^{t,x}_s = x + \int_0^s b(t-r, R^{t,x}_r)\,dr + \sigma U_s \quad \text{for all } t \in [0,T],\ s \in [0,t] \text{ and } x \in \mathbb{R}^d. \tag{69}$$

Consider the mappings $\mathcal{U}\colon [0,T]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$ and $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ given by

$$\mathcal{U}(t,x) = \varphi(R^{t,x}_t)\exp\Big(\int_0^t B(t-s, R^{t,x}_s)\,ds\Big) \quad \text{and} \quad u(t,x) = \mathbb{E}\big[\mathcal{U}(t,x)\big] \tag{70}$$

for $t \in [0,T]$ and $x \in \mathbb{R}^d$. Then

(i) $u \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ *and*

(ii) $u(t,x)$

$$= \varphi(x) + \int_0^t \left[\tfrac{1}{2} \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(s,x)\big) + \langle b(s,x), (\nabla_x u)(s,x)\rangle_{\mathbb{R}^d} + B(s,x)u(s,x) \right] ds \tag{71}$$

for all $t \in [0,T]$ *and* $x \in \mathbb{R}^d$.

Proof. Let $\mathcal{V}\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ be given by

$$\mathcal{V}(t,x) = \varphi(x) + \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \Big[\tfrac{1}{2} \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_s^{t,x})\big) + \langle (\nabla\varphi)(R_s^{t,x}), b(t-s, R_s^{t,x})\rangle_{\mathbb{R}^d} + B(t-s, R_s^{t,x})\varphi(R_s^{t,x}) \Big] ds \quad \text{for } t \in [0,T] \text{ and } x \in \mathbb{R}^d. \tag{72}$$

It follows from Itô's formula and (69) that

$$\varphi(R_s^{t,x}) = \varphi(x) + \int_0^s \langle (\nabla\varphi)(R_r^{t,x}), \sigma\, dU_r\rangle_{\mathbb{R}^d} + \int_0^s \Big[\langle (\nabla\varphi)(R_r^{t,x}), b(t-r, R_r^{t,x})\rangle_{\mathbb{R}^d} + \tfrac{1}{2} \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_r^{t,x})\big) \Big] dr \quad \mathbb{P}\text{-a.s.} \tag{73}$$

for all $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. In addition, one has

$$\exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) = 1 + \int_0^s \exp\Big(\int_0^r B(t-w, R_w^{t,x})\, dw\Big)\, B(t-r, R_r^{t,x})\, dr \tag{74}$$

for all $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Combining this, (70), (72), (73) and, again Itô's formula, shows that

$$\begin{aligned} \mathcal{U}(t,x) &= \varphi(R_t^{t,x}) \exp\Big(\int_0^t B(t-s, R_s^{t,x})\, ds\Big) \\ &= \varphi(x) + \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \langle (\nabla\varphi)(R_s^{t,x}), \sigma\, dU_s\rangle_{\mathbb{R}^d} \\ &\quad + \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \langle (\nabla\varphi)(R_s^{t,x}), b(t-s, R_s^{t,x})\rangle_{\mathbb{R}^d}\, ds \\ &\quad + \tfrac{1}{2} \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_s^{t,x})\big)\, ds \\ &\quad + \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) B(t-s, R_s^{t,x})\varphi(R_s^{t,x})\, ds \\ &= \mathcal{V}(t,x) + \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \langle (\nabla\varphi)(R_s^{t,x}), \sigma\, dU_s\rangle_{\mathbb{R}^d} \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{75}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Now, observe that Lemma 2.4 and the assumption that $\varphi \in C^3(\mathbb{R}^d,[0,\infty))$ has at most polynomially growing partial derivatives of first, second and third order imply that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\big(|\varphi(R_s^{t,x})|^p + \|(\tfrac{\partial}{\partial x}\varphi)(R_s^{t,x})\|^p_{L(\mathbb{R}^d,\mathbb{R})} + \|(\tfrac{\partial^2}{\partial x^2}\varphi)(R_s^{t,x})\|^p_{L^{(2)}(\mathbb{R}^d,\mathbb{R})} + \|(\tfrac{\partial^3}{\partial x^3}\varphi)(R_s^{t,x})\|^p_{L^{(3)}(\mathbb{R}^d,\mathbb{R})}\big)\Big] < \infty \quad \text{for all } p \in (0,\infty). \tag{76}$$

Similarly, Lemma 2.4 and the assumption that the partial derivative of $B \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ with respect to t as well as its first and second order partial derivatives with respect to the x-variables are at most polynomially growing ensure that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\big(|B(t-s,R_s^{t,x})|^p + |(\tfrac{\partial}{\partial t}B)(t-s,R_s^{t,x})|^p + \|(\tfrac{\partial}{\partial x}B)(t-s,R_s^{t,x})\|^p_{L(\mathbb{R}^d,\mathbb{R})} + \|(\tfrac{\partial^2}{\partial x^2}B)(t-s,R_s^{t,x})\|^p_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}\big)\Big] < \infty \quad \text{for all } p \in (0,\infty). \tag{77}$$

Furthermore, Lemma 2.4 and the assumption $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty$ imply that

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\Big|\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\Big|^p\Bigg] < \infty \quad \text{for all } p \in (0,\infty). \tag{78}$$

From this, (76), Lemma 2.4, and Hölder's inequality one obtains

$$\mathbb{E}\bigg[\int_0^t \Big\|\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\sigma^*(\nabla\varphi)(R_s^{t,x})\Big\|^2_{\mathbb{R}^d}\,ds\bigg] < \infty \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^d. \tag{79}$$

Next, note that it follows from the assumptions on $b \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ that it grows at most linearly. Combining this with (70), (75), (76)–(78), and Fubini's theorem gives

$$\begin{aligned}
u(t,x) &= \mathbb{E}\big[\mathcal{U}(t,x)\big] = \mathbb{E}\big[\mathcal{V}(t,x)\big] \\
&= \varphi(x) + \int_0^t \mathbb{E}\Big[\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\langle(\nabla\varphi)(R_s^{t,x}), b(t-s,R_s^{t,x})\rangle_{\mathbb{R}^d}\Big]\,ds \\
&+ \tfrac{1}{2}\int_0^t \mathbb{E}\Big[\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_s^{t,x})\big)\Big]\,ds \\
&+ \int_0^t \mathbb{E}\Big[\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\,B(t-s,R_s^{t,x})\varphi(R_s^{t,x})\Big]\,ds \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^d.
\end{aligned} \tag{80}$$

From (69), the assumption that $b \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ has bounded partial derivatives, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]), one obtains that for all $\omega \in \Omega$ and $s \in [0,T]$, the mapping $[s,T]\times\mathbb{R}^d \ni (t,x) \mapsto R_s^{t,x} \in \mathbb{R}^d$ is in $C^{1,0}([s,T]\times\mathbb{R}^d,\mathbb{R}^d)$ with

$$\tfrac{\partial}{\partial t}R_s^{t,x} = \int_0^s \Big[(\tfrac{\partial}{\partial t}b)(t-r,R_r^{t,x}) + \big[(\tfrac{\partial}{\partial x}b)(t-r,R_r^{t,x})\big]\big(\tfrac{\partial}{\partial t}R_r^{t,x}\big)\Big]\,dr \tag{81}$$

(cf. also, e.g., [29, Theorem 4.6.5]). This and the assumption that $b \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives imply that

$$\begin{aligned}
&\|\tfrac{\partial}{\partial t} R_s^{t,x}\|_{\mathbb{R}^d} \leq T \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial t} b)(r,y)\|_{\mathbb{R}^d} \\
&\quad + \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \int_0^s \|\tfrac{\partial}{\partial t} R_r^{t,x}\|_{\mathbb{R}^d}\, dr \quad \text{for all } t\in[0,T],\ s\in[0,t] \text{ and } x \in \mathbb{R}^d.
\end{aligned} \tag{82}$$

Gronwall's integral inequality (cf., e.g., [24, Lemma 2.11]) therefore ensures that

$$\begin{aligned}
&\sup_{t\in[0,T]} \sup_{s\in[0,t]} \sup_{x\in\mathbb{R}^d} \|\tfrac{\partial}{\partial t} R_s^{t,x}\|_{\mathbb{R}^d} \\
&\leq T\big[\sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial t} b)(r,y)\|_{\mathbb{R}^d}\big] \exp\Big(T \sup_{(r,y)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\Big) < \infty.
\end{aligned} \tag{83}$$

Next observe that (81), Lemma 2.4, the assumptions on b, φ, B, the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) imply that

(a)
$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \big[(\tfrac{\partial}{\partial x}\varphi)(R_s^{t,x})\big]\big(b(t-s, R_s^{t,x})\big)\, ds \bigg) \\
&= \exp\Big(\int_0^t B(t-r, R_r^{t,x})\, dr\Big) \big[(\tfrac{\partial}{\partial x}\varphi)(R_t^{t,x})\big]\big(b(0, R_t^{t,x})\big) \\
&+ \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \big[(\tfrac{\partial}{\partial x}\varphi)(R_s^{t,x})\big]\big(b(t-s, R_s^{t,x})\big) \\
&\quad \times \int_0^s \Big[(\tfrac{\partial}{\partial t} B)(t-r, R_r^{t,x}) + \big[(\tfrac{\partial}{\partial x} B)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial}{\partial t} R_r^{t,x}\big)\Big]\, dr\, ds \\
&+ \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \big[(\tfrac{\partial^2}{\partial x^2}\varphi)(R_s^{t,x})\big]\big(\tfrac{\partial}{\partial t} R_s^{t,x}, \mathfrak{b}(t-s, R_s^{t,x})\big)\, ds \\
&+ \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \\
&\quad \times \big[(\tfrac{\partial}{\partial x}\varphi)(R_s^{t,x})\big]\Big[(\tfrac{\partial}{\partial t} b)(t-s, R_s^{t,x}) + \big[(\tfrac{\partial}{\partial x} b)(t-s, R_s^{t,x})\big]\big(\tfrac{\partial}{\partial t} R_s^{t,x}\big)\Big]\, ds
\end{aligned} \tag{84}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$,

(b)
$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_s^{t,x})\big)\, ds \bigg) \\
&= \exp\Big(\int_0^t B(t-r, R_r^{t,x})\, dr\Big) \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_t^{t,x})\big) \\
&+ \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}\varphi)(R_s^{t,x})\big) \\
&\quad \times \Big[\int_0^s \Big((\tfrac{\partial}{\partial t} B)(t-r, R_r^{t,x}) + \big[(\tfrac{\partial}{\partial x} B)(t-r, R_r^{t,x})\big]\big(\tfrac{\partial}{\partial t} R_r^{t,x}\big) \Big)\, dr\Big]\, ds \\
&+ \int_0^t \exp\Big(\int_0^s B(t-r, R_r^{t,x})\, dr\Big) \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*\big[(\tfrac{\partial}{\partial x} \operatorname{Hess}\varphi)(R_s^{t,x})\big]\big(\tfrac{\partial}{\partial t} R_s^{t,x}\big)\big)\, ds
\end{aligned} \tag{85}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, and

(c)

$$\begin{aligned}
&\frac{d}{dt}\biggl(\int_0^t \exp\Bigl(\int_0^s B(t-r, R_r^{t,x})\,dr\Bigr)\, B(t-s, R_s^{t,x})\,\varphi(R_s^{t,x})\,ds\biggr)\\
&= \exp\Bigl(\int_0^t B(t-r, R_r^{t,x})\,dr\Bigr) B(0, R_t^{t,x})\,\varphi(R_t^{t,x})\\
&+ \int_0^t \exp\Bigl(\int_0^s B(t-r, R_r^{t,x})\,dr\Bigr) B(t-s, R_s^{t,x})\,\varphi(R_s^{t,x})\\
&\times \int_0^s \Bigl[(\tfrac{\partial}{\partial t}B)(t-r, R_r^{t,x}) + \bigl[(\tfrac{\partial}{\partial x}B)(t-r, R_r^{t,x})\bigr]\bigl(\tfrac{\partial}{\partial t}R_r^{t,x}\bigr)\Bigr]\,dr\,ds \qquad (86)\\
&+ \int_0^t \exp\Bigl(\int_0^s B(t-r, R_r^{t,x})\,dr\Bigr)\varphi(R_s^{t,x})\\
&\times \Bigl[(\tfrac{\partial}{\partial t}B)(t-s, R_s^{t,x}) + \bigl[(\tfrac{\partial}{\partial x}B)(t-s, R_s^{t,x})\bigr]\bigl(\tfrac{\partial}{\partial t}R_s^{t,x}\bigr)\Bigr]\,ds\\
&+ \int_0^t \exp\Bigl(\int_0^s B(t-r, R_r^{t,x})\,dr\Bigr) B(t-s, R_s^{t,x})\bigl[(\tfrac{\partial}{\partial x}\varphi)(R_s^{t,x})\bigr]\bigl(\tfrac{\partial}{\partial t}R_s^{t,x}\bigr)\,ds
\end{aligned}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

Combining (72), (76)–(78), (83), and the assumption that $b \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives with (a)–(c) and Hölder's inequality shows that

$$\mathbb{E}\Bigl[\sup_{t\in[0,T]} \sup_{x\in[-p,p]^d} |(\tfrac{\partial}{\partial t}\mathcal{V})(t,x)|^p\Bigr] < \infty \quad \text{for all } p \in (0,\infty). \qquad (87)$$

This, (80), (a)–(c), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]), and the fundamental theorem of calculus ensure that

(A) for all $x \in \mathbb{R}^d$, the mapping $[0,T] \ni t \mapsto u(t,x) \in \mathbb{R}$ is in $C^1([0,T], \mathbb{R})$,

(B) the mapping $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto (\frac{\partial}{\partial t}u)(t,x) \in \mathbb{R}$ is in $C([0,T] \times \mathbb{R}^d, \mathbb{R})$, and

(C) $(\frac{\partial}{\partial t}u)(t,x) = \mathbb{E}[(\frac{\partial}{\partial t}\mathcal{V})(t,x)]$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$,

which, together with Lemma 2.8.(i), implies (i).

Next, note that it follows from the Markov property of $(R_s^{t,x})_{s\in[0,t]}$ that

$$\mathbb{E}\Bigl[G((R_{h+s}^{t,x})_{s\in[0,t-h]})\mathbb{1}_A(R_h^{t,x})\Bigr] = \int_A \mathbb{E}\Bigl[G((R_s^{t-h,y})_{s\in[0,t-h]})\Bigr] R_h^{t,x}(\mathbb{P})(dy) \qquad (88)$$

for all Borel subsets $A \subseteq \mathbb{R}^d$, $t \in [0,T]$, $h \in [0,t]$, $x \in \mathbb{R}^d$ and bounded functions $G \in C(C([0,t-$

$h],\mathbb{R}^d),\mathbb{R})$, which together with (70), implies that

$$\begin{aligned}
u(t,x) &= \mathbb{E}\big[\mathcal{U}(t,x)\big] = \mathbb{E}\Big[\varphi(R^{t,x}_t)\exp\Big(\int_0^t B(t-s,R^{t,x}_s)\,ds\Big)\Big] \\
&= \mathbb{E}\Big[\exp\Big(\int_0^h B(t-s,R^{t,x}_s)\,ds\Big)\mathbb{E}\Big[\varphi(R^{t,x}_t)\exp\Big(\int_h^t B(t-s,R^{t,x}_s)\,ds\Big)\,\Big|\,\mathfrak{S}(R^{t,x}_s, 0\le s\le h)\Big]\Big] \\
&= \mathbb{E}\Big[\exp\Big(\int_0^h B(t-s,R^{t,x}_s)\,ds\Big)\mathbb{E}\Big[\varphi(R^{t,x}_t)\exp\Big(\int_h^t B(t-s,R^{t,x}_s)\,ds\Big)\,\Big|\,\mathfrak{S}(R^{t,x}_h)\Big]\Big] \\
&= \mathbb{E}\Big[\exp\Big(\int_0^h B(t-s,R^{t,x}_s)\,ds\Big)\mathbb{E}\Big[\varphi(R^{t-h,y}_{t-h})\exp\Big(\int_0^{t-h} B(t-h-s,R^{t-h,y}_s)\,ds\Big)\Big]\Big|_{y=R^{t,x}_h}\Big] \\
&= \mathbb{E}\Big[\exp\Big(\int_0^h B(t-s,R^{t,x}_s)\,ds\Big)u(t-h,R^{t,x}_h)\Big].
\end{aligned} \tag{89}$$

Furthermore, (69), (i), and Itô's formula assure that

$$\begin{aligned}
u\big(t-h,R^{t,x}_h\big) &= u(t,x) - \int_0^h (\tfrac{\partial}{\partial t}u)(t-s,R^{t,x}_s)\,ds + \int_0^h \big[(\tfrac{\partial}{\partial x}u)(t-s,R^{t,x}_s)\big]\big(b(t-s,R^{t,x}_s)\big)\,ds \\
&\quad + \int_0^h \langle(\tfrac{\partial}{\partial x}u)(t-s,R^{t,x}_s),\sigma\,dU_s\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\int_0^h \operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(t-s,R^{t,x}_s)\big)\,ds \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{90}$$

for all $t\in[0,T]$, $h\in[0,t]$ and $x\in\mathbb{R}^d$. Combining this, (74), and Itô's formula ensures that

$$\begin{aligned}
&\exp\Big(\int_0^h B(t-s,R^{t,x}_s)\,ds\Big)u\big(t-h,R^{t,x}_h\big) \\
&= u(t,x) - \int_0^h \exp\Big(\int_0^s B(t-r,R^{t,x}_r)\,dr\Big)(\tfrac{\partial}{\partial t}u)(t-s,R^{t,x}_s)\,ds \\
&+ \int_0^h \exp\Big(\int_0^s B(t-r,R^{t,x}_r)\,dr\Big)\big[(\tfrac{\partial}{\partial x}u)(t-s,R^{t,x}_s)\big]\big(b(t-s,R^{t,x}_s)\big)\,ds \\
&+ \int_0^h \exp\Big(\int_0^s B(t-r,R^{t,x}_r)\,dr\Big)\big[(\tfrac{\partial}{\partial x}u)(t-s,R^{t,x}_s)\big]\sigma\,dU_s \\
&+ \tfrac{1}{2}\int_0^h \exp\Big(\int_0^s B(t-r,R^{t,x}_r)\,dr\Big)\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(t-s,R^{t,x}_s)\big)\,ds \\
&+ \int_0^h \exp\Big(\int_0^s B(t-r,R^{t,x}_r)\,dr\Big)B(t-s,R^{t,x}_s)\,u(t-s,R^{t,x}_s)\,ds \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{91}$$

for all $t\in[0,T]$, $h\in[0,t]$ and $x\in\mathbb{R}^d$. Moreover, it follows from Lemma 2.4, Lemma 2.7, Lemma 2.8, (a)–(c), (A)–(C), and the assumptions on b, φ and B that

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\Big(|u(t-s,R_s^{t,x})|^p+\|(\tfrac{\partial}{\partial x}u)(t-s,R_s^{t,x})\|_{L(\mathbb{R}^d,\mathbb{R})} + \|(\tfrac{\partial^2}{\partial x^2}u)(t-s,R_s^{t,x})\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}\Big)\Bigg]<\infty \quad \text{for all } p\in(0,\infty). \tag{92}$$

This ensures that

$$\mathbb{E}\bigg[\int_0^t\Big\|\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\sigma^*(\tfrac{\partial}{\partial t}u)(t-s,R_s^{t,x})\Big\|_{\mathbb{R}^d}^2\,ds\bigg]<\infty \quad \text{for all } t\in[0,T] \text{ and } x\in\mathbb{R}^d, \tag{93}$$

which, together with (89), (91), (92) and Fubini's theorem yields

$$\begin{aligned}0&=\mathbb{E}\Big[\exp\Big(\int_0^h B(t-s,R_s^{t,x})\,ds\Big)u\big(t-h,R_h^{t,x}\big)-u(t,x)\Big]\\&=\int_0^h\mathbb{E}\Big[\exp\Big(\int_0^s B(t-r,R_r^{t,x})\,dr\Big)\Big(-(\tfrac{\partial}{\partial t}u)(t-s,R_s^{t,x})+\tfrac12\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(t-s,R_s^{t,x})\big)\\&\qquad+\big[(\tfrac{\partial}{\partial x}u)(t-s,R_s^{t,x})\big]\big(u(t-s,R_s^{t,x})\big)+B(t-s,R_s^{t,x})\,u(t-s,R_s^{t,x})\Big)\Big]\,ds\end{aligned} \tag{94}$$

for all $t\in(0,T]$, $h\in[0,t]$ and $x\in\mathbb{R}^d$. This, (i), (76)–(78), (92), Lemma 2.5, and Lemma 2.7 imply that

$$\begin{aligned}0&=\lim_{h\downarrow 0}\frac1h\mathbb{E}\Big[\exp\Big(\int_0^h B(t-s,R_s^{t,x})\,ds\Big)u\big(t-h,R_h^{t,x}\big)-u(t,x)\Big]\\&=-(\tfrac{\partial}{\partial t}u)(t,x)+\tfrac12\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(t,x)\big)+\langle b(t,x),(\tfrac{\partial}{\partial t}u)(t,x)\rangle_{\mathbb{R}^d}+B(t,x)u(t,x)\end{aligned} \tag{95}$$

for all $t\in(0,T]$ and $x\in\mathbb{R}^d$. Moreover, note that (69) and (70) ensure that $u(0,x)=\varphi(x)$ for all $x\in\mathbb{R}^d$. Combining this with (95) and (i) proves (ii), which completes the proof. □

Proposition 3.2. *Let $T,c,\alpha\in(0,\infty)$, $d\in\mathbb{N}$, and $\sigma\in\mathbb{R}^{d\times d}$. Let $b\in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ have bounded partial derivatives of first and second order with respect to the x-variables, let $\varphi\in C^2(\mathbb{R}^d,[0,\infty))$ have at most polynomially growing partial derivatives of first and second order, and let $B\in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ have at most polynomially growing partial derivatives of first and second order with respect to the x-variables. In addition, assume that*

$$\|b(t,x)-b(s,x)\|_{\mathbb{R}^d}+\|(\tfrac{\partial^n}{\partial x^n}b)(t,x)-(\tfrac{\partial^n}{\partial x^n}b)(s,x)\|_{L^{(n)}(\mathbb{R}^d,\mathbb{R}^d)}\le c|t-s|^\alpha, \tag{96}$$

$$B(t,x)\le c(1+\|x\|_{\mathbb{R}^d}),\ \textit{and} \tag{97}$$

$$|B(t,x)-B(s,x)|+\|(\tfrac{\partial^n}{\partial x^n}B)(t,x)-(\tfrac{\partial^n}{\partial x^n}B)(s,x)\|_{L^{(n)}(\mathbb{R}^d,\mathbb{R})}\le c(1+\|x\|_{\mathbb{R}^d})|t-s|^\alpha \tag{98}$$

for all $n \in \{1,2\}$, $s,t \in [0,T]$ and $x \in \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion $U\colon [0,T]\times\Omega \to \mathbb{R}^d$ with continuous sample paths. Consider stochastic processes $R^{t,x} = (R^{t,x})_{s\in[0,t]}\colon [0,t]\times\Omega \to \mathbb{R}^d$, $t \in [0,T]$, $x \in \mathbb{R}^d$, satisfying

$$R_s^{t,x} = x + \int_0^s b(t-r, R_r^{t,x})\,dr + \sigma U_s \tag{99}$$

for all $t \in [0,T]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$, and let the function $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be given by

$$u(t,x) = \mathbb{E}\bigg[\varphi(R_t^{t,x})\exp\Big(\int_0^t B(t-s, R_s^{t,x})\,ds\Big)\bigg] \quad \textit{for } t \in [0,T] \textit{ and } x \in \mathbb{R}^d. \tag{100}$$

Then

(i) *u belongs to $C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, and*

(ii) $u(t,x)$

$$= \varphi(x) + \int_0^t \Big[\tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(s,x)\big) + \langle b(s,x), (\nabla_x u)(s,x)\rangle_{\mathbb{R}^d} + B(s,x)\,u(s,x)\Big]\,ds \tag{101}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

Proof. Throughout this proof fix a $q \in [1,\infty)$ such that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\left[\frac{|\varphi(t,x)| + \|(\frac{\partial}{\partial x}\varphi)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} + \|(\frac{\partial^2}{\partial x^2}\varphi)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}}{(1+\|x\|_{\mathbb{R}^d})^q}\right] < \infty. \tag{102}$$

and

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\left[\frac{|B(t,x)| + \|(\frac{\partial}{\partial x}B(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} + \|(\frac{\partial^2}{\partial x^2}B)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}}{(1+\|x\|_{\mathbb{R}^d})^q}\right] < \infty. \tag{103}$$

Let the mappings $\varphi_n\colon \mathbb{R}^d \to [0,\infty)$ and $b_n, B_n\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{N}$, be given by

$$\varphi_n(x) = \big(\tfrac{n}{2\pi}\big)^{d/2}\int_{\mathbb{R}^d}\varphi(y)\exp\big(-\tfrac{n}{2}\|x-y\|_{\mathbb{R}^d}^2\big)\,dy, \tag{104}$$

$$b_n(t,x) = \big(\tfrac{n}{2\pi}\big)^{1/2}\int_{-\infty}^{\infty} b\big(\min\{T,\max\{s,0\}\},x\big)\exp\Big(\tfrac{-n(t-s)^2}{2}\Big)\,ds, \tag{105}$$

$$B_n(t,x) = \big(\tfrac{n}{2\pi}\big)^{1/2}\int_{-\infty}^{\infty} B\big(\min\{T,\max\{s,0\}\},x\big)\exp\Big(\tfrac{-n(t-s)^2}{2}\Big)\,ds \tag{106}$$

for all $n \in \mathbb{N}$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. Now, consider $R^{n,t,x} = (R_s^{n,t,x})_{s\in[0,t]}\colon [0,t]\times\Omega \to \mathbb{R}^d$, $n \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $\mathcal{U}_n\colon [0,T]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$, $n \in \mathbb{N}_0$, and $u_n\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$, $n \in \mathbb{N}$, given by

$$R_s^{n,t,x} = x + \int_0^s b_n(t-r, R_r^{n,t,x})\,dr + \sigma U_s, \tag{107}$$

$$\mathcal{U}_0(t,x) = \varphi(R_t^{t,x}) \exp\Big(\int_0^t B(t-s, R_s^{t,x})\, ds\Big), \quad \mathcal{U}_n(t,x) = \varphi_n(R_t^{n,t,x}) \exp\Big(\int_0^t B_n(t-s, R_s^{n,t,x})\, ds\Big), \tag{108}$$

and

$$u_n(t,x) = \mathbb{E}[\mathcal{U}_n(t,x)] \tag{109}$$

for $n \in \mathbb{N}$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. (102) and (104) imply that $(\varphi_n)_{n\in\mathbb{N}} \subseteq C^3(\mathbb{R}^d, \mathbb{R})$ as well as

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \left[\frac{|\varphi_n(t,x)| + \|(\frac{\partial}{\partial x}\varphi_n)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} + \|(\frac{\partial^2}{\partial x^2}\varphi_n)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}}{(1+\|x\|_{\mathbb{R}^d})^q} \right] < \infty. \tag{110}$$

Furthermore, since $\exp(-s^2/2)$ is even in $s \in \mathbb{R}$, it follows from (105) that

$$\begin{aligned} (\tfrac{\partial}{\partial t} b_n)(t,x) &= \int_{-\infty}^{\infty} b\big(\min\{T, \max\{s,0\}\}, x\big) n(s-t) \exp\big(\tfrac{-n(t-s)^2}{2}\big)\, ds \\ &= \big(\tfrac{n}{2\pi}\big)^{1/2} \int_{-\infty}^{\infty} \Big[b\big(\min\{T,\max\{s,0\}\},x\big) - b\big(t,x\big)\Big] n(s-t) \exp\big(\tfrac{-n(t-s)^2}{2}\big)\, ds. \end{aligned} \tag{111}$$

Moreover, one obtains from (96) and the fact that $|\min\{T,\max\{s,0\}\} - t| \le |t-s|$ for all $t \in [0,T]$ and $s \in \mathbb{R}$ that

$$\begin{aligned} |(\tfrac{\partial}{\partial t} b_n)(t,x)| &\le \big(\tfrac{n}{2\pi}\big)^{1/2} \int_{-\infty}^{\infty} c\,|\min\{T,\max\{s,0\}\} - t|^{\alpha}\, n|s-t| \exp\big(\tfrac{-n(t-s)^2}{2}\big)\, ds \\ &\le c\big(\tfrac{n}{2\pi}\big)^{1/2} \int_{-\infty}^{\infty} n|s-t|^{1+\alpha} \exp\big(\tfrac{-n(t-s)^2}{2}\big)\, ds \le \tfrac{c}{\sqrt{2\pi}} n^{(1-\alpha)/2} \int_{-\infty}^{\infty} |z|^{1+\alpha} \exp\big(\tfrac{-|z|^2}{2}\big)\, dz \end{aligned} \tag{112}$$

for all $n \in \mathbb{N}$, $t \in [0,T]$ and $x \in \mathbb{R}^d$. Combining this with the assumption that $b \in C^{0,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ has bounded partial derivatives of first and second order with respect to the x-variables and (105) ensures that

(a) for all $n \in \mathbb{N}$, $b_n \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ has a bounded partial derivative with respect to t and bounded partial derivatives of first and second order with respect to the x-variables,

(b)
$$\sup_{n\in\mathbb{N}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b_n)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} \le \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial}{\partial x} b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} < \infty,$$

(c)
$$\sup_{n\in\mathbb{N}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial^2}{\partial x^2} b_n)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} \le \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|(\tfrac{\partial^2}{\partial x^2} b)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} < \infty.$$

Next, note that it follows from (106) and the assumption that $B(t,x) \le c(1 + \|x\|_{\mathbb{R}^d})$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$ that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \left[\frac{B_n(t,x)}{1+\|x\|_{\mathbb{R}^d}}\right] \le \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} \left[\frac{B(t,x)}{1+\|x\|_{\mathbb{R}^d}}\right] < \infty, \tag{113}$$

In addition, one obtains from (103) and (106) that $(B_n)_{n\in\mathbb{N}} \subseteq C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ and

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}\left[\frac{|B_n(t,x)| + \|(\frac{\partial}{\partial x}B_n)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} + \|(\frac{\partial^2}{\partial x^2}B_n)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}}{(1+\|x\|_{\mathbb{R}^d})^q}\right] < \infty. \tag{114}$$

Now, note that it follows from Lemma 2.8 and the assumptions on φ, b, and B that

(a') $u \in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$,

(b') $(\frac{\partial}{\partial x}u)(t,x) = \mathbb{E}[(\frac{\partial}{\partial x}\mathcal{U}_0)(t,x)]$ for all $t\in[0,T]$ and $x\in\mathbb{R}^d$, and

(c') $(\frac{\partial^2}{\partial x^2}u)(t,x) = \mathbb{E}[(\frac{\partial^2}{\partial x^2}\mathcal{U}_0)(t,x)]$ for all $t\in[0,T]$ and $x\in\mathbb{R}^d$.

By Lemma 2.8, Lemma 3.1, (a)–(c) and (114)–(110), one has

(A) $u_n \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ for all $n\in\mathbb{N}$,

(B) $(\frac{\partial}{\partial x}u_n)(t,x) = \mathbb{E}[(\frac{\partial}{\partial x}\mathcal{U}_n)(t,x)]$ for all $n\in\mathbb{N}$, $t\in[0,T]$ and $x\in\mathbb{R}^d$,

(C) $(\frac{\partial^2}{\partial x^2}u_n)(t,x) = \mathbb{E}[(\frac{\partial^2}{\partial x^2}\mathcal{U}_n)(t,x)]$ for all $n\in\mathbb{N}$, $t\in[0,T]$ and $x\in\mathbb{R}^d$, and

(D)
$$\begin{aligned} u_n(t,x) &= \varphi_n(x) \\ &+ \int_0^t \Big[\tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u_n)(s,x)\big) + \langle b_n(s,x),(\nabla_x u_n)(s,x)\rangle_{\mathbb{R}^d} + B_n(s,x)\,u_n(s,x)\Big]\,ds \end{aligned} \tag{115}$$

for all $n\in\mathbb{N}$, $t\in[0,T]$ and $x\in\mathbb{R}^d$. From Lemma 2.1.(i) together with (96), (105), (b) and (c) one obtains

(A') $\limsup_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|b_n(t,x)-b(t,x)\|_{\mathbb{R}^d} = 0$,

(B') $\limsup_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|(\frac{\partial}{\partial x}b_n)(t,x)-(\frac{\partial}{\partial x}b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)} = 0$, and

(C') $\limsup_{n\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|(\frac{\partial^2}{\partial x^2}b_n)(t,x)-(\frac{\partial^2}{\partial x^2}b)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} = 0$.

Combining Lemma 2.6.(i) with (A) gives

$$\begin{aligned} &\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in\mathbb{R}^d}\|R_s^{n,t,x}-R_s^{t,x}\|_{\mathbb{R}^d} \\ &\le T\exp\big(T\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\big\|(\tfrac{\partial}{\partial x}b)(r,y)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\big) \\ &\quad\times\limsup_{n\to\infty}\sup_{(r,y)\in[0,T]\times\mathbb{R}^d}\|b_n(r,y)-b(r,y)\|_{\mathbb{R}^d} = 0. \end{aligned} \tag{116}$$

Hence, we obtain from Lemma 2.4 that

$$\begin{aligned} &\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\exp\big(p\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|R_s^{n,t,x}\|_{\mathbb{R}^d}\big)\Big] \\ &\qquad\le\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\exp\big(p\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|R_s^{n,t,x}-R_s^{t,x}\|_{\mathbb{R}^d}\big) \\ &\qquad\times\exp\big(p\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|R_s^{t,x}\|_{\mathbb{R}^d}\big)\Big] < \infty \quad \text{for all } p\in(0,\infty). \end{aligned} \tag{117}$$

Moreover, Lemma 2.2, (B), and (116) yield

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\big\|(\tfrac{\partial}{\partial x}b_n)(t-s,R_s^{n,t,x})-(\tfrac{\partial}{\partial x}b)(t-s,R_s^{t,x})\big\|_{L(\mathbb{R}^d,\mathbb{R}^d)}=0 \tag{118}$$

for all $p\in(0,\infty)$, which together with Lemma 2.6.(ii),(b) and (117) shows that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|\tfrac{\partial}{\partial x_i}R_s^{n,t,x}-\tfrac{\partial}{\partial x_i}R_s^{t,x}\|_{\mathbb{R}^d}=0 \tag{119}$$

for all $i\in\{1,2,\dots,d\}$ and $p\in(0,\infty)$. Furthermore, it follows from Lemma 2.5.(iii) and (b) that

$$\begin{aligned}&\sup\nolimits_{n\in\mathbb{N}}\sup\nolimits_{t\in[0,T]}\sup\nolimits_{s\in[0,t]}\sup\nolimits_{x\in\mathbb{R}^d}\Big[\|\tfrac{\partial}{\partial x_i}R_s^{t,x}\|_{\mathbb{R}^d}+\|\tfrac{\partial}{\partial x_i}R_s^{n,t,x}\|_{\mathbb{R}^d}\Big]\\&\leq 2\exp\big(T\sup\nolimits_{(t,x)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\big)<\infty\quad\text{for all } i\in\{1,2,\dots,d\}.\end{aligned} \tag{120}$$

By Lemma 2.2, (C), and (116), we obtain that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\big\|(\tfrac{\partial^2}{\partial x^2}b_n)(t-s,R_s^{n,t,x})-(\tfrac{\partial^2}{\partial x^2}b)(t-s,R_s^{t,x})\big\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}=0 \tag{121}$$

for all $p\in(0,\infty)$. Lemma 2.6.(iii) (b), (c), and (118) hence assure that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{s\in[0,t]}\sup_{x\in[-p,p]^d}\|\tfrac{\partial^2}{\partial x_i\partial x_j}R_s^{n,t,x}-\tfrac{\partial^2}{\partial x_i\partial x_j}R_s^{t,x}\|_{\mathbb{R}^d}=0 \tag{122}$$

for all $i,j\in\{1,2,\dots,d\}$ and $p\in(0,\infty)$. Moreover, it follows from Lemma 2.5.(v), (b) and (c) that

$$\begin{aligned}&\sup\nolimits_{n\in\mathbb{N}}\sup\nolimits_{t\in[0,T]}\sup\nolimits_{s\in[0,t]}\sup\nolimits_{x\in\mathbb{R}^d}\big[\|\tfrac{\partial^2}{\partial x_i\partial x_j}R_s^{t,x}\|_{\mathbb{R}^d}+\|\tfrac{\partial^2}{\partial x_i\partial x_j}R_s^{n,t,x}\|_{\mathbb{R}^d}\big]\\&\leq 2T\big[\sup\nolimits_{(t,x)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial^2}{\partial x^2}b)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}\big]\exp\big(3T\sup\nolimits_{(t,x)\in[0,T]\times\mathbb{R}^d}\|(\tfrac{\partial}{\partial x}b)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R}^d)}\big)\\&<\infty\quad\text{for all } i,j\in\{1,2,\dots,d\}.\end{aligned} \tag{123}$$

In the next step, we note that Lemma 2.1, Lemma 2.2, (116), and (117) ensure that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}\sup_{x\in[-p,p]^d}|\mathcal{U}_n(t,x)-\mathcal{U}_0(t,x)|=0\quad\text{for all } p\in(0,\infty). \tag{124}$$

Moreover, it follows from Lemma 2.4 together with (102), (113), (110) and (117) that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{U}_n(t,x)-\mathcal{U}_0(t,x)|^p\right]<\infty\quad\text{for all } x\in\mathbb{R}^d \text{ and } p\in(0,\infty). \tag{125}$$

Combining this and (124) with the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]) and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) implies that

$$\limsup_{n\to\infty}\sup_{t\in[0,T]}|u_n(t,x)-u(t,x)|\leq\limsup_{n\to\infty}\mathbb{E}\left[\sup\nolimits_{t\in[0,T]}|\mathcal{U}_n(t,x)-\mathcal{U}_0(t,x)|\right]=0 \tag{126}$$

for all for $x \in \mathbb{R}^d$. Next, note that Lemma 2.1, Lemma 2.2, Lemma 2.7.(ii), (98), (102), (116), and (119) ensure that

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{x\in[-p,p]^d} \|(\tfrac{\partial}{\partial x}\mathcal{U}_n)(t,x) - (\tfrac{\partial}{\partial x}\mathcal{U}_0)(t,x)\|_{L(\mathbb{R}^d,\mathbb{R})} = 0 \quad \text{for all } p \in (0,\infty). \tag{127}$$

Moreover, (102), (113), (110), (117), (120), Lemma 2.4, and Lemma 2.7.(ii) imply that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \|(\tfrac{\partial}{\partial x}\mathcal{U}_n)(t,x) - (\tfrac{\partial}{\partial x}\mathcal{U}_0)(t,x)\|^p_{L(\mathbb{R}^d,\mathbb{R})}\right] < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in (0,\infty). \tag{128}$$

It therefore follows from (a)–(b), (A)–(B), (127), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) that

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \|(\tfrac{\partial}{\partial x}u_n)(t,x) - (\tfrac{\partial}{\partial x}u)(t,x)\|_{\mathbb{R}^d} = 0 \quad \text{for all } x \in \mathbb{R}^d. \tag{129}$$

From Lemma 2.1, Lemma 2.2, Lemma 2.7.(iii), (98), (102), (116), (119), and (122) one obtains that

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{x\in[-p,p]^d} \|(\tfrac{\partial^2}{\partial x^2}\mathcal{U}_n)(t,x) - (\tfrac{\partial^2}{\partial x^2}\mathcal{U}_0)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} = 0 \quad \text{for all } p \in (0,\infty). \tag{130}$$

Moreover, observe that (102), (113), (110), (117), (120), (123), Lemma 2.4, and Lemma 2.7.(iii) show that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \|(\tfrac{\partial^2}{\partial x^2}\mathcal{U}_n)(t,x) - (\tfrac{\partial^2}{\partial x^2}\mathcal{U}_0)(t,x)\|^p_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)}\right] < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in (0,\infty). \tag{131}$$

(a), (c), (A), (C), (130), the de la Vallée Poussin theorem (cf., e.g., [28, Corollary 6.21]), and the Vitali convergence theorem (cf., e.g., [28, Theorem 6.25]) hence imply that

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \|(\tfrac{\partial^2}{\partial x^2}u_n)(t,x) - (\tfrac{\partial^2}{\partial x^2}u)(t,x)\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R}^d)} = 0 \quad \text{for all } x \in \mathbb{R}^d. \tag{132}$$

Moreover, note that (104) and Lemma 2.1.(ii) ensure that

$$\limsup_{n\to\infty} |\varphi_n(x) - \varphi(x)| = 0 \quad \text{for all } x \in \mathbb{R}^d. \tag{133}$$

It therefore follows from (D), (126), (129), and (132) that

$$\begin{aligned} &u(t,x) \\ &= \varphi(x) + \int_0^t \Big[\tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x u)(s,x)\big) + \langle u(s,x), (\nabla_x u)(s,x)\rangle_{\mathbb{R}^d} + B(s,x)\,u(s,x)\Big]\,ds \end{aligned} \tag{134}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, which establishes (ii). Moreover, (ii) and (a) imply (i), which finishes the proof of the proposition. □

3.2 Proof of Theorem 1.1

Throughout this proof, let

$$I = (I(t,x))_{t\in[0,T],x\in\mathbb{R}^d} = (I(t,x,\omega))_{t\in[0,T],x\in\mathbb{R}^d,\omega\in\Omega}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R} \tag{135}$$

be given by

$$I(t,x) = \langle h(x), V_t\rangle_{\mathbb{R}^d}, \quad t\in[0,T],\, x\in\mathbb{R}^d, \tag{136}$$

and $\mathcal{L}\colon C^2(\mathbb{R}^d,\mathbb{R})\to C(\mathbb{R}^d,\mathbb{R})$ by

$$(\mathcal{L}u)(x) = \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess} u)(x)\big) - \langle \mu(x), (\nabla u)(x)\rangle_{\mathbb{R}^d}, \quad u\in C^2(\mathbb{R}^d,\mathbb{R}),\, x\in\mathbb{R}^d. \tag{137}$$

Consider the mappings $b_v\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $v\in C([0,T],\mathbb{R}^d)$, given by

$$b_v(t,x) = \sigma\sigma^*[h'(x)]^* v(t) - \mu(x), \quad v\in C([0,T],\mathbb{R}^d),\, t\in[0,T],\, x\in\mathbb{R}^d, \tag{138}$$

and define

$$\mathfrak{u} = (\mathfrak{u}(t,x))_{t\in[0,T],x\in\mathbb{R}^d} = (\mathfrak{u}(t,x,\omega))_{t\in[0,T],x\in\mathbb{R}^d,\omega\in\Omega}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R} \tag{139}$$

by

$$\mathfrak{u}(t,x,\omega) = u_{V(\omega),Y(\omega)}(t,x), \quad t\in[0,T],\, x\in\mathbb{R}^d,\, \omega\in\Omega. \tag{140}$$

Observe that it follows from (4)–(8) that for all $t\in[0,T]$ and $x\in\mathbb{R}^d$, the mapping $X_t(x)\colon\Omega\to\mathbb{R}$ is $\mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable, which establishes (i).

Next, note that the assumptions on $\mu\in C^3(\mathbb{R}^d,\mathbb{R}^d)$ and $h\in C^4(\mathbb{R}^d,\mathbb{R}^d)$ together with (6) imply that

(a) for every $v\in C([0,T],\mathbb{R}^d)$, $b_v\in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ has bounded partial derivatives of first and second order with respect to the x-variables,

(b) $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{B_{v,y}(t,x)}{1+\|x\|_{\mathbb{R}^d}}<\infty$ for all $v,y\in C([0,T],\mathbb{R}^d)$, and

(c) for all $v,y\in C([0,T],\mathbb{R}^d)$, the mapping $B_{v,y}\in C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ has at most polynomially growing partial derivatives of first and second order with respect to the x-variables.

Moreover, note that it follows from (4) that

$$\inf_{\alpha\in(0,1)}\sup_{s\neq t\in[0,T]}\tfrac{\|Y_t-Y_s\|_{\mathbb{R}^d}}{|t-s|^\alpha} + \inf_{\alpha\in(0,1)}\sup_{s\neq t\in[0,T]}\tfrac{\|V_t-V_s\|_{\mathbb{R}^d}}{|t-s|^\alpha} < \infty \quad \mathbb{P}\text{-a.s.} \tag{141}$$

This, (6), and (138) assure that for $\mathbb{P}$-a.a. $\omega \in \Omega$ the following hold:

$$\inf_{\alpha\in(0,1)} \sup_{s\neq t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{\|b_{V(\omega)}(t,x) - b_{V(\omega)}(s,x)\|_{\mathbb{R}^d}}{|t-s|^\alpha} < \infty, \tag{142}$$

$$\inf_{\alpha\in(0,1)} \sup_{n\in\{1,2\}} \sup_{s\neq t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{\|(\frac{\partial^n}{\partial x^n} b_{V(\omega)})(t,x) - (\frac{\partial^n}{\partial x^n} b_{V(\omega)})(s,x)\|_{L^{(n)}(\mathbb{R}^d,\mathbb{R}^d)}}{|t-s|^\alpha} < \infty, \tag{143}$$

$$\inf_{\alpha\in(0,1)} \sup_{s\neq t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{|B_{V(\omega),Y(\omega)}(t,x) - B_{V(\omega),Y(\omega)}(s,x)|}{(1+\|x\|_{\mathbb{R}^d})|t-s|^\alpha} < \infty, \tag{144}$$

$$\inf_{\alpha\in(0,1)} \sup_{n\in\{1,2\}} \sup_{s\neq t\in[0,T]} \sup_{x\in\mathbb{R}^d} \frac{\|(\frac{\partial^n}{\partial x^n} B_{V(\omega),Y(\omega)})(t,x) - (\frac{\partial^n}{\partial x^n} B_{V(\omega),Y(\omega)})(s,x)\|_{L^{(n)}(\mathbb{R}^d,\mathbb{R})}}{(1+\|x\|_{\mathbb{R}^d})|t-s|^\alpha} < \infty, \tag{145}$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \tfrac{B_{V(\omega),Y(\omega)}(t,x)}{1+\|x\|_{\mathbb{R}^d}} < \infty. \tag{146}$$

(a)–(c), the assumption that $\varphi \in C^2(\mathbb{R}^d,[0,\infty))$ has at most polynomially growing derivatives up to the second order, Lemma 2.8, and Proposition 3.2 (with $b \leftarrow b_{V(\omega)}$, $R^{t,x} \leftarrow R^{V(\omega),t,x}$, $\varphi \leftarrow \varphi$, $B \leftarrow B_{V(\omega),Y(\omega)}$, $u \leftarrow u_{V(\omega),Y(\omega)}$ in the notation of Lemma 2.8 and Proposition 3.2) ensure the following:

(A) for all $\omega \in \Omega$, the mapping $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mathfrak{u}(t,x,\omega) \in \mathbb{R}$ is in $C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$,

(B) for $\mathbb{P}$-a.a. $\omega \in \Omega$, the mapping $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mathfrak{u}(t,x,\omega) \in \mathbb{R}$ is in $C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, and

(C)
$$\begin{aligned}\mathfrak{u}(t,x,\omega) = u_{V(\omega),Y(\omega)}(t,x) = \varphi(x) + \int_0^t \Big[&\tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess}_x \mathfrak{u})(s,x,\omega)\big) \\ &+ \big\langle \sigma\sigma^*[h'(x)]^* V_s(\omega) - \mu(x), (\nabla_x \mathfrak{u})(s,x,\omega)\big\rangle_{\mathbb{R}^d} + B_{V(\omega),Y(\omega)}(s,x)\mathfrak{u}(s,x,\omega)\Big]\,ds \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{147}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

It follows from the assumptions on $h \in C^4(\mathbb{R}^d,\mathbb{R}^d)$ that for all $\omega \in \Omega$, the mapping $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto I(t,x,\omega) \in \mathbb{R})$ is in $C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$. (A), (8), and the fact that $X_t(x) = e^{I(t,x)}\mathfrak{u}(t,x)$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$ imply that for every $\omega \in \Omega$, the mapping $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto X_t(x,\omega) \in \mathbb{R}$ is in $C^{0,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$. This establishes (ii).

In the next step, we note that the fact that

$$(\nabla_x I)(t,x) = [h'(x)]^* V_t \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^d, \tag{148}$$

(137), and (C) show that

$$\mathfrak{u}(t,x) = \varphi(x) + \int_0^t \big[(\mathcal{L}_x \mathfrak{u})(s,x) + \langle \sigma\sigma^*(\nabla_x I)(s,x), (\nabla_x \mathfrak{u})(s,x)\rangle_{\mathbb{R}^d} + B_{V,Y}(s,x)\,\mathfrak{u}(s,x)\big]\,ds \quad \mathbb{P}\text{-a.s.} \tag{149}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Now, consider the random field

$$\mathfrak{v} = (\mathfrak{v}(t,x))_{t\in[0,T],x\in\mathbb{R}^d} = (\mathfrak{v}(t,x,\omega))_{t\in[0,T],x\in\mathbb{R}^d,\omega\in\Omega} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R} \tag{150}$$

given by

$$\mathfrak{v}(t,x) = e^{-I(t,x)}(\mathcal{L}X_t)(x), \quad t \in [0,T],\, x \in \mathbb{R}^d. \tag{151}$$

The fact that $X_t(x) = e^{I(t,x)}\mathfrak{u}(t,x)$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, (A), and Lemma 3.3 below (applied for every $t \in [0,T]$, $\omega \in \Omega$ with $a \leftarrow \sigma\sigma^*$, $\nu \leftarrow \mu$, $u \leftarrow (\mathbb{R}^d \ni x \mapsto \mathfrak{u}(t,x,\omega) \in \mathbb{R}$ in the notation of Lemma 3.3) hence yield

$$\begin{aligned} \mathfrak{v}(t,x) &= (\mathcal{L}_x\mathfrak{u})(t,x) + \langle \sigma\sigma^*(\nabla_x I)(t,x), (\nabla_x \mathfrak{u})(t,x)\rangle_{\mathbb{R}^d} \\ &\quad + \big[\tfrac{1}{2}\langle \sigma\sigma^*(\nabla_x I)(t,x), (\nabla_x I)(t,x)\rangle_{\mathbb{R}^d} + (\mathcal{L}_x I)(t,x)\big]\mathfrak{u}(t,x) \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^d. \end{aligned} \tag{152}$$

So it follows from (B) and (149) that

$$\begin{aligned} &\mathfrak{u}(t,x) \\ &= \varphi(x) + \int_0^t \big[\mathfrak{v}(s,x) + \big(B_{V,Y}(s,x) - \tfrac{1}{2}\langle \sigma\sigma^*(\nabla_x I)(s,x), (\nabla_x I)(s,x)\rangle_{\mathbb{R}^d} + (\mathcal{L}_x I)(s,x)\big)\mathfrak{u}(s,x)\big]\, ds \end{aligned} \tag{153}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Combining this and (6) with the fact that $(\nabla_x I)(t,x) = [h'(x)]^* V_t$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$ shows that

$$\mathfrak{u}(t,x) = \varphi(x) + \int_0^t \big[\mathfrak{v}(s,x) + \big(\langle h(x), h(Y_s)\rangle_{\mathbb{R}^d} - \tfrac{1}{2}\|h(x)\|^2_{\mathbb{R}^d} - (\operatorname{div}\mu)(x)\big)\mathfrak{u}(s,x)\big]\, ds \quad \mathbb{P}\text{-a.s.} \tag{154}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Next, note that Itô's formula and the fact that $I(t,x) = \langle h(x), V_t\rangle_{\mathbb{R}^d}$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$ ensure that

$$e^{I(t,x)} = e^{I(0,x)} + \int_0^t e^{I(s,x)}\Big(\langle h(x), dV_s\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\|h(x)\|^2_{\mathbb{R}^d}\, ds\Big) \quad \mathbb{P}\text{-a.s for all } t \in [0,T] \text{ and } x \in \mathbb{R}^d. \tag{155}$$

Moreover, since V is a Brownian motion and $I(t,x) = \langle h(x), V_t\rangle_{\mathbb{R}^d}$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, it follows from (5), (7), and (8) that

$$X_0(x) = e^{I(0,x)}u_{V,Y}(0,x) = \mathfrak{u}(0,x) = \varphi(x) \quad \text{for all } x \in \mathbb{R}^d. \tag{156}$$

Itô's formula, the fact that $X_t(x) = e^{I(t,x)}u(t,x)$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, (154), and (155) show that

$$\begin{aligned} X_t(x) = \varphi(x) &+ \int_0^t e^{I(s,x)}\Big[\mathfrak{v}(s,x) + \Big(\langle h(x), h(Y_s)\rangle_{\mathbb{R}^d} - \tfrac{1}{2}\|h(x)\|^2_{\mathbb{R}^d} - (\operatorname{div}\mu)(x)\Big)\mathfrak{u}(s,x)\Big]\, ds \\ &+ \int_0^t \mathfrak{u}(s,x)\, e^{I(s,x)}\Big(\langle h(x), dV_s\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\|h(x)\|^2_{\mathbb{R}^d}\, ds\Big) \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{157}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. Since $X_t(x) = e^{I(t,x)}\mathfrak{u}(t,x)$ for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, it therefore follows from (151) that

$$\begin{aligned} X_t(x) = \varphi(x) + \int_0^t \big[(\mathcal{L}X_s)(x) - X_s(x)(\operatorname{div}\mu)(x)\big]\,ds \\ + \int_0^t X_s(x)\langle h(x), h(Y_s)\rangle_{\mathbb{R}^d}\,ds + \int_0^t X_s(x)\langle h(x), dV_s\rangle_{\mathbb{R}^d} \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{158}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. This, (ii), the fact that

$$(\operatorname{div}(\mu X_s))(x) = \langle \mu(x), (\nabla X_s)(x)\rangle_{\mathbb{R}^d} + X_s(x)(\operatorname{div}\mu)(x) \quad \text{for all } s \in [0,T] \text{ and } x \in \mathbb{R}^d, \tag{159}$$

(4), and (137) demonstrate that

$$X_t(x) = \varphi(x) + \int_0^t \Big[\tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma\sigma^*(\operatorname{Hess} X_s)(x)\big) - (\operatorname{div}(\mu X_s))(x)\Big]\,ds + \int_0^t X_s(x)\langle h(x), dZ_s\rangle_{\mathbb{R}^d} \tag{160}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and $x \in \mathbb{R}^d$. This establishes (iii) and completes the proof of Theorem 1.1. □

Lemma 3.3. *Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and consider the mapping $\mathcal{L}\colon C^2(\mathbb{R}^d,\mathbb{R}) \to C(\mathbb{R}^d,\mathbb{R})$ given by*

$$(\mathcal{L}u)(x) = \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(a(\operatorname{Hess} u)(x)\big) - \langle \nu(x), (\nabla u)(x)\rangle_{\mathbb{R}^d}, \quad u \in C^2(\mathbb{R}^d,\mathbb{R}),\ x \in \mathbb{R}^d, \tag{161}$$

where $a \in \mathbb{R}^{d\times d}$ is a symmetric matrix and ν a function in $C^1(\mathbb{R}^d,\mathbb{R}^d)$. Then

$$e^{-I(x)}\big(\mathcal{L}(e^I u)\big)(x) = (\mathcal{L}u)(x) + \langle a(\nabla I)(x), (\nabla u)(x)\rangle_{\mathbb{R}^d} + \big[\tfrac{1}{2}\langle a(\nabla I)(x), (\nabla I)(x)\rangle_{\mathbb{R}^d} + (\mathcal{L}I)(x)\big]u(x) \tag{162}$$

for all $I \in C^2(\mathbb{R}^d,\mathbb{R})$, $u \in C^2(\mathbb{R}^d,\mathbb{R})$ and $x \in \mathbb{R}^d$.

Proof. Let $I \in C^2(\mathbb{R}^d,\mathbb{R})$ and $u \in C^2(\mathbb{R}^d,\mathbb{R})$. One obtains from the product rule and chain rule that

$$\begin{aligned} \big(\nabla(e^I u)\big)(x) = u(x)(\nabla(e^I))(x) + e^{I(x)}(\nabla u)(x) = u(x)e^{I(x)}(\nabla I)(x) + e^{I(x)}(\nabla u)(x) \\ = e^{I(x)}\big[u(x)(\nabla I)(x) + (\nabla u)(x)\big] \quad \text{for all } x \in \mathbb{R}^d. \end{aligned} \tag{163}$$

A further application of the product rule and chain rule gives

$$\begin{aligned} \big(\tfrac{\partial^2}{\partial x_i \partial x_j}(e^I u)\big)(x) = e^{I(x)}(\tfrac{\partial}{\partial x_i}I)(x)\Big[(\tfrac{\partial}{\partial x_j}I)(x)\,u(x) + (\tfrac{\partial}{\partial x_j}u)(x)\Big] \\ + e^{I(x)}\Big[(\tfrac{\partial^2}{\partial x_i \partial x_j}I)(x)\,u(x) + (\tfrac{\partial}{\partial x_j}I)(\tfrac{\partial}{\partial x_i}u)(x) + (\tfrac{\partial^2}{\partial x_i \partial x_j}u)(x)\Big] \end{aligned} \tag{164}$$

for all $i,j \in \{1,2,\dots,d\}$ and $x \in \mathbb{R}^d$, which implies that

$$\begin{aligned} & e^{-I(x)}\big(\operatorname{Hess}(e^I u)\big)(x) \\ & = (\operatorname{Hess} u)(x) + (\nabla I)(x) \otimes (\nabla u)(x) + (\nabla u)(x) \otimes (\nabla I)(x) + u(x)\big[(\nabla I)(x) \otimes (\nabla I)(x) + (\operatorname{Hess} I)(x)\big] \end{aligned} \tag{165}$$

for all $x \in \mathbb{R}^d$. Since a is symmetric, one has $\operatorname{Trace}_{\mathbb{R}^d}(av \otimes w) = \langle av, w \rangle_{\mathbb{R}^d} = \langle aw, v \rangle_{\mathbb{R}^d} = \operatorname{Trace}_{\mathbb{R}^d}(aw \otimes v)$ vor all $v, w \in \mathbb{R}^d$. Therefore, one obtains from (161) and (163) that

$$\begin{aligned}
&e^{-I(x)}\big(\mathcal{L}(e^I u)\big)(x) = \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(ae^{-I(x)}(\operatorname{Hess}(e^I u))(x)\big) - \langle \nu(x), e^{-I(x)}(\nabla(e^I u))(x)\rangle_{\mathbb{R}^d} \\
&= \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(a(\operatorname{Hess} u)(x) + a(\nabla I)(x) \otimes (\nabla u)(x) + a(\nabla u)(x) \otimes (\nabla I)(x) \\
&\quad + au(x)(\nabla I)(x) \otimes (\nabla I)(x) + u(x)a(\operatorname{Hess} I)(x)\big) - \langle \nu(x), u(x)(\nabla I)(x) + (\nabla u)(x)\rangle_{\mathbb{R}^d} \\
&= \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(a(\operatorname{Hess} u)(x)\big) - \langle \nu(x), (\nabla u)(x)\rangle_{\mathbb{R}^d} \\
&\quad + \langle a(\nabla I)(x), (\nabla u)(x)\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\langle a(\nabla I)(x), (\nabla I)(x)\rangle_{\mathbb{R}^d} u(x) \\
&\quad + \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(a(\operatorname{Hess} I)(x)\big)u(x) - \langle \nu(x), (\nabla I)(x)\rangle_{\mathbb{R}^d} u(x) \\
&= (\mathcal{L}u)(x) + \langle a(\nabla I)(x), (\nabla u)(x)\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\langle a(\nabla I)(x), (\nabla I)(x)\rangle_{\mathbb{R}^d} u(x) + (\mathcal{L}I)(x)\, u(x),
\end{aligned} \tag{166}$$

for all $x \in \mathbb{R}^d$, which proves the lemma. □

4 Numerical experiments

Together with time-discretization, the trapezoidal rule, and Monte Carlo sampling, the Feynman–Kac type representation formula of Theorem 1.1 can be used to approximate the solution of a given Zakai equation of the form (2) along a realization of the observation process Z.

We illustrate this in the following example: Choose $\alpha, T \in (0, \infty)$, $\gamma \in \mathbb{R}$, $d \in \mathbb{N}$, and let $\sigma \in \mathbb{R}^{d \times d}$ be given by $\sigma_{ij} = d^{-1/2}$ for all $i, j \in \{1, \dots, d\}$. Consider a d-dimensional signal process with dynamics

$$Y_t = Y_0 + \int_0^t \gamma Y_s\big(1 + \|Y_s\|^2_{\mathbb{R}^d}\big)^{-1}\, ds + \sigma W_t, \qquad t \in [0, T], \tag{167}$$

for a random initial condition $Y_0 \colon \Omega \to \mathbb{R}^d$ with density $\varphi(x) = \big(\frac{\alpha}{2\pi}\big)^{d/2}\exp\big(-\frac{\alpha}{2}\|x\|^2_{\mathbb{R}^d}\big)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an independent standard Brownian motion $W \colon [0, T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths. Assume the observation process is given by

$$Z_t = \int_0^t \beta Y_s\, ds + V_t, \quad t \in [0, T], \tag{168}$$

for a constant $\beta \in \mathbb{R}$ and a standard Brownian motion $V \colon [0, T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths independent of Y_0 and W. Let $U \colon [0, T] \times \Omega \to \mathbb{R}^d$ be another standard Brownian motion with continuous sample paths and assume the stochastic processes $R^{v,t,x} \colon [0, t] \times \Omega \to \mathbb{R}^d$, $v \in C([0, T], \mathbb{R}^d)$, $t \in [0, T]$, $x \in \mathbb{R}^d$ satisfy

$$R^{v,t,x}_s = x + \int_0^s \Big[\sigma\sigma^* \beta v(t - r) - \gamma R^{v,t,x}_r\big(1 + \|R^{v,t,x}_r\|^2_{\mathbb{R}^d}\big)^{-1}\Big]\, dr + \sigma U_s \tag{169}$$

for all $v \in C([0, T], \mathbb{R}^d)$, $t \in [0, T]$, $s \in [0, t]$ and $x \in \mathbb{R}^d$. Let the mappings $B_{v,y}, u_{v,y} \colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$, $v, y \in C([0, T], \mathbb{R}^d)$ be given by

$$\begin{aligned}
B_{v,y}(t, x) =& \tfrac{1}{2}\langle \sigma\sigma^* \beta v(t), \beta v(t)\rangle_{\mathbb{R}^d} + \beta^2\langle x, y(t)\rangle_{\mathbb{R}^d} - \tfrac{\beta^2}{2}\|x\|^2_{\mathbb{R}^d} - \beta\gamma(1 + \|x\|^2_{\mathbb{R}^d})^{-1}\langle x, v(t)\rangle_{\mathbb{R}^d} \\
&- d\gamma(1 + \|x\|^2_{\mathbb{R}^d})^{-1} - 2\gamma\|x\|^2_{\mathbb{R}^d}(1 + \|x\|^2_{\mathbb{R}^d})^{-2}
\end{aligned} \tag{170}$$

and

$$u_{v,y}(t,x) = \mathbb{E}\Big[\varphi\Big(R_t^{v,t,x}\Big)\exp\Big(\int_0^t B_{v,y}(t-s, R_s^{v,t,x})\, ds\Big)\Big], \tag{171}$$

$v, y \in C([0,T], \mathbb{R}^d)$, $t \in [0,T]$, $x \in \mathbb{R}^d$. By Theorem 1.1, the random field $X\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ given by

$$X_t(x,\omega) = u_{V(\omega),Y(\omega)}(t,x)\exp\big(\langle \beta x, V_t(\omega)\rangle_{\mathbb{R}^d}\big), \quad t \in [0,T],\ x \in \mathbb{R}^d,\ \omega \in \Omega, \tag{172}$$

solves the Zakai equation

$$\begin{aligned} X_t(x) = \varphi(x) + \int_0^t X_s(x)\langle \beta x, dZ_s\rangle_{\mathbb{R}^d} \\ + \int_0^t \Big[\tfrac{1}{2}\textstyle\sum_{i,j=1}^d \big(\tfrac{\partial^2}{\partial x_i \partial x_j} X_s\big)(x) - \sum_{i=1}^d \tfrac{\partial}{\partial x_i}\big(\gamma x_i [1 + \|x\|_{\mathbb{R}^d}^2]^{-1} X_s(x)\big)\Big]\, ds \quad \mathbb{P}\text{-a.s.,} \end{aligned} \tag{173}$$

$t \in [0,T]$, $x \in \mathbb{R}^d$, corresponding to the dynamics (167)–(168).

We use formula (172) to approximate $X_T(x)$ for a given realization $(z(t))_{t\in[0,T]}$ of $(Z_t)_{t\in[0,T]}$. But instead of $Y(\omega)$ and $V(\omega)$, which in a typical filtering application, are not observable, we plug artificial realizations of $(Y_t)_{t\in[0,T]}$ and $(V_t)_{t\in[0,T]}$ that are consistent with $(z(t))_{t\in[0,T]}$ into (172). First, we create a typical path of $(Y_t)_{t\in[0,T]}$, by solving (167) with $Y_0 = 0$ and $W \equiv 0$. This gives $y(t) = 0$, $t \in [0,T]$. By (168), the corresponding realization of $(V_t)_{t\in[0,T]}$ then needs to be $v(t) = z(t)$, $t \in [0,T]$.

For numerical purposes we choose an $N \in \mathbb{N}$ and consider the following discretized versions of (167)–(168):

$$\mathcal{Y}_0 = Y_0, \quad \mathcal{Y}_n = \mathcal{Y}_{n-1} + \gamma \mathcal{Y}_{n-1}(1 + \|\mathcal{Y}_{n-1}\|_{\mathbb{R}^d}^2)^{-1}\tfrac{T}{N} + \sigma(W_{nT/N} - W_{(n-1)T/N}), \tag{174}$$

$$\mathcal{Z}_0 = 0, \quad \mathcal{Z}_n = \beta \sum_{m=1}^n \frac{\mathcal{Y}_{m-1} + \mathcal{Y}_m}{2}\frac{T}{N} + V_{nT/N},\ n \in \{1,\dots,N\}. \tag{175}$$

Let $U^{(i)}\colon [0,T] \times \Omega \to \mathbb{R}^d$, $i \in \mathbb{N}$, be i.i.d. standard Brownian motions independent of Y_0, W, V, and consider $R_n^{(x,i)}\colon \Omega \to \mathbb{R}^d$, $n \in \{0,1,\dots,N\}$, $x \in \mathbb{R}^d$, $i \in \mathbb{N}$, given by $R_0^{(i,x)} = x$ and

$$R_n^{(x,i)} = R_{n-1}^{(x,i)} + \Big[\sigma\sigma^*\beta\mathcal{Z}_{N-n+1} - \gamma R_{n-1}^{(x,i)}(1 + \|R_{n-1}^{(x,i)}\|_{\mathbb{R}^d}^2)^{-1}\Big]\frac{T}{N} + \sigma\Big(U_{nT/N}^{(i)} - U_{(n-1)T/N}^{(i)}\Big) \tag{176}$$

for $x \in \mathbb{R}^d$, $i \in \mathbb{N}$ and $n \in \{1,2,\dots,N\}$. Now, define the mappings $B_n, u_M\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$, $n \in \{0,1,\dots,N\}$, $M \in \mathbb{N}$, by

$$\begin{aligned} B_n(x) =& \tfrac{1}{2}\langle \sigma\sigma^*\beta\mathcal{Z}_n, \beta\mathcal{Z}_n\rangle_{\mathbb{R}^d} - \tfrac{\beta^2}{2}\|x\|_{\mathbb{R}^d}^2 - \beta\gamma(1 + \|x\|_{\mathbb{R}^d}^2)^{-1}\langle x, \mathcal{Z}_n\rangle_{\mathbb{R}^d} \\ & - d\gamma(1 + \|x\|_{\mathbb{R}^d}^2)^{-1} - 2\gamma\|x\|_{\mathbb{R}^d}^2(1 + \|x\|_{\mathbb{R}^d}^2)^{-1}, \quad n \in \{1,\dots,n\},\ x \in \mathbb{R}^d, \end{aligned} \tag{177}$$

and

$$u_M(x) = \frac{1}{M}\sum_{i=1}^M \varphi\big(R_N^{(x,i)}\big)\exp\left(\sum_{n=1}^N \frac{T}{2N}\Big[B_{N-n}(R_n^{(x,i)}) + B_{N-n+1}(R_{n-1}^{(x,i)})\Big]\right), \tag{178}$$

$M \in \mathbb{N}$, $x \in \mathbb{R}^d$. It follows from the law of large numbers that, for $M \to \infty$,

$$\mathcal{X}_M(x) = u_M(x) \exp(\langle \beta x, \mathcal{Z}_N \rangle_{\mathbb{R}^d}) \overset{\mathbb{P}\text{-a.s.}}{\longrightarrow} \mathbb{E}[\mathcal{X}_1(x) \mid \mathcal{Z}], \tag{179}$$

which approximates $X_T(x)$.

Table 1 below shows point estimates and 95% confidence intervals for $\mathbb{E}[\mathcal{X}_1(0) \mid \mathcal{Z}]$ for different realizations of $\mathcal{Z}$, $\alpha = 2\pi$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{4}$, $T = 1$ and $N = 100$. For every $d \in \{1, 2, 5, 10, 20, 50, 75, 100\}$ we simulated five realizations of $\mathcal{Z}$, for each of which we computed an estimate of $\mathbb{E}[\mathcal{X}_1(0) \mid \mathcal{Z}]$ by simulating a realization of $\mathcal{X}_M(0) = u_M(0)$ for $M = 4{,}096{,}000$. The 95% confidence intervals were approximated, using the central limit theorem, with

$$\left[\mathcal{X}_M(0) - \frac{s_M}{\sqrt{M}}\, q_{0.975}\,,\, \mathcal{X}_M(0) + \frac{s_M}{\sqrt{M}}\, q_{0.975}\right], \tag{180}$$

where $q_{0.975}$ is the 97.5%-quantile of the standard normal distribution and s_M^2 the sample variance of (178) given by

$$s_M^2 = \frac{1}{M-1} \sum_{i=1}^{M} \left\{ \varphi\left(R_N^{(x,i)}\right) \exp\left(\sum_{n=1}^{N} \frac{T}{2N} \left[B_{N-n}\left(R_n^{(x,i)}\right) + B_{N-n+1}\left(R_{n-1}^{(x,i)}\right)\right]\right) - u_M(0) \right\}^2 .$$

The reported runtimes are averages of the times needed to compute $\mathcal{X}_M(0)$ for five different realizations of $\mathcal{Z}$.

The plots in Figure 1 show $\mathcal{X}_M(x)$ for a realization of $\mathcal{Z}$ and different $x \in \mathbb{R}^d$ for $d \in \{1, 2\}$. Figures 2–4 show $\mathcal{X}_M(x, 0, \ldots, 0)$ for a realization of $\mathcal{Z}$ and different $x \in \mathbb{R}$ for $d \in \{5, 10, 20, 50, 75, 100\}$.

The numerical experiments presented in this section were implemented in Python using TensorFlow on a NVIDIA GeForce RTX 2080 Ti GPU. The underlying system was an AMD Ryzen 9 3950X CPU with 64 GB DDR4 memory running Tensorflow 2.1 on Ubuntu 19.10. The Python source codes can be found in the GitHub repository `https://github.com/seb-becker/zakai`.

Acknowledgements

A.J. acknowledges funding by the German Research Foundation (DFG) under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure. A.N. acknowledges funding by the Nanyang Assistant Professorship Grant (NAP Grant) Machine Learning Based Algorithms in Finance and Insurance.

d	Point est.	95% CI	avg. run time
1	0.332416	[0.332118, 0.332713]	8.1s
	0.334207	[0.333905, 0.334508]	
	0.334263	[0.333964, 0.334562]	
	0.337189	[0.336884, 0.337494]	
	0.336953	[0.336647, 0.337258]	
2	0.204586	[0.204337, 0.204835]	8.4s
	0.204517	[0.204270, 0.204764]	
	0.208730	[0.208479, 0.208982]	
	0.213430	[0.213136, 0.213724]	
	0.205862	[0.205615, 0.206108]	
5	0.083024	[0.082884, 0.083164]	9.4s
	0.086099	[0.085954, 0.086245]	
	0.084513	[0.084369, 0.084658]	
	0.084619	[0.084478, 0.084760]	
	0.083317	[0.083178, 0.083455]	
10	0.032556	[0.032480, 0.032633]	9.3s
	0.033489	[0.033408, 0.033570]	
	0.034450	[0.034367, 0.034533]	
	0.032063	[0.031993, 0.032133]	
	0.039904	[0.039794, 0.040014]	
20	0.009978	[0.009943, 0.010012]	9.2s
	0.010046	[0.010016, 0.010077]	
	0.010071	[0.010041, 0.010101]	
	0.009250	[0.009221, 0.009279]	
	0.009700	[0.009667, 0.009732]	
50	0.001833	[0.001822, 0.001845]	13.1s
	0.001593	[0.001581, 0.001606]	
	0.001246	[0.001230, 0.001261]	
	0.002024	[0.002011, 0.002037]	
	0.020861	[0.020402, 0.021320]	
75	0.000490	[0.000484, 0.000495]	19.4s
	0.000809	[0.000801, 0.000818]	
	0.000341	[0.000338, 0.000343]	
	0.000768	[0.000755, 0.000781]	
	0.000329	[0.000326, 0.000333]	
100	0.001034	[0.001025, 0.001044]	25.8s
	0.000170	[0.000167, 0.000173]	
	0.000159	[0.000156, 0.000163]	
	0.087215	[0.085490, 0.088941]	
	0.000571	[0.000534, 0.000609]	

Table 1: Approximations of $X_1(0)$ for the Zakai equation (173) for $d \in \{1, 2, 5, 10, 20, 50, 75, 100\}$

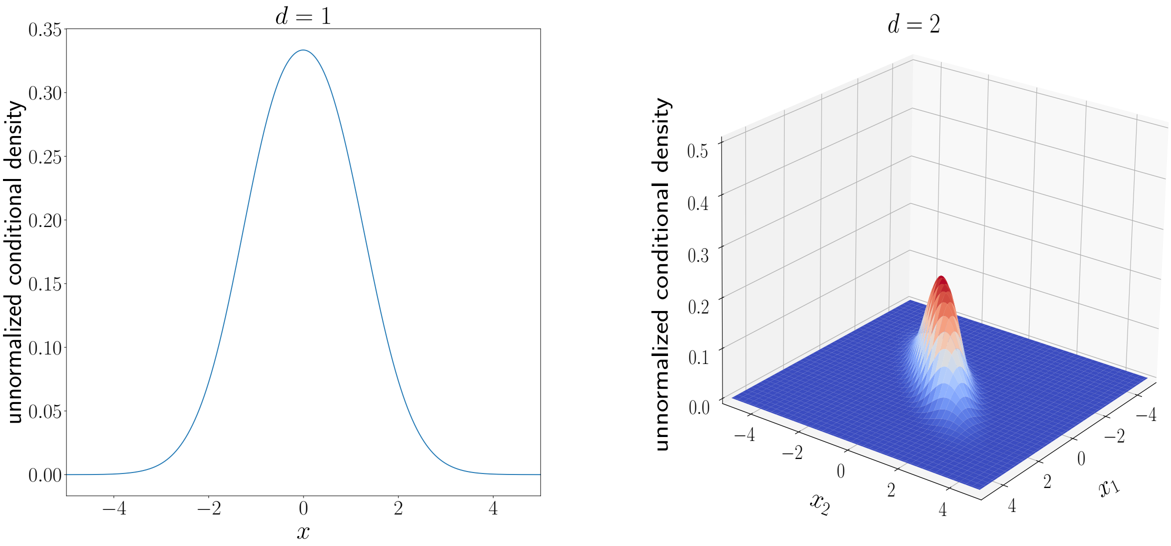

Figure 1: Approximations of $X_1(x)$, $x \in \mathbb{R}^d$, for $d \in \{1, 2\}$

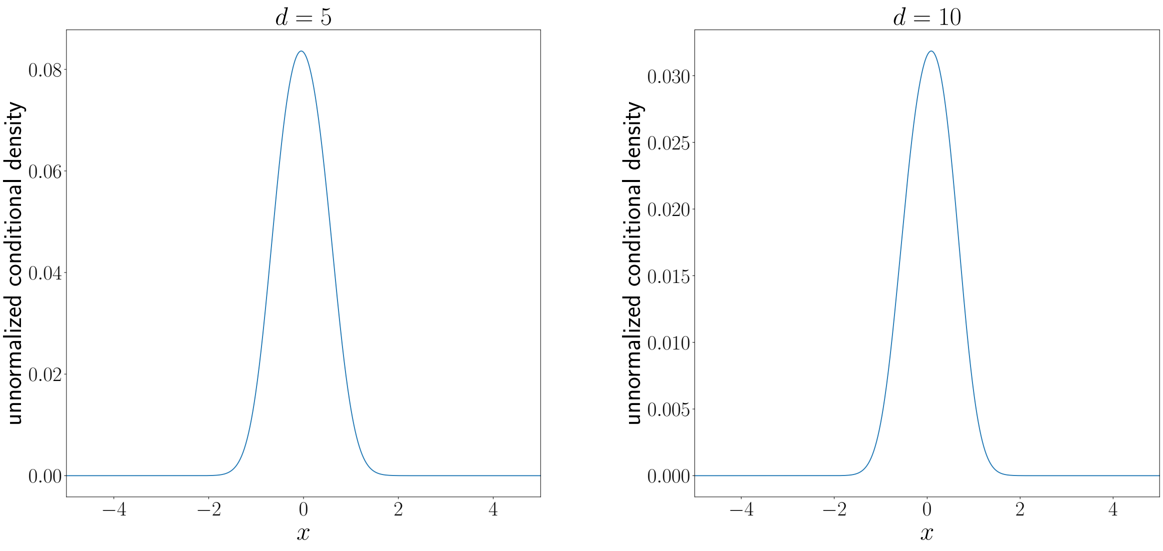

Figure 2: Approximations of $X_1(x, 0 \ldots, 0)$, $x \in \mathbb{R}$, for $d \in \{5, 10\}$

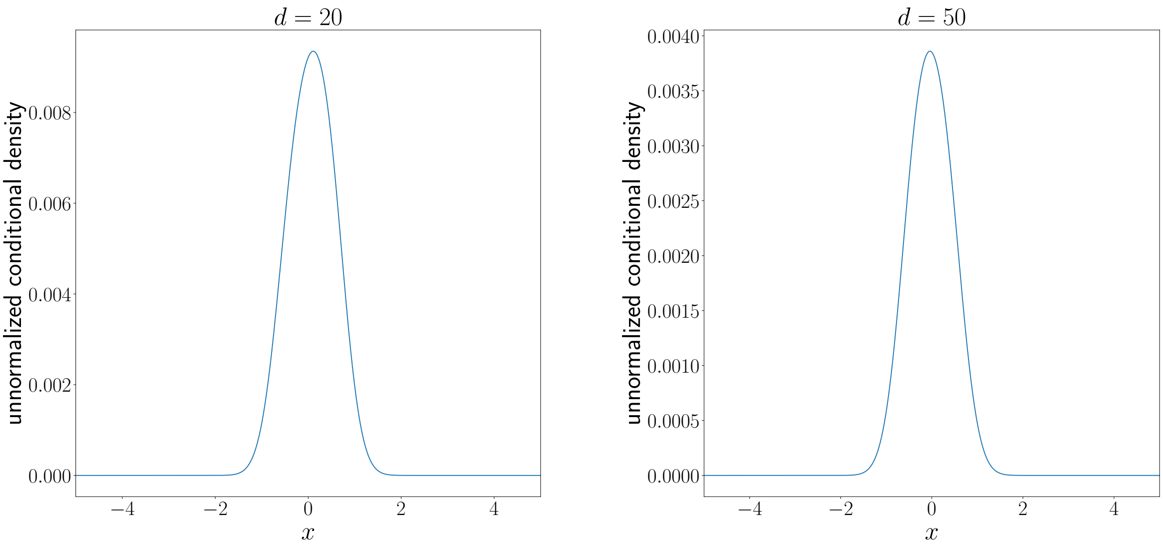

Figure 3: Approximations of $X_1(x, 0, \dots, 0)$, $x \in \mathbb{R}$, for $d \in \{20, 50\}$

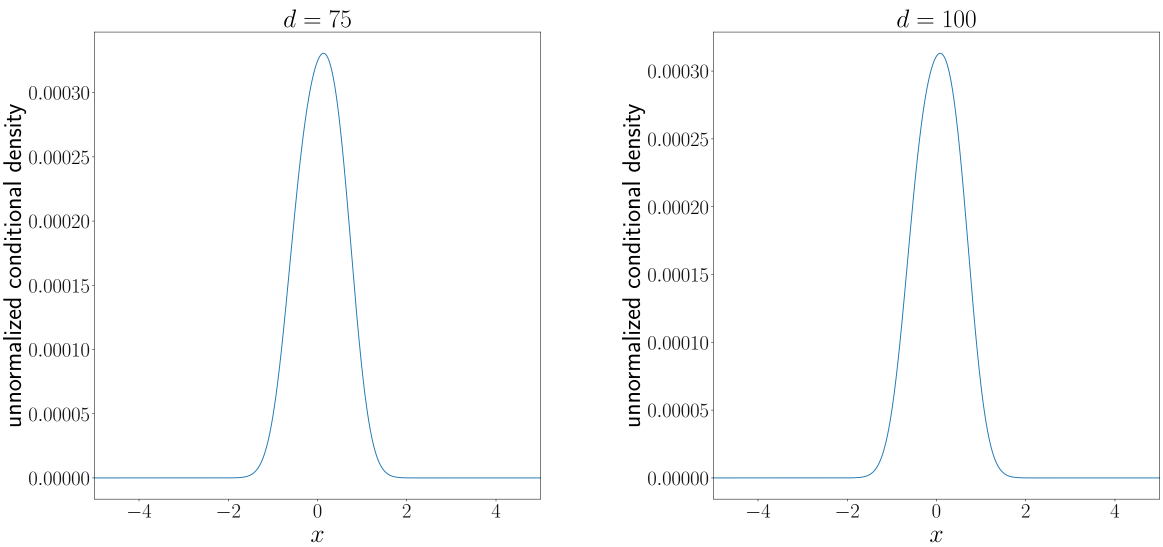

Figure 4: Approximations of $X_1(x, 0 \dots, 0)$, $x \in \mathbb{R}$, for $d \in \{75, 100\}$

References

[1] Beck, C., Becker, S., Cheridito, P., Jentzen, A., and Neufeld, A. Deep learning based numerical approximation algorithms for stochastic partial differential equations and high-dimensional nonlinear filtering problems *arXiv:2012.01194* (2020).

[2] Boufoussi, B., Van Casteren, J., and Mrhardy, N. Generalized backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions. *Bernoulli 13*, 2 (2007), 423–446.

[3] Brigo, D., and Hanzon, B. On some filtering problems arising in mathematical finance. *Insurance: Mathematics and Economics 22*, 3 (1998), 53–64.

[4] Buckdahn, R., and Ma, J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. part i. *Stochastic processes and their applications 93*, 2 (2001), 181–204.

[5] Buckdahn, R., and Ma, J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. part ii. *Stochastic processes and their applications 93*, 2 (2001), 205–228.

[6] Buckdahn, R., and Ma, J. Pathwise stochastic control problems and stochastic HJB equations. *SIAM journal on control and optimization 45*, 6 (2007), 2224–2256.

[7] Budman, H., Holcomb, T., and Morari, M. PLS-based robust inferential control for a packed-bed reactor. *1991 American Control Conference* (1991), 256–261.

[8] Buehner, M., McTaggart-Cowan, R., and Heilliette, S. An ensemble Kalman filter for numerical weather prediction based on variational data assimilation: Varenkf. *Monthly Weather Review 145*, 2 (2017), 617–635.

[9] Cassola, F., and Burlando, M. Wind speed and wind energy forecast through Kalman filtering of numerical weather prediction model output. *Applied Energy 99* (2012), 154–166.

[10] Ceci, C., and Colaneri, K. Recent advances in nonlinear filtering with a financial application to derivatives hedging under incomplete information. *Bayesian Inference, Javier Prieto Tejedor (Ed.), InTech, DOI: 10.5772/intechopen.70060.* (2017).

[11] Che, Y., Peng, X., Delle Monache, L., Kawaguchi, T., and Xiao, F. A wind power forecasting system based on the weather research and forecasting model and Kalman filtering over a wind-farm in Japan. *Journal of Renewable and Sustainable Energy 8*, 1 (2016), 013302.

[12] Chen, C.-Y., and Sun, C.-C. Adaptive inferential control of packed-bed reactors. *Chemical Engineering Science 46*, 4 (1991), 1041–1054.

[13] Coculescu, D., Geman, H., and Jeanblanc, M. Valuation of default sensitive claims under imperfect information. *Finance and Stochastics 12*, 2 (2008), 195–218.

[14] Crisan, D. Exact rates of convergence for a branching particle approximation to the solution of the Zakai equation. *Ann. Probab. 31*, 2 (2003), 693–718.

[15] Crisan, D., Gaines, J., and Lyons, T. Convergence of a branching particle method to the solution of the Zakai equation. *SIAM J. Appl. Math. 58*, 5 (1998), 1568–1590.

[16] Doss, H. Liens entre équations différentielles stochastiques et ordinaires. *Annales de l'IHP Probabilités et statistiques 13*, 2 (1977), 99–125.

[17] Duc, L., Kuroda, T., Saito, K., and Fujita, T. Ensemble Kalman filter data assimilation and storm surge experiments of tropical cyclone Nargis. *Tellus A: Dynamic Meteorology and Oceanography 67*, 1 (2015), 25941.

[18] Duffie, D., and Lando, D. Term structures of credit spreads with incomplete accounting information. *Econometrica 69* (2001), 633–664.

[19] Falissard, F. Genuinely multi-dimensional explicit and implicit generalized Shapiro filters for weather forecasting, computational fluid dynamics and aeroacoustics. *Journal of Computational Physics 253* (2013), 344–367.

[20] Frey, R., and Runggaldier, W. Pricing credit derivatives under incomplete information: a nonlinear-filtering approach. *Finance and Stochastics 14*, 4 (2010), 495–526.

[21] Frey, R., and Schmidt, T. Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *Finance and Stochastics 16*, 1 (2012), 105–133.

[22] Gobet, E., Pagès, G., Pham, H., and Printems, J. *Discretization and simulation for a class of SPDEs with applications to Zakai and McKean-Vlasov equations.* Preprint LPMA 958, Université Paris 6-Paris 7, Paris, 2005.

[23] Gobet, E., Pagès, G., Pham, H., and Printems, J. Discretization and simulation of the Zakai equation. *SIAM J. Numer. Anal. 44*, 6 (2006), 2505–2538.

[24] Grohs, P., Hornung, F., Jentzen, A., and Von Wurstemberger, P. A proof that artificial neural networks overcome the curse of dimensionality in the numerical approximation of Black–Scholes partial differential equations. *arXiv:1809.02362* (2018).

[25] Jentzen, A., Kuckuck, B., Müller-Gronbach, T., and Yaroslavtseva, L. On the strong regularity of degenerate additive noise driven stochastic differential equations with respect to their initial values. *arXiv:1904.05963* (2019), 59 pages.

[26] Kalman, R. E. A new approach to linear filtering and prediction problems. *Trans. ASME Ser. D. J. Basic Engrg. 82*, 1 (1960), 35–45.

[27] Kalman, R. E., and Bucy, R. S. New results in linear filtering and prediction theory. *Trans. ASME Ser. D. J. Basic Engrg. 83* (1961), 95–108.

[28] Klenke, A. *Probability Theory*, second ed. Universitext. Springer, London, 2014. A comprehensive course.

[29] Kunita, H. *Stochastic flows and stochastic differential equations*, vol. 24. Cambridge university press, 1997.

[30] Kushner, H. J. On the differential equations satisfied by conditional probabilitity densities of Markov processes, with applications. *Journal of the Society for Industrial and Applied Mathematics, Series A: Control 2*, 1 (1964), 106–119.

[31] Kutschireiter, A., Surace, S. C., and Pfister, J.-P. The Hitchhiker's guide to nonlinear filtering. *J. Math. Psych. 94* (2020), 102307, 21.

[32] Ledoux, M., and Talagrand, M. Probability in Banach Spaces. Isoperimetry and Processes. Springer-Verlag, Berlin, 1991.

[33] Pelosi, A., Medina, H., Van den Bergh, J., Vannitsem, S., and Chirico, G. B. Adaptive Kalman filtering for postprocessing ensemble numerical weather predictions. *Monthly Weather Review 145*, 12 (2017), 4837–4854.

[34] Rutzler, W. Nonlinear and adaptive parameter estimation methods for tubular reactors. *Ind. Eng. Chem. Res. 26*, 2 (1987), 325–333.

[35] Seinfeld, J. H., Gavalas, G. R., and Hwang, M. Nonlinear filtering in distributed parameter systems. *J. Dyn. Sys., Meas., Control 93*, 3 (1971), 157–163.

[36] Soliman, M. A., and Ray, W. H. Non-linear filtering for distributed parameter systems having a small parameter. *International Journal of Control 30*, 5 (1979), 757–772.

[37] Sussmann, H. J. On the gap between deterministic and stochastic ordinary differential equations. *The Annals of Probability* (1978), 19–41.

[38] Windes, L. C., Cinar, A., and Ray, W. H. Dynamic estimation of temperature and concentration profiles in a packed bed reactor. *Chemical Engineering Science 44*, 10 (1989), 2087–2106.

[39] Zakai, M. On the optimal filtering of diffusion processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 11*, 3 (1969), 230–243.