APPENDIX

A. PROOF OF LEMMA 1

We define $\overline{\mathcal{R}_{k-1}} = \bigcup_{i=1}^{n} W(r_i)$. Obviously, $\overline{\mathcal{R}_{k-1}} \subseteq \mathcal{R}_{k-1}$. For each item $r \in \overline{\mathcal{R}_{k-1}}$, let $W^{-1}(r)$ denote the set of all the items r' in $\mathcal{R}'_k$ such that $r \in W(r')$. Figure 9 shows an example of $W^{-1}(r)$. It follows from the earlier explanation that each item in $W^{-1}(r)$ must have a duration no longer than that of r. Moreover, by definition, $I(r)$ overlaps with each item in $W^{-1}(r)$.

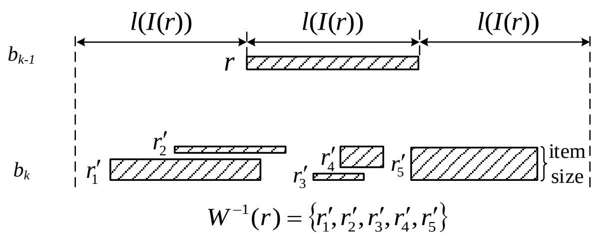

Figure 9: Definition of $W^{-1}(r)$

For each item $r' \in W^{-1}(r)$, it can be shown that $I(r')^- \geq I(r)^- - l\big(I(r)\big)$. Assume on the contrary that $I(r')^- < I(r)^- - l\big(I(r)\big)$. Since the duration of r' is no longer than that of r, it follows that $I(r')^+ = I(r')^- + l\big(I(r')\big) \leq I(r')^- + l\big(I(r)\big) < I(r)^-$, which contradicts to the fact that $I(r)$ overlaps with $I(r')$. Similarly, it can be shown that $I(r')^+ \leq I(r)^+ + l\big(I(r)\big)$. Note that the X-periods of all the items in $\mathcal{R}'_k$ are disjoint. Therefore,

$$\begin{aligned}\sum_{r' \in W^{-1}(r)} l\big(X(r')\big) &\leq \max_{r' \in W^{-1}(r)} I(r')^+ - \min_{r' \in W^{-1}(r)} I(r')^- \\ &\leq I(r)^+ + l\big(I(r)\big) - I(r)^- + l\big(I(r)\big) \\ &= 3 \cdot l\big(I(r)\big).\end{aligned}$$

Thus, we have

$$\begin{aligned} d_k^* &= \sum_{i=1}^{n} \Big(\sum_{r \in W(r_i)} s(r) \cdot l\big(X(r_i)\big) \Big) \\ &= \sum_{r \in \overline{\mathcal{R}_{k-1}}} s(r) \cdot \Big(\sum_{r' \in W^{-1}(r)} l\big(X(r')\big) \Big) \\ &\leq \sum_{r \in \overline{\mathcal{R}_{k-1}}} s(r) \cdot 3 \cdot l\big(I(r)\big) \\ &= 3 \cdot d(\overline{\mathcal{R}_{k-1}}) \\ &\leq 3 \cdot d(\mathcal{R}_{k-1}). \end{aligned}$$

Hence, the lemma is proven. □

B. PROOF OF LEMMA 2

We prove this claim by showing that when the examination of an altitude h completes, all the area between altitude h and the next altitude h^- to examine ($h^- < h$) is colored.

Consider the coloring status of the horizontal line at altitude h after h has been examined. It is obvious that this line must be fully colored since by definition, the examination terminates only when the uncolored interval set is empty. Thus, after the examination, the horizontal line at altitude h consists of blue intervals and red intervals only. A blue interval b may be colored during the examination of altitude h or an earlier examination of another altitude h^+ ($h^+ > h$) (for example, $[t_4, t_5)$ in Figure 10). Note that blue

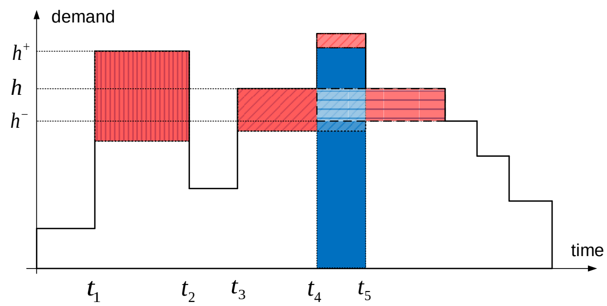

Figure 10: Item placement in demand chart

coloring can only be done by step 18 of Phase 1 which colors blue all the area below the altitude under examination to the bottom of the demand chart. Thus, in either case, all the area below the blue interval b at altitude h must have been colored blue.

Likewise, a red interval e may be colored during the examination of altitude h (for example, $[t_3, t_4)$ in Figure 10) or an earlier examination of another altitude h^+ ($h^+ > h$) (for example, $[t_1, t_2)$ in Figure 10). Note that red coloring can only be done due to item placement (step 9). Suppose that e is colored due to the placement of an item r. This implies that r's lower boundary (at altitude $h - s(r)$ or $h^+ - s(r)$) must be below h. According to step 16 of Phase 1, the altitude of r's low boundary would be added to M for examination. Since h^- is the next altitude to examine after h, it must hold that $h - s(r) \leq h^-$ or $h^+ - s(r) \leq h^-$. Therefore, the area below the red interval e between altitudes h and h^- are colored red when the examination of altitude h completes.

By similar arguments, when the examination of the last (lowest) altitude h completes, all the area between altitude h and the bottom of the demand chart is colored. We remark that, strictly speaking, the bottom line of the demand chart at altitude 0 is not colored. But this does not affect the correctness of the following analysis for approximation ratio. □

C. PROOF OF LEMMA 3

It is obvious from the algorithm that r's left boundary (at time $I(r)^-$), right boundary (at time $I(r)^+$) and upper boundary (at altitude h) are within the demand chart. Thus, we only need to check r's lower boundary (at altitude $h - s(r)$).

Based on the proof of Lemma 2, at the beginning of an altitude h's examination, all the area above altitude h has been colored. According to the algorithm, if an item r is allowed to be placed at altitude h, its interval $I(r)$ must overlap with some uncolored interval I_u at altitude h. It can be inferred that the area above the intersection $I(r) \cap I_u$ at height h cannot be colored blue. This is because when blue coloring is performed over some interval, all the area below the interval to the bottom of the demand chart is colored blue (step 18). Thus, the area above the intersection $I(r) \cap I_u$ at height h can only be all colored red.

Consider a time $t \in I(r) \cap I_u$. Let $S_S(t)$ denote the total size of all active small items at time t. Recall that the height of the demand chart at time t is $S_S(t)$. Thus, at time t, the distance from altitude h to the ceiling of the demand chart is $S_S(t) - h$. Since this range has all been colored red, by the algorithm definition, among all the small items that have been placed before examining altitude h, the total size of those active at time t is $S_S(t) - h$. The new item r to be placed at height h is also active at time t. If $h < s(r)$, it implies that the total size of all active small items at time t is at least $S_S(t) - h + s(r) > S_S(t)$, which contradicts the definition of $S_S(t)$. Hence, $h \geq s(r)$ and the lemma is proven. □

D. PROOF OF LEMMA 4

Assume on the contrary that there is an item r left not placed at the end of Phase 1. We check the coloring status of its active interval $I(r)$ at altitude $s(r)$, where $s(r)$ is the size of r. According to Lemma 2, $I(r)$ is colored at altitude $s(r)$.

If some part of $I(r)$ is colored red, let's consider a point $(t, s(r))$ that is colored red where $t \in I(r)$. Suppose that $(t, s(r))$ is colored red due to the placement of another item r' at altitude h'. Then, we have $t \in I(r')$ and $h' \geq s(r) > h' - s(r')$. It follows similar arguments to the proof of Lemma 3 that among all the small items that have been placed before examining altitude h', the total size of those active at time t is $S_S(t) - h'$. Note that items r and r' are also active at time t. Thus, the total size of all active small items at time t is at least $S_S(t) - h' + s(r') + s(r) > S_S(t)$, which contradicts the definition of $S_S(t)$.

On the other hand, if the entire interval $I(r)$ is colored blue at altitude $s(r)$, let's consider the maximum altitude h ($h \geq s(r)$) at which the whole interval $I(r)$ is blue. Then, one or more parts of $I(r)$ at altitude h are colored blue during the examination of altitude h. Assume that the last part of $I(r)$ is colored blue due to an uncolored interval I_u satisfying $I(r) \cap I_u \neq \emptyset$. Then, before the last part of $I(r)$ is colored blue, r is an item satisfying the condition for placement (step 7) since $I(r) \cap I_u \neq \emptyset$ and $I(r) \backslash I_u$ has all been colored blue. Thus, instead of coloring I_u (and the area below it) blue, r should have been placed at altitude h, which again leads to contradiction. Hence, the lemma is proven. □

E. PROOF OF LEMMA 5

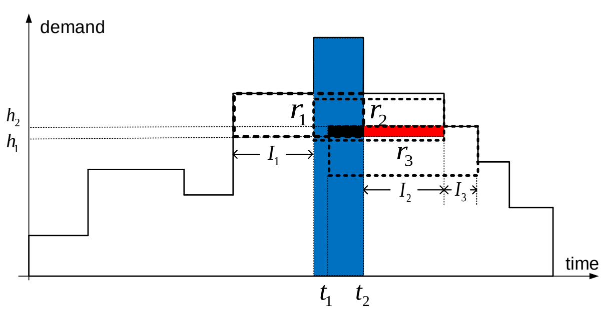

Figure 11: Item overlapping in demand chart

Lemma 2 has shown that the entire demand chart is colored after Phase 1. According to the algorithm definition, any red area cannot be further covered by any new item after it is colored red. Thus, no two items can overlap in red areas. Now, it is left to show that no three items can overlap together in blue areas. Assume on the contrary that a blue area $[t_1, t_2) \times (h_1, h_2]$ (where $t_1 < t_2$ and $h_1 < h_2$) is covered by three items r_1, r_2 and r_3 in their placement as shown in Figure 11. Suppose that the placement of each item r_i causes an uncolored interval I_i to be colored red at some altitude. Obviously, I_1, I_2 and I_3 cannot intersect with one another along the time dimension. Moreover, since the area $[t_1, t_2) \times (h_1, h_2]$ is colored blue, we have $I_i \cap [t_1, t_2) = \emptyset$ ($i = 1, 2, 3$). Thus, at least two of I_1, I_2 and I_3 must be on the same side of $[t_1, t_2)$. Without loss of generality, assume that I_2 and I_3 are both on the right side of $[t_1, t_2)$, i.e., $I_2^- \geq t_2$ and $I_3^- \geq t_2$. Since I_2 and I_3 do not intersect, assume without loss of generality that I_2 is on the left side of I_3. Then, it follows that $t_2 \leq I_2^- < I_2^+ \leq I_3^- < I_3^+$. As a result, between r_2 and r_3, the first item placed would have colored the area $[I_2^-, I_2^+) \times (h_1, h_2]$ red, making it impossible for the second item to be placed. This leads to a contradiction. Therefore, no three items can overlap together in blue areas. Hence, the lemma is proven. □

F. PROOF OF THEOREM 2

Let $S(t)$ denote the total size of all active items at time t. Then, we have $S(t) = S_L(t) + S_S(t)$ at any time t.

At any time t when there are only large items active, it is obvious that

$$\lfloor 2S_L(t) \rfloor = \lfloor 2S(t) \rfloor \leq \lceil 4S(t) \rceil \leq 4\lceil S(t) \rceil.$$

At any time t when there are only small item active, it is apparent that

$$2\lceil 2S_S(t) \rceil - 1 = 2\lceil 2S(t) \rceil - 1 \leq 2 \cdot 2 \cdot \lceil S(t) \rceil - 1 < 4\lceil S(t) \rceil.$$

Now consider any time t when there are both large and small items active. Apparently, $0 \leq \lceil 2S_S(t) \rceil - 2 \cdot S_S(t) < 1$. If $0 \leq \lceil 2S_S(t) \rceil - 2 \cdot S_S(t) \leq \frac{1}{2}$, we can rewrite the total number of open bins as follows.

$$\begin{aligned}
&\lfloor 2S_L(t) \rfloor + 2\lceil 2S_S(t) \rceil - 1 \\
&= \lfloor 2S_L(t) \rfloor + 2 \cdot \big(2 \cdot S_S(t) + (\lceil 2S_S(t) \rceil - 2 \cdot S_S(t))\big) - 1 \\
&= \lfloor 2S_L(t) \rfloor + 4 \cdot S_S(t) + 2 \cdot (\lceil 2S_S(t) \rceil - 2 \cdot S_S(t)) - 1 \\
&\leq \lfloor 2S_L(t) \rfloor + 4 \cdot S_S(t) + 1 - 1 \\
&= \lfloor 2S_L(t) \rfloor + 4 \cdot S_S(t) \\
&\leq 4 \cdot S_L(t) + 4 \cdot S_S(t) \\
&\leq 4\lceil S_L(t) + S_S(t) \rceil \\
&= 4\lceil S(t) \rceil.
\end{aligned}$$

If $\frac{1}{2} < \lceil 2S_S(t) \rceil - 2 \cdot S_S(t) < 1$, we have $\lceil 4S_S(t) \rceil = 2\lceil 2S_S(t) \rceil - 1$. As a result,

$$\begin{aligned}
\lfloor 2S_L(t) \rfloor + 2\lceil 2S_S(t) \rceil - 1 &= \lfloor 2S_L(t) \rfloor + \lceil 4S_S(t) \rceil \\
&\leq \lceil 2S_L(t) + 4S_S(t) \rceil \\
&\leq \lceil 4S_L(t) + 4S_S(t) \rceil \\
&\leq 4\lceil S_L(t) + S_S(t) \rceil \\
&= 4\lceil S(t) \rceil.
\end{aligned}$$

In summary, at any time t, the total number of open bins is bounded by $4\lceil S(t) \rceil$. Therefore,

$$\begin{aligned}
&\int_{\cup_{r \in \mathcal{R}_L} I(r)} \lfloor 2S_L(t) \rfloor \, \mathrm{d}t + \int_{\cup_{r \in \mathcal{R}_S} I(r)} (2\lceil 2S_S(t) \rceil - 1) \, \mathrm{d}t \\
&\leq \int_{\cup_{r \in \mathcal{R}} I(r)} 4\lceil S(t) \rceil \, \mathrm{d}t \\
&\leq 4 \cdot OPT_{total}(\mathcal{R}),
\end{aligned}$$

where the last inequality follows from the bound given in Proposition 3. Hence, the theorem is proven. □

G. PROOF OF LEMMA 6

According to First Fit packing, every time a new bin is opened, the sum of the levels of the new bin and any existing bin must exceed 1 (the bin capacity). Since the items arriving in the second stage do not depart in this stage (by definition, all the items depart in the interval $(t, t+\rho]$ which is part of the third stage), the level of any open bin would never decrease over time. Thus, the sum of the levels of any two open bins must always exceed 1 at any moment in the second stage. Consider any moment in the second stage. Suppose there are n open bins at that time. Let x_i $(i = 1, 2, 3, \dots, n)$ denote the level of the i-th open bin. Based on the above reasoning, for any $1 \le i < j \le n$, we have

$$x_i + x_j > 1. \tag{11}$$

Adding up the inequalities (11) for all $1 \le i < j \le n$, we obtain

$$(n-1) \cdot \sum_{i=1}^{n} x_i > \frac{n(n-1)}{2}.$$

It follows that

$$\frac{1}{n} \sum_{i=1}^{n} x_i > \frac{1}{2}. \tag{12}$$

Hence, the lemma is proven. □

H. PROOF OF LEMMA 7

By definition, the supplier bin of b_{j_1} must have an index lower than j_1. Since b_{j_1} and b_{j_2} share the same supplier bin, the supplier bin of b_{j_2} also has an index lower than j_1. Note that $j_1 < j_2$, if bin b_{j_1} is closed after time a_{j_2}, by definition, the supplier bin of b_{j_2} must have an index no less than j_1. Thus, it can be inferred that bin b_{j_1} is closed no later than a_{j_2}, i.e., $I^+_{j_1} \le a_{j_2}$. Similarly, we also have $I^+_{j_2} \le a_{j_3}$, $I^+_{j_3} \le a_{j_4}$, ..., $I^+_{j_{s-1}} \le a_{j_s}$. Therefore, all the periods $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ do not overlap and it holds that $a_{j_1} < I^+_{j_1} \le a_{j_2} < I^+_{j_2} \le \cdots < I^+_{j_{s-1}} \le a_{j_s}$.

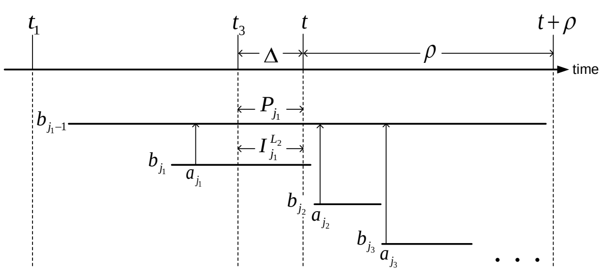

Figure 12: Case 1 of r_{j_1}'s arrival

If r_{j_1} is an instance of Case 1 (see Figure 12), the length of the supplier period P_{j_1} is $t - t_3 = t - (t - \Delta) = \Delta$. Since $a_{j_1} \le t_3 < a_{j_2} < a_{j_3} < \cdots < a_{j_s} < t + \rho$, all the periods $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ fall in the time interval from t_3 to $t+\rho$. Since they do not overlap, their total length is bounded by $t + \rho - t_3 = t + \rho - (t - \Delta) = \rho + \Delta$. Thus, the total length of P_{j_1}, $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ is bounded by $\Delta + \rho + \Delta = \rho + 2\Delta$.

If r_{j_1} is an instance of Case 2 (see Figure 13), the length of the supplier period P_{j_1} is $a_{j_1} + \Delta - t_3$. Since $t_3 < a_{j_1} < a_{j_2} < a_{j_3} < \cdots < a_{j_s} < t + \rho$, all the periods $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ fall in the time interval from a_{j_1} to $t+\rho$. So, their total length is bounded by $t + \rho - a_{j_1}$. As a result, the total

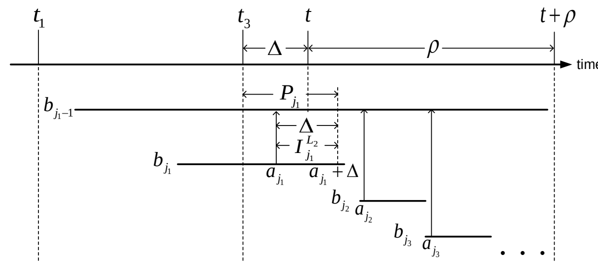

Figure 13: Case 2 of r_{j_1}'s arrival

length of P_{j_1}, $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ is bounded by $a_{j_1} + \Delta - t_3 + t + \rho - a_{j_1} = \rho + \Delta + t - t_3 = \rho + 2\Delta$.

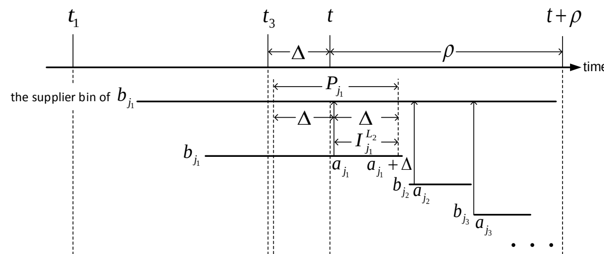

Figure 14: Case 3 of r_{j_1}'s arrival

If r_{j_1} is an instance of Case 3 (see Figure 14), the length of the supplier period P_{j_1} is $(a_{j_1} + \Delta) - (a_{j_1} - \Delta) = 2\Delta$. Since $t \le a_{j_1} < a_{j_2} < a_{j_3} < \cdots < a_{j_s} < t + \rho$, all the periods $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ fall in the time interval from t to $t+\rho$. So, their total length is bounded by ρ. Therefore, the total length of P_{j_1}, $I^{L2}_{j_1} \cup I^{L3}_{j_1}$, $I^{L2}_{j_2} \cup I^{L3}_{j_2}$, ..., $I^{L2}_{j_s} \cup I^{L3}_{j_s}$ is bounded by $\rho + 2\Delta$.

Hence, the lemma is proven. □

I. PROOF OF INEQUALITY (7)

For each bin group $H \in \mathcal{G}$, let $I^{L2}_{f(H)} - \cup_{G \in \mathcal{G}} P_{f(G)}$ denote the part of the period $I^{L2}_{f(H)}$ that does not overlap with any supplier period of the flag bins of all bin groups, and let $I^{L2}_{f(H)} \cap (\cup_{G \in \mathcal{G}} P_{f(G)})$ denote the part of the period $I^{L2}_{f(H)}$ that overlaps with the supplier periods.

Since $s(r_{f(H)}) < \frac{\Delta}{\rho + 2\Delta}$, it follows that

$$\begin{aligned} & s(r_{f(H)}) \cdot l\big(I^{L2}_{f(H)} \cap (\cup_{G \in \mathcal{G}} P_{f(G)})\big) \\ & < \frac{\Delta}{\rho + 2\Delta} \cdot l\big(I^{L2}_{f(H)} \cap (\cup_{G \in \mathcal{G}} P_{f(G)})\big). \end{aligned}$$

Therefore, based on (5) and (6), we have

$$\begin{aligned} & d(P_{f(H)}) + s(r_{f(H)}) \cdot l\big(I^{L2}_{f(H)} - \cup_{G \in \mathcal{G}} P_{f(G)}\big) \\ & = d(P_{f(H)}) + s(r_{f(H)}) \cdot l\big(I^{L2}_{f(H)}\big) \\ & \quad - s(r_{f(H)}) \cdot l\big(I^{L2}_{f(H)} \cap (\cup_{G \in \mathcal{G}} P_{f(G)})\big) \\ & > \Delta - \frac{\Delta}{\rho + 2\Delta} \cdot l\big(I^{L2}_{f(H)} \cap (\cup_{G \in \mathcal{G}} P_{f(G)})\big) \\ & \ge \frac{\Delta}{\rho + 2\Delta} \cdot \Big(l(P_{f(H)}) + \sum_{b_j \in H} l(I^{L2}_j \cup I^{L3}_j)\Big) \end{aligned}$$

$$-\frac{\Delta}{\rho+2\Delta}\cdot l\big(I_{f(H)}^{L_2}\cap(\cup_{G\in\mathcal{G}}P_{f(G)})\big)$$
$$=\frac{\Delta}{\rho+2\Delta}\cdot\Big(l(P_{f(H)})+l\big(I_{f(H)}^{L_2}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)+I_{f(H)}^{L_3}\Big)$$
$$+\frac{\Delta}{\rho+2\Delta}\cdot\sum_{b_j\in H,j\neq f(H)}l(I_j^{L_2}\cup I_j^{L_3})$$
$$\geq\frac{\Delta}{\rho+2\Delta}\cdot\Big(l(P_{f(H)})+l\big(I_{f(H)}^{L_2}\cup I_{f(H)}^{L_3}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)\Big)$$
$$+\frac{\Delta}{\rho+2\Delta}\cdot\sum_{b_j\in H,j\neq f(H)}l(I_j^{L_2}\cup I_j^{L_3}). \tag{13}$$

By definition, different groups of bins have different supplier bins. Thus, the supplier periods of the flag bins of all bin groups, i.e., $P_{f(H)}$'s, do not intersect with each other. In addition, the L_2-periods and L_3-periods of all the bins do not intersect with one another as they are associated with different bins. Therefore, any time point associated with each bin can simultaneously belong to at most one supplier period and one L_2-period or one L_3-period.

As a result,

$$\bigcup_{H\in\mathcal{G}}\Big(P_{f(H)}\cup(\bigcup_{b_j\in H}I_j^{L_2}\cup I_j^{L_3})\Big)$$
$$=\Big(\bigcup_{H\in\mathcal{G}}P_{f(H)}\Big)\bigcup\Big(\bigcup_{b_j\in H,H\in\mathcal{G}}(I_j^{L_2}\cup I_j^{L_3})\Big)$$

can be broken into the following disjoint periods: $P_{f(H)}$ for each bin group $H\in\mathcal{G}$, and $(I_j^{L_2}\cup I_j^{L_3}-\bigcup_{G\in\mathcal{G}}P_{f(G)})$ for each bin b_j in each bin group $H\in\mathcal{G}$. Also note that item r_j is active throughout the period $I_j^{L_2}$.

Hence,

$$d\Big(\bigcup_{H\in\mathcal{G}}(P_{f(H)}\cup(\bigcup_{b_j\in H}(I_j^{L_2}\cup I_j^{L_3})))\Big)$$
$$=\sum_{H\in\mathcal{G}}d(P_{f(H)})+\sum_{b_j\in H,H\in\mathcal{G}}d\big(I_j^{L_2}\cup I_j^{L_3}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)$$
$$\geq\sum_{H\in\mathcal{G}}d(P_{f(H)})+\sum_{b_j\in H,H\in\mathcal{G}}d\big(I_j^{L_2}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)$$
$$\geq\sum_{H\in\mathcal{G}}d(P_{f(H)})+\sum_{b_j\in H,H\in\mathcal{G}}s(r_j)\cdot l\big(I_j^{L_2}-\cup_{G\in\mathcal{G}}P_{f(G)}\big), \tag{14}$$

and

$$l\Big(\bigcup_{H\in\mathcal{G}}(P_{f(H)}\cup(\bigcup_{b_j\in H}I_j^{L_2}\cup I_j^{L_3}))\Big)$$
$$=\sum_{H\in\mathcal{G}}l(P_{f(H)})+\sum_{b_j\in H,H\in\mathcal{G}}l\big(I_j^{L_2}\cup I_j^{L_3}-\cup_{G\in\mathcal{G}}P_{f(G)}\big). \tag{15}$$

Combining (13), (14) and (15), we have

$$d\Big(\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{b_j\in H,H\in\mathcal{G}}(I_j^{L_2}\cup I_j^{L_3}))\Big)$$
$$=d\Big(\bigcup_{H\in\mathcal{G}}(P_{f(H)}\cup(\bigcup_{b_j\in H}(I_j^{L_2}\cup I_j^{L_3})))\Big)$$
$$\geq\sum_{H\in\mathcal{G}}d(P_{f(H)})+\sum_{b_j\in H,H\in\mathcal{G}}s(r_j)\cdot l\big(I_j^{L_2}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)$$
$$\geq\sum_{H\in\mathcal{G}}d(P_{f(H)})+\sum_{H\in\mathcal{G}}s(r_{f(H)})\cdot l\big(I_{f(H)}^{L_2}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)$$
$$=\sum_{H\in\mathcal{G}}\Big(d(P_{f(H)})+s(r_{f(H)})\cdot l\big(I_{f(H)}^{L_2}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)\Big)$$
$$>\frac{\Delta}{\rho+2\Delta}\cdot\sum_{H\in\mathcal{G}}l(P_{f(H)})$$
$$+\frac{\Delta}{\rho+2\Delta}\cdot\sum_{H\in\mathcal{G}}l\big(I_{f(H)}^{L_2}\cup I_{f(H)}^{L_3}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)$$
$$+\frac{\Delta}{\rho+2\Delta}\cdot\sum_{H\in\mathcal{G}}\sum_{b_j\in H,j\neq f(H)}l(I_j^{L_2}\cup I_j^{L_3})$$
$$\geq\frac{\Delta}{\rho+2\Delta}\cdot\sum_{H\in\mathcal{G}}l\big(P_{f(H)}\big)$$
$$+\frac{\Delta}{\rho+2\Delta}\cdot\sum_{b_j\in H,H\in\mathcal{G}}l\big(I_j^{L_2}\cup I_j^{L_3}-\cup_{G\in\mathcal{G}}P_{f(G)}\big)$$
$$=\frac{\Delta}{\rho+2\Delta}\cdot l\Big(\bigcup_{H\in\mathcal{G}}(P_{f(H)}\cup(\bigcup_{b_j\in H}(I_j^{L_2}\cup I_j^{L_3})))\Big)$$
$$=\frac{\Delta}{\rho+2\Delta}\cdot l\Big(\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{b_j\in H,H\in\mathcal{G}}(I_j^{L_2}\cup I_j^{L_3}))\Big). \tag{16}$$

Recall that the bins in the bin groups of $\mathcal{G}$ include all the bins b_i where $I_i^L\neq\emptyset$ and an item of size less than $\frac{\Delta}{\rho+2\Delta}$ is placed in b_i by the end of I_i^L. According to Proposition 5, for all the other bins, $I_i^{L_2}=\emptyset$ and $I_i^{L_3}=\emptyset$. Thus,

$$\bigcup_{b_j\in H,H\in\mathcal{G}}(I_i^{L_2}\cup I_i^{L_3})=\bigcup_{i=1}^{m}(I_i^{L_2}\cup I_i^{L_3}).$$

As a result, (16) can be rewritten as

$$d\Big(\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{i=1}^{m}(I_j^{L_2}\cup I_j^{L_3}))\Big)$$
$$\geq\frac{\Delta}{\rho+2\Delta}\cdot l\Big(\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{i=1}^{m}(I_j^{L_2}\cup I_j^{L_3}))\Big). \tag{17}$$

The remaining periods in the left bin usage time, i.e., $\bigcup_{i=1}^{m}I_i^L-\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup\big(\bigcup_{i=1}^{m}I_j^{L_2}\cup I_j^{L_3}\big)\subseteq\bigcup_{i=1}^{m}I_i^{L_1}$, all fall in the L_1-periods of the bins. By Proposition 4, the bin levels are at least $\frac{\Delta}{\rho+2\Delta}$ high in these periods. Therefore, we have the following relation between the length and the time-space demand of these periods:

$$\frac{\Delta}{\rho+2\Delta}\cdot l\Big(\bigcup_{i=1}^{m}I_i^L-\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{i=1}^{m}(I_j^{L_2}\cup I_j^{L_3}))\Big)$$
$$\leq d\Big(\bigcup_{i=1}^{m}I_i^L-\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{i=1}^{m}(I_j^{L_2}\cup I_j^{L_3}))\Big). \tag{18}$$

Combining (17) and (18), we obtain

$$\frac{\Delta}{\rho+2\Delta}\cdot\sum_{i=1}^{m}l(I_i^L)$$
$$=\frac{\Delta}{\rho+2\Delta}\cdot l(\bigcup_{i=1}^{m}I_i^L)$$
$$\leq\frac{\Delta}{\rho+2\Delta}\cdot l\Big(\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{i=1}^{m}(I_j^{L_2}\cup I_j^{L_3}))\Big)$$
$$+\frac{\Delta}{\rho+2\Delta}\cdot l\Big(\bigcup_{i=1}^{m}I_i^L-\bigcup_{H\in\mathcal{G}}P_{f(H)}\cup(\bigcup_{i=1}^{m}(I_j^{L_2}\cup I_j^{L_3}))\Big)$$

$$
\begin{aligned}
&\leq d\Big(\bigcup_{H\in\mathcal{G}} P_{f(H)} \cup \big(\bigcup_{i=1}^{m}(I_j^{L_2} \cup I_j^{L_3})\big)\Big) \\
&\quad + d\Big(\bigcup_{i=1}^{m} I_i^{L} - \bigcup_{H\in\mathcal{G}} P_{f(H)} \cup \big(\bigcup_{i=1}^{m}(I_j^{L_2} \cup I_j^{L_3})\big)\Big) \\
&\leq d\Big(\bigcup_{i=1}^{m} I_i\Big) \\
&= \sum_{i=1}^{m} d(I_i).
\end{aligned}
$$

□