Construction of Relative Difference Sets and Hadamard Groups

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Abstract

There exist normal $(2m, 2, 2m, m)$ relative difference sets and thus Hadamard groups of order $4m$ for all m of the form

$$m = x2^{a+t+u+w+\delta-\epsilon+1}6^b 9^c 10^d 22^e 26^f \prod_{i=1}^{s} p_i^{4a_i} \prod_{i=1}^{t} q_i^2 \prod_{i=1}^{u} ((r_i+1)/2)r_i^{v_i}) \prod_{i=1}^{w} s_i$$

under the following conditions: $a, b, c, d, e, f, s, t, u, w$ are nonnegative integers, $a_1, \ldots, a_r$ and $v_1, ..., v_u$ are positive integers, $p_1, \ldots, p_s$ are odd primes, $q_1, \ldots, q_t$ and $r_1, \ldots, r_u$

are prime powers with $q_i \equiv 1 \pmod 4$ and $r_i \equiv 1 \pmod 4$ for all i, $s_1, ..., s_w$ are integers with $1 \leq s_i \leq 33$ or $s_i \in \{39, 43\}$ for all i, x is a positive integer such that $2x-1$ or $4x-1$ is a prime power. Moreover, $\delta = 1$ if $x > 1$ and $c + s > 0$, $\delta = 0$ otherwise, $\epsilon = 1$ if $x = 1$, $c + s = 0$, and $t + u + w > 0$, $\epsilon = 0$ otherwise.

We also obtain some necessary conditions for the existence of $(2m, 2, 2m, m)$ relative difference sets in central products of $\mathbb{Z}_4$ with abelian groups, and provide a table cases for which $m \leq 100$ and the existence of such relative difference sets is open.

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1 Introduction

Let G be a group of order mn, and let N be a subgroup of G of order n. An $\mathbf{(m, n, k, \boldsymbol{\lambda})}$ **relative difference set R in G relative to N** is a k-subset of G such that every $g \in G \setminus N$ has exactly λ representations $g = r_1 r_2^{-1}$ with $r_1, r_2 \in R$, and no nonidentity element of N has such a representation. If N is a normal subgroup of G, we say that R is a **normal** relative difference set.

A relative difference set in a group G is equivalent to a divisible design on which G acts as a Singer group. We refer the reader to [17] for an introduction to this subject; further background can be found in [3, 22].

An **Hadamard matrix** is a square matrix with entries ± 1 only whose row vectors are pairwise orthogonal. The Hadamard Conjecture asserts that there exists an Hadamard matrix of order $4t$ for every positive integer t.

In this paper, we construct normal relative difference sets with parameters $(2m, 2, 2m, m)$. These relative difference sets are closely related to Hadamard matrices, as following specialization of a result of Jungnickel [17, Cor. 7.2] shows.

Result 1.1 *Suppose there is a normal* $(2m, 2, 2m, m)$ *difference set. Then there is an Hadamard matrix of order* $2m$*. In particular,* m *is even if* $m > 1$*.*

According to Ito [13, 14, 15, 16], a group of order $4m$ containing a normal $(2m, 2, 2m, m)$ relative difference set is called an **Hadamard group**. Ito showed that the dicyclic group

$$Q_{8t} = \langle a, b | a^{4t} = b^4 = 1, a^{2t} = b^2, b^{-1}ab = a^{-1} \rangle$$

is an Hadamard group for all t such that $2t-1$ or $4t-1$ is a prime power. He conjectured in [16] that Q_{8t} is an Hadamard group for every positive integer t. In view of Result 1.1, Ito's conjecture is a strengthening of the Hadamard conjecture. In [11], the connection between Hadamard groups and cocyclic Hadmard matrices was clarified, and Ito's conjecture was verified by computer for $t \leq 11$. In [19, 20] asymptotic existence results for cocyclic Hadmard matrices and thus for $(2m, 2, 2m, m)$ relative difference sets were obtained.

The following type of Hadamard matrices H (cf. [2]) is relevant to Ito's conjecture.

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}, \tag{1}$$

where A, B, C, D are circulant matrices. The following result from [23] implies the validity of Ito's conjecture for $t \leq 46$.

Result 1.2 *Let s be a positive integer such that $2s-1$ or $4s-1$ is a prime power or s is odd and there is an Hadamard matrix of type (1) of order $4s$. Then there are $(4t, 2, 4t, 2t)$ relative difference sets in Q_{8t} for all t of the form*

$$t = 2^a 10^b 26^c s$$

with $a, b, c \geq 0$.

Williamson matrices provide examples of Hadamard matrices of type (1), which can be used in Result 1.2. We refer to [2, 8, 24, 25, 26] for constructions of such matrices.

In this paper, we extend Result 1.2 by utilizing real and complex Golay pairs, Williamson matrices, and building sets to recursively construct $(2m, 2, 2m, m)$ relative difference sets in central products of $\mathbb{Z}_4$ with abelian groups.

2 Preliminaries

To study relative difference sets, it is convenient to use the group ring notation. We identify a subset A of a group G with the element $\sum_{g\in A} g$ of the integral group ring $\mathbb{Z}[G]$. For $B = \sum_{g\in G} b_g g$ in $\mathbb{Z}[G]$, we write $B^{(-1)} = \sum_{g\in G} b_g g^{-1}$. In the group ring language, the definition of relative difference sets reads as follows.

Lemma 2.1 *A k-subset R of a group G of order mn is an (m, n, k, λ) relative difference set in G relative to N if and only if R satisfies*

$$RR^{(-1)} = k + \lambda(G - N)$$

in $\mathbb{Z}[G]$.

For an abelian group G, we denote the group of complex characters of G by $\hat{G}$. The trivial character of G is the character sending all elements of G to 1. The following is a standard result [3, Chapter VI, Lemma 3.5].

Result 2.2 *Let G be a finite abelian group and $D = \sum_{g\in G} d_g g \in \mathbb{C}[G]$. Then*

$$d_g = \frac{1}{|G|} \sum_{\chi\in\hat{G}} \chi(Dg^{-1})$$

for all $g \in G$.

We will need the following result of Kronecker. See [4, Section 2.3, Thm. 2] for a proof.

Result 2.3 *An nonzero algebraic integer all of whose conjugates have absolute value at most 1 is a root of unity.*

The following is due to Ma [21].

Result 2.4 *Let p be a prime and let G be a finite abelian group with a cyclic Sylow p-subgroup. If $Y \in \mathbb{Z}[G]$ satisfies $\chi(Y) \equiv 0 \bmod p^a$ for all nontrivial characters χ of G, then there exist $X_1, X_2 \in \mathbb{Z}[G]$ such that*

$$Y = p^a X_1 + P X_2,$$

where P is the unique subgroup of order p of G.

The next result follows from a lemma given in [9, p. 53], which is an extension of a result of Lal, McFarland, and Odoni [18].

Result 2.5 *Let $p \equiv 3 \pmod 4$ be a prime, let a, b be positive integers, $\zeta = \exp(2\pi i/p^a)$. If $|X|^2 + |Y|^2 \equiv 0 \pmod{p^{2b}}$ for $X, Y \in \mathbb{Z}[\zeta]$, then $X \equiv 0 \pmod{p^b}$ and $Y \equiv 0 \pmod{p^b}$.*

3 Golay Transversals and Relative Difference Sets

We first define the groups in which we will construct relative $(2m, 2, 2m, m)$ difference sets.

Definition 3.1 Let H be an abelian group of order $2m$, and let g be an element of order 2 in H. Let $K = \langle y \rangle$ be a cyclic group of order 4. We denote the partial semidirect product (c.f. [10, Chapter 2.5]) of H and K defined by $y^2 = g$ and $y^{-1}hy = h^{-1}$ for all $h \in H$ by $Q(H, g)$.

Remark 3.2 Using the notation above, let G be the semidirect product of H and K defined by $H \trianglelefteq G$ and $y^{-1}hy = h^{-1}$ for all $h \in H$. Then $Q(H, g)$ is isomorphic to $G/\langle y^2 g \rangle$, see [10, Chapter 2.5]. The group $Q(H, g)$ is denoted by $\mathrm{gr}(H, g)$ in [1].

Example 3.3 Let $H = \langle x \rangle$ be a cyclic group of order $2m$ and $g = x^m$. Then $Q(H, g)$ is the dicyclic group of order $4m$:

$$Q(H, g) = \langle x, y : x^{2m} = y^4 = 1, x^m = y^2, y^{-1}xy = x^{-1} \rangle = \mathrm{Dic}_m.$$

The existence of a $(2m, 2, 2m, m)$ relative difference set in $Q(H, g)$ is equivalent to the existence of two elements A, B of the group ring $\mathbb{Z}[H]$ satisfying a character condition closely related to Golay sequences. Thus we introduce the following terminology.

Definition 3.4 Let H be an abelian group of even order, and let g be an element of order 2 of H. Let $g_1, ..., g_m$ be a complete system of coset representatives of $\langle g \rangle$ in H. Let $A = \sum_{i=1}^m a_i g_i$, $B = \sum_{i=1}^m b_i g_i$ with $a_i, b_i \in \{-1, 1\}$. We call (A, B) a **Golay transversal of *H* with respect to *g*** if

$$|\chi(A)|^2 + |\chi(B)|^2 = |H|$$

for all characters χ of H with $\chi(g) = -1$.

Remark 3.5 The notion of a Golay transversal is equivalent to that of a *pair of complementary relative difference sets* used in [1]. The formulation in terms of Golay transversals, however, is more convenient for the recursive constructions of relative difference sets we will establish.

The following observation is fundamental for the rest of the paper.

Proposition 3.6 *Let H be an abelian group of even order $2m$, and let g be an element of order 2 of H. A $(2m, 2, 2m, m)$ difference set in $Q(H, g)$ relative to $\langle g\rangle$ exists if and only if there is a Golay transversal of H with respect to g.*

Proof Suppose that (A, B) is a Golay transversal of H with respect to g. Let $h_1, ..., h_m$ be a complete system of coset representatives of $\langle g\rangle$ in H. Write $A = \sum_{i=1}^{m}(-1)^{c_i}h_i$, $B = \sum_{i=1}^{m}(-1)^{d_i}h_i$ with $c_i, d_i \in \{0, 1\}$, $X = \sum_{i=1}^{m} g^{c_i}h_i$, $Y = \sum_{i=1}^{m} g^{d_i}h_i$, and $R = X+Yy$, where y is the element of $Q(H, g)$ given in Definition 3.1. Then R is a $2m$-subset of $Q(H, g)$. Note that $yay^{-1} = a^{-1}$ and thus $ya = a^{-1}y$ for all $a \in H$, and thus $yX^{(-1)} = Xy$ and similarly $y^{-1}Y^{(-1)} = Yy^{-1}$. We compute

$$\begin{aligned} RR^{(-1)} &= XX^{(-1)} + YY^{(-1)} + Xy^{-1}Y^{(-1)} + YyX^{(-1)} \\ &= XX^{(-1)} + YY^{(-1)} + XY(y + y^{-1}). \end{aligned}$$

Note that $y + y^{-1} = (1 + y^2)y = (1 + g)y$. Thus, Y is a complete set of coset representatives of $\langle g\rangle$ in H, we have $XY(y+y^{-1}) = mHy$. Hence, by Lemma 2.1, R is a $(2m, 2, 2m, m)$ difference set in $Q(H, g)$ relative to $\langle g\rangle$ if and only if

$$XX^{(-1)} + YY^{(-1)} = 2m + m(H - \langle g\rangle). \tag{2}$$

By Result 2.2, (2) holds if and only if

$$|\chi(X)|^2 + |\chi(Y)|^2 = 2m + m\chi(H - \langle g\rangle) \tag{3}$$

for all $\chi \in \hat{H}$. Let χ_0 be the trivial character of H. Then $\chi_0(X) = \chi_0(Y) = m$, $\chi_0(H) = 2m$, and $\chi_0(\langle g\rangle) = 2$. Hence (3) holds for $\chi = \chi_0$. Now suppose that $\chi \neq \chi_0$ is a character of H with $\chi(g) = 1$. Then $\chi(X) = \chi(Y) = 0$, as X and Y are both complete sets of coset representatives of $\langle g\rangle$ in H. Moreover, $\chi(H) = 0$ and $\chi(\langle g\rangle) = 2$. Hence (3) holds in this case, too.

Finally, suppose that χ is a character of H with $\chi(g) = -1$. Then $\chi(H) = \chi(\langle g \rangle) = 0$ and

$$|\chi(X)|^2 + |\chi(Y)|^2 = |\chi(A)|^2 + |\chi(B)|^2 = 2m,$$

as we assume that (A, B) is a Golay transversal of H with respect to g. Hence (3) holds in this case, too, and thus for all $\chi \in \hat{H}$. Thus R is a $(2m, m, 2m, m)$ relative difference set in $Q(H, g)$ relative to $\langle g \rangle$.

We have shown that the existence of a Golay transversal of H with respect to g implies the existence of a $(2m, m, 2m, m)$ relative difference set in $Q(H, g)$ relative to $\langle g \rangle$. The proof of the converse is similar. □

Corollary 3.7 *Let t be a positive integer such that $2t - 1$ or $4t - 1$ is a prime power. Then there is a Golay transversal of $\mathbb{Z}_{4t}$.*

Proof This follows from Proposition 3.6, since it is known that there are $(4t, 2, 4t, 2t)$ relative difference sets in Dic_{2t} in these cases, see [13, 14, 23]. □

4 Golay Transversals from Williamson Quadruples and Building Sets

Williamson matrices are a useful tool for the construction of $(2m, 2, 2m, m)$ relative difference sets. Constructions of such matrices are contained in [8, 24, 25, 26]. The term "Williamson matrix" has been used with many different meanings in the literature. For the latest state of the art concerning the search for "original" (in the sense of Williamson) Williamson matrices, see [12].

As "Williamson matrices" are not uniquely defined in the literature, we need to introduce a new yet another term to avoid confusion.

Definition 4.1 Let G be an abelian group of order v. A $v \times v$ matrix $A = (a_{g,h})_{g,h \in G}$ is called **G-invariant** if $a_{gk,hk} = a_{g,h}$ for all $g, h, k \in G$.

Definition 4.2 Let G be a finite abelian group, and let $A_1, ..., A_4 \in \mathbb{Z}[G]$ with coefficients ± 1 only. If $\sum_{i=1}^{4} A_i A_i^{(-1)} = 4|G|$ and $A_i = A_i^{(-1)}$ for all i, we call $(A_1, ..., A_4)$ a **Williamson quadruple** over G.

The following is well known and straightforward to verify.

Result 4.3 *Let G be a finite abelian group. Assume that*

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix} \tag{4}$$

is a Hadamard matrix and A, B, C, D are symmetric and G-invariant. Write $A = (a_{g,h})_{g,h \in G}$, $A' = \sum_{g \in G} a_{g,1} g$ etc. Then $(A', ..., D')$ is a Williamson quadruple.

Let q be a prime power. We denote the elementary abelian group of order q by $\mathrm{EA}(q)$. The following Williamson quadruples are known.

Result 4.4 *Let $q \equiv 1 \pmod 4$ be a prime power, and let r be a nonnegative integer. There are Williamson quadruples over the following groups.*

(a) $\mathrm{EA}(q^2)$,

(b) $\mathbb{Z}_{(q+1)/2} \times \mathrm{EA}(q^r)$,

(c) $\mathbb{Z}_m$ *for* $1 \leq m \leq 33$ *and* $m \in \{39, 43\}$.

Proof This follows from Result 4.3, since matrices of the form (4) are known to exist in these cases, see [8, 24, 25, 26]. □

Theorem 4.5 *Suppose there exists a Williamson quadruple over an abelian group G, and there exists a Golay transversal of an abelian group H with respect to h.*

(a) *If $\langle h \rangle$ has a complement in H, i.e., $H = K \times \langle h \rangle$ for some $K \leq H$, then there is a Golay transversal of $G \times K \times \mathbb{Z}_4$ with respect to h, where $\mathbb{Z}_4 = \langle y \rangle$ and $h = y^2$.*

(b) *If $\langle h \rangle$ has no complement in H, then there is a Golay transversal of $G \times H \times \mathbb{Z}_2$ with respect to h.*

Proof Let $(A_1, \ldots, A_4)$ be a Williamson quadruple over G and let (A, B) be a Golay transversal of H with respect to h.

(a) As above, write $H = K \times \langle h \rangle$ and let y be an element with $y^2 = h$. Define

$$\begin{aligned} f &= \frac{1}{2}\left[A(A_1 + A_2y + A_3 + A_4y) + B^{(-1)}(A_1 + A_2y - A_3 - A_4y)\right], \\ g &= \frac{1}{2}\left[B(A_1 + A_2y + A_3 + A_4y) + A^{(-1)}(-A_1 - A_2y + A_3 + A_4y)\right]. \end{aligned}$$

It is straightforward to verify that f and g have coefficients $0, \pm1$ only and that the elements of $G \times K \times \mathbb{Z}_4$ with nonzero coefficient in f, respectively g, form a complete system of coset representatives of $\langle h \rangle$ in $G \times K \times \mathbb{Z}_4$. We compute

$$\begin{aligned} ff^{(-1)} + gg^{(-1)} &= \frac{1}{2}(AA^{(-1)} + BB^{(-1)})[(A_1 + A_2y)(A_1 + A_2y)^{(-1)} \\ &\quad +(A_3 + A_4y)(A_3 + A_4y)^{(-1)}] \\ &= \frac{1}{2}(AA^{(-1)} + BB^{(-1)})\left(\left(\sum_{i=1}^{4} A_i^2\right) + (y + y^{-1})(A_1A_2 + A_3A_4)\right). \end{aligned}$$

Now let χ be a character of $G \times K \times \mathbb{Z}_4$ with $\chi(h) = -1$. Then $\chi(y + y^{-1}) = \chi(y)(\chi(1 + h)) = 0$. Moreover, $\chi\left(\sum A_i^2\right) = 4|G|$ as $(A_1, \ldots, A_4)$ is a Williamson quadruple over G and $\chi(AA^{(-1)} + BB^{(-1)}) = |H|$ as (A, B) is a Golay transversal of H with respect to h. Hence $|\chi(f)|^2 + |\chi(g)|^2 = 2|G||H| = |G \times K \times \mathbb{Z}_4|$, which shows that (f, g) is a Golay transversal of $G \times K \times \mathbb{Z}_4$ with respect to h.

(b) As h has no complement in H, there is $z \in H$ with $z^2 = h$. Let x be a generator of $\mathbb{Z}_2$ and set $y = xz$. Note that $y^2 = h$. Now the same construction as in part (a) shows that there is a Golay transversal of $G \times H \times \mathbb{Z}_2$ with respect to h. □

In the following result, we obtain Golay transversals from building sets given in [7].

Proposition 4.6 *Let p_1,...,p_k be any odd primes (not necessarily distinct), and let b_1, ..., b_r be positive integers ($k = 0$ or $r = 0$ is allowed here). Furthermore, let t be a positive integer such that $2t - 1$ or $4t - 1$ is a prime power. Let*

$$G = \mathbb{Z}_{3^{b_1}}^2 \times \cdots \times \mathbb{Z}_{3^{b_r}}^2 \times \mathbb{Z}_{p_1}^4 \times \cdots \times \mathbb{Z}_{p_k}^4.$$

Then there are Golay transversals of $\mathbb{Z}_4 \times G$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times G$, and $\mathbb{Z}_2 \times \mathbb{Z}_{4t} \times G$.

Proof Write $|G| = u^2$. By [7, Thm. 6.6] there are subsets $D_1, ..., D_4$ of G with $|D_1| = |D_2| = |D_3| = (u^2 - u)/2$, $|D_4| = (u^2 + u)/2$, such that, for every nontrivial character χ of G, we have $|\chi(D_i)|^2 = u^2$ for one i and $\chi(D_j) = 0$ for $j \neq i$. Set $E_i = 2D_i - G$, $i = 1, ..., 4$. Then the E_i have coefficients ± 1 only and, for every nontrivial character χ of G, we have $|\chi(E_i)|^2 = 4u^2$ for one i and $\chi(E_j) = 0$ for $j \neq i$.

Now let $H = \mathbb{Z}_4 \times G$ or $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times G$, let g be an element of order 2 of H, and let $y \in H \setminus \{G\langle g\rangle\}$. Set $X = E_1 + yE_2$ and $Y = E_3 + yE_4$. Then $|\chi(X)|^2 + |\chi(Y)|^2 = 4u^2$ for all nontrivial characters χ of H. Hence (X, Y) is the required Golay transversal of H.

It remains to show that $K = \mathbb{Z}_2 \times \mathbb{Z}_{4t} \times G$ has a Golay transversal. By Corollary 3.7, there exists a Golay transversal (A, B) of $\mathbb{Z}_{4t}$. Write $\mathbb{Z}_2 = \langle z\rangle$ and set $X = E_1 + zE_2$ and $Y = E_3 + zE_4$ where the E_i's are as above. Define

$$\begin{aligned} f &= \frac{1}{2}\left[A(X+Y) + B^{(-1)}(X-Y)\right], \\ g &= \frac{1}{2}\left[B(X+Y) + A^{(-1)}(-X+Y)\right]. \end{aligned}$$

It is straightforward to verify that f and g have coefficients $0, \pm 1$ only and that the elements of K with nonzero coefficient in f, respectively g, form a complete system of coset representatives of $\langle h\rangle$ in K, where h is the element of order 2 in $\mathbb{Z}_{4t}$. We compute

$$ff^{(-1)} + gg^{(-1)} = \frac{1}{2}(AA^{(-1)} + BB^{(-1)})(XX^{(-1)} + YY^{(-1)})).$$

Hence

$$|\chi(f)|^2 + |\chi(g)|^2 = \frac{1}{2}|\mathbb{Z}_{4t}|4u^2 = 8tu^2 = |K|$$

for all characters χ of K with $\chi(h) = -1$. $\square$

5 Golay Transversals from Pairs of Golay Sequences

Let $a = (a_0, ..., a_{v-1})$, $b = (b_0, ..., b_{v-1})$ be sequences, and write $f = \sum_{i=0}^{v-1} a_i x^i$, $g = \sum_{i=0}^{v-1} b_i x^i$. If $a_j, b_j \in \{\pm 1\}$ and

$$f(x)f(x^{-1}) + g(x)g(x^{-1}) = 2v,$$

then (a, b) is called a **Golay pair of length v**. Golay pairs of length $2^a 10^b 26^c$ are known to exist [25].

If $a_j, b_j \in \{\pm 1, \pm i\}$, where $i = \sqrt{-1}$, and

$$f(x)\overline{f(x^{-1})} + g(x)\overline{g(x^{-1})} = 2v,$$

then (a, b) is called a **complex Golay pair of length v**. Complex Golay pairs of length $2^{a+u}3^b5^c11^d13^e$, where $a, b, c, d, e, u \geq 0, b + c + d + e \leq a + 2u + 1, u \leq c + e$, are shown to exist in [5].

The following essentially is [1, Thm. 4.2]. We include a proof for the convenience of the reader.

Result 5.1 *Let H be an abelian group and assume that there is a Golay transversal of H with respect to h. Let G be an abelian group containing H such that that G/H is cyclic, and assume that a pair of Golay sequences of length $|G/H|$ exists. Then there is a Golay transversal of G with respect to h.*

Proof Write $n = |G/H|$ and let $a \in G$ such that $1, a, ..., a^{n-1}$ represent all cosets of H in G. Let f, g be the polynomials corresponding to a complex Golay pair of length n, and let (A, B) be a Golay transversal of H with respect to h. Define

$$\begin{aligned} C &= \frac{1}{2}\left[A(f(a) + g(a)) + B^{(-1)}(f(a) - g(a))\right], \\ D &= \frac{1}{2}\left[B(f(a) + g(a)) + A^{(-1)}(-f(a) + g(a))\right]. \end{aligned}$$

Then $CC^{(-1)} + DD^{(-1)} = n(AA^{(-1)} + BB^{(-1)})$. Hence $|\chi(C)|^2 + |\chi(D)|^2 = n|H| = |G|$ for all characters χ of G with $\chi(h) = -1$.

$\square$

Lemma 5.2 *Let H be an abelian group containing a subgroup $\langle y \rangle \cong \mathbb{Z}_4$ such that there is a Golay transversal of H with respect to y^2. Let G be an abelian group containing H such that G/H is cyclic, and that a complex Golay pair of length $\frac{1}{2}|G/H|$ exists. Then there is a Golay transversal of G with respect to y^2.*

Proof Write $n = \frac{1}{2}|G/H|$ and let $a \in G$ such that $1, a, \ldots, a^{2n-1}$ represent all cosets of H in G. Let (A, B) be a Golay transversal of H with respect to y^2. Let f, g be the polynomials corresponding to a Golay pair of length n, with each occurrence of imaginary unit, i, is replaced by y. Define

$$\begin{aligned} C &= Af(a^2) - aB^{(-1)}g(a^2), \\ D &= aA^{(-1)}g(a^2) + Bf(a^2). \end{aligned}$$

Then

$$CC^{(-1)} + DD^{(-1)} = (AA^{(-1)} + BB^{(-1)})(f(a^2)f(a^{-2}) + g(a^2)g(a^{-2})).$$

Let χ be a character of G with $\chi(y^2) = -1$. Then $\chi(y) = \pm i$ and thus $\chi(f(a^2)f(a^{-2}) + g(a^2)g(a^{-2})) = |G/H|$ by the definition of complex Golay pair. Hence $|\chi(C)|^2 + |\chi(D)|^2 = 2n|H| = |G|$ for all characters χ of G with $\chi(y^2) = -1$. $\square$

6 Main Results

Theorem 6.1 *Let r, s, t, u, w be nonnegative integers and let*

- $a_1, \ldots, a_r$ *and* $v_1, ..., v_u$ *be nonnegative integers,*
- $p_1, \ldots, p_s$ *be odd primes (not necessarily distinct),*
- $q_1 \ldots, q_t$ *and* $r_1, \ldots, r_u$ *be prime powers with* $q_i \equiv 1 \pmod 4$ *and* $r_i \equiv 1 \pmod 4$ *for all* i,
- $s_1, ..., s_w$ *be integers with* $1 \leq s_i \leq 33$ *or* $s_i \in \{39, 43\}$ *for all* i,
- x *be a positive integer such that* $2x - 1$ *or* $4x - 1$ *is a prime power,*
- $\delta = 1$ *if* $x > 1$ *and* $r + s > 0$, $\delta = 0$ *otherwise,*
- $\epsilon = 1$ *if* $x = 1$, $r + s = 0$, *and* $t + u + w > 0$, $\epsilon = 0$ *otherwise,*

and write

$$K = \mathbb{Z}_{4x} \times \mathbb{Z}_2^{t+u+w+\delta-\epsilon} \prod_{i=1}^{r} \mathbb{Z}_{3^{a_i}}^2 \prod_{i=1}^{s} \mathbb{Z}_{p_i}^4 \prod_{i=1}^{t} \mathrm{EA}(q_i^2) \prod_{i=1}^{u} \left(\mathbb{Z}_{(r_i+1)/2} \times \mathrm{EA}(r_i^{v_i})\right) \prod_{i=1}^{w} \mathbb{Z}_{s_i}$$

Let a, b, c, d, e be nonnegative integers and let H be any abelian group which contains K as a subgroup such that

$$|H| = 2^a 6^b 10^c 22^d 26^e |K|.$$

Then there is an involution $g \in H$ such that $Q(H, g)$ contains an $(|H|, 2, |H|, |H|/2)$ relative difference set.

Proof Write $G = \prod_{i=1}^{r} \mathbb{Z}_{3^{a_i}}^2 \prod_{i=1}^{s} \mathbb{Z}_{p_i}^4$. First suppose $\delta = 1$. Then $\epsilon = 0$ and there is a Golay transversal of

$$\mathbb{Z}_{4x} \times \mathbb{Z}_2^{\delta-\epsilon} \times G = \mathbb{Z}_{4x} \times \mathbb{Z}_2 \times G$$

by Proposition 4.6. Hence, by repeated application of Theorem 4.5, there is a Golay transversal of K.

Now suppose $\epsilon = 1$. Then $x = 1$, $\delta = 0$, and $r + s = 0$. Now let $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. As there is a Golay transversal of $\mathbb{Z}_2$, there is a Golay transversal of

$$\mathbb{Z}_4 \times \prod_{i=1}^{\alpha_1} \mathrm{EA}(q_i^2) \prod_{i=1}^{\alpha_2} \left(\mathbb{Z}_{(r_i+1)/2} \times \mathrm{EA}(r_i^{v_i})\right) \prod_{i=1}^{\alpha_3} \mathbb{Z}_{s_i}$$

by Theorem 4.5. Thus, by repeated application of Theorem 4.5, there is a Golay transversal of

$$K = \mathbb{Z}_4 \times \mathbb{Z}_2^{t+u+w-1} \prod_{i=1}^{t} \mathrm{EA}(q_i^2) \prod_{i=1}^{u} \left(\mathbb{Z}_{(r_i+1)/2} \times \mathrm{EA}(r_i^{v_i})\right) \prod_{i=1}^{w} \mathbb{Z}_{s_i}.$$

Finally, suppose $\delta = \epsilon = 0$. Then $x = 1$ or $r + s = 0$. If $x = 1$, then there is a Golay transversal of $\mathbb{Z}_4 \times G$ by Proposition 4.6 and thus a Golay transversal of K by a repeated application of Theorem 4.5. If $x > 1$, then $r + s = 0$ and there is a Golay transversal of $\mathbb{Z}_{4x}$ by Corollary 3.7. Thus there is a Golay transversal of K by a repeated application of Theorem 4.5.

In summary, we have shown that there is a Golay transversal of K in all cases. By a repeated application of Result 5.1 and Lemma 5.2, there is a Golay transversal of H. Now the assertion follows from Proposition 3.6. □

Note that the exponent of all groups H covered by Theorem 6.1 is divisible by 4. When we attempt to construct Golay transversals of groups whose exponent is not divisible by 4, the group structure will be more restricted. The following result deals with this case.

Theorem 6.2 *Let r, s be nonnegative integers, let $a_1, \ldots, a_r$ be positive integers, and let $p_1, \ldots, p_s$ be odd primes (not necessarily distinct), and write*

$$K = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \prod_{i=1}^{r} \mathbb{Z}_{3^{a_i}}^2 \prod_{i=1}^{s} \mathbb{Z}_{p_i}^4.$$

Let a, b, c be nonnegative integers and let H be any abelian group which contains K as a subgroup such that

$$|H| = 2^a 10^b 26^c |K|.$$

Then there is an involution $g \in H$ such that $Q(H, g)$ contains an $(|H|, 2, |H|, |H|/2)$ relative difference set.

Proof By Proposition 4.6 there is a Golay transversal of K. By a repeated application of Result 5.1, there is a Golay transversal of H. Hence the assertion follows from Proposition 3.6. □

Corollary 6.3 *There exist normal $(2m, 2, 2m, m)$ relative difference sets for all m of the form*

$$m = x 2^{a+t+u+w+\delta-\epsilon+1} 6^b 9^c 10^d 22^e 26^f \prod_{i=1}^{s} p_i^{4a_i} \prod_{i=1}^{t} q_i^2 \prod_{i=1}^{u} ((r_i+1)/2) r_i^{v_i}) \prod_{i=1}^{w} s_i$$

under the following conditions: $a, b, c, d, e, f, s, t, u, w$ are nonnegative integers, $a_1, \ldots, a_r$ and $v_1, ..., v_u$ are positive integers, $p_1, \ldots, p_s$ are odd primes, $q_1, \ldots, q_t$ and $r_1, \ldots, r_u$ are prime powers with $q_i \equiv 1 \pmod 4$ and $r_i \equiv 1 \pmod 4$ for all i, $s_1, ..., s_w$ are integers with $1 \le s_i \le 33$ or $s_i \in \{39, 43\}$ for all i, x is a positive integer such that $2x - 1$ or $4x - 1$ is a prime power. Moreover, $\delta = 1$ if $x > 1$ and $c + s > 0$, $\delta = 0$ otherwise, $\epsilon = 1$ if $x = 1$, $c + s = 0$, and $t + u + w > 0$, $\epsilon = 0$ otherwise.

Proof This follows from Theorem 6.1. □

7 Necessary Conditions for the Existence of Golay Transversals

We have seen that it is easy to construct Golay transversal over a large variety of finite abelian groups. As we will show in this section, however, there are many finite abelian groups that do not admit Golay transversals. We remark that some of the results in this section are similar to results in [1].

Lemma 7.1 *Let H be an abelian group of even order. If $|H| > 2$ and there is a Golay transversal of H, then $|H| \equiv 0 \pmod 4$.*

Proof This follows from Result 1.1 and Proposition 3.6. □

Lemma 7.2 *Let H be an abelian group, and assume there is a Golay transversal of H with respect to g such that $\langle g \rangle$ has a complement in H. Then $|H|$ is a sum of two squares.*

Proof Let (A, B) be a Golay transversal of H with respect to g. By assumption there is a character χ of H of order 2 with $\chi(g) = -1$. Then $\chi(A)$ and $\chi(B)$ are integers and $|\chi(A)|^2 + |\chi(B)|^2 = |H|$. □

Lemma 7.3 *Let $p \equiv 3 \pmod 4$ be a prime, and let U be an elementary abelian 2-group. Suppose $H = U \times P$ where P is an abelian p-group. If $\exp(P) > \sqrt{|P|}$, then there is no Golay transversal of H.*

Proof Suppose there is a Golay transversal (A, B) of H with respect to g. By Lemma 7.2, $|P|$ is a square, say $|P| = p^{2b}$ some integers b. Write $H = G \times \langle g \rangle$, $\exp(P) = p^c$, and let Q be a subgroup of P of order p^{2b-c} such that P/Q is cyclic. Let $\rho : \mathbb{Z}[H] \to \mathbb{Z}[G/Q]$ be the epimorphism determined by $\rho(g) = -1$ and $\rho(h) = hQ$ for all $h \in G$. Write $C = \rho(A)$ and $D = \rho(B)$. Note that all coefficients of C and D are bounded by $|Q| = p^{2b-c}$ in absolute value. By the definition of a Golay transversal, we have

$$|\chi(C)|^2 + |\chi(D)|^2 = |H| \equiv 0 \pmod{p^{2b}}$$

for all characters χ of G/Q. Write $\zeta = \exp(2\pi i/p^c)$. Since $\chi(C), \chi(D) \in \mathbb{Z}[\zeta]$, we conclude $\chi(C) \equiv 0 \pmod{p^b}$ and $\chi(D) \equiv 0 \pmod{p^b}$ for all characters χ of G/Q by Result 2.5. Hence Result 2.4 implies

$$C = p^b X + SY$$

with $X, Y \in \mathbb{Z}[G/Q]$ where S is the subgroup of order p of G/Q. Note that $p^c = \exp(P) > \sqrt{|P|} = p^b$ by assumption and thus $c > b$. If X was not a multiple of S, then there would be two coefficients of C whose difference is at least p^b. This is impossible since the coefficients of C are bounded by $p^{2b-c} \leq p^{b-1}$ in absolute value. Hence X and thus C is a multiple of S. By the same argument, D is a multiple of S. But this implies $|\chi(A)|^2 + |\chi(B)|^2 = 0$ for all characters χ of H with $\chi(g) = -1$ which are trivial on Q and have order $2p^c$, a contradiction. □

Let u be a positive integer, and let G be an abelian group of order $4u^2$. A subset D of G with $|D| = 2u^2 - u$ and $DD^{(-1)} = u^2 + (u^2 - u)G$ is called an **Hadamard difference set**. It is well known [3, Lem. 3.12] that a subset D of G with $|D| = 2u^2 - u$ is an Hadamard difference set in G if and only if $|\chi(D)|^2 = u^2$ for all nontrivial characters χ of G.

Lemma 7.4 *Let $p \equiv 3 \pmod 4$ be a prime. Suppose that there is a Golay transversal of $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times K$ where K is an abelian p-group. Then $|H|$ is a square and there is a Hadamard difference set in H and in $\mathbb{Z}_4 \times K$.*

Proof Suppose there is a Golay transversal (A, B) of H with respect to g. By Lemma 7.2, $|H|$ is a square, say $|H| = 4p^{2d}$. Write $H = G \times \langle g \rangle$ and let $\rho : \mathbb{Z}[H] \to \mathbb{Z}[G]$ be the epimorphism determined by $\rho(g) = -1$ and $\rho(h) = h$ for all $h \in G$. Write $C = \rho(A)$ and $D = \rho(B)$. Note that C and D have coefficients ± 1 only. By the definition of a Golay transversal, we have

$$|\chi(C)|^2 + |\chi(D)|^2 = 4p^{2d}. \tag{5}$$

for all characters χ of G. If χ has order 1 or 2, then (5) implies $\chi(C) = 0$ or $\chi(D) = 0$ since $4p^{2d} = (\pm 2p^d)^2$ are the only representations of $4p^{2d}$ as a sum of two squares.

Now let χ be character of G of order divisible by p. Since C and D have coefficients ± 1 only, $\chi(C)$ and $\chi(D)$ are divisible by 2. Write $X = \chi(C)/2$ and $Y = \chi(D)/2$. Then

$$|X|^2 + |Y|^2 = p^{2d}. \tag{6}$$

Write $\exp(H) = 2p^e$ and $\zeta = \exp(2\pi i/p^e)$. Note $X, Y \in \mathbb{Z}[\zeta]$. By Result 2.5, we have $X \equiv 0 \pmod{p^d}$ and $Y \equiv 0 \pmod{p^d}$. Write $X = p^d x$, $Y = p^d y$. Then $x, y \in \mathbb{Z}[\zeta]$ and $|x|^2 + |y|^2 = 1$. Thus all conjugates of x and y have absolute value at most 1. This implies that one of x, y is a root of unity and the other is 0. This shows that $\chi(C) = 0$ or $\chi(D) = 0$ for all characters χ of order divisible by p.

In summary, we have shown

$$\chi(C) = 0 \text{ or } \chi(D) = 0 \tag{7}$$

for all characters χ of G. Now let M be an abelian group of order $4p^{2d}$ containing G as a subgroup and let $y \in M \setminus G$. Set $E = C + Dy$. Then $|\psi(E)|^2 = 4p^{2d}$ for all characters ψ of K by (5) and (7). Hence $F = (E + M)/2$ is a Hadamard difference set in M. $\square$

Corollary 7.5 *Let $p \equiv 3 \pmod 4$ be a prime, let H be an abelian p-group and suppose that there is a Golay transversal of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times H$. Then $|H|$ is a square and $\exp(H) \leq \sqrt{|H|}$. Furthermore, if $H = \mathbb{Z}_{p^a} \times K$ with $|K| = p^a$ for some a, then $p = 3$ and K is cyclic.*

Proof This follows from Lemma 7.4 and known results on Hadamard difference sets [6, Thms. 2.11, 2.12]. $\square$

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References

[1] K. T. Arasu, Y. Q. Chen, A. Pott: Hadamard and conference matrices. *J. Algebraic Combin.* **14** (2001), 103–117.

[2] L. D. Baumert, M. Hall: Hadamard matrices of the Williamson type. *Math. Comp.* **19** (1965), 442–447.

[3] T. Beth, D. Jungnickel, H. Lenz: *Design Theory* (second edition). Cambridge University Press 1999.

[4] Z. I. Borevich, I. R. Shafarevich: *Number Theory.* Academic Press 1966.

[5] R. Craigen, W. Holzmann, H. Kharaghani: Complex Golay sequences: structure and applications. *Discrete Math.* **252** (2002), 73–89

[6] J. A. Davis and J. Jedwab: A survey of Hadamard difference sets. In: K.T. Arasu et al., eds., *Groups, Difference Sets and the Monster*, de Gruyter 1996, 145–156.

[7] J.A. Davis and J. Jedwab: A unifying construction of difference sets. *J. Combin. Theory Ser. A* **80** (1997), 13–78.

[8] D. Z. Dokovic: Williamson matrices of order $4n$ for $n = 33, 35, 39$. *Discrete Math.* **115** (1993), 267–271.

[9] S. Eliahou, M. Kervaire, B. Saffari: A new restriction on the length of Golay complementary sequences. *J. Comb. Theory Ser. A 55* (1990), 49–59.

[10] D. Gorenstein: *Finite Groups* (second edition). Chelsea Publishing Co., New York, 1980.

[11] D. L. Flannery: Cocyclic Hadamard Matrices and Hadamard Groups are Equivalent. *J. Algebra* **192** (1997), 749–779.

[12] W. H. Holzmann, H. Kharaghani, B. Tayfeh-Rezaie: Williamson matrices up to order 59. *Des. Codes Cryptogr.* **46** (2008), 343–352.

[13] N. Ito: On Hadamard groups. *J. Algebra* **168** (1994), 981–987.

[14] N. Ito: On Hadamard groups, II. *J. Algebra* **169** (1994), 936–942.

[15] N. Ito: Some remarks on Hadamard groups. In: *Groups–Korea 94*, de Gruyter 1995, 149–155.

[16] N. Ito: On Hadamard groups, II. *Rep. Fac. Sci. Technol. Meijo Univ.* **37** (1997), 1–7.

[17] D. Jungnickel: On automorphism groups of divisible designs. *Canad. J. Math.* **34** (1982), 257–297.

[18] S. Lal, R. L. McFarland, R. W. K. Odoni: On $|\alpha|^2 + |\beta|^2 = p^t$ in certain cyclotomic fields. In: *Number theory and algebra*, Academic Press 1977, 195–197.

[19] W. de Launey: On the asymptotic existence of Hadamard matrices. *J. Combin. Theory Ser. A* **116** (2009), 1002–1008.

[20] W. de Launey, Warwick, M. J. Smith: Cocyclic orthogonal designs and the asymptotic existence of cocyclic Hadamard matrices and maximal size relative difference sets with forbidden subgroup of size 2. *J. Combin. Theory Ser. A* **93** (2001), 37–92.

[21] S. L. Ma: *Polynomial addition sets*. Ph.D. thesis. University of Hong Kong 1985.

[22] A. Pott: A survey on relative difference set. K. T. Arasu et al. (eds.), *Groups, Difference Sets and the Monster*, de Gruyter 1996, 195–232.

[23] B. Schmidt: Williamson Matrices and a Conjecture of Ito's. *Des. Codes Cryptogr.* **17** (1999), 61–68.

[24] E. Spence: An Infinite Family of Williamson Matrices. *J. Austral. Math. Soc.* **24** (1977), 252–256.

[25] R. J. Turyn: An infinite class of Williamson matrices. *J. Combin. Theory Ser. A* **12** (1972), 319–321.

[26] A.L. Whiteman: An infinite family of Hadamard matrices of Williamson type. *J. Combin. Theory Ser. A* **14** (1973), 334–340.

A Appendix

The following table presents all abelian groups of order divisible by 4, up to 400, which do not admit Golay transversals or for which the existence of such transversals is open. In the case of nonexistence, a reference to the proof is provided. Groups that admit Golay transverals by Theorems 6.1 and 6.2 are *not* listed.

Order	Group	Exist?
12	(2)(2)(3)	Lem 7.3
24	(2)(2)(2)(3)	Lem 7.3
28	(2)(2)(7)	Lem 7.3
36	(2)(2)(9)	Lem 7.3
44	(2)(2)(11)	Lem 7.3
48	(2)(2)(2)(2)(3)	Lem 7.3
56	(2)(2)(2)(7)	Lem 7.3
60	(2)(2)(3)(5)	Lem 7.2
68	(2)(2)(17)	?
72	(2)(2)(2)(9)	Lem 7.3
76	(2)(2)(19)	Lem 7.3
84	(2)(2)(3)(7)	Lem 7.2
88	(2)(2)(2)(11)	Lem 7.3
92	(2)(2)(23)	Lem 7.3
96	(2)(2)(2)(2)(2)(3)	Lem 7.3
100	(2)(2)(25)	?
100	(2)(2)(5)(5)	?
108	(2)(2)(27)	Lem 7.3
108	(3)(4)(9)	?
108	(2)(2)(3)(9)	Lem 7.3

Order	Group	Exist?
108	(3)(3)(3)(4)	?
108	(2)(2)(3)(3)(3)	Lem 7.2
112	(2)(2)(2)(2)(7)	Lem 7.3
116	(2)(2)(29)	?
120	(2)(2)(2)(3)(5)	Lem 7.2
124	(2)(2)(31)	Lem 7.3
132	(2)(2)(3)(11)	Lem 7.2
136	(2)(2)(2)(17)	?
140	(2)(2)(5)(7)	Lem 7.2
144	(2)(2)(2)(2)(9)	Lem 7.3
148	(2)(2)(37)	?
152	(2)(2)(2)(19)	Lem 7.3
156	(2)(2)(3)(13)	Lem 7.2
164	(2)(2)(41)	?
168	(2)(2)(2)(3)(7)	Lem 7.2
172	(2)(2)(43)	Lem 7.3
176	(2)(2)(2)(2)(11)	Lem 7.3
180	(2)(2)(5)(9)	?
180	(2)(2)(3)(3)(5)	?
184	(2)(2)(2)(23)	Lem 7.3
188	(4)(47)	?
188	(2)(2)(47)	Lem 7.3
192	(2)(2)(2)(2)(2)(2)(3)	Lem 7.3
196	(2)(2)(49)	Cor 7.5
196	(4)(7)(7)	?

Order	Group	Exist?
196	(2)(2)(7)(7)	Cor 7.5
204	(2)(2)(3)(17)	Lem 7.2
212	(2)(2)(53)	?
216	(2)(2)(2)(27)	Lem 7.3
216	(2)(2)(2)(3)(9)	Lem 7.3
216	(2)(2)(2)(3)(3)(3)	Lem 7.2
220	(2)(2)(5)(11)	Lem 7.2
224	(2)(2)(2)(2)(2)(7)	Lem 7.3
228	(2)(2)(3)(19)	Lem 7.2
232	(2)(2)(2)(29)	?
236	(4)(59)	?
236	(2)(2)(59)	Lem 7.3
240	(2)(2)(2)(2)(3)(5)	Lem 7.2
244	(2)(2)(61)	?
248	(2)(2)(2)(31)	Lem 7.3
252	(2)(2)(7)(9)	Lem 7.2
252	(3)(3)(4)(7)	?
252	(2)(2)(3)(3)(7)	Lem 7.2
260	(4)(5)(13)	?
260	(2)(2)(5)(13)	?
264	(2)(2)(2)(3)(11)	Lem 7.2
268	(4)(67)	?
268	(2)(2)(67)	Lem 7.3
272	(2)(2)(2)(2)(17)	?
276	(2)(2)(3)(23)	Lem 7.2

Order	Group	Exist?
280	(2)(2)(2)(5)(7)	Lem 7.2
284	(2)(2)(71)	Lem 7.3
288	(2)(2)(2)(2)(2)(9)	Lem 7.3
292	(4)(73)	?
292	(2)(2)(73)	?
296	(2)(2)(2)(37)	?
300	(2)(2)(3)(25)	Lem 7.2
300	(2)(2)(3)(5)(5)	Lem 7.2
304	(2)(2)(2)(2)(19)	Lem 7.3
308	(2)(2)(7)(11)	Lem 7.2
312	(2)(2)(2)(3)(13)	Lem 7.2
316	(2)(2)(79)	Lem 7.3
324	(4)(81)	?
324	(2)(2)(81)	Lem 7.3
324	(3)(4)(27)	?
324	(2)(2)(3)(27)	Lem 7.3
324	(3)(3)(4)(9)	?
324	(2)(2)(3)(3)(9)	Cor 7.5
328	(2)(2)(2)(41)	?
332	(2)(2)(83)	Lem 7.3
336	(2)(2)(2)(2)(3)(7)	Lem 7.2
340	(2)(2)(5)(17)	?
344	(2)(2)(2)(43)	Lem 7.3
348	(2)(2)(3)(29)	Lem 7.2
352	(2)(2)(2)(2)(2)(11)	Lem 7.3

Order	Group	Exist?
356	(4)(89)	?
356	(2)(2)(89)	?
360	(2)(2)(2)(5)(9)	?
364	(2)(2)(7)(13)	Lem 7.2
368	(2)(2)(2)(2)(23)	Lem 7.3
372	(3)(4)(31)	?
372	(2)(2)(3)(31)	Lem 7.2
376	(8)(47)	?
376	(2)(4)(47)	?
376	(2)(2)(2)(47)	Lem 7.3
380	(2)(2)(5)(19)	Lem 7.2
384	(2)(2)(2)(2)(2)(2)(2)(3)	Lem 7.3
388	(2)(2)(97)	?
392	(2)(2)(2)(49)	Lem 7.3
392	(7)(7)(8)	?
392	(2)(2)(2)(7)(7)	?
396	(2)(2)(9)(11)	Lem 7.2
396	(3)(3)(4)(11)	?
396	(2)(2)(3)(3)(11)	Lem 7.2