Quantal Response Equilibrium for Sponsored Search Auctions: Computation and Inference

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Sponsored search auctions have attracted much research attention in recent years and different equilibrium concepts have been studied to understand advertisers' bidding strategies. However, the assumption that bidders are perfectly rational in these studies is unrealistic in the real world. In this work, we investigate the quantal response equilibrium (QRE) solution concept for sponsored search auctions, in which advertisers choose strategies based on their relative expected utilities. QRE is powerful in characterizing the bounded rationality in the sense that it only assumes that an advertiser chooses a better strategy with a larger probability instead assuming that the advertiser chooses the best strategy deterministically. We first propose a homotopy-based method, which is potential to be globally convergent, to compute QRE for sponsored search auctions. We show that this method is effective in finding a QRE. Then we fit the model into the datasets from a commercial search engine and develop an estimator to infer the values of advertisers and click-through rates of their advertisements. Our experiments on several search phrases indicate that the model works quite well for certain queries and the values we estimated are consistent with the basic property of sponsored search auctions.

1. INTRODUCTION

Sponsored search has become a major monetization means for commercial search engines. When a user issues a query to a search engine, in addition to several relevant webpages, a set of relevant advertisements will also be displayed on the search result page. To show his/her ad on the search result page, an advertiser/bidder[1] is required

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[1]In this paper, we use "advertiser" and "bidder" interchangeably.

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to submit a bid for the query.[2] Most of the time, there are many more advertisers bidding for the query than the number of available ad slots. Hence, the search engines need a mechanism to decide which ads should be shown on the result page, how to allocate the slots to the shown ads, and how to charge an advertiser if his/her ad is clicked by the user. The most popular mechanism used by commercial search engines is the Generalized Second Price (GSP) mechanism, in which the slots are allocated to advertisers according to a ranking rule, and an advertiser will be charged a price to maintain his/her current rank position if his/her ad is clicked.

Given its critical role in sponsored search, the GSP mechanism has attracted much research attention recently and there have been many exciting studies [Jansen and Mullen 2008; Qin et al. 2014] in this field. Among those studies, equilibrium analysis is a hot topic to understand advertisers' behaviors. Varian [2007] studied the concept of symmetric Nash equilibria for GSP auctions and proved the existence of a symmetric Nash equilibrium which can achieve the maximal revenue among all Nash equilibria. Edelman et al. [2005] defined a subset of Nash equilibria called locally envy-free equilibria which are essentially equivalent to the symmetric Nash equilibria. Borgers et al. [2013] further stated the existence of multiple Nash equilibria in GSP auctions.

A limitation of those studies on equilibrium concepts is that they assume the complete rationality of advertisers. That is, advertisers are very smart; they can find their optimal strategies and take optimal actions. However, in the real world, rationality of human beings is limited by different kinds of constraints, including the information they collect, the cognitive limitations of their minds, and the finite amount of time available to make a decision. Therefore, it is necessary to study the equilibrium concepts for sponsored search auctions under the assumption of bounded rationality, which is exactly the focus of this work.

In this paper, we introduce the *quantal response equilibrium* [McKelvey and Palfrey 1995; Goeree et al. 2005] into sponsored search auctions considering that it can deal with limited rationality situations and has presented very good performance in general normal form games. QRE is a mixed strategy equilibrium, in which strategies with higher utilities are more likely to be chosen than those with lower utilities, but the best is not chosen with certainty due to the limited rationality of participants. We assume that all bidders take actions according to the quantal response model and investigate QRE for sponsored search auctions from two aspects: computation and inference.

1.1. Our Work

The first part of our work is about computing a QRE. We show that this problem is equivalent to solving a set of non-linear equations, which can be further transformed into finding a solution of a continuous non-linear function. Basic Newton-type algorithms are usually locally convergent and work well only when we could provide a good starting point. Unfortunately, it is difficult to find a good starting point in sponsored search auctions. To address this problem, we introduce the *homotopy* principle [Allgower and Georg 1990], which has been successfully used for equilibrium computation [Turocy 2005; Herings and Peeters 2010]. Advantages of homotopy based methods include their numerical stability and potential to be globally convergent. To design a homotopy based method for QRE computation, we first define a problem with a unique easily-computed solution. Then we build a continuous transformation from the easy to solve artificial problem into the original problem we want to solve. Next, a predictor-corrector procedure is performed to the problems of the continuous transformation step by step until finally the problem of interest has been solved. We further show that

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[2]For simplicity, we only consider exact match between a query and keywords. For broad match, please refer to [Mahdian and Wang 2009; Dhangwatnotai 2012; Chen et al. 2014].

there are many nice properties of the sponsored search auctions which can be used to refine the computational procedure.

The second part of our work is to fit the QRE model into the real world sponsored search auction data and infer the parameters of the model, including the values of bidders and the click-through rates (CTRs) of their ads. We develop a learning algorithm based on the commonly used Maximum Likelihood Estimation (MLE) method to infer those parameters. Our experimental results indicate that the QRE model fits well into the data.

To sum up, our contributions lie in three aspects. First, comparing with Nash equilibria, the QRE we introduced into sponsored search auctions is more practical since it can model bounded rationality of bidders. Second, we design a homotopy-based algorithm to effectively compute a QRE. Third, we develop an estimator to infer the parameters of QRE for sponsored search auctions.

1.2. Notations and Assumptions

Consider a keyword auction with a set of N bidders denoted by $I = \{1, 2, \ldots, N\}$ competing for a set of K ad slots denoted by $S = \{1, 2, \ldots, K\}$. Usually, we have more advertisers than slots: $N \geq K$. Let v_i denote the private value of bidder i, which expresses the maximum per-click price he/she is willing to pay. We use a vector $v = (v_1, v_2, \ldots, v_N)$ to represent the value profile of all the bidders. Let b_i denote the bid submitted by advertiser i to participate in the auction. We use a vector $b = (b_1, b_2, \ldots, b_N)$ to represent the bid profile of all the bidders. Let θ_{ik} denote the CTR of bidder i's ad when it is placed at slot k, which is usually assumed to be the product of the ad CTR α_i and the slot CTR β_k. We use $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_K)$ to represent the profile of ad and slot CTRs respectively. Without loss of generality, we assume $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_K$.

The GSP mechanism consists of a ranking rule and a pricing rule. The ranking rule determines the allocation of the ad slots to the bidders and the pricing rule determines the payment of each bidder when his/her ad is clicked. The rank-by-revenue allocation rule is widely used, in which bidders are ranked in the descending order of the products of their bids and ad CTRs. If an ad is clicked, the payment of the corresponding bidder is the minimum amount that maintains his/her current rank position.

1.3. Organization

The rest of this paper is organized as follows. In Section 2, we introduce the concept of QRE. We propose a homotopy-based algorithm to compute a QRE in Section 3. In Section 4, we discuss how to estimate the parameters of the QRE model. Experimental results are reported in Section 5 and conclusions are given in the last section.

2. QUANTAL RESPONSE EQUILIBRIUM

In this section we introduce the concept of quantal response equilibria for sponsored search auctions.

2.1. Motivation

Given that GSP is not a dominant-strategy mechanism, Nash equilibrium solutions become an important means to understand how bidders behave in sponsored search auctions. While there exist quite a few Nash equilibrium concepts proposed and studied for sponsored search auctions, symmetric Nash equilibrium [Varian 2007] and locally envy-free Nash equilibrium [Edelman et al. 2005] are the most famous two: the symmetric Nash equilibrium captures the notion that there should be no incentive for any pair of bidders to swap their slots, and the locally-envy free Nash equilibrium coptures the notion that there should be no incentive for any bidder to exchange bids with the bidder ranked one position above him/her.

While those equilibrium concepts have many nice properties, a common limitation of them is that they assume the perfect rationality of bidders, i.e., bidders have good knowledge about their utilities and take optimal actions to maximize their utilities. However, the assumption of perfect rationality of bidders is too good to be true in real-world sponsored search auctions. As pointed out in [Xu et al. 2013], in practice, an advertiser may be unwilling, incapable, or constrained to choose "best response" strategies. Therefore, a natural question arises: how can we weaken the perfect-rationality assumption and still obtain some meaningful solution concept for sponsored search auctions?

Observing that in real-world sponsored search auctions, an advertiser usually has uncertainty about his/her utility and is more likely to choose a strategy with higher utility instead of always choosing best strategies. We introduce quantal response equilibrium to model the bounded rationality of bidders in the following subsection.

2.2. Quantal Response Equilibrium

Let $O = \{o_1, o_2, \ldots, o_M\}$ denote the global bid space of all the advertisers and $B_i \subseteq O$ denote the bid space of an individual advertiser i. Let $B_{-i} = B_1 \times B_2 \times \cdots \times B_{(i-1)} \times B_{(i+1)} \times \cdots \times B_N$ be the joint bid space of all the other advertisers excluding i. Let b_{-i} be a realization of B_{-i}, and $b_{-i,k}$ is a realization of B_k in B_{-i}.

Let $r_i(o_j, b_{-i})$ denote the slot allocated to bidder i and $p_i(o_j, b_{-i})$ denote the payment of advertiser i when he/she bids $b_i = o_j$ and the bid profile of all the other advertisers is b_{-i}. Then the utility of advertiser i is

$$u_i(o_j, b_{-i}) = (v_i - p_i(o_j, b_{-i}))\alpha_i \beta_{r_i(o_j, b_{-i})}. \tag{1}$$

Let σ_i denote advertiser i's mixed strategy over B_i and $\sigma_i(o_j)$ denote the probability that advertiser i bids o_j. We use $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ and σ_{-i} to represent the mixed strategy profile of all bidders and that of others excluding advertiser i respectively. Then the expected utility of bidder i choosing o_j, given σ_{-i}, is

$$\overline{u}_i(o_j, \sigma_{-i}) = \sum_{b_{-i} \in B_{-i}} P(b_{-i}) u_i(o_j, b_{-i}), \tag{2}$$

where $P(b_{-i}) = \prod_{l \in I \setminus \{i\}} \sigma_l(b_{-i,l})$.

The quantal response π_{ij} of bidder i to others' mixed strategies σ_{-i} is defined as

$$\pi_{ij}(\overline{u}_i(\sigma_{-i})) = \frac{e^{\overline{u}_i(o_j, \sigma_{-i})\lambda_i}}{\sum_{k=1}^M e^{\overline{u}_i(o_k, \sigma_{-i})\lambda_i}}, \tag{3}$$

where $\overline{u}_i(\sigma_{-i})$ is bidder i's utility profile and λ_i is the precision parameter of bidder i, which is a non-negative real number. Normally, we use the vectors $u = (\overline{u}_1(\sigma_{-1}), \overline{u}_2(\sigma_{-2}), \ldots, \overline{u}_N(\sigma_{-N}))$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ to denote the utility and precision profiles respectively.

For simplicity, we define $\sigma_{ij} = \sigma_i(o_j)$, and then a QRE [McKelvey and Palfrey 1995; Goeree et al. 2005] can be defined as follows.

DEFINITION 1 (QUANTAL RESPONSE EQUILIBRIUM). *A quantal response equilibrium is a mixed strategy profile σ such that for any $i \in I$ and $o_j \in B_i$, $\sigma_{ij} = \pi_{ij}(\overline{u}_i(\sigma_{-i}))$.*

A QRE is proven to exist in any normal-form games with homogeneous precision parameters [McKelvey and Palfrey 1995]. Although our model has heterogeneous precision parameters, we can easily convert it to a model with homogeneous precision parameters by defining a new utility function $\overline{U}_i = \overline{u}_i(o_j, \sigma_{-i})\lambda_i$. Since the sponsored search auction with the GSP mechanism is a normal-form game, there always exists a QRE.

3. COMPUTING QRE FOR SPONSORED SEARCH AUCTIONS

In this section, we discuss how to compute a QRE given v, λ, α and β.

From Definition 1 and Eq. (3), we can see that σ is a QRE if it satisfies the following set of equations:

$$\sigma_{ij} = \frac{e^{\overline{u}_i(o_j, \sigma_{-i})\lambda_i}}{\sum_{o_k \in B_i} e^{\overline{u}_i(o_k, \sigma_{-i})\lambda_i}}, \forall i \in I, o_j \in B_i. \tag{4}$$

Let $U = \sum_{i \in I} |B_i|$ denote the number of equations. By rearranging Eq. (4) we obtain a continuous function $F : R^U \mapsto R^U$ as below:

$$
\begin{aligned}
F_{ij}(\sigma) &= \left(\frac{\sum_{o_k \in B_i} e^{\overline{u}_i(o_k, \sigma_{-i})\lambda_i}}{e^{\overline{u}_i(o_j, \sigma_{-i})\lambda_i}} \right)^{-1} - \sigma_{ij} \\
&= \left(\sum_{o_k \in B_i} e^{(\overline{u}_i(o_k, \sigma_{-i}) - \overline{u}_i(o_j, \sigma_{-i}))\lambda_i} \right)^{-1} - \sigma_{ij}, i \in I, o_j \in B_i.
\end{aligned}
\tag{5}
$$

Now we can see that computing a QRE of sponsored search auctions is equivalent to finding a zero point of the nonlinear function $F(\sigma)$. If a good initial point which is close to a zero point of F is available, starting from this initial point, we can directly apply Newton-style iteration methods to find a zero point of F and so a QRE of the auction. However, it is very difficult to find such a good initial point because of the non-linearity of F. As pointed by Allgower and Georg [1990], Newton-style iteration methods often fail because poor start points are very likely to be chosen. In this work, we make use of the homotopy principle, which has been widely used for equilibrium computation, and design an efficient homotopy-based algorithm by leveraging the special properties of sponsored search auctions. We first describe the algorithm in Section 3.1 and then show how to speed up the algorithm based on the peculiarities of sponsored search auctions in Section 3.2.

3.1. The Homotopy-based Algorithm

The basic idea of the homotopy principle is composed by two steps: given a problem we want to solve, (1) deform the problem to one with a unique easily-computed solution and build a continuous transformation from the original problem into the deformed problem that is easy to solve; (2) reverse the deformation to trace solutions of the associated problems during the deformation until finally finding a solution of the original problem.

To design a homotopy-based algorithm, our first step is to find a degenerated form of the original equations. In our problem F has a degenerated form G by setting $\lambda_i = 0, \forall i$:

$$G_{ij}(\sigma) = \frac{1}{|B_i|} - \sigma_{ij}, i \in I, o_j \in B_i. \tag{6}$$

Then we define a homotopy function between $F(\sigma)$ and $G(\sigma)$ as $H : R^U \times [0, 1] \mapsto R^U$ with $H(\sigma, 0) = G(\sigma)$ and $H(\sigma, 1) = F(\sigma)$:

$$H_{ij}(\sigma, t) = \left(\sum_{o_k \in B_i} e^{(\overline{u}_i(o_k, \sigma_{-i}) - \overline{u}_i(o_j, \sigma_{-i}))\lambda_i t} \right)^{-1} - \sigma_{ij}, i \in I, o_j \in J. \tag{7}$$

It is easy to get that $H(\sigma, 0)$ has a unique zero point $\sigma_{ij} = \frac{1}{|B_i|}$. Further, we define the solution set of H as

$$H^{-1}(0) = \{(\sigma, t) | H(\sigma, t) = 0\}. \tag{8}$$

Since a QRE always exists in normal-form games, for a given t there exists a $\sigma(t)$ such that $H(\sigma(t), t)) = 0$.

The remaining problem is to trace out a path consisting of zeros $(\sigma(t), t) \in H^{-1}(0)$, which starts at $(\sigma(0), 0)$ and ends at $(\sigma(1), 1)$. Considering the possibility of the existence of turning points [Judd 1998], increasing t monotonically when tracing the path may lead to points far away from zero points of H. A common practice to avoid the disturbance of turning points is to view the σ and t as functions of an implicit parameter a simultaneously and to compute a parametric path $c(a) = (\sigma(t(a)), t(a))$ which satisfies

$$H(c(a)) = 0. \tag{9}$$

The method we use to trace the path is called predictor-corrector (PC) [Allgower and Georg 1990], the basic idea of which is to numerically trace the path c by generating a sequence of points $v_i = (\sigma, t)_i, i = 1, 2, \ldots$ along the path satisfying a chosen tolerance criterion, say $\|H(v_i)\| \le \varepsilon$ for some $\varepsilon > 0$. In particular, given that we have found a point v_i on the path c, an Euler predictor step is used to predict the next point v_{i+1} on c:

$$v_{i+1} = v_i + \Delta \cdot c'(a)\big|_{c(a)=v_i}, \tag{10}$$

where $\Delta > 0$ is the step length. Then a corrector phase is necessary to refine the accuracy of v_{i+1}. We make use of Gauss-Newton method as presented below:

$$v_{i+1}^r = v_{i+1} - H'(v_{i+1})^+ H(v_{i+1}), \tag{11}$$

where $H'(v_{i+1})^+$ is the Moore-Penrose inverse[3] of the Jacobian matrix $H'(v_{i+1})$ of H at point v_{i+1} and v_{i+1}^r is the refined point of v_{i+1}. If $\|H(v_{i+1}^r)\| > \varepsilon$, we will substitute v_{i+1}^r into the right side of Eq. (11) to find another refined point and check whether it satisfies the tolerance criterion. This procedure is performed again and again until we find an acceptable point which will be used in the predictor phase to infer the next point. The PC method, starting with $(\sigma(0), 0)$, is applied step by step until the $(\sigma(1), 1)$ is reached.

The derivative $c'(a)$ of c in Eq. (10) and the Jacobian matrix $H'(v_{i+1})$ of H in Eq. (11) are unknown at this stage. We first consider how to calculate $c'(a)$. By differentiating Eq. (9) we get the following equation.

$$H'(c(a))c'(a) = 0. \tag{12}$$

The solution of Eq. (12) is

$$c'(a) = \pm \begin{pmatrix} (-1)^1 \cdot det\left((H'(c(a)))_{-1}\right) \\ (-1)^2 \cdot det\left((H'(c(a)))_{-2}\right) \\ \vdots \\ (-1)^d \cdot det\left((H'(c(a)))_{-d}\right) \\ \vdots \\ (-1)^{U+1} \cdot det\left((H'(c(a)))_{-(U+1)}\right) \end{pmatrix}, d = 1, \cdots, U+1, \tag{13}$$

where $(H'(c(a)))_{-d}$ denotes the Jacobian matrix $H'(c(a))$ with the d-th[4] column removed and $det\left((H'(c(a)))_{-d}\right)$ is its determinant. We know from Eq. (13) that once

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[3]The Moore-Penrose inverse of a matrix A is defined by $A^+ = A^T(AA^T)^{-1}$.

[4]σ_{ij} is assumed to be assigned to c in lexicographic order, namely, the mixed strategies of bidder 1 are listed first, in ascending order, followed by those of bidder 2, and so on. The last component of the vector corresponds to t.

$H'(c(a))$ is given, the value of $c'(a)$ could be obtained directly. Eq. (11) also involves the problem of computing H's Jacobian matrix, which is explained next.

We use $\widetilde{u}_{ij}$ to represent $\overline{u}_i(o_j, \sigma_{-i})\lambda_i$ for simplicity in the remaining part of this paper. Since function H is a mapping from $R^U \times [0,1]$ to R^U, its Jacobian matrix contains $U \cdot (U+1)$ partial derivatives, which can be divided into four cases:

Case 1. $i \in I$ and $o_j \in B_i$,

$$\frac{\partial H_{ij}}{\partial \sigma_{ij}} = -1; \tag{14}$$

Case 2. $i \in I$, o_j and $o_k \in B_i$, but $j \neq k$,

$$\frac{\partial H_{ij}}{\partial \sigma_{ik}} = 0; \tag{15}$$

Case 3. i and $l \in I$, $o_j \in B_i$ and $o_m \in B_l$, but $i \neq l$,

$$\frac{\partial H_{ij}}{\partial \sigma_{lm}} = -\left(te^{-\widetilde{u}_{ij}t} \sum_{o_k \in B_i} e^{\widetilde{u}_{ik}t} \left(\frac{\partial \widetilde{u}_{ik}}{\partial \sigma_{lm}} - \frac{\partial \widetilde{u}_{ij}}{\partial \sigma_{lm}} \right) \right)^{-2}; \tag{16}$$

Case 4. $i \in I$ and $o_j \in B_i$,

$$\frac{\partial H_{ij}}{\partial t} = -\left(e^{-\widetilde{u}_{ij}t} \sum_{o_k \in B_i} e^{\widetilde{u}_{ik}t}(\widetilde{u}_{ik} - \widetilde{u}_{ij}) \right)^{-2}. \tag{17}$$

Note that the sign of c' is not determined in Eq. (13). We choose the sign of c' to ensure the derivative of t to a, which corresponds to the $(U+1)$-th component in c', is positive at $(\sigma(0), 0)$, i.e.,

$$\mu \cdot (-1)^{U+1} det\left((H'(c(a)))_{-(U+1)} \right) > 0, \tag{18}$$

where $\mu = 1$ or -1 is the sign of c' to be chosen. Substituting $t = 0$ into Eq. (18) and combining with Eqs. (14)-(17) we get

$$\mu \cdot (-1)^{U+1} \cdot (-1)^U > 0, \tag{19}$$

which indicates that

$$(-1)^{2U+1}\mu > 0. \tag{20}$$

Hence we could conclude that $\mu = -1$ at $(\sigma(0), 0)$.

By now we have discussed how to compute Eqs. (10) and (11). Then we can use the PC method to find the point $(\sigma(1), 1) \in H^{-1}(0)$ in which $\sigma(1)$ is exactly the solution of F.

The completed process of our proposed method is summarized in Algorithm 1. In lines 1 and 2 we initialize t with 0 and the starting point v_1 with $(\sigma(t), t)$. Lines 3-9 use the PC method to generate a set of points $v_i, i = 2, 3, \ldots$ along the path until eventually the point $(\sigma(1), 1)$ is found. Line 4 is the Euler predictor step which computes the next point v_{i+1} given v_i according to Eq. (10). The Gauss-Newton corrector step is performed repeatedly in lines 5-7 to improve the accuracy of the point predicted by the predictor phase till it is within the tolerance ε. t is updated in line 8 and the result is returned in line 10.

3.2. Efficient Computation for Sponsored Search Auctions

Algorithm 1 and Eq. (13) indicate that we need to compute the Jacobian matrix H' of H at each predictor and corrector step when tracing the path with PC method. Since

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ALGORITHM 1: Computing a QRE

Input: Bidder set I, slot set S, advertiser effect profile α, position effect profile β, each bidder's
         bid strategy space, precision profile λ, step length Δ, and tolerance criterion ε.
Output: A QRE strategy profile σ.

1 $t \leftarrow 0$;
2 $v \leftarrow (\sigma(t), t)$;
3 **while** $t \neq 1$ **do**
4 $v \leftarrow v + \Delta \cdot c'(a)\big|_{c(a)=v}$;
5   **while** $\|H(v)\| > \varepsilon$ **do**
6 $v \leftarrow v - H'(v)^+ H(v)$;
7   **end**
8 $t \leftarrow$ the last component of v;
9 **end**
10 $(\sigma, t) \leftarrow v$;

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the dimension of H could be large, the efficiency of calculating H' will significantly affect the speed of the algorithm. In this subsection, we discuss how to speed up the proposed algorithm through efficient calculation of H' by leveraging the properties of sponsored search auctions.

The elements in H' are classified into four cases as shown in Eqs. (14)-(17). The partial derivatives in the first two cases are quite easy to work out, and that in the last two cases involve computing $\widetilde{u}_{ij} = \overline{u}_i(o_j, b_{-i}) \cdot \lambda_i$ and $\frac{\partial \widetilde{u}_{ij}}{\partial \sigma_{lm}} = \frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}} \cdot \lambda_i$. So the main effort on computing H' is to solve $\overline{u}_i(o_j, b_{-i})$ and $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$ for all the bidders.

Eq. (2) implies that $\overline{u}_i(o_j, b_{-i})$ is bidder i's expected utility over B_{-i} when he/she chooses o_j. A naive approach is to traverse the space B_{-i} with $\prod_{l \in I \setminus \{i\}} |B_l|$ realizations when computing $\overline{u}_i(o_j, b_{-i})$. The computation of $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$ is similar except that bidder l's strategy is given by o_m and we just consider the remaining $N - 2$ bidders' joint bid space. Clearly, the naive traversal method (TM for short) for computing H' is very time-consuming.

Fortunately, the expected utilities in sponsored search auctions with the GSP mechanism have many special properties that could be utilized to reduce the computational complexity. Here we take bidder i and his/her bid o_j as an example to show the properties of his/her utility based on the GSP mechanism.

Let I_g denote the set of other bidders $l \neq i$ such that $\alpha_l \cdot b_l > \alpha_i \cdot b_i$. Similarly, $I_e(I_l)$ denotes the set of other bidders with their bids multiplied by the corresponding ad CTRs equal to (less than) $\alpha_i \cdot b_i$. We assume the tie is broken randomly. It thus follows that:

(1) When $|I_g| \geq K$, i's utility is zero.
(2) When $|I_g| < K$ and $|I_g| + |I_e| \geq K$, bidder i has a probability $\frac{1}{|I_e|+1}$ to be allocated at a slot ranging from $|I_g| + 1$ to K and his/her expected payment according to our assumption on the tie is $p_i = \frac{1}{|I_e| \cdot \alpha_i} \sum_{l \in I_e} \alpha_l b_l$. The utility of i in such case is

$$\frac{1}{|I_e| + 1}(v_i - p_i) \sum_{s=|I_g|+1}^{K} \alpha_i \beta_s.$$

(3) When $|I_g| + |I_e| < K$, i's location will be any one from slot $|I_g| + 1$ to slot $|I_g| + |I_e| + 1$ with an identical probability $\frac{1}{|I_e|+1}$. His/her payment will be $p_i = \frac{1}{|I_e| \cdot \alpha_i} \sum_{l \in I_e} \alpha_l b_l$ if ranked at the slot between $|I_g| + 1$ and $|I_g| + |I_e|$. On the other hand, his/her

payment will be the maximum $(\alpha_k \cdot b_k)/\alpha_i$ for $k \in I_l$ if allocated at slot $|I_g| + |I_e| + 1$ respectively. We use p_{max} to represent this payment and then his/her final utility is

$$\frac{1}{|I_e|+1}\left((v_i - p_i)\sum_{s=|I_g|+1}^{|I_g|+|I_e|}\alpha_i\beta_s + (v_i - p_{max})\alpha_i\beta_{|I_g|+|I_e|+1}\right),$$

When computing the $\overline{u}_i(o_j, b_{-i})$ and $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$, instead of traversing the joint strategy space, we just assign all the other bidders into I_g, I_e and I_l, which will lead to at most $\binom{N-1}{|I_g|} \cdot \binom{N-1-|I_g|}{|I_e|}$ results[5]. Since we only care about the situations where $|I_g| < K$, the most total number of cases we take into account in sponsored search auctions is

$$\sum_{|I_g|=0}^{K-1}\sum_{|I_e|=0}^{N-1-|I_g|}\binom{N-1}{|I_g|} \cdot \binom{N-1-|I_g|}{|I_e|} = \sum_{|I_g|=0}^{K-1}\binom{N-1}{|I_g|} \cdot 2^{N-1-|I_g|} \tag{21}$$

$$= 2^{N-1} + \sum_{|I_g|=1}^{K-1}\binom{N-1}{|I_g|} \cdot 2^{N-1-|I_g|}$$

$$< 2^{N-1} + \sum_{|I_g|=1}^{K-1}\frac{(N-1)^{|I_g|}}{2^{|I_g|-1}}2^{N-1-|I_g|} \tag{22}$$

$$= 2^{N-1} + \sum_{|I_g|=1}^{K-1}(N-1)^{|I_g|}2^{N-2|I_g|}$$

$$= 2^{N-1} + \frac{(N-1)^K 2^{N-2K+2} - (N-1)2^N}{N-5} \tag{23}$$

$$= \frac{2^{N-1}[2^{3-2K}(N-1)^K - N - 7]}{N-5}. \tag{24}$$

Generally, there is always a large number of bidders competing for several fixed slots, so the amount of the situations is mainly based on 2^N which is a reduction of $\prod_{l \in I \setminus \{i\}} |B_l|$ since $|B_l| \geq 2$ in most cases. Besides, a direct deduction of the properties is

$$\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lk}} = \frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}, l \neq i; \alpha_l o_m, \alpha_l o_k > \alpha_i o_j, \tag{25}$$

which can be used to further avoid redundancy.

There is an obvious relationship between $\overline{u}_i(o_j, b_{-i})$ and $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$ in all normal form games including sponsored search auctions such that

$$\overline{u}_i(o_j, b_{-i}) = \sum_{o_m \in B_l}\sigma_{lm}\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}, l \neq i, \tag{26}$$

which implies that once $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$ is obtained, $\overline{u}_i(o_j, b_{-i})$ could be calculated accordingly.

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[5]We first determine the bidders in I_g, and then I_e. The rest of bidders are all in I_l.

We make a summary of our improved method (IM for short). First we take advantage of the three properties in sponsored search auctions to efficiently compute $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$, then we further reduce redundant calculations on $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$ by Eq. (25), next we use Eq. (26) to compute $\overline{u}_i(o_j, b_{-i})$, and finally by combining $\overline{u}_i(o_j, b_{-i})$ and $\frac{\partial \overline{u}_i(o_j, b_{-i})}{\partial \sigma_{lm}}$ with Eqs. (14)-(17) we get H' which is frequently used in Algorithm 1.

4. PARAMETER ESTIMATION

In this section, we investigate how to infer the valuation profile v and the precision profile λ of the advertisers together with the ad CTR profile α and slot CTR profile β given that a QRE σ has been reached. We propose an algorithm based on the principle of likelihood maximization, which is very popular in machine learning domain for parameter estimation.

Given a QRE strategy σ, the logarithmic likelihood of the unknown parameters v, λ, α and β is

$$L(v, \lambda, \alpha, \beta | \sigma) \ = \ log\left(\prod_{i \in I} \prod_{o_j \in B_i} (\pi_{ij}(\overline{u}_i(\sigma_{-i})))^{\sigma_{ij}}\right) \tag{27}$$

$$= \sum_{i \in I} \sum_{o_j \in B_i} \sigma_{ij} log(\pi_{ij}(\overline{u}_i(\sigma_{-i}))). \tag{28}$$

Combining with Eq. (3), we have

$$L(v, \lambda, \alpha, \beta | \sigma) = \sum_{i \in I}(\lambda_i \cdot \sum_{o_j \in B_i} \sigma_{ij} \overline{u}_i(o_j, \sigma_{-i})) - \sum_{i \in I} log(\sum_{o_k \in B_i} e^{\overline{u}_i(o_k, \sigma_{-i})\lambda_i}). \tag{29}$$

Then the parameters can be estimated by maximizing the likelihood as shown in the following optimization problem.

$$\max_{v, \lambda, \alpha, \beta} L(v, \lambda, \alpha, \beta | \sigma) \tag{30}$$

$$s.t. \begin{cases} v_i > 0, \\ \lambda_i \geq 0, \\ 0 < \alpha_i < 1, \\ 0 < \beta_s < 1, \\ \beta_s \geq \beta_{s+1} \end{cases} \tag{31}$$

Since in sponsored search auctions, the allocation rule is not a continuous function of ad CTR profile α, the utility of a bidder is not continuous with respect to α; as a result, the likelihood defined in Eq. (29) is not continuous with respect to α.

To address the discontinuity of the likelihood function, we split the unknown parameters into two groups and sequentially optimize them in each iteration: we treat v, λ, β as a group and α as the other group; in each iteration, we first optimize v, λ, β and then α.

The function $L(v, \lambda, \alpha, \beta | \sigma)$ in Eq. (32) is continuous with respect to the parameters in the first group. We can learn a better set of v, λ, β by solving the following sub optimization problem.

$$\max_{v, \lambda, \beta} L(v, \lambda, \alpha, \beta | \sigma) \tag{32}$$

$$s.t. \begin{cases} v_i > 0 \\ \lambda_i \geq 0 \\ 0 < \beta_s < 1 \\ \beta_s \geq \beta_{s+1} \end{cases} \tag{33}$$

Since the above optimization problem is non-convex, it is difficult to find the global maximum. We turn to find a set of local maxima with different starting points and then choose the best one to improve the possibility of reaching the global maximum of the sub problem.

As aforementioned, the likelihood function is not continuous with respect to α. Here we do not optimize bidders' ad CTRs simultaneously. Instead, we deal with them one by one. Let us take the ad CTR α_i of bidder i as an example and keep $\alpha_j, \forall j \neq i$ fixed. Given all the other parameters are fixed, it is easy to know that the likelihood has at most K discontinuous points at

$$\{\frac{\alpha_{(1)}b_{(1)}}{b_i}, \frac{\alpha_{(2)}b_{(2)}}{b_i}, \cdots, \frac{\alpha_{(K)}b_{(K)}}{b_i}\} \cap (0, 1),$$

where $\alpha_{(k)}b_{(k)}$ denotes the k-th largest ranking score among all the other $N-1$ bidders except i. Then we can partition the feasible domain of α_i into at most $K+1$ intervals and the likelihood function L is continuous with respect to α_i in each interval. Let I_k denote the k-th interval (e.g., $I_k = [\frac{\alpha_{(5)}b_{(5)}}{b_i}, \frac{\alpha_{(4)}b_{(4)}}{b_i})$). Thus, by solving at most $K+1$ optimization problem (see Eq.(34) and (35)), we can find a better α_i given all the other parameters.

$$\max_{\alpha_i} \ L(v, \lambda, \alpha, \beta | \sigma) \tag{34}$$

$$s.t. \ \alpha_i \in I_k \tag{35}$$

Similarly, the above optimization problem is not convex. To avoid being tracked into a bad local maximum, we can also find a set of local maxima with different starting points and choose the best one.

The whole procedure is presented in Algorithm 2. In line 1 we initialize the likelihood of the original optimization problem with negative infinity. Line 2 sets an initial α. Lines 3-15 iteratively optimize the two groups of parameters. Line 4 fixes α and updates (v, λ, β). Lines 5-7 fix (v, λ, β) and update α. Line 8 checks the performance of the updated parameters. Lines 9-14 control the optimization process: if we make progress in this iteration, we continue the optimization; otherwise, we terminate the optimization and return the latest parameters. Again, to avoid a bad local maximum, we run the algorithm multiple times with different initial α's in Line 2 and choose the best learned parameters as the final output in our experiments.

5. EXPERIMENTAL EVALUATION

We conducted a set of experiments to test the proposed algorithm for QRE computation and the algorithm for parameter estimation. We report the experimental results in this section.

5.1. Effectiveness of the Homotopy Method

We first make a comparison between the TM and the IM declared in Section 3.2 for computing H' with 24 different size of games as depicted in Table I. We let the total number of slots $K = \lfloor N/2 \rfloor$ and assumed all bidders share the same bid strategy space. For each game, we applied the two methods to 100 instances with randomly generated

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ALGORITHM 2: Parameter estimation

Input: A QRE profile σ, the stopping criterion δ

Output: Estimated value profile v, precision profile λ, ad CTR profile α, position CTR profile β, and logarithmic likelihood L.

1 $L \leftarrow -\infty$;
2 Randomly generate an ad CTR profile α;
3 **while** *True* **do**
4   Fix α and update v, λ, β by solving the optimization problem shown in Eq. (32) and (33);
5   **for** $i \leftarrow 1, 2, \cdots, N$ **do**
6   Fix $\alpha_j, \forall j \neq i, v, \lambda, \beta$ and update α_i by solving at most $K + 1$ sub problem as shown in Eq. (34) and (35);
7   **end**
8   Calculate the likelihood $\hat{L}$ with the updated parameters $v, \lambda, \alpha, \beta$;
9   **if** $\hat{L} - L > \delta$ **then**
10 $L \leftarrow \hat{L}$;
11   **end**
12   **else**
13   Return the learned parameters $v, \lambda, \alpha, \beta$;
14   **end**
15 **end**

Table I. Runtime of TM and IM for computing H'

Runtime(sec)			M					
			5	6	7	8	9	10
		TM	0.789	1.458	3.040	5.876	10.648	16.220
	5	IM	0.670	0.937	1.236	1.850	2.114	2.647
		Improvement	11.59%	35.75%	59.33%	68.53%	80.15%	83.68%
		TM	4.218	11.079	25.525	54.882	110.122	202.182
	6	IM	3.554	4.731	6.588	8.712	11.615	13.859
N		Improvement	15.76%	57.29%	74.18%	84.12%	89.45%	93.15%
		TM	23.891	79.180	215.892	528.704	1170.764	2396.645
	7	IM	11.164	16.020	22.159	29.414	37.864	46.299
		Improvement	53.27%	79.77%	89.74%	94.44%	96.76%	98.07%
		TM	155.292	593.852	1835.082	5022.691	12286.726	29958.058
	8	IM	53.607	80.960	111.254	148.542	190.038	234.271
		Improvement	65.48%	86.37%	93.94%	97.04%	98.45%	99.22%

v, α, β and computed their average runtime. We also calculated the improvement[6] of the runtime of IM compared with that of TM.

From Table I we know that both of the two methods are efficient and the improvement of IM is not apparent when game size is small. When N gradually increases from 5 to 8, the runtime of IM increases by about 90 times with M fixed. However, the speed of TM slows down dramatically with N increasing especially when M is large, say by 1847 times when $M = 10$. This result is expected, because Section 3.2 demonstrates that IM only need to consider about 2^N situations but IM will traverse about M^N cases. We illustrate the runtime of TM and IM in Figure 1 with N fixed and M increasing, which clearly represents the sharp degradation of TM's performance when M becomes larger. The efficiency of IM is not affected obviously, which further confirms the conclusions in Section 3.2.

Next we will evaluate the runtime performance of our proposed homotopy method for computing a QRE for sponsored search auctions. We conducted our experiments on 32 games with N ranging from 3 to 6 and M increasing from 3 to 10. Again, we made

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[6]The improvement is calculated by the formula $|$(runtime of IM) $-$ (runtime of TM)$|$/(runtime of TM).

(a) N=5   (b) N=6

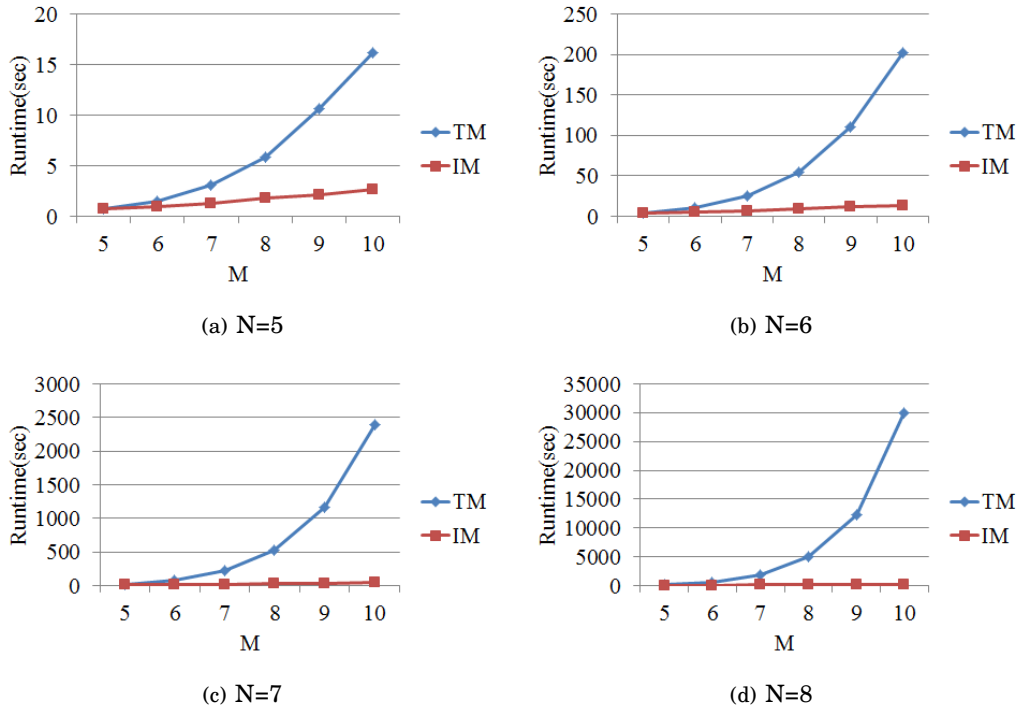(c) N=7   (d) N=8

Fig. 1.   Runtime of TM and IM with N fixed and M increasing.

Table II. Runtime of Algorithm 1 for different size of games

Runtime(sec)		M							
		3	4	5	6	7	8	9	10
N	3	0.101	0.163	0.243	0.299	0.408	0.478	0.701	0.777
	4	0.386	0.759	1.186	1.766	2.588	3.565	4.745	5.695
	5	2.197	4.568	7.461	12.100	18.977	23.026	28.185	39.778
	6	7.928	18.095	31.454	51.560	71.750	94.026	118.073	160.464

$K = \lfloor N/2 \rfloor$ and assumed all participants share the same strategy space. The average runtime of 100 instances randomly generated for each game is shown in Table II.

We see from Table II that when N and M are small, Algorithm 1 runs very fast. We find that the runtime in the last column of Table II is always about 20 times that in the first column. In contrast, when $M = 3$, the runtime in the last row is 70 times that of in the first row, but when $M = 10$, this proportion soars to about 210. We can conclude through the above observation that the speed of Algorithm 1 slows down more significantly with N increasing than with M increasing since in the predictor and corrector phase we need to calculate the Jacobian matrix of H whose computation cost is mainly related with N as declared in Section 3.2. Although M does not visibly affect the cost for computing the components of H', the size of H' and σ will largen correspondingly with M growing which results in the increasement of the runtime of Algorithm 1.

In the future work, we will conduct more extensive experiments to evaluate the algorithm's performance on large-scale games and propose new approaches to improve the scalability of our algorithm.

Table III. Estimated parameters in query 1

	N			L
	1	2	3	
v	2996	2048	4548	
λ	.4064	.7766	.0579	-3.3526
α	.0463	.0990	.0906	
β	(.0998, .0364, .0363)			

Table IV. Estimated parameters in query 2

	N			L
	1	2	3	
v	1237	769	400	
λ	1.6487	12.5776	1.8914	-1.8529
α	.0446	.0517	.0721	
β	(.0844, .0842, .0840)			

Table V. Estimated parameters in query 3

	N			L
	1	2	3	
v	628	130	513	
λ	29.2835	558.8019	24.3247	-2.6095
α	.0364	.0752	.0410	
β	(.0501, .0343, .0285)			

5.2. Evaluating the Parameter Estimation Algorithm

In this subsection we present the experimental evaluation of Algorithm 2 on real data. We first estimate the parameters $v, \lambda, \alpha, \beta$ using Algorithm 2 and then check whether bidders' real strategies keep consistent with those predicted by the QRE model given those estimated parameters.

Our experiments are based on a log file of a commercial search engine over three months, which contains the information about advertisers' bids, ranks and CTRs. For ease of analysis, we tested our algorithm on some queries with a relative small number of bidders. We further removed the bidders who give a very high or low bid and never make a change since these bidders seem not to participate in the competition. We use three representative queries here to show the estimation results. After the data preprocessing, we finally got three bidders in each query. Bidders' real mixed strategy profile σ was computed with the log file. Then we employed Algorithm 2 to infer the parameters for each query and obtained the following three sets of results as depicted in Tables III-V.

By substituting the estimated parameters into Eq. (3) we got the predicted mixed strategies of bidders for each query as depicted in Tables VI-VIII[7]. σ_i is bidder i's real mixed strategy profile over his/her bid space and π_i corresponds to that predicted by the QRE model. The error represents the absolute difference between σ_i and π_i.

We learn from the estimation results that bidders' precision parameters differ from each other dramatically. An observation of bidders' values and their strategies shows that most of the time bidders will not overbid in the auction which is consistent with our experience. We see that the QRE model fits the data in query 2 and query 3 very well. One possible reason for the deviation of the estimation for query 1 is that Algorithm 2 may find a not-very-good local optimal solution. Overall, the QRE model works well for the reality.

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[7]The symbol "–" at row i and column j in the tables represents that o_j is not an element of B_i. Besides, we use π_{ij} to denote $\pi_{ij}(\overline{u}_i(\sigma_{-i}))$ for briefness.

Table VI. Real and predicted mixed strategies for query 1

	$O = \{1000, 1500, 2000, 2500, 3000, 3500\}$					
	o_1	o_2	o_3	o_4	o_5	o_6
σ_1	.1261	.3025	.4202	–	.1513	–
π_1	.2829	.2829	.2829	–	.1513	–
error	.1569	.0196	.1373	–	.0000	–
σ_2	.2975	–	.5950	.1074	–	–
π_2	.2972	–	.5952	.1076	–	–
error	.0003	–	.0002	.0001	–	–
σ_3	–	.2609	–	.3565	–	.3826
π_3	–	.2603	–	.3662	–	.3735
error	–	.0005	–	.0097	–	.0091

Table VII. Real and predicted mixed strategies for query 2

	$O = \{400, 650, 800, 850, 1000\}$				
	o_1	o_2	o_3	o_4	o_5
σ_1	–	.6417	–	–	.3583
π_1	–	.6417	–	–	.3583
error	–	.0000	–	–	.0000
σ_2	–	–	.7660	–	.2340
π_2	–	–	.7660	–	.2340
error	–	–	.0000	–	.0000
σ_3	.6348	–	–	.3652	–
π_3	.6348	–	–	.3652	–
error	.0000	–	–	.0000	–

Table VIII. Real and predicted mixed strategies for query 3

	$O = \{100, 200, 250, 300, 350, 400, 450, 500\}$							
	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
σ_1	–	.2927	–	.2439	–	–	.4634	–
π_1	–	.2929	–	.2437	–	–	.4634	–
error	–	.0002	–	.0002	–	–	.0000	–
σ_2	.6897	–	.1466	.1207	–	–	–	.0431
π_2	.6889	–	.1037	.1037	–	–	–	.1037
error	.0007	–	.0429	.0170	–	–	–	.0606
σ_3	–	–	.0522	–	.1304	.8174	–	–
π_3	–	–	.0386	–	.1560	.8054	–	–
error	–	–	.0135	–	.0256	.0120	–	–

We next evaluate the estimated CTRs of our model by compare the clicks computed with the parameters in Tables III-V and the real clicks in the log file. Let τ_i and τ_i^* be the estimated and the real clicks of bidder i respectively. Then we calculated the relative error as follows:

$$RE_i = \frac{|\tau_i - \tau_i^*|}{\tau_i^*} \tag{36}$$

The results are illustrated in Figure 2, which indicates that the estimated CTRs are acceptable since the largest relative error is no more than 10%.

6. CONCLUSION

In this paper we introduced the quantal response equilibrium concept into sponsored search auctions to model bidders' bounded rationality. In addition to introducing QRE, this paper provided two other key contributions. First, we proposed a homotopy-based algorithm to compute a QRE and analyze how to leverage the special properties of sponsored search auctions to make the algorithm more efficient. Second, we fitted the QRE model into real-world sponsored search auction data to infer bidders' values,
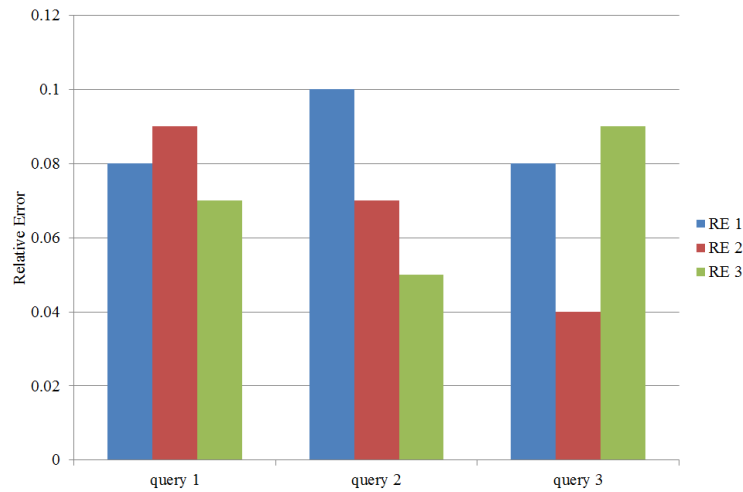
Fig. 2.   Clicks prediction.

CTRs and the parameters representing bidders' rationalities. Experimental results showed that the algorithm for computing QRE is promising and the QRE model fits the real data well.

As for future work, we plan to improve the scalability of the proposed algorithms and apply the QRE model to large-scale datasets to check its effectiveness. We assumed that queries exactly match keywords in this work. We will consider broad match auctions in the future.

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