# Bilateral Bargaining with One-Sided Two-Type Uncertainty

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## Abstract

*It is a challenging problem to find agents' rational strategies in bargaining with incomplete information. In this paper we perform a game theoretic analysis of agents' rational strategies in finite horizon bilateral bargaining with one-sided uncertainty regarding agents' reserve prices. The negotiation setting considered in this paper has four features: alternating-offers bargaining protocol, finite horizon, two-type uncertainty about agents' reserve prices, and discount factors. The main contribution of this paper is the development of a novel algorithm to find a pure strategy sequential equilibrium in the setting we study. Our algorithm is based on the combination of game theoretic analysis and search techniques which finds agents' equilibrium in pure strategies when they exist.*

## 1. Introduction

The research of automated negotiation has received prominent attention in recent years and has been widely used for allocating resources. In this paper, we focus on the most common bargaining protocol, i.e., the Rubinstein's alternating-offers [11], which captures the most important features of bargaining: bargaining consists of a sequence of offers and decisions to accept or reject these offers. We analyze the situation with one-sided uncertain reserve prices and where agents have deadlines. This problem is customarily modeled as a Bayesian extensive-form games with infinite number of actions. The appropriate solution concept for such a class of game is *sequential equilibrium* [8], which specifies a pair: a *system of beliefs* that prescribes how agents' beliefs must be updated during the game and *strategies* that prescribe how agents should act. In a sequential equilibrium there is a sort of circularity between belief system and strategies: strategies must be *sequentially rational* given the belief system and belief system must be *consistent* with respect to strategies. The study of bargaining with uncertain information is well known to be a challenging problem because of this circularity. There is no generally applicable algorithm for such problem in the literature. Operational research inspired algorithms such as Miltersen-Sorensen [9] work only on games with finite number of strategies, and therefore cannot be applied to bargaining. Several attempts (e.g., [4], [5]) to extend the backward induction method [6] have been tried, but the strategies found by the backward induction approach is not guaranteed to be sequentially rational given the designed system of beliefs [7]. This is because in the computation of the equilibrium they break down the circularity between strategies and belief system.

The microeconomic literature provides a number of closed form results with very narrow uncertainty settings. For instance, Rubinstein [12] considered bilateral bargaining with uncertainty over two possible discount factors. Gatti *et al*. [7] analyzed bilateral bargaining with one-sided uncertain deadlines. The *only* known result about bargaining with uncertain reserve prices is due to Chatterjee and Samuelson [1], [2] where they studied bilateral infinite horizon bargaining with two-type uncertainty over the reservation values. The absence of agents' deadlines makes these two results non applicable to the situation we study in the paper.

The main contribution of this paper is the development of a novel algorithm to find a pure strategy sequential equilibrium in the setting we study. Our algorithm combines together game theoretic analysis with state space search techniques and it is sound and complete. The approach is novel in the literature. Our approach is based on the following two observations: 1) with pure strategies, the possible classes of agents' choice rules are finite (a class specifies only whether different agent types will behave in the same way or in different ways at a decision making point), and 2) given a system of choice rule classes (each time point is assigned a class of choice rule) we are able to derive theoretically the agents' optimal strategies (by a Bayesian extension of backward induction) and to check whether or not a sequential equilibrium exits with such belief system (this makes the algorithm sound). Our algorithm enumerates over the possible systems of choice rules and for each one checks the existence of the equilibrium (this makes the algorithm complete).[1] To make this enumeration efficient, we employ state space search techniques.

The rest of this paper proceeds as follows: We start with complete information negotiation in Section 2. Section 3 discusses bargaining with two possible types of reserve prices. Section 4 concludes this paper and outlines the ideas of extending our analysis to bargaining with finitely many possible types of reserve prices.

1. Enumeration based methods were used in [10] to compute Nash equilibria. They enumerate the agents' strategy supports. In our problem, this approach cannot be applied because of the infinite number of supports.

## 2. Bargaining with Complete Information

We follow [7] to describe the non-cooperative bargaining problem between a buyer $\mathbf{b}$ and a seller $\mathbf{s}$. All the agents enter the market at time 0. The seller agent wants to sell a single indivisible good for some money. The buyer agent wants to buy the indivisible good provided by the seller. The characteristics of a transaction that are relevant to an agent are the price $x$ and the number of periods $t$ after the agent's entry into the market that the transaction is concluded.

We study a discrete time (indexed by integers $0, 1, 2, \ldots$) bilateral negotiation in this paper. A finite horizon alternating-offers bargaining protocol is utilized for the negotiation on one continuous issue (price of a good). Formally, the buyer $\mathbf{b}$ and the seller $\mathbf{s}$ can act at times $t \in \mathbb{N}$. The player function $\iota : \mathbb{N} \rightarrow \{\mathbf{b}, \mathbf{s}\}$ returns the agent that acts at time $t$ and is such that $\iota(t) \neq \iota(t+1)$, i.e., a pair of agents bargain by making offers in alternate fashion. This paper focuses on single-issue negotiation. However, our model can be easily extended to handle multi-issue negotiation as in [7].

Possible actions $\sigma^t_{\iota(t)}$ of agent $\iota(t)$ at any time $t > 0$ are: 1) $\mathit{offer}[x]$, where $x \in \mathbb{R}$ is the proposed price for the good; 2) $\mathit{exit}$, which implies that negotiation between $\mathbf{b}$ and $\mathbf{s}$ fails; and 3) $\mathit{accept}$, which implies that $\mathbf{b}$ and $\mathbf{s}$ make an agreement. At time point $t = 0$ the only allowed actions are 1) and 2). If $\sigma^t_{\iota(t)} = \mathit{accept}$ the bargaining stops and the outcome is $(x, t)$, where $x$ is the value such that $\sigma^{t-1}_{\iota(t-1)} = \mathit{offer}[x]$. This is to say that the agents agree on the value $x$ at time point $t$. If $\sigma^t_{\iota(t)} = \mathit{exit}$ the bargaining stops and the outcome is $FAIL$. Otherwise the bargaining continues to the next time point.

Each agent $\mathbf{a} \in \{\mathbf{b}, \mathbf{s}\}$ has a utility function $U_{\mathbf{a}} : (\mathbb{R} \times \mathbb{N}) \cup FAIL \rightarrow \mathbb{R}$, which represents its gain over the possible bargaining outcomes. Each utility function $U_{\mathbf{a}}$ depends on $\mathbf{a}$'s reserve price $\mathrm{RP}_{\mathbf{a}} \in \mathbb{R}^+$, temporal discount factor $\delta_{\mathbf{a}} \in (0, 1]$, and deadline $T_{\mathbf{a}} \in \mathbb{N}, T_{\mathbf{a}} > 0$. If the bargaining outcome is $(x, t)$, then the utility function $U_{\mathbf{a}}$ is defined as:

$$U_{\mathbf{a}}(x,t) = \begin{cases} (\mathrm{RP}_{\mathbf{a}} - x) \cdot \delta^t_{\mathbf{a}} & \text{if } t \leq T_{\mathbf{a}} \text{ and } \mathbf{a} \text{ is a buyer} \\ (x - \mathrm{RP}_{\mathbf{a}}) \cdot \delta^t_{\mathbf{a}} & \text{if } t \leq T_{\mathbf{a}} \text{ and } \mathbf{a} \text{ is a seller} \\ \epsilon < 0 & \text{otherwise} \end{cases}$$

If the outcome is $FAIL$, $U_{\mathbf{a}}(FAIL) = 0$. Notice that the assignment of a strictly negative value to $U_{\mathbf{a}}$ after $\mathbf{a}$'s deadline allows one to capture the essence of the deadline: an agent, after its deadline, strictly prefers to exit the negotiation rather than to reach any agreement. Finally, we assume the feasibility of the problem, i.e., $\mathrm{RP}_{\mathbf{b}} \geq \mathrm{RP}_{\mathbf{s}}$, and the rationality of the agents, i.e., each agent acts to maximize its utility. $[\mathrm{RP}_{\mathbf{s}}, \mathrm{RP}_{\mathbf{b}}]$ is the zone of potential agreements.

With complete information the appropriate solution concept for the game we are dealing with is the subgame perfect equilibrium. In subgame perfect equilibrium, agents' strategies are in equilibrium in every possible subgame. Such a solution can be found by backward induction [7].

Initially, it is determined the time $T = \min(T_{\mathbf{b}}, T_{\mathbf{s}})$ where the game rationally stops. The equilibrium outcome of every subgame starting from $t \geq T$ is $FAIL$, since at least one agent will make $exit$. Therefore, at $t = T$ agent $\iota(T)$ would accept any offer $x$ which gives it a utility not worse than $FAIL$, namely, any offer $x$ such that $U_{\iota(T)}(x, T) \geq 0$. From $t = T - 1$ back to $t = 0$ it is possible to find the optimal offer agent $\iota(t)$ can make at $t$, if it makes an offer, and the offers that it would accept. $x^*(t)$ denotes the optimal offer of agent $\iota(t)$ at $t$. $x^*(t)$ is the offer such that, if $t < T - 1$, agent $\iota(t+1)$ is indifferent at $t+1$ between accepting it and rejecting it to make its optimal offer $x^*(t+1)$ and, if $t = T-1$, agent $\iota(t+1)$ is indifferent at $t+1$ between accepting it and exiting from negotiation. Formally, $x^*(t)$ is such that $U_{\iota(t+1)}(x^*(t)), t) = U_{\iota(t+1)}(x^*(t+1), \text{t+1})$ if $t < T - 1$ and $U_{\iota(t+1)}(x^*(t), t) = 0$ if $t = T - 1$. The offers agent $\iota(t)$ would accept at $t$ are all those offers that give it a utility no worse than the utility given by offering $x^*(t)$. The equilibrium strategy of any sub-game starting from $0 \leq t < T$ prescribes that agent $\iota(t)$ offers $x^*(t)$ at $t$ and agent $\iota(t+1)$ accepts it at $t+1$.

Backward propagation is used to provide a recursive formula for $x^*(t)$: given value $x$ and agent $\mathbf{a}$, we call backward propagation of value $x$ for agent $\mathbf{a}$ the value $y$ such that $U_{\mathbf{a}}(y, t-1) = U_{\mathbf{a}}(x, t)$; we employ the arrow notation $x_{\leftarrow \mathbf{a}}$ for backward propagations. Formally, $x_{\leftarrow \mathbf{b}} = \mathrm{RP}_{\mathbf{b}} - (\mathrm{RP}_{\mathbf{b}} - x) \cdot \delta_{\mathbf{b}}$ and $x_{\leftarrow \mathbf{s}} = \mathrm{RP}_{\mathbf{s}} + (x - \mathrm{RP}_{\mathbf{s}}) \cdot \delta_{\mathbf{s}}$. If a value $x$ is backward propagated $n$ times for agent $\mathbf{a}$, we write $x_{\leftarrow n[\mathbf{a}]}$, e.g. $x_{\leftarrow 2[\mathbf{a}]} = (x_{\leftarrow \mathbf{a}})_{\leftarrow \mathbf{a}}$. If a value is backward propagated for more than one agent, we list them left to right in the subscript, e.g., $x_{\leftarrow \mathbf{b}2[\mathbf{s}]} = ((x_{\leftarrow \mathbf{b}})_{\leftarrow \mathbf{s}})_{\leftarrow \mathbf{s}}$. The values of $x^*(t)$ can be calculated recursively from $t = T-1$ back to $t = 0$ as follows:

$$x^*(t) = \begin{cases} \mathrm{RP}_{\iota(t+1)} & \text{if } t = T-1 \\ (x^*(t+1))_{\leftarrow \iota(t+1)} & \text{if } t < T-1 \end{cases}$$

It can be observed that $x_{\leftarrow \mathbf{b}} \geq x$ as $x_{\leftarrow \mathbf{b}} - x = \mathrm{RP}_{\mathbf{b}} - (\mathrm{RP}_{\mathbf{b}} - x) \cdot \delta_{\mathbf{b}} - x = (1 - \delta_{\mathbf{b}})(\mathrm{RP}_{\mathbf{b}} - x) \geq 0$, and $x_{\leftarrow \mathbf{s}} \leq x$ as $x_{\leftarrow \mathbf{s}} - x = \mathrm{RP}_{\mathbf{s}} + (x - \mathrm{RP}_{\mathbf{s}}) \cdot \delta_{\mathbf{s}} - x = (\delta_{\mathbf{s}} - 1)(x - \mathrm{RP}_{\mathbf{s}}) \leq 0$.

Finally, the equilibrium strategies of $\mathbf{b}$ can be defined as (the equilibrium strategies of $\mathbf{s}$ can be defined analogously)

$$\sigma^*_{\mathbf{b}}(t) = \begin{cases} t=0 & \mathit{offer}[x^*(0)] \\ 0<t<T & \begin{cases} \text{if } \sigma_{\mathbf{s}}(t-1) = \mathit{offer}[x] \text{ with } x \leq x^*(t)_{\leftarrow \mathbf{b}} & \mathit{accept} \\ \text{otherwise} & \mathit{offer}[x^*(t)] \end{cases} \\ T \leq t \leq T_{\mathbf{b}} & \begin{cases} \text{if } \sigma_{\mathbf{s}}(t-1) = \mathit{offer}[x] \text{ with } x \leq \mathrm{RP}_{\mathbf{b}} & \mathit{accept} \\ \text{otherwise} & \mathit{exit} \end{cases} \\ T_{\mathbf{b}} < t & \mathit{exit} \end{cases}$$

Therefore, at equilibrium, the two agents will reach an agreement at time $t = 1$ and the agreement price is $x^*(0)$.

## 3. Two-Types of Reserve Prices

This section considers the case where there are two possible reserve prices for the buyer.

### 3.1. Introducing Uncertainty

With uncertain information, the appropriate solution concept for an extensive-form game is Kreps and Wilson's *sequential* *equilibrium* [8]. A sequential equilibrium is a pair $a = \langle \mu, \sigma \rangle$ (also called an assessment) where $\mu$ is a belief system that specifies how agents' beliefs evolve during the game and $\sigma$ specifies agents' strategies. At an equilibrium $\mu$ must be *consistent* with respect to $\sigma$ and $\sigma$ must be *sequentially rational* given $\mu$.

In this section we assume that the buyer $\mathbf{b}$ can be of two types: buyer $\mathbf{b}_h$ with a reserve price $\mathrm{RP}_h$ and buyer $\mathbf{b}_l$ with a reserve price $\mathrm{RP}_l$ such that $\mathrm{RP}_h > \mathrm{RP}_l$. The initial belief of $\mathbf{s}$ on $\mathbf{b}$ is $\mu(0) = \langle \Delta^0_{\mathbf{b}}, P^0_{\mathbf{b}} \rangle$ where $\Delta^0_{\mathbf{b}} = \{\mathbf{b}_h, \mathbf{b}_l\}$ and $P^0_{\mathbf{b}} = \{\omega^0_{\mathbf{b}_h}, \omega^0_{\mathbf{b}_l}\}$ where $\omega^0_{\mathbf{b}_h}$ ($\omega^0_{\mathbf{b}_l}$, respectively) is the *priori* probability that $\mathbf{b}$ is of type $\mathbf{b}_h$ ($\mathbf{b}_l$, respectively). It follows that $\omega^0_{\mathbf{b}_h} + \omega^0_{\mathbf{b}_l} = 1$. The belief of $\mathbf{s}$ on the type of $\mathbf{b}$ at time $t$ is $\mu(t)$. The probability assigned by $\mathbf{s}$ to $\mathbf{b} = \mathbf{b}_h$ at time $t$ is denoted $\omega^t_{\mathbf{b}_h}$; the probability assigned to $\mathbf{b} = \mathbf{b}_l$ is $\omega^t_{\mathbf{b}_l} = 1 - \omega^t_{\mathbf{b}_h}$. Given an assessment $a = \langle \mu, \sigma \rangle$, there are two possible bargaining outcomes: outcome $o_{\mathbf{b}_h}$ if $\mathbf{b} = \mathbf{b}_h$ and $o_{\mathbf{b}_l}$ if $\mathbf{b} = \mathbf{b}_l$. We denote bargaining outcome as $o = \langle o_{\mathbf{b}_h}, o_{\mathbf{b}_l} \rangle$.

With pure strategies, buyer types' possible behaviors regarding whether they behave in the same way on the equilibrium path at each decision making node are finite. We use the term "choice rule" to characterize agents' strategies regarding whether they behave in the same way at a specific decision making point. Easily, at a decision making node $\mathbf{b}_l$ and $\mathbf{b}_h$ can make the same offer (in this case, choice rules are said *pooling*) or can make different offers (in this case, choice rules are said *separating*). On the basis of this consideration, we can make some assumptions over the belief system without loosing generality. On the equilibrium path $\mu(t) = \langle \Delta^t_{\mathbf{b}}, P^t_{\mathbf{b}} \rangle$ of $\mathbf{s}$ on $\mathbf{b}$ at any time $t$ is one the following. After a time point $t$ where buyer types' choice rule is pooling, $\mu(t+1) = \mu(t)$, i.e., $\Delta^{t+1}_{\mathbf{b}} = \Delta^t_{\mathbf{b}}$ and $P^{t+1}_{\mathbf{b}} = P^t_{\mathbf{b}}$. After a time point $t$ where buyer types' choice rule is separating, there could be two possible beliefs: if the equilibrium offer of $\mathbf{b}_h$ has been observed, then $\Delta^{t+1}_{\mathbf{b}} = \{\mathbf{b}_h\}$ ($\mathbf{s}$ believes $\mathbf{b} = \mathbf{b}_h$ with certainty), which implies $\omega^{t+1}_{\mathbf{b}_h} = 1$ and $\omega^{t+1}_{\mathbf{b}_l} = 0$; if instead the equilibrium offer of $\mathbf{b}_l$ has been observed, $\Delta^{t+1}_{\mathbf{b}} = \{\mathbf{b}_l\}$ ($\mathbf{s}$ believes $\mathbf{b} = \mathbf{b}_l$ with certainty), which implies $\omega^t_{\mathbf{b}_h} = 0$ and $\omega^{t+1}_{\mathbf{b}_l} = 1$. We need also specify the belief system off the equilibrium path, i.e., when an agent makes an action that is not optimal. We use the *optimistic conjectures* [12]. That is, when $\mathbf{b}$ acts off the equilibrium strategy, agent $\mathbf{s}$ will believe that agent $\mathbf{b}$ is of its "weakest" type, i.e., of the type that would gain the least in a complete information game. This choice is directed to assure the existence of the equilibrium for the largest subset of the space of the parameters. In our case, the weakest type is $\mathbf{b}_h$ (we prove it in the following section). We can therefore specify $\mu(t)$ by specifying $\Delta^t_{\mathbf{b}}$. We will write $\mu(t) = \{\mathbf{b}_h, \mathbf{b}_l\}$, or $\mu(t) = \{\mathbf{b}_h\}$, or $\mu(t) = \{\mathbf{b}_l\}$.

### 3.2. Off the Equilibrium Path Optimal Strategies

Before analyzing equilibrium strategies when the buyer can be of two types, we provide the optimal strategies in the situations $\mathbf{s}$ believes the buyer of one single type. There are two cases: 1) Seller $\mathbf{s}$ has the right belief about the type of the buyer $\mathbf{b}$. In this case, agents' equilibrium strategies are the equilibrium strategies of the corresponding complete information bargaining discussed in Section 2. Let $x^*_{\mathbf{b}_h}(t)$ ($x^*_{\mathbf{b}_l}(t)$, respectively) be any agent optimal offer at time $t$ when $\mathbf{b}$ is of type $\mathbf{b}_h$ ($\mathbf{b}_l$, respectively) in this case. 2) Seller $\mathbf{s}$ has the wrong belief about the type of the buyer $\mathbf{b}$, i.e., $\mathbf{b}_h$ is believed to be $\mathbf{b}_l$ and $\mathbf{b}_l$ is believed to be $\mathbf{b}_h$.

*Lemma 1:* $x^*_{\mathbf{b}_h}(t) \geq x^*_{\mathbf{b}_l}(t)$.

The proofs are omitted for reasons of space. We can see that $\mathbf{b}_h$ is weaker than $\mathbf{b}_l$ in terms of its offering price at each time point in complete information bargaining.

*Theorem 2:* If seller $\mathbf{s}$ has the wrong belief about the type of $\mathbf{b}$, the optimal strategies of agent $\mathbf{s}$ are those in complete information bargaining. The optimal strategies $\sigma^*_{\mathbf{b}_h}(t)|\{\mathbf{b}_l\}$ of buyer $\mathbf{b}_h$ when it's believed to be $\mathbf{b}_l$ are:

$$\sigma^*_{\mathbf{b}_h}(t)|\{\mathbf{b}_l\} = \begin{cases} \textit{accept } y & \text{if } y \leq (x^*_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_h} \\ \textit{offer } x^*_{\mathbf{b}_l}(t) & \textit{otherwise} \end{cases}$$

The optimal strategies $\sigma^*_{\mathbf{b}_l}(t)|\{\mathbf{b}_h\}$ of the buyer $\mathbf{b}_l$ when it's believed to be $\mathbf{b}_h$ are:

- If $\iota(T) = \mathbf{b}$, accept $y$ if $y \leq \min\{(x^*_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_l}, \mathrm{RP}_l\}$. Otherwise, offer $\min\{x^*_{\mathbf{b}_h}(t), \mathrm{RP}_l\}$.
- If $\iota(T) = \mathbf{s}$, accept $y$ if $y \leq \min\{(x^*_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_l}, (\mathrm{RP}_{\mathbf{s}})_{\leftarrow (T-t)[\mathbf{b}_l]}$. Otherwise, offer $\min\{x^*_{\mathbf{b}_h}(t), (\mathrm{RP}_{\mathbf{s}})_{\leftarrow (T-1-t)[\mathbf{b}_l]}\}$.

*Proof:* Case 1 ($\mathbf{b}_h$ is believed to be $\mathbf{b}_l$). If the seller offers $x^*_{\mathbf{b}_l}(t-1)$, buyer $\mathbf{b}_h$'s optimal strategy is to accept it as the minimum price that the seller would accept at time $t+1$, i.e., $x^*_{\mathbf{b}_l}(t)$, gives $\mathbf{b}_h$ a utility lesser than $x^*_{\mathbf{b}_l}(t-1)$ since $(x^*_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_h} > (x^*_{\mathbf{b}_l}(t))_{\leftarrow \mathbf{b}_l} = x^*_{\mathbf{b}_l}(t-1)$. If the seller acts off the equilibrium path and offers a price $y$ lower than $x^*_{\mathbf{b}_l}(t-1)$, the optimal strategy of $\mathbf{b}_h$ is obviously to accept $y$. If the seller offers a price $y$ greater than $x^*_{\mathbf{b}_l}(t-1)$, the optimal strategy of $\mathbf{b}_h$ is to accept $y$ only if $y \leq (x^*_{\mathbf{b}_l}(t)_{\leftarrow \mathbf{b}_h}$, otherwise $\mathbf{b}_h$'s optimal strategy is to reject $y$ and to offer $x^*_{\mathbf{b}_l}(t)$. Note that $x^*_{\mathbf{b}_h}(t) \leq \mathrm{RP}_h$ and $x^*_{\mathbf{b}_l}(t) \leq \mathrm{RP}_h$.

Case 2 ($\mathbf{b}_l$ is believed to be $\mathbf{b}_h$). This case is more complicated as seller's optimal offer $x^*_{\mathbf{b}_h}(t-1)$ on its equilibrium path is not acceptable to $\mathbf{b}_l$ as when $\mathbf{b}_l$ offers $x^*_{\mathbf{b}_h}(t)$ at time $t$, $(x^*_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_l} < (x^*_{\mathbf{b}_h}(t))_{\leftarrow \mathbf{b}_h} = x^*_{\mathbf{b}_h}(t-1)$. In addition, $\mathbf{b}_l$ may not offer $x^*_{\mathbf{b}_h}(t)$ if it's advantageous to wait for the agreement at time $T$. There are two situations: 1) $\iota(T) = \mathbf{b}$. In this case, $\mathbf{s}$ will propose $\mathrm{RP}_h$ at time $T-1$, which is not acceptable to buyer $\mathbf{b}_l$ as $\mathrm{RP}_h$ is higher than $\mathbf{b}_l$'s reserve price. Therefore, $\mathbf{b}_l$'s optimal offer at time $t$ is $\min\{x^*_{\mathbf{b}_h}(t), \mathrm{RP}_l\}$. Note that $x^*_{\mathbf{b}_l}(t)$ is always not acceptable to $\mathbf{s}$. 2) $\iota(T) = \mathbf{s}$. In this case, $\mathbf{b}_l$ will propose $\mathrm{RP}_{\mathbf{s}}$ at time $T-1$, which will be accepted by seller $\mathbf{s}$ at time $T$. Therefore, $\mathbf{b}_l$'s optimal offer at time $t$ is $\min\{x^*_{\mathbf{b}_h}(t), (\mathrm{RP}_{\mathbf{s}})_{\leftarrow (T-1-t)[\mathbf{b}_l]}\}$. □

### 3.3. Solving Algorithm

Our algorithm combines game theoretic analysis and state space search techniques and it is sound and complete. By applying state space search, we enumerate all possible choice systems. A choice system specifies buyer types' choice rule at

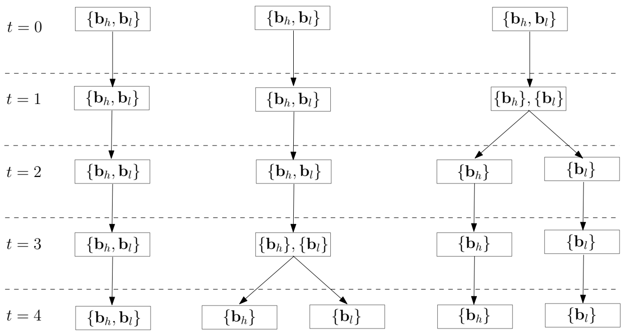

Fig. 1. Choice systems for bilateral bargaining with two-type uncertainty where $\iota(0) = \mathbf{b}$ and $T = 4$.

all decision making points along the negotiation horizon. By exploiting game theoretic analysis we design a pair composed of choice rules and belief system for each possible choice rules. More precisely, we design a pair for pooling choice rule and a pair for separating choice rule. These pairs are parameterized: agents' optimal offers and acceptance at time $t$ depend on the agents' strategies in the following time points till the end of the bargaining. Furthermore, we assign each pair some conditions: if they are satisfied, then there is a sequential equilibrium in the subgame starting from time $t$. For each choice system, we employ a Bayesian extension of backward induction to derive agents' optimal strategies. Agents' optimal strategies at time $t$ is built on agents' equilibrium strategies from time $t+1$ to $T$. In summary, we employ a forward-backward approach to find sequential equilibria: we search forward to find all the choice systems and we construct backward agents' equilibrium strategies and belief systems.

Given $\mathbf{s}$'s belief on the type of $\mathbf{b}$, different buyer types can choose different choice rules: either behave in the same way or behave in different ways. While it is very involved to compute sequential equilibria considering all the options at each decision making point, we explicitly fix the choice rule at each decision making point and then compute the sequential equilibrium of the bilateral game where buyer types' choices are specified in the choice system. To guarantee the completeness of our approach, we enumerate all possible choice systems. Our approach can be treated as a way of shifting the difficulty of finding a sequential equilibrium in a bargaining game where the buyer has multiple choices to finding a sequential equilibrium in multiple bargaining games in which the buyer's choice is fixed.

We explain our approach through a bilateral bargaining example where $\iota(0) = \mathbf{b}$ and $T = 4$. As there are two types, there exist only two choice rules: 1) different buyer types behave in the same way (i.e., make the same offer) or 2) different buyer types behave in different ways (i.e., make different offers). Once the buyer chooses to differentiate its two types at time point $t$, the later bargaining becomes complete information bargaining. At time $t = 0$, the belief of $\mathbf{s}$ on the type of $\mathbf{b}$ is $\{\mathbf{b}_h, \mathbf{b}_l\}$ and $\mathbf{b}$'s different types can choose to make the same offer or make different offers. If $\mathbf{b}$ chooses the separating rule, seller $\mathbf{s}$ will update its belief at time $t = 1$ and bargaining from time $t = 1$ to the deadline becomes complete information bargaining. If $\mathbf{b}$ chooses the pooling rule at time $t = 0$, seller $\mathbf{s}$'s belief at time $t = 1$ will still be $\{\mathbf{b}_h, \mathbf{b}_l\}$. Then at time $t = 2$, buyer types can still choose to behave in the same way or behave in different ways. No matter what the choice rule is at time $t = 2$, $\mathbf{b}$ has no choice at its deadline $t = 4$. There are totally three choice systems: 1) different buyer types always use the pooling choice rule; 2) different buyer types apply the pooling choice rule at time $t = 0$ and apply the separating choice rule at time $t = 2$; and 3) different buyer types apply the separating choice rule at time $t = 0$. Fig. 1 shows the three choice systems for the bilateral bargaining. A node $\Delta_{\mathbf{b}}$ (e.g., $\{\mathbf{b}_h\}$) at time $t$ implies that $\mathbf{s}$'s belief at the beginning of $t$ is $\Delta_{\mathbf{b}}$. A node $\{\Delta^a_{\mathbf{b}}, \Delta^r_{\mathbf{b}}\}$ (e.g., $\{\{\mathbf{b}_h\}, \{\mathbf{b}_l\}\}$) at time $t$ implies that 1) $\mathbf{s}$'s belief at the beginning of time $t$ is either $\Delta^a_{\mathbf{b}}$ or $\Delta^r_{\mathbf{b}}$; 2) the offers of buyer types $\Delta^a_{\mathbf{b}}$ and $\Delta^r_{\mathbf{b}}$ are different at time $t-1$.

Given a choice system, we adopt a modified backward induction approach to compute the sequential equilibrium. In the rest of this section, we first consider two special situations where buyer types always choose the pooling choice rule or always choose the separating choice rule. Then we construct sequential equilibria for general cases based on our analysis of the two special situations.

### 3.4. Always Use Pooling Choice Rule

In this section we study agents' equilibrium strategies when different buyer types always behave in the same way at each decision making point. Accordingly, seller's belief will always be its initial belief. We start considering a bargaining with deadline $T = 2$. There are two situations: $\iota(T) = \mathbf{s}$ or $\iota(T) = \mathbf{b}$. The former case is simple as both $\mathbf{b}_h$ and $\mathbf{b}_l$ will propose $\mathbf{s}$'s reserve price $\mathrm{RP}_{\mathbf{s}}$ at $t = 1$ and $\mathbf{s}$ will propose its best offer based on its initial belief at $t = 0$. $\mathbf{s}$ has two choices: $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}$ with an expected utility $EU_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}) = U_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}, 1)$ and $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}$ with an expected utility $EU_{\mathbf{s}}(\mathrm{RP}_h) = \omega^0_{\mathbf{b}_h} U_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}, 1) + \omega^0_{\mathbf{b}_l} U_{\mathbf{s}}(\mathrm{RP}_{\mathbf{s}}, 2)$. The bargaining outcome is $\langle ((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}, 1), ((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}, 1) \rangle$ if $\omega^0_{\mathbf{b}_h} \leq ((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l} - \mathrm{RP}_{\mathbf{s}})/((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h} - \mathrm{RP}_{\mathbf{s}})$. Otherwise, the bargaining outcome is $\langle ((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}, 1), (\mathrm{RP}_{\mathbf{s}}, 2) \rangle$.

Now consider the case $\iota(T) = \mathbf{b}$. At time $t = 1$, agent $\mathbf{s}$ can choose between the offer $\mathrm{RP}_h$ and $\mathrm{RP}_l$. While offering $\mathrm{RP}_h$, it can get an expected utility $EU_{\mathbf{s}}(\mathrm{RP}_h) = \omega^0_{\mathbf{b}_h} U_{\mathbf{s}}(\mathrm{RP}_h, T)$ as $\mathbf{b}_l$ will not accept the offer $\mathrm{RP}_h$. While offering $\mathrm{RP}_l$, it can get an expected utility $EU_{\mathbf{s}}(\mathrm{RP}_l) = U_{\mathbf{s}}(\mathrm{RP}_l, T)$. Let $e^1$ the equivalent price of the optimal offer of agent $\mathbf{s}$ in the subgame beginning from time 1 where it begins to bargaining. Formally, $e^1$ is a price such that $U_{\mathbf{s}}(e^1, T) = \max\{EU_{\mathbf{s}}(\mathrm{RP}_h), EU_{\mathbf{s}}(\mathrm{RP}_l)\}$. Accordingly, $e^1_{\leftarrow \mathbf{s}}$ is the lowest offer agent $\mathbf{s}$ would accept at time $t = 1$ with the initial belief. Therefore, the optimal offer of the buyer is $e^1_{\leftarrow \mathbf{s}}$. Note that if $e^1_{\leftarrow \mathbf{s}} > \mathrm{RP}_l$, it is not rational for $\mathbf{b}_l$ to offer $e^1_{\leftarrow \mathbf{s}}$. In this case, there is no sequential rational strategy while always using the pooling choice rule.

*Theorem 3:* Assume that $T = 2$ and $\iota(T) = \mathbf{b}$, if $\mathrm{RP}_l \geq e^1_{\leftarrow \mathbf{s}}$, there is one and only one sequentially rational pure

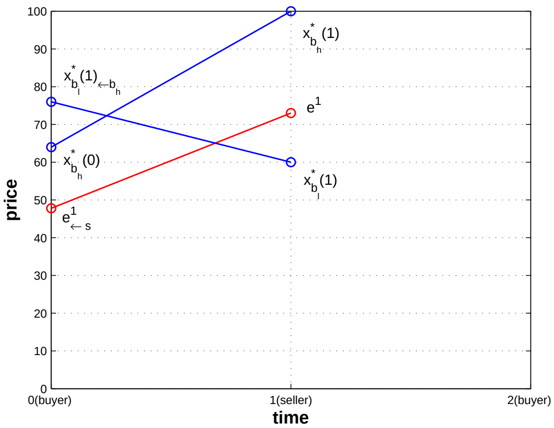


Fig. 2. Backward induction construction with $T = 2$, $\iota(T) = \mathbf{b}$, $\mathrm{RP}_h = 100$, $\omega^0_{\mathbf{b}_h} = 0.7$, $\mathrm{RP}_l = 60$, $\omega^0_{\mathbf{b}_l} = 0.3$, $\mathrm{RP}_{\mathbf{s}} = 10$, $\delta_{\mathbf{s}} = \delta_{\mathbf{b}} = 0.6$.

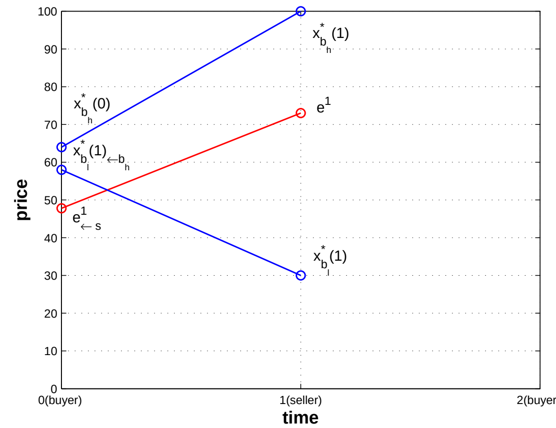


Fig. 3. Backward induction construction with the same setting as in Fig. 2 except $\mathrm{RP}_l = 30$.

strategy profile given the system of beliefs

$$\mu(1) = \begin{cases} \Delta^0_{\mathbf{b}} & \text{if } \sigma^*_{\mathbf{b}}(0) = \textit{offer } e^1_{\leftarrow\mathbf{s}} \\ \{\mathbf{b}_h\} & \textit{otherwise} \end{cases}$$

The strategies $\sigma^*_{\mathbf{b}_h}(0)$, $\sigma^*_{\mathbf{b}_l}(0)$, and $\sigma^*_{\mathbf{s}}(1)$ are: $\sigma^*_{\mathbf{b}_h}(0) = \sigma^*_{\mathbf{b}_l}(0) = \textit{offer } e^1_{\leftarrow\mathbf{s}}$, and $\sigma^*_{\mathbf{s}}(1) = \textit{accept } y$ if $y \geq e^1_{\leftarrow\mathbf{s}}$. The bargaining outcome is $\langle(e^1_{\leftarrow\mathbf{s}}, 1), (e^1_{\leftarrow\mathbf{s}}, 1)\rangle$.

*Proof:* We first analyze the strategies on the equilibrium path. We assume that the buyer behaves according to the prescribed equilibrium strategies and we analyze th optimal strategy of the seller. The seller can accept $e^1_{\leftarrow\mathbf{s}}$ and gain $U_{\mathbf{s}}(e^1_{\leftarrow\mathbf{s}}, T-1)$ or reject it and make an offer. However, the maximum expected utility the seller can have from the subgame from time 1 is just $U_{\mathbf{s}}(e^1, T) = U_{\mathbf{s}}(e^1_{\leftarrow\mathbf{s}}, T-1)$. Thus, seller's optimal strategy is to accept $e^1_{\leftarrow\mathbf{s}}$.

We assume that the seller behaves according to the prescribed equilibrium strategies and we analyze the optimal strategy of the buyer. We start considering the strategy of $\mathbf{b}_h$. If $\mathbf{b}_h$ offers $e^1_{\leftarrow\mathbf{s}}$, it gains $U_{\mathbf{b}_h}(e^1_{\leftarrow\mathbf{s}}, T-1)$. If $\mathbf{b}_h$ offers a price $y$ no less than $\sigma^*_{\mathbf{b}_h}(T-1)_{\leftarrow\mathbf{s}}$, the seller will accept it and $\mathbf{b}_h$ gains a utility lower than $U_{\mathbf{b}_h}(e^1_{\leftarrow\mathbf{s}}, T-1)$ since $\sigma^*_{\mathbf{b}_h}(T-1)_{\leftarrow\mathbf{s}} > e^1_{\leftarrow\mathbf{s}}$. If $\mathbf{b}_h$ offers a price $y \neq e^1_{\leftarrow\mathbf{s}}$ which is less than $\sigma^*_{\mathbf{b}_h}(T-1)_{\leftarrow\mathbf{s}}$, the seller will reject it and will propose $\sigma^*_{\mathbf{b}_h}(T-1)$ at time $T-1$ which will give $\mathbf{b}_h$ a utility lower than $U_{\mathbf{b}_h}(e^1_{\leftarrow\mathbf{s}}, T-1)$ since $\sigma^*_{\mathbf{b}_h}(T-1)_{\leftarrow\mathbf{b}_h} > e^1_{\leftarrow\mathbf{s}}$. Similarly, we can get that $\mathbf{b}_l$ has no incentive to propose a price not equal to $e^1_{\leftarrow\mathbf{s}}$.

There is no sequential rational pure strategy when $\mathrm{RP}_l < e^1_{\leftarrow\mathbf{s}}$ since it supposes that both the buyer's types behave in the same way, whereas the optimal strategy of $\mathbf{b}_l$ is to not propose $e^1_{\leftarrow\mathbf{s}}$ as it will get a negative utility by doing so. □

Fig. 2 shows an example of backward induction construction with $T = 2$, $\iota(T) = \mathbf{b}$, $\mathrm{RP}_h = 100$, $\omega^0_{\mathbf{b}_h} = 0.7$, $\mathrm{RP}_l = 60$, $\omega^0_{\mathbf{b}_l} = 0.3$, $\mathrm{RP}_{\mathbf{s}} = 10$, $\delta_{\mathbf{s}} = \delta_{\mathbf{b}} = 0.6$. At time $t = 1$, $\mathbf{s}$ can offer either 60 or 100: If it offers 60, its expected utility is $(60 - 10)0.6^2 = 18$; If it offers 100, its expected utility is $0.7(100 - 10)0.6^2 = 22.68$. Thus, the optimal offer of $\mathbf{s}$ at time $t = 1$ is 100 and the equivalent price is $e^1 = 73$. Then we have $e^1_{\leftarrow\mathbf{s}} = 47.8$. As $\mathrm{RP}_l > e^1_{\leftarrow\mathbf{s}}$, there is a sequential equilibrium within the belief system while always using the pooling choice rule. If we change $\mathrm{RP}_l$ to 30 (Fig. 3). At time $t = 1$, $\mathbf{s}$ can offer either 30 or 100. If it offers 30, its expected utility is $(30 - 10)0.6^2 = 7.2$. Thus, the optimal offer of $\mathbf{s}$ at time $t = 1$ is 100 and the equivalent price is $e^1 = 73$. Then we have $e^1_{\leftarrow\mathbf{s}} = 47.8$. As $\mathrm{RP}_l < e^1_{\leftarrow\mathbf{s}}$, there is no sequential rational strategy within the belief system while always using the pooling choice rule.

In a signaling game, there are often multiple equilibrium outcomes and these equilibria are not equivalent from the point of view of utility of each agent and social welfare (i.e., the sum of utilities of all agents). The multiplicity of equilibria means that, without refinement, equilibrium theory provides few clear predictions. A number of refinements (e.g., pareto efficiency) has been proposed. In this work, we do not consider the pooling choice rule in which all buyer types make a rejectable offer since, given the same equilibrium strategies in the subgame, the equilibrium outcome when all buyer types make an acceptable offer pareto dominates the equilibrium outcome when all buyer types make a rejectable offer.

We now consider an arbitrary deadline $T$. We apply the backward induction starting from deadline $T$ and inductively determine agents' equilibrium strategies. Let $e^t$ be the equivalent price of the optimal offer of $\mathbf{s}$ at time $t$. First we consider the case $\iota(T) = \mathbf{b}$. At any time $t$, $\mathbf{b}$'s optimal offer is $e^{t+1}_{\leftarrow\mathbf{s}}$. At time $T-1$, $\mathbf{s}$'s equivalent price $e^{T-1}$ of the optimal offer satisfies $U_{\mathbf{s}}(e^{T-1}, T) = \max\{EU_{\mathbf{s}}(\mathrm{RP}_h), EU_{\mathbf{s}}(\mathrm{RP}_l)\}$ where $EU_{\mathbf{s}}(\mathrm{RP}_h) = \omega^0_{\mathbf{b}_h} U_{\mathbf{s}}(\mathrm{RP}_h, T)$ and $EU_{\mathbf{s}}(\mathrm{RP}_l) = U_{\mathbf{s}}(\mathrm{RP}_l, T)$. At time $t < T-1$, $\mathbf{s}$'s equivalent price $e^t$ of its optimal offer satisfies $U_{\mathbf{s}}(e^t, t+1) = \max\{EU_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}\mathbf{b}_l}), EU_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}\mathbf{b}_h})\}$ where $EU_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}\mathbf{b}_l}) = U_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}\mathbf{b}_l}, t+1)$ and $EU_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}\mathbf{b}_h}) = \omega^0_{\mathbf{b}_h} U_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}\mathbf{b}_h}, t+1) + \omega^0_{\mathbf{b}_l} U_{\mathbf{s}}(e^{t+2}_{\leftarrow\mathbf{s}}, t+2)$. Following Theorem 3, the condition of existence is that $\mathrm{RP}_l \geq e^{t+1}_{\leftarrow\mathbf{s}}$ at any time $t < T-1$ such that $\iota(t) = \mathbf{b}$.

Now we consider the case $\iota(T) = \mathbf{s}$. $\mathbf{b}$'s optimal offer at time $T-1$ is $\mathrm{RP}_{\mathbf{s}}$. $\mathbf{s}$ chooses between $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}$ and $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}$ at time $T$–2. $\mathbf{s}$'s equivalent price $e^{T-2}$ of its optimal offer satisfies $U_{\mathbf{s}}(e^{T-2}, T-1) = \max\{EU_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}), EU_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l})\}$ where $EU_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}) = U_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_l}, T-1)$ and $EU_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}) = \omega^0_{\mathbf{b}_h} U_{\mathbf{s}}((\mathrm{RP}_{\mathbf{s}})_{\leftarrow \mathbf{b}_h}, T-1) + \omega^0_{\mathbf{b}_l} U_{\mathbf{s}}(\mathrm{RP}_{\mathbf{s}}, T)$. We can construct the value of $e^t$ in the same way for $t < T-1$. The condition of existence is that $e^{t+1}_{\leftarrow \mathbf{s}} \leq \min\{\mathrm{RP}_l, (\mathrm{RP}_{\mathbf{s}})_{\leftarrow (T-1-t)[\mathbf{b}_l]}\}$ at any time $t < T-1$ such that $\iota(t) = \mathbf{b}$. Note that $(\mathrm{RP}_{\mathbf{s}})_{\leftarrow (T-1-t)[\mathbf{b}_l]}$ is the maximum offer buyer $\mathbf{b}_l$ is willing to pay.

*Theorem 4:* The bilateral bargaining has a unique sequentially rational pure strategy profile given the following belief system where $\mu(t+1)$ is given by

- If $\mu(t) = \{\mathbf{b}_h\}$ or $\mu(t) = \{\mathbf{b}_l\}$, $\mu(t+1) = \mu(t)$.
- $\mu(t) = \mu(0)$ and there are four cases. 1) If $t = 0$ and $\sigma_{\mathbf{b}}(t) = \mathit{offer}\, e^{t+1}_{\leftarrow \mathbf{s}}$, $\mu(t+1) = \mu(0) = \{\mathbf{b}_h, \mathbf{b}_l\}$; 2) If $t > 0$ and $\mathbf{b}$ rejects $y \in (e^{t+1}_{\leftarrow \mathbf{s}\mathbf{b}_h}, +\infty]$ and $\sigma_{\mathbf{b}}(t) = \mathit{offer}\, e^{t+1}_{\leftarrow \mathbf{s}}$, $\mu(t+1) = \mu(0) = \{\mathbf{b}_h, \mathbf{b}_l\}$; 3) If $t > 0$ and $\mathbf{b}$ rejects $y \in (e^{t+1}_{\leftarrow \mathbf{s}\mathbf{b}_l}, e^{t+1}_{\leftarrow \mathbf{s}\mathbf{b}_h}]$ and $\sigma_{\mathbf{b}}(t) = \mathit{offer}\, e^{t+1}_{\leftarrow \mathbf{s}}$, $\mu(t+1) = \{\mathbf{b}_l\}$; 4) otherwise, $\mu(t+1) = \{\mathbf{b}_h\}$.

if $\mathrm{RP}_l \geq e^{t+1}_{\leftarrow \mathbf{s}}$ when $\iota(T) = \mathbf{b}$ and $e^{t+1}_{\leftarrow \mathbf{s}} \geq (\mathrm{RP}_{\mathbf{s}})_{\leftarrow (T-1-t)[\mathbf{b}_l]}$ when $\iota(T) = \mathbf{s}$. The equilibrium strategies $\sigma^*_{\mathbf{s}}(t)|\{\mathbf{b}_h, \mathbf{b}_l\}$ of agent $\mathbf{s}$ are:

$$\begin{cases} \mathit{accept}\ y & \text{if } y \leq e^t_{\leftarrow \mathbf{s}} \\ \mathit{offer}\ y = \arg\max_{y \in \{e^{t+2}_{\leftarrow \mathbf{s}\mathbf{b}_h}, e^{t+2}_{\leftarrow \mathbf{s}\mathbf{b}_l}\}} EU_{\mathbf{s}}(y) & \mathit{otherwise} \end{cases}$$

and the equilibrium strategies of the buyer are:

$$\sigma^*_{\mathbf{b}_h}(t)|\{\mathbf{b}_h, \mathbf{b}_l\} = \begin{cases} \mathit{accept}\ y & \text{if } y \leq e^{t+1}_{\leftarrow \mathbf{s}\mathbf{b}_h} \\ \mathit{offer}\ \leq e^{t+1}_{\leftarrow \mathbf{s}} & \mathit{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{b}_l}(t)|\{\mathbf{b}_h, \mathbf{b}_l\} = \begin{cases} \mathit{accept}\ y & \text{if } y \leq e^{t+1}_{\leftarrow \mathbf{s}\mathbf{b}_l} \\ \mathit{offer}\ \leq e^{t+1}_{\leftarrow \mathbf{s}} & \mathit{otherwise} \end{cases}$$

Agents' equilibrium strategies when $\mu(t)$ is a singleton is given by Section 3.2. The bargaining outcome is 1) $\langle (e^1_{\leftarrow \mathbf{s}}, 1), (e^1_{\leftarrow \mathbf{s}}, 1) \rangle$ if $\iota(0) = \mathbf{b}$; 2) $\langle (e^2_{\leftarrow \mathbf{s}\mathbf{b}_l}, 1), (e^2_{\leftarrow \mathbf{s}\mathbf{b}_l}, 1) \rangle$ if $\iota(0) = \mathbf{s}$ and $\omega^0_{\mathbf{b}_h} \leq (e^2_{\leftarrow \mathbf{s}\mathbf{b}_l} - \mathrm{RP}_{\mathbf{s}})/(e^2_{\leftarrow \mathbf{s}\mathbf{b}_h} - e^2_{\leftarrow \mathbf{s}}\delta_{\mathbf{s}} - (1-\delta_{\mathbf{s}})\mathrm{RP}_{\mathbf{s}})$; 3) $\langle (e^2_{\leftarrow \mathbf{s}\mathbf{b}_h}, 1), (e^2_{\leftarrow \mathbf{s}}, 2) \rangle$ if $\iota(0) = \mathbf{s}$ and $\omega^0_{\mathbf{b}_h} > (e^2_{\leftarrow \mathbf{s}\mathbf{b}_l} - \mathrm{RP}_{\mathbf{s}})/(e^2_{\leftarrow \mathbf{s}\mathbf{b}_h} - e^2_{\leftarrow \mathbf{s}}\delta_{\mathbf{s}} - (1-\delta_{\mathbf{s}})\mathrm{RP}_{\mathbf{s}})$.

*Proof:* The sequential rationality is easily seen from the backward construction. Consistency can be proved by the assessment sequence $a_n = (\mu_n, \sigma_n)$ where $\sigma_n$ is the fully mixed strategy profile such that for the seller and $\mathbf{b}_h$ there is probability $1 - 1/n$ of performing the action prescribed by the equilibrium strategy profile and the remaining probability $1/n$ is uniformly distributed among the other allowed actions; while for $\mathbf{b}_l$, there is probability $1 - 1/n^2$ of performing the action prescribed by the equilibrium strategy profile and the remaining probability $1/n^2$ is uniformly distributed among the other allowed actions, and $\mu_n$ is the system of beliefs obtained applying Bayes rule starting from the same *priori* probability distribution $P^0_{\mathbf{b}}$. As $n \rightarrow \infty$, the above mixed strategy profile converges to the equilibrium strategy profile. In addition, the beliefs generated by the mixed strategy profile converges to the priori probability distribution. Thus, the assessment is consistent. □

### 3.5. Always Use Separating Choice Rule

Here we consider the belief system in which two buyer types behave in different ways. Then the seller will learn the buyer's type after it observes the buyer's first offer. Therefore, if $\iota(0) = \mathbf{b}$ ($\iota(0) = \mathbf{s}$, respectively), agent $\mathbf{s}$ will learn $\mathbf{b}$'s type at beginning of time point $t = 1$ ($t = 2$, respectively) and the later bargaining is complete information bargaining.

We start considering a bargaining with an arbitrary deadline $T$ and $\iota(0) = \mathbf{b}$. Different from the approach in the previous section where we start backward induction from the deadline, we move forward from time $t = 0$. Let the equilibrium offers of $\mathbf{b}_h$ and $\mathbf{b}_l$ at time 0 be $x$ and $y$ such that $x \neq y$. If agent $\mathbf{s}$ accepts both offers $x$ and $y$, at least one type has an incentive to offer $\min\{x, y\}$. Therefore, the offer $\min\{x, y\}$ will be rejected by $\mathbf{s}$. There are two cases: $x > y$ and $x < y$. First we consider the case $x < y$. Then $\mathbf{b}_h$ will make a low offer (e.g., $\mathrm{RP}_{\mathbf{s}}$) which be rejected by $\mathbf{s}$. Then at time 1, $\mathbf{s}$ will make the offer $x^*_{\mathbf{b}_h}(1)$ and $\mathbf{b}_h$ will accept it. The optimal offer of $\mathbf{b}_l$ at time $t = 0$ is the lowest price agent $\mathbf{s}$ would accept at time 1 while believing $\mathbf{b}$ to be $\mathbf{b}_l$, i.e., $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{s}} = x^*_{\mathbf{b}_l}(0)$. $\mathbf{b}_h$ has no incentive to behave as $\mathbf{b}_l$ if $(x^*_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h} \leq x^*_{\mathbf{b}_l}(0)$. However, according to Lemma 1 and the definition of backward propagation, we have 1) $(x^*_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h} \geq (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$ and 2) $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h} > (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{s}} = x^*_{\mathbf{b}_l}(0)$. It follows that $(x^*_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h} > x^*_{\mathbf{b}_l}(0)$. Thus, the equilibrium is not sequential rational as $\mathbf{b}_h$ has an incentive to behave as $\mathbf{b}_l$.

Then we consider the case $x > y$. The optimal offer of $\mathbf{b}_h$ is the lowest price agent $\mathbf{s}$ would accept at time 1 believing its opponent $\mathbf{b}_h$ is obviously $(x^*_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{s}} = x^*_{\mathbf{b}_h}(0)$. For agent $\mathbf{b}_l$, any offer $y$ such that $y \in [\mathrm{RP}_{\mathbf{s}}, x^*_{\mathbf{b}_h}(0))$ will be rejected by seller. By convention that the equilibrium offer of $\mathbf{b}_h$ is $\mathrm{RP}_{\mathbf{s}}$. The existence of a such equilibrium depends on two conditions: $\mathbf{b}_h$ must have no incentive to behave as $\mathbf{b}_l$, i.e., $x^*_{\mathbf{b}_h}(0) \leq (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$, and $\mathbf{b}_l$ must have no incentive to behave as $\mathbf{b}_h$, i.e., $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} \leq x^*_{\mathbf{b}_h}(0)$.

*Theorem 5:* Bilateral bargaining such that $\iota(0) = \mathbf{b}$ has one and only one stationary sequential equilibrium profile in pure strategies given the system of beliefs:

$$\mu(1) = \begin{cases} \{\mathbf{b}_l\} & \text{if } \sigma_{\mathbf{b}}(0) = \mathit{offer}\, \mathrm{RP}_{\mathbf{s}} \\ \{\mathbf{b}_h\} & \mathit{otherwise} \end{cases}$$

if $x^*_{\mathbf{b}_h}(0) \leq (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$ and $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} \leq x^*_{\mathbf{b}_h}(0)$. The equilibrium strategies of agent $\mathbf{b}$ are: $\sigma^*_{\mathbf{b}_h}(0)|\{\mathbf{b}_h, \mathbf{b}_l\} = \mathit{offer}\ x^*_{\mathbf{b}_h}(0)$, $\sigma^*_{\mathbf{b}_l}(0)|\{\mathbf{b}_h, \mathbf{b}_l\} = \mathit{offer}\ \mathrm{RP}_{\mathbf{s}}$. Agents' strategies when $\mu(t)$ is singleton are specified in Section 3.2. The bargaining outcome is $\langle (x^*_{\mathbf{b}_h}(0), 1), (x^*_{\mathbf{b}_l}(1), 2) \rangle$.

Now we consider the case $\iota(0) = \mathbf{s}$. Agent $\mathbf{s}$ knows that at time $t = 1$ 1) $\mathbf{b}_h$ will offer $x^*_{\mathbf{b}_h}(1)$ and 2) $\mathbf{b}_l$ will offer offer $\mathrm{RP}_{\mathbf{s}}$. Therefore, $\mathbf{s}$'s optimal offer to $\mathbf{b}_h$ at time $t = 0$ is $(x^*_{\mathbf{b}_h}(1))_{\leftarrow \mathbf{b}_h} = x^*_{\mathbf{b}_h}(0)$. $\mathbf{s}$'s optimal offer to $\mathbf{b}_l$ at time $t = 0$ is $(x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}$. $\mathbf{s}$'s expected utility while offering $x^*_{\mathbf{b}_h}(0)$ is $EU_{\mathbf{s}}(x^*_{\mathbf{b}_h}(0)) =$

$\omega^0_{\mathbf{b}_h} U_{\mathbf{s}}(x^*_{\mathbf{b}_h}(0),1) + \omega^0_{\mathbf{b}_l} U_{\mathbf{s}}(x^*_{\mathbf{b}_l}(2),3)$. $\mathbf{s}$'s expected utility while offering $(x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}$ is $EU_{\mathbf{s}}((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}) = U_{\mathbf{s}}((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]},1)$. $\mathbf{s}$ will choose the offer which gives it the highest expected utility at time 0.

*Theorem 6:* Assume that the following belief system is used: if $\mathbf{b}$ rejects $\mathbf{s}$'s offer $y \in ((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}, +\infty)$ and offers $\mathrm{RP}_{\mathbf{s}}$ at time 1, then $\mu(2) = \{\mathbf{b}_l\}$. Otherwise, $\mu(2) = \{\mathbf{b}_h\}$. Bilateral bargaining such that $\iota(0) = \mathbf{s}$ has a unique stationary sequential equilibrium profile in pure strategies if $x^*_{\mathbf{b}_h}(1) \le (x^*_{\mathbf{b}_l}(2))_{\leftarrow \mathbf{b}_h}$ and $(x^*_{\mathbf{b}_l}(2))_{\leftarrow \mathbf{b}_l} \le x^*_{\mathbf{b}_h}(1)$. The equilibrium strategies $\sigma^*_s$ of agent $\mathbf{s}$ are:

$$\sigma^*_{\mathbf{s}}(0)|\{\mathbf{b}_h,\mathbf{b}_l\} = \textit{offer} \arg\max_{y \in \{x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]}, x^*_{\mathbf{b}_h}(0)\}} EU_{\mathbf{s}}(y)$$

$$\sigma^*_{\mathbf{b}_h}(1)|\{\mathbf{b}_h,\mathbf{b}_l\} = \begin{cases} \textit{accept } y & \text{if } y \le x^*_{\mathbf{b}_h}(0) \\ \textit{offer } x^*_{\mathbf{b}_h}(1) & \textit{otherwise} \end{cases}$$

$$\sigma^*_{\mathbf{b}_l}(1)|\{\mathbf{b}_h,\mathbf{b}_l\} = \begin{cases} \textit{accept } y & \text{if } y \le x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]} \\ \textit{offer } \mathrm{RP}_{\mathbf{s}} & \textit{otherwise} \end{cases}$$

Agents' strategies when $\mu(t)$ is singleton are those in complete information bargaining. The equilibrium outcome is $\langle(((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]},1),((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]},1)\rangle$ if $\omega^0_{\mathbf{b}_h} \le ((x^*_{\mathbf{b}_l}(2))_{\leftarrow 2[\mathbf{b}_l]} - \mathrm{RP}_{\mathbf{s}} - (x^*_{\mathbf{b}_l}(2) - \mathrm{RP}_{\mathbf{s}})\delta^2_{\mathbf{b}})/(x^*_{\mathbf{b}_h}(0) - \mathrm{RP}_{\mathbf{s}} - (x^*_{\mathbf{b}_l}(2) - \mathrm{RP}_{\mathbf{s}})\delta^2_{\mathbf{b}})$. Otherwise, the equilibrium outcome is $\langle(x^*_{\mathbf{b}_h}(0),1),(x^*_{\mathbf{b}_l}(2),3)\rangle$.

We can observe that the conditions for the existence of the above equilibrium are defined at the beginning of bargaining and the existence of the above equilibrium does not require the existence of a such equilibrium in later negotiation. Consider the sequential equilibrium when buyer types always choose different actions in the bilateral bargaining in Fig. 2. We have $x^*_{\mathbf{b}_h}(0) = 64$ and $x^*_{\mathbf{b}_l}(1) = 60$. $\mathbf{b}_h$ has no incentive to behave as $\mathbf{b}_l$ since $x^*_{\mathbf{b}_h}(0) = 64 < 76 = (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$, and $\mathbf{b}_l$ has no incentive to behave as $\mathbf{b}_h$ since $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} = 60 < 64 = x^*_{\mathbf{b}_h}(0)$. However, there is no sequential equilibrium within the belief system for the bilateral bargaining in Fig. 3. We have $x^*_{\mathbf{b}_h}(0) = 64$ and $x^*_{\mathbf{b}_l}(1) = 30$. However, this strategy is not rational for $\mathbf{b}_h$ as it has an incentive to behave as prescribed for $\mathbf{b}_l$ since $x^*_{\mathbf{b}_h}(0) = 64 \ge 58 = (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$.

### 3.6. Combining Pooling and Separating Choice Rules

In this section we consider agents' equilibrium strategies while employing a belief system which combines the pooling choice rule and the separating choice rule. The only reasonable combination is to employ the pooling choice rule from time 0 to some time $\tau \le T$ and to employ the separating choice rule from time $\tau$ to the deadline $T$. The reason is simple: once different buyer types behave in different ways, seller $\mathbf{s}$ will learn the type of the buyer and then agents conduct complete information bargaining. Then we have the following result and its proof is trivial.

*Theorem 7:* For a finite horizon bargaining with two possible reserve prices of the buyer, if there is no sequential equilibrium in pure strategies, there is no sequential equilibrium in pure strategies within the belief system which employs both pooling and separating choice rules.

*Theorem 8:* For a finite horizon bargaining with two possible reserve prices of the buyer, there may be no sequential equilibrium in pure strategies.

*Proof:* As the deadline of the bilateral bargaining in Fig. 3 is 2, if there is a sequential equilibrium in pure strategies, the choice rule at time $t = 0$ can only be pooling or separating. As there is no pure strategy sequential equilibrium in both cases, there is no pure strategy sequential equilibrium while applying both choice rules for the bilateral bargaining in Fig. 3. Thus, there is no pure strategy sequential equilibrium for the bilateral bargaining in Fig. 3 (Theorem 7). □

There may be more than one sequential equilibrium for a bilateral bargaining problem with two possible types of reserve price. For example, there are two sequential equilibria for the bilateral bargaining in Fig. 2: one with only using the pooling choice rule and one with only using the separating choice rule.

If there is a pure strategy sequential equilibrium, there should be a time point $\tau$ such that there is a sequential equilibrium for subgame $\Gamma^{[\tau,T]}$ which uses the separating choice rule and a sequential equilibrium for subgame $\Gamma^{[0,\tau]}$ which only uses the pooling choice rule. Let the system of beliefs and equilibrium strategies for subgame $\Gamma^{[\tau,T]}$ be $\mu^{[\tau,T]}$ and $\sigma^{*,[\tau,T]}$, respectively. Let the system of beliefs and equilibrium strategies for subgame $\Gamma^{[0,\tau]}$ be $\mu^{[0,\tau]}$ and $\sigma^{*,[0,\tau]}$, respectively. The two equilibria form a sequential equilibrium.

*Theorem 9:* If there is a $\tau$ such that

1) $\iota(\tau) = \mathbf{s}$;
2) There is a separating choice rule based sequential equilibrium $\langle\mu^{[\tau,T]},\sigma^{*,[\tau,T]}\rangle$ for subgame $\Gamma^{[\tau,T]}$. Let $e^\tau$ be $\mathbf{s}$'s equivalent price at time $\tau$;
3) There is a pooling choice rule based sequential equilibrium $\langle\mu^{[0,\tau]},\sigma^{*,[0,\tau]}\rangle$ for subgame $\Gamma^{[0,\tau]}$ such that $\mathbf{s}$ accepts $(e^\tau)_{\leftarrow \mathbf{s}}$ at time $\tau$;

then $\langle\{\mu^{[0,\tau]},\mu^{[\tau,T]}\},\{\sigma^{*,[0,\tau]},\sigma^{*,[\tau,T]}\}\rangle$ form a pure strategy sequential equilibrium.

The proof is omitted: Sequential rationality is obvious given the backward induction construction and consistency can be proved in the same way as in Theorem 4.

If there is a sequential equilibrium for such a $\tau$ value in Theorem 9, the equilibrium is unique for the specific $\tau$ given the backward induction process. Therefore, to find out a sequential equilibrium, we just need to search all the possible values of $\tau \le T$. If there is no sequential equilibrium for all values of $\tau$, we can conclude that there is no sequential equilibrium. The computational complexity of finding a sequential equilibrium for a specific value of $\tau$ is $\mathcal{O}(T)$. Thus, the computational complexity of finding a sequential equilibrium for a bilateral bargaining with two possible types of reserve price is $\mathcal{O}(T^2)$.

We show how to compute agents' equilibrium offers on the equilibrium path while using both the pooling choice rule and separating choice rule. We use the example in Fig. 2 and change the deadline to $T = 4$. Fig. 1 shows that there are three choice systems for the bilateral bargaining game. First we consider the choice system in which the pooling choice rule is used at both time $t = 0$ and $t = 2$. The optimal offer of $\mathbf{s}$ at time $t = 3$ is 100 and the equivalent price is $e^3 = 73$. At $t = 2$, both buyer type will offer $e^3_{\leftarrow \mathbf{s}} = 47.8$. At $t = 1$, $\mathbf{s}$ can offer 1) $(47.8)_{\leftarrow \mathbf{b}_h} = 68.68$, which will give $\mathbf{s}$ an expected utility with $0.7(68.68 - 10)0.6^2 + 0.3(47.8 - 10)0.6^3 = 17.2368$; 2) $(47.8)_{\leftarrow \mathbf{b}_l} = 52.68$, which will give $\mathbf{s}$ an expected utility with $(52.68 - 10)0.6^2 = 15.3648$. Therefore, the optimal offer of $\mathbf{s}$ at $t = 1$ is $(47.8)_{\leftarrow \mathbf{b}_h} = 68.68$ and the equivalent price is $e^1 = 57.88$. At $t = 0$, both buyer type will offer $e^1_{\leftarrow \mathbf{s}} = 38.728$. It's easy to see that all equilibrium existence conditions are satisfied. Thus, there is a sequential equilibrium with the choice system.

Next we consider the choice system in which the separating choice rule is used at time $t = 0$. First we assume the existence of sequential equilibrium and we have $x^*_{\mathbf{b}_h}(0) = 51.04$ and $x^*_{\mathbf{b}_l}(1) = 48$. $\mathbf{b}_h$ has no incentive to behave as $\mathbf{b}_l$ since $x^*_{\mathbf{b}_h}(0) = 51.04 < 68.8 = (x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_h}$. However, $\mathbf{b}_l$ has an incentive to behave as $\mathbf{b}_h$ since $(x^*_{\mathbf{b}_l}(1))_{\leftarrow \mathbf{b}_l} = 52.8 > 51.04 = x^*_{\mathbf{b}_h}(0)$. Therefore, there is no sequential equilibrium with this choice system.

Finally, we consider the choice system in which the pooling choice rule is used at $t = 0$ and the separating choice rule is used at $t = 2$. We first consider the subgame starting from $t = 2$, which is equivalent to the bargaining game in Fig. 2. Thus, there is a sequential equilibrium for the subgame with the separting choice rule in which $\mathbf{b}_h$'s optimal offer at time $t = 2$ is $x^*_{\mathbf{b}_h}(2) = 64$, $\mathbf{b}_l$'s optimal offer at time $t = 2$ is $\mathrm{RP}_{\mathbf{s}}$, and $\mathbf{s}$ will offer $x^*_{\mathbf{b}_l}(3) = 60$ at time $t = 3$ if it receives offer $\mathrm{RP}_{\mathbf{s}}$ at time $t = 2$. Then we consider the subgame from the beginning to time $t = 2$. At $t = 1$, $\mathbf{s}$ can offer 1) $(64)_{\leftarrow \mathbf{b}_h} = 78.4$, which will give $\mathbf{s}$ an expected utility $0.7(78.4 - 10)0.6^2 + 0.3(60 - 10)0.6^4 = 19.1808$ (note that if $\mathbf{b}$ is of type $\mathbf{b}_l$, it will offer $\mathrm{RP}_{\mathbf{s}}$ at time $t = 2$ and make an agreement with $\mathbf{s}$ at time $t = 4$); 2) $(x^*_{\mathbf{b}_l}(3))_{\leftarrow 2[\mathbf{b}_l]} = 60$, which will give $\mathbf{s}$ an expected utility $(60 - 10)0.6^2 = 18$. Therefore, the optimal offer of $\mathbf{s}$ at $t = 1$ is 78.4 and the equivalent price is $e^1 = 63.28$. At $t = 0$, both buyer types will offer $e^1_{\leftarrow \mathbf{s}} = 41.968$, which is lower than both types' reserve prices. Thus, there is a sequential equilibrium within this choice system.

## 4. Conclusion

This paper analyzes agents' rational behavior in alternating-offers bilateral bargaining with one-sided uncertainty on reserve prices. Our approach searches all choice systems and compute sequential equilibria employing a Bayesian extension of backward induction. Our approach is sound, complete, and can be applied to other uncertainty settings.

Our approach can be extended to handle the setting where there is finitely many types of reserve prices. When there are only two types and the two buyer types behave in different ways at a time point, the only choice rule is that the type with higher reserve price offers an acceptable price and the other type offer a price that will be rejected. With more types, the buyer has more options (i.e., choice rules) of differentiating its types. To find sequential equilibrium, we just need to search all possible choice systems (each specifying the buyer's choice rule at each time point when the buyer is the offering agent, i.e., whether different buyer types will behave in the same way or not) and compute agents' optimal (equilibrium) strategies for each choice system. The presence of many types increases the computational complexity of the procedure to find equilibrium strategies and requires more stringent equilibrium existence conditions.

One major motivation of the study of bargaining theory is designing successful bargaining agents in practical applications where agents often have incomplete information. One future research direction is to experimentally evaluate the performance of the derived fully rational equilibrium strategies as compared with other heuristics based negotiation strategies (e.g., negotiation decision functions [3]). Another future research direction is finding mixed equilibrium strategies for scenarios in which there is no pure strategy equilibrium. Another challenging topic is finding sequential equilibrium for bargaining with two-sided uncertainty.


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