

Maximum k -Biplex Search on Bipartite Graphs: A Symmetric-BK Branching Approach

Kaiqiang Yu
Nanyang Technological University
Singapore
kaiqiang002@e.ntu.edu.sg

Cheng Long*
Nanyang Technological University
Singapore
c.long@ntu.edu.sg

ABSTRACT

Enumerating maximal k -biplexes (MBPs) of a bipartite graph has been used for applications such as fraud detection. Nevertheless, there usually exists an exponential number of MBPs, which brings up two issues when enumerating MBPs, namely the effectiveness issue (many MBPs are of low values) and the efficiency issue (enumerating all MBPs is not affordable on large graphs). Existing proposals of tackling this problem impose constraints on the number of vertices of each MBP to be enumerated, yet they are still not sufficient (e.g., they require to specify the constraints, which is often not user-friendly, and cannot control the number of MBPs to be enumerated directly). Therefore, in this paper, we study the problem of finding K MBPs with the most edges called MaxBPs, where K is a positive integral user parameter. The new proposal well avoids the drawbacks of existing proposals (i.e., the number of MBPs to be enumerated is directly controlled and the MBPs to be enumerated tend to have high values since they have more edges than the majority of MBPs). We formally prove the NP-hardness of the problem. We then design two *branch-and-bound* algorithms, among which, the better one called FastBB improves the worst-case time complexity to $O^*(\gamma_k^n)$, where O^* suppresses the polynomials, γ_k is a real number that relies on k and is *strictly* smaller than 2, and n is the number of vertices in the graph. For example, for $k = 1$, γ_k is equal to 1.754. We further introduce three techniques for boosting the performance of the branch-and-bound algorithms, among which, the best one called PBIE can further improve the time complexity to $O^*(\gamma_k^{d^3})$ for large sparse graphs, where d is the maximum degree of the graph (note that $d \ll n$ for sparse graphs). We conduct extensive experiments on both real and synthetic datasets, and the results show that our algorithm is up to four orders of magnitude faster than all baselines and finding MaxBPs works better than finding all MBPs for a fraud detection application.

CCS CONCEPTS

• **Mathematics of computing** → Graph algorithms.

*Cheng Long is the corresponding author.

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KEYWORDS

bipartite graph; maximum k -biplex; maximum subgraph search

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1 INTRODUCTION

Bipartite graph is an important type of data structure where vertices are divided into two disjoint sets at two sides and each edge connects a vertex at one side and another at the other side. Bipartite graphs have been widely used for modeling the interactions between two types of entities in many real applications, with the entities being vertices and the interactions being edges. Some examples include commenting interactions between users and articles in social media [47], purchasing interactions between customers and products in E-commerce [33], visiting interactions between users and websites in web applications [3], etc.

A bipartite subgraph is called a k -biplex if each vertex at one side *disconnects* at most k vertices at the other side, where k is often a small positive integer [30, 44, 45]. k -biplex has been used to capture *dense/cohesive* subgraphs in a given bipartite graph for solving practical problems such as anomaly detection [11, 43], online recommendation [13, 28] and community search [14, 36]. For example, in e-commerce platforms, some agents are paid to promote the ranks of certain products by coordinating a group of fake users to post fake comments. The subgraphs induced by these fake users and the products they promote would be dense and likely to be k -biplexes [11].

Motivations. There are a few studies on the problem of enumerating *maximal k -biplexes* (MBPs) [30, 44, 45]. Nevertheless, there usually exist an exponential number of MBPs, which brings two issues of enumerating all MBPs. First, not all MBPs carry essential information (e.g., those MBPs with few vertices are often deemed not interesting [30]). Second, the process of enumerating all MBPs is costly (e.g., according to existing studies [44], enumerating all MBPs on large graphs is not affordable). To mitigate these issues, existing studies [45] impose some constraints on the number of vertices at each of the two sides of a MBP to be enumerated, e.g., they enumerate only MBPs with *at least* a certain number vertices at each side (which we call large MBPs). While this strategy makes it possible to control the number of MBPs to be enumerated, it achieves the goal only in an indirect way and introduces an additional issue of requiring to set proper thresholds of the number of vertices. In cases where users have no prior knowledge about

the thresholds, they would find it not user-friendly. They can try different thresholds, but then the enumeration processes would run multiple times and bring up the time costs.

Motivated by these issues, in this paper, we study the problem of finding K MBPs with the most edges among large MBPs, where K is an integral parameter. We call each of these MBPs a *maximum k -biplex* (MaxBP). Compared with existing studies on k -biplexes, the problem of finding MaxBPs enjoys three advantages. First, each MaxBP to be found has more edges than those that are not returned, and thus the found MBP is of more significance. In our experiments, we verify this via a case study, which shows that a method based on MaxBPs provides F1 score up to 0.99 for a fraud detection task. Second, it provides a *direct* control on the number of MBPs to be found without the need of making multiple trials of enumeration. Third, compared with alternative proposals, e.g., finding the *first* K MBPs (as existing studies [45] do) or finding K MBPs with the most vertices, our solution to the problem of finding MaxBPs would return MBPs that are more balanced, which are deemed to be superior over imbalanced structures [23].

Baseline Methods. Given the fact that MaxBPs are the MBPs with the most edges, we can adapt the existing algorithms of enumerating large MBPs, namely iMB [44] and iTraversal [45], to find the MaxBPs and yield the following two baseline methods. The first one called iMBadp adapts iMB [44] (a branch-and-bound algorithm) by incorporating additional pruning techniques which prune those branches that cannot hold any MBP with more edges than the K MaxBPs found so far. The second baseline called iTradp simply runs iTraversal [45] (a reverse search based algorithm) and returns K MaxBPs since iTraversal cannot be equipped with additional pruning techniques easily. Nevertheless, iMBadp has its efficiency highly rely on the pruning techniques and iTradp needs to explore all large MBPs, which is time-consuming. Both of them have the worst-case time complexity of $O^*(2^n)$, where n is the number of vertices in the given bipartite graph and O^* suppresses polynomials. Furthermore, we can adapt those existing algorithms of enumerating large maximal k -plexes, which are counterparts of MBPs on general graphs. Here, a k -plex is a subgraph with each vertex disconnecting at most k vertices in the subgraph. Therefore, the third baseline method called FPAdp adapts FaPlexen [51] (the algorithm for enumerating large maximal k -plexes with the number of vertices at least a threshold) with some additional pruning techniques that are tailored for MBPs. Still, FPAdp is inferior to the new algorithms we will introduce in this paper both theoretically and empirically.

New Methods. We first introduce a *branch-and-bound* algorithm called BasicBB, which is based on a conventional and widely-used branching strategy that we call *Bron-Kerbosch (BK) branching* [4]. The BK branching recursively partitions the search space (i.e., the set of all possible MBPs) to multiple sub-spaces via *branching*. BasicBB has the worst-case time complexity $O(n \cdot d \cdot 2^n)$ (i.e., $O^*(2^n)$), where d is the maximum degree of the graph. This time complexity is the same as those of the baseline methods. We then introduce a new branching strategy called *Symmetric-BK (Sym-BK) branching*, which is symmetric to the BK branching but better suits our problem of finding MaxBPs. We further present our method for determining an ordering of vertices, which is critical for Sym-BK

branching. We finally introduce a new branch-and-bound algorithm called FastBB, which is based on the Sym-BK branching. FastBB has its worst-case time complexity $O(n \cdot d \cdot \gamma_k^n)$ (i.e., $O^*(\gamma_k^n)$), where γ_k is *strictly* smaller than 2 and depends on the setting of k . For example, when $k = 1$, γ_k is 1.754. This is a remarkable theoretical improvement over the prior solutions given that many existing algorithms of enumerating subgraphs are based on BK branching and have the worst-case time complexity of $O^*(2^n)$ [30, 39, 44].

In addition, we adapt two existing techniques for boosting the efficiency and scalability of the branch-and-bound (BB) algorithms including BasicBB and FastBB. They share the idea of constructing *multiple* problem instances of finding MaxBPs each on a smaller subgraph. Specifically, the first technique, called *progressive bounding* (PB), is adapted from an existing study of finding the biclique with the most edges [23]. The second technique, called *inclusion-exclusion* (IE), is adapted from the *decomposition* technique, which has been widely used for enumerating and finding subgraph structures [5, 7, 40]. PB improves the practical performance of a BB algorithm only while IE improves both the theoretical time complexity (for certain sparse graphs) and the practical performance. We then present PBIE, which combines PB and IE naturally. PBIE enjoys the benefits of both PB and IE. We note that all these techniques are orthogonal to the BB algorithms, i.e., any of these techniques, namely PB, IE, and PBIE, can be used to boost the efficiency and/or scalability of BasicBB and FastBB. To the best of our knowledge, this is first time that PB and IE are combined naturally.

Contributions. Our major contributions are summarized below.

- We study the problem of finding MaxBPs, and formally prove the NP-hardness of the problem.
- We propose an efficient branch-and-bound algorithm, called FastBB, which is based on a novel Sym-BK branching strategy. In particular, FastBB achieves the state-of-the-art worst-case time complexity $O(n \cdot d \cdot \gamma_k^n)$ with $\gamma_k < 2$.
- We further introduce a combined framework, called PBIE, to further boost the performance of FastBB. PBIE combines two adapted frameworks, namely the progressive bounding framework PB and the inclusion-exclusion based framework IE. When PBIE is used with FastBB, the worst-time time complexity becomes $O(d^4 \cdot \gamma_k^{d^3})$. Note that this is better than that of FastBB on certain graphs (e.g., those sparse graphs with $d \ll n$).
- We conduct extensive experiments using both real and synthetic datasets, and the results show that (1) the proposed algorithms are up to four orders of magnitude faster than all baselines and (2) finding MaxBPs work better in a fraud detection task than enumerating MBPs.

Roadmap. The rest of this paper is organized below. Section 2 defines the problem and shows its NP-hardness. Section 3 presents the branch-and-bound algorithms BasicBB and FastBB. Section 4 presents the frameworks PB, IE and PBIE. We conduct extensive experiments in Section 5. Section 6 reviews the related work and Section 7 concludes the paper.

2 PROBLEM DEFINITION

Let $G = (L \cup R, E)$ be an undirected and unweighted bipartite graph, where L and R are two disjoint vertex sets and E is an edge set. For the graph G , we use $V(G)$, $L(G)$, $R(G)$, and $E(G)$ to denote its set

of vertices, left side, right side and set of edges, respectively, i.e., $V(G) = L \cup R$, $L(G) = L$, $R(G) = R$, and $E(G) = E$. Given $X \subseteq L$ and $Y \subseteq R$, we use $G[X \cup Y]$ to denote the induced (bipartite) graph of G , i.e., $G[X \cup Y]$ includes the set of vertices $X \cup Y$ and the set of edges between X and Y . All subgraphs considered in this paper are induced subgraphs. We use H or (X, Y) as a shorthand of $H = G[X \cup Y]$.

Given $v \in L$, we use $\Gamma(v, R)$ (resp. $\bar{\Gamma}(v, R)$) to denote the set of neighbours (resp. non-neighbours) of v in R , i.e., $\Gamma(v, R) = \{u \mid (v, u) \in E \text{ and } u \in R\}$ (resp. $\bar{\Gamma}(v, R) = \{u \mid (v, u) \notin E \text{ and } u \in R\}$). We define $\delta(v, R) = |\Gamma(v, R)|$ and $\bar{\delta}(v, R) = |\bar{\Gamma}(v, R)|$. We use d to denote the maximum degree of vertex in G . We have symmetric definitions for each vertex $u \in R$. Next, we review the definition of k -biplex [44].

DEFINITION 1 (k -BIPLEX [44]). Given a graph $G = (L, R, E)$, a positive integer k , $X \subseteq L$ and $Y \subseteq R$, a subgraph $G[X \cup Y]$ is said to be a k -biplex if $\bar{\delta}(v, Y) \leq k$, $\forall v \in X$ and $\delta(u, X) \leq k$, $\forall u \in Y$.

A k -biplex H is said to be maximal if there is no other k -biplex H' containing H , i.e., $V(H) \subseteq V(H')$. Large real graphs usually involve numerous maximal k -biplexes and most of them highly overlap. In this paper, we aim to find K maximal k -biplexes with the most edges, where K is a positive integral user-parameter. In addition, we consider two size constraints θ_L and θ_R on each maximal k -biplex H to be found, namely $|L(H)| \geq \theta_L$ and $|R(H)| \geq \theta_R$. These constraints would help to filter out some skewed maximal k -biplexes, i.e., the number of vertices at one side is extremely larger than that at the other side. To guarantee that all found maximal k -biplexes are connected, we further require $\theta_L \geq 2k + 1$ and $\theta_R \geq 2k + 1$ based on the following lemma.

LEMMA 1. A k -biplex H is connected if $|L(H)| \geq 2k + 1$ and $|R(H)| \geq 2k + 1$.

PROOF. This can be proved by contradiction. Suppose H is not connected and is partitioned into two connected components, namely H_1 and H_2 . We derive the contradiction by showing that H is not a k -biplex: for a vertex v_1 in $L(H_1)$, it disconnects more than k vertices, i.e., $\bar{\delta}(v_1, R(H)) \geq |R(H_2)| \geq k + 1$. Specifically, we derive $\bar{\delta}(v_1, R(H)) \geq |R(H_2)|$ since v_1 from H_1 disconnects all vertices in H_2 based on the assumption, and we derive $|R(H_2)| \geq k + 1$ by (1) for a vertex v_2 in L_2 , $|R(H_2)| \geq \delta(v_2, R(H))$ since all neighbours of v_2 within H reside in $R(H_2)$ based on the assumption and (2) $\delta(v_2, R(H)) \geq R(H) - k \geq 2k + 1 - k \geq k + 1$ since v_2 disconnects at most k vertices (H is a k -biplex) and $R(H) \geq 2k + 1$. \square

We formalize the problem studied in this paper as follows.

PROBLEM 1 (MAXIMUM k -BIPLEX SEARCH). Given a bipartite graph $G = (L \cup R, E)$, four positive integers $K > 0$, $k > 0$, $\theta_L \geq 2k + 1$ and $\theta_R \geq 2k + 1$, the maximum k -biplex search problem aims to find K maximal k -biplexes such that each found maximal k -biplex H satisfies that $|L(H)| \geq \theta_L$, $|R(H)| \geq \theta_R$ and $|E(H)|$ is larger than $|E(H')|$ for any other maximal k -biplex H' that is not returned.

In this paper, we use MBP and MaxBP as a shorthand of a maximal k -biplex and one of the K maximal k -biplexes with the most edges, respectively.

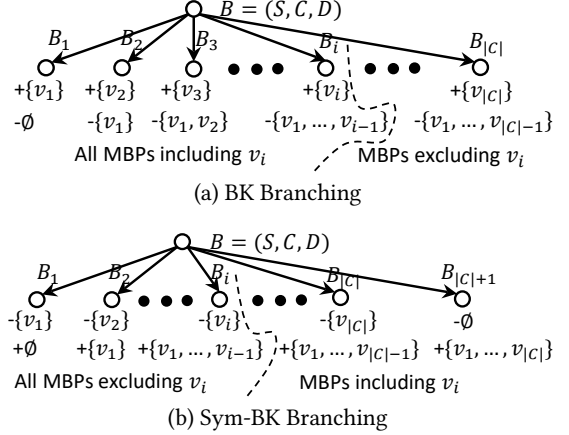


Figure 1: Illustration of two branching strategies. Each node denotes a branch (S, C, D) . The notation “+” means to include a vertex by adding it S and “-” means to exclude a vertex by adding it to D .

NP-hardness. The maximum k -biplex search problem is NP-hard, which we present in the following lemma (with its proof provided in Section 8).

LEMMA 2. The maximum k -biplex search problem is NP-hard.

Remarks. In the following sections (Section 3 and Section 4), we focus on the setting of $K = 1$ (i.e., the problem becomes to find a MBP with the maximum number of edges) when presenting the algorithms for ease of presentation. We note that these algorithms can be naturally extended for general settings of K with minimal efforts (i.e., we maintain K MaxBPs instead of 1 MaxBP found so far throughout the algorithm for pruning) and are tested in Section 5.

3 BRANCH-AND-BOUND ALGORITHMS

We first introduce a *branch-and-bound* algorithm called BasicBB, which is based on a conventional and widely-used branching strategy that we call *Bron-Kerbosch (BK) branching* [4], in Section 3.1. BasicBB has the worst-case time complexity $O(|V| \cdot d \cdot 2^{|V|})$. We then introduce a new branching strategy called *Symmetric-BK (Sym-BK) branching*, which is symmetric to the BK branching but better suits our problem of finding MaxBP, in Section 3.2. We further present our method for determining an ordering of vertices, which is critical for Sym-BK branching, in Section 3.3. We finally introduce a new branch-and-bound algorithm called FastBB, which is based on Sym-BK branching, and analyze its time complexity in Section 3.4. FastBB has its worst-case time complexity $O(|V| \cdot d \cdot \gamma_k^{|V|})$, where γ_k is strictly smaller than 2 and depends on the setting k . For example, when $k = 1$, γ_k is 1.754.

3.1 A BK Branching based Branch-and-Bound Algorithm: BasicBB

Our first attempt is to adapt the seminal *Bron-Kerbosch (BK) algorithm* [4]. It recursively partitions the search space (i.e., the set of all possible MBPs) to multiple sub-spaces via *branching*. Specifically, each sub-space is represented by a triplet of three sets (S, C, D) as explained below.

- **Partial set** S . Set of the vertices that *must* be included in every MBP within the space.
- **Candidate set** C . Set of the vertices that *can* be included in S in order to form a MBP within the space.
- **Exclusion set** D . Set of vertices that *must not* be included in any MBP within the space.

We further denote by S_L , C_L , and D_L the left side of S , C , and D , respectively, and define S_R , C_R , and D_R similarly for the right side.

The recursive process of the BK algorithm starts from the full search space with $S = \emptyset$, $C = V$, and $D = \emptyset$. Consider the branching step at a current branch $B = (S, C, D)$. Let $\langle v_1, v_2, \dots, v_{|C|} \rangle$ be a sequence of the vertices in C . The branching step would partition the space to $|C|$ sub-spaces (and correspondingly $|C|$ branches), where the i^{th} branch, denoted by $B_i = (S_i, C_i, D_i)$, includes S and v_i and excludes v_1, v_2, \dots, v_{i-1} . Formally, for $1 \leq i \leq |C|$, we have

$$S_i = S \cup \{v_i\}; D_i = D \cup \{v_1, v_2, \dots, v_{i-1}\}; C_i = C - \{v_1, v_2, \dots, v_i\} \quad (1)$$

For illustration, consider Figure 1(a). Note that C_i and D_i can be further refined by removing those vertices that cannot be included to $G[S_i]$ to form a k -biplex.

We call the above branching strategy *BK branching*. BK branching essentially corresponds to a recursive *binary* branching process. It first splits the current branch into two branches, one including v_1 (this is the branch B_1) and the other excluding v_1 . Then, it further splits the latter into two branches, one including v_2 (this is the branch B_2) and the other excluding v_2 . It continues the process until the last branch, which excludes $v_1, v_2, \dots, v_{|C|-1}$ and includes $v_{|C|}$ (this corresponds to the branch $B_{|C|}$), is formed. In particular, the branches B_1, B_2, \dots, B_i cover all MBPs including v_i and branches $B_{i+1}, \dots, B_{|C|}$ cover those excluding v_i , as indicated by the dashed line in Figure 1(a).

We note that BK branching relies on an ordering of vertices in the candidate set C , i.e., $\langle v_1, v_2, \dots, v_{|C|} \rangle$, for producing branches. In this paper, we follow the existing studies [1, 48] and use the *non-decreasing vertex degree ordering* (where vertices are ranked in a non-decreasing order of their degrees in $S \cup C$, i.e., $\delta(v_i, S \cup C) \leq \delta(v_j, S \cup C)$ for any $i < j$) since this would help with effective pruning as shown empirically. We note that normally the ordering does not affect the worst-case theoretical time complexity of the algorithm based on BK branching.

During the recursive search process, some pruning techniques can be applied. Let $B = (S, C, D)$ be a branch. First, branch B can be pruned if S is not a k -biplex since (1) each partial set in the search space corresponding to this branch is a *superset* of S and (2) based on the hereditary property of k -biplex, any superset of a non- k -biplex is not a k -biplex. Second, branch B can be pruned if an upper bound of the left side (resp. the right side) of a k -biplex in the space is smaller than θ_L (resp. θ_R) based on the problem definition. Third, branch B can be pruned if an upper bound of the number of edges in a k -biplex in the space is smaller than the largest one of a k -biplex known so far. Fourth, branch B can be pruned if there exists a vertex in D such that including this vertex to each k -biplex in the space would still result in a k -biplex. We will elaborate on these pruning rules in detail in Section 3.4. Finally, the recursive process of the BK algorithm terminates at a branch $B = (S, C, D)$ if $G[S \cup C]$ is a k -biplex since $G[S \cup C]$ would be the MaxBP within the space of the branch.

We call this BK algorithm, which is a *branch-and-bound* algorithm based on BK branching and the aforementioned four pruning techniques, *BasicBB*, and present its pseudo-code in Algorithm 1. Similar to many existing algorithms that are based on BK branching, the worst-case time complexity of *BasicBB* is $O(|V| \cdot d \cdot 2^{|V|})$ (i.e., $O^*(2^{|V|})$) [30, 44], though its practical performance can be boosted by the pruning techniques.

Algorithm 1: The branch-and-bound algorithm based on BK branching: *BasicBB*

Input: A graph $G(L \cup R, E)$, k , θ_L and θ_R
Output: The maximal k -biplex H^* with the most edges

```

1  $H^* \leftarrow G[\emptyset]$ ; // Global variable
2 BasicBB-Rec( $\emptyset, L \cup R, \emptyset$ ); return  $H^*$ ;
3 Procedure BasicBB-Rec( $S, C, D$ )
   /* Termination */
4   if  $G[S \cup C]$  is  $k$ -biplex then
5      $H^* \leftarrow G[S \cup C]$  if  $|E[G[S \cup C]]| > |E(H^*)|$ ; return;
   /* Pruning */
6   if any of pruning conditions is satisfied (details in Section 3.4)
     then return;
   /* BK Branching */
7   Create  $|C|$  branches  $B_i = (S_i, C_i, D_i)$  based on Equation (1);
8   for each branch  $B_i$  do
9      $\lfloor$  BasicBB-Rec( $S_i, C_i, D_i$ );

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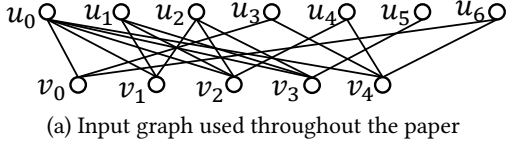
3.2 A New Branching Strategy: Sym-BK Branching

We observe that there exists a branching strategy, which is natural and *symmetric* to BK branching. Specifically, consider the branching step at a current branch $B = (S, C, D)$. Let $\langle v_1, v_2, \dots, v_{|C|} \rangle$ be a sequence of the vertices in C . This branching step would partition the space to $(|C| + 1)$ sub-spaces (and correspondingly $(|C| + 1)$ branches), where the i^{th} branch, denoted by $B_i = (S_i, C_i, D_i)$, includes S and v_1, v_2, \dots, v_{i-1} and excludes v_i . Here, v_0 and $v_{|C|+1}$ both correspond to null. Formally, for $1 \leq i \leq |C| + 1$, we have

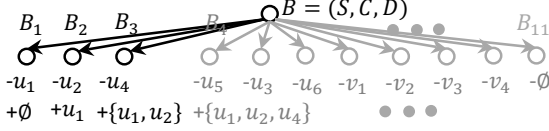
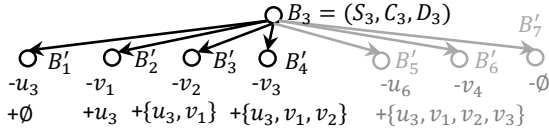
$$S_i = S \cup \{v_1, v_2, \dots, v_{i-1}\}; D_i = D \cup \{v_i\}; C_i = C - \{v_1, v_2, \dots, v_i\} \quad (2)$$

For illustration, consider Figure 1(b). Note that C_i and D_i can be further refined by removing those vertices that cannot be included to $G[S_i]$ to form a k -biplex.

We call the above branching strategy *symmetric-BK (Sym-BK) branching*. Sym-BK branching corresponds to another recursive *binary* branching process, which is symmetric to that of the BK branching. Specifically, it first splits the current branch into two branches, one excluding v_1 (this is the branch B_1) and the other including v_1 . Then, it further splits the latter into two branches, one excluding v_2 (this is the branch B_2) and the other including v_2 . It continues the process until the last branch, which includes $v_1, v_2, \dots, v_{|C|}$ (this corresponds to the branch $B_{|C|+1}$), is formed. In particular, the branches B_1, B_2, \dots, B_i cover all MBPs *excluding* v_i and branches $B_{i+1}, \dots, B_{|C|+1}$ cover those *including* v_i , as indicated by the dashed line in Figure 1(b).



(a) Input graph used throughout the paper

(b) Case 1 ($S = \{u_0, v_0\}$, $C = \{u_1, u_2, u_4, u_5, u_3, u_6, v_1, v_2, v_3, v_4\}$ and $D = \emptyset$)(c) Case 2 ($S_3 = \{u_0, v_0, u_1, u_2\}$, $C_3 = \{u_3, v_1, v_2, v_3, u_6, v_4\}$ and $D_3 = \{u_4, u_5\}$)**Figure 2: Illustration of Sym-BK branching ($k = 2$).**

Sym-BK branching vs. BK branching. They are symmetric to each other and both of them are natural branching strategies. They differ in that among two branches formed by a binary branching based on a vertex, BK branching recursively partitions the branch *excluding* the vertex while Sym-BK recursively partitions one more branch than BK branching at each branching step (i.e., $|C| + 1$ branches vs. $|C|$ branches), but this difference of one extra branch is negligible given that there can be many branches produced at the branching step. Compared with BK branching, Sym-BK branching has the following advantages when adopted for our problem of finding the MaxBP.

First, at each branching step, it would produce branches with *bigger* partial sets S_i (note that the i^{th} branch by Sym-BK branching involves $|S| + (i - 1)$ vertices in the partial set while that by BK branching involves $|S| + 1$ vertices). Consequently, the produced branch would have a larger chance to be pruned due to the hereditary property of k -biplex (if a set of vertices is not a k -biplex, then none of its supersets is, but not vice versa). **Second**, the partial set of the j^{th} branch, i.e., S_j , is always a superset of that of the i^{th} branch, i.e., S_i , for any $j > i$. Consequently, if S_i is not a k -biplex (which means the branch B_i can be pruned), then all branches following B_i can be pruned (since their partial sets are supersets of S_i and thus they are not k -biplexes either based on the hereditary property).

To illustrate, consider the example in Figure 2. One branching step of Sym-BK branching is shown in Figure 2(b). The fourth branch has the partial set of $S_4 = \{u_0, v_0, u_1, u_2, u_4\}$, which is not a k -biplex. All the following branches have their partial sets as supersets of S_4 , and thus they can be pruned immediately (as indicated by the shaded color in the figure).

In fact, with Sym-BK branching and a carefully-designed ordering of vertices (details will be introduced in Section 3.3), our new branch-and-bound algorithm FastBB would have the worst-case time complexity of $O(|V| \cdot d \cdot \gamma_k^{|V|})$ with $\gamma_k < 2$, which is strictly smaller than that of the Bas3CBB algorithm based on BK branching (details will be introduced in Section 3.4).

Remarks. In [51], the authors design a branching strategy, which performs one of two branching operations at a branch depending on the situation: (1) generating two branches based on a *single* vertex in C (i.e., one including and the other not including this vertex) and (2) generating at most $(|C| + 1)$ branches based on all vertices in C with pruning. We call this branching strategy *hybrid branching*. Our Sym-BK branching is superior over hybrid branching for our problem in two aspects. First, Sym-BK branching has a more simplified form. Second, the branch-and-bound algorithm based on Sym-BK branching has lower worst-case time complexity *theoretically* (details are in Section 3.4) and runs faster *empirically* (details are in Section 5).

3.3 Sym-BK Branching: Ordering of Vertices

The Sym-BK branching relies on an ordering of the vertices in C . Recall that in the branches $B_1, B_2, \dots, B_{|C|+1}$ generated by Sym-BK branching, the partial set of a branch B_j is always a superset of that of a preceding branch B_i ($i < j$). Therefore, our idea is to figure out a small subset C' of vertices in C such that including them *collectively* to the partial set S would violate the k -biplex definition. We then put these vertices *before* other vertices in the ordering. In this way, the branch with the partial set of $S \cup C'$ and all the following branches can be pruned directly. We elaborate on this idea in detail next.

We notice that $G[S \cup C]$ is not a k -biplex since otherwise the recursion would terminate at this branch. It means that there exists at least a vertex in $S \cup C$, which has more than k disconnections within $G[S \cup C]$. Without loss of generality, we assume that the vertex is from the left side and denote it by $\hat{v} \in S_L \cup C_L$. Consider the set of vertices that disconnect \hat{v} in C_R , i.e., $\bar{\Gamma}(\hat{v}, C_R)$. We know that including $\bar{\Gamma}(\hat{v}, C_R)$ to S collectively would violate the k -biplex definition. Specifically, if \hat{v} is already in S_L , i.e., $\hat{v} \in S_L$, we can include at most $k - \bar{\delta}(\hat{v}, S_R)$ vertices from $\bar{\Gamma}(\hat{v}, C_R)$ to S without violating the k -biplex definition; and if \hat{v} is not yet in S_L , i.e., $\hat{v} \in C_L$, we can include \hat{v} together with at most $k - \bar{\delta}(\hat{v}, S_R)$ vertices from $\bar{\Gamma}(\hat{v}, C_R)$ to S without violating the k -biplex definition. For the simplicity of notations, we define

$$a = k - \bar{\delta}(\hat{v}, S_R); \quad b = \bar{\delta}(\hat{v}, C_R). \quad (3)$$

Intuitively, a means the greatest possible number of disconnections that \hat{v} can have when including more vertices from C to S for forming MBPs. Note that we have $0 \leq a \leq k$ (since S is a k -biplex and thus $\bar{\delta}(\hat{v}, S_R) \leq k$) and $a < b$ (since $b - a = \bar{\delta}(\hat{v}, S_R \cup C_R) - k > 0$). Based on \hat{v} , we define an ordering of the vertices in C .

Case 1: $\hat{v} \in S_L$. In this case, we define the ordering as follows.

$$\langle u_1, u_2, \dots, u_b, u_{b+1}, \dots, u_{|C|} \rangle, \quad (4)$$

where u_1, u_2, \dots, u_b are vertices from $\bar{\Gamma}(\hat{v}, C_R)$ in any order and $u_{b+1}, u_{b+2}, \dots, u_{|C|}$ are vertices from $C - \bar{\Gamma}(\hat{v}, C_R)$ in any order. Based on this ordering, the branch B_{a+2} would have the partial set $S \cup \{u_1, u_2, \dots, u_{a+1}\}$, which is not a k -biplex (since \hat{v} would have more than k disconnections). Therefore, branches $B_{a+2}, B_{a+3}, \dots, B_{|C|+1}$ can be pruned and only the first $(a + 1)$ branches, namely B_1, B_2, \dots, B_{a+1} , would be kept. To illustrate, we consider a branch with $S = \{u_0, v_0\}$, $D = \emptyset$ and $C = \{u_1, u_2, u_3, u_4, u_5, u_6, v_1, v_2, v_3, v_4\}$ for finding a MaxBP with $k = 2$ from the input graph in Figure 2(a).

Based on v_0 in S that disconnects 3 vertices, i.e., $\{u_1, u_2, u_4\}$, we define the ordering $\langle u_1, u_2, u_4, u_5, u_3, u_6, v_1, v_2, v_3, v_4 \rangle$ for Sym-BK branching as shown in Figure 2(b). The branches B_4, \dots, B_{11} can be pruned since B_4 has the partial set $\{u_0, u_1, u_2, u_4, v_0\}$ not a k -biplex (v_0 has more than $k = 2$ disconnections).

Case 2: $\hat{v} \in C_L$. In this case, we define the ordering as follows.

$$\langle \hat{v}, u_1, u_2, \dots, u_b, u_{b+2}, \dots, u_{|C|} \rangle, \quad (5)$$

where u_1, u_2, \dots, u_b are vertices from $\bar{\Gamma}(\hat{v}, C_R)$ in any order and $u_{b+2}, u_{b+3}, \dots, u_{|C|}$ are vertices from $C - \bar{\Gamma}(\hat{v}, C_R) - \{\hat{v}\}$ in any order. Based on this ordering, the branch B_{a+3} would have the partial set as $S \cup \{\hat{v}, u_1, u_2, \dots, u_{a+1}\}$, which is not a k -biplex (since \hat{v} would have more than k disconnections). Therefore, branches $B_{a+3}, B_{a+4}, \dots, B_{|C|+1}$ can be pruned and only the first $(a + 2)$ branches, namely B_1, B_2, \dots, B_{a+2} , would be kept. To illustrate, we consider another branch B_3 with $S_3 = \{u_0, u_1, u_2, v_0\}$, $C_3 = \{u_3, u_6, v_1, v_2, v_3, v_4\}$ and $D_3 = \{u_4, u_5\}$ in Figure 2(c). Based on u_3 in C that disconnects 3 vertices, i.e., $\{v_1, v_2, v_3\}$, we define the ordering $\langle u_3, v_1, v_2, v_3, u_6, v_4 \rangle$ for Sym-BK branching as shown in Figure 2(c). The branches B'_5, B'_6 and B'_7 can be pruned since B'_5 has the partial set as $\{u_0, u_1, u_2, u_3, v_0, v_1, v_2, v_3\}$ not a k -biplex (u_3 has more than $k = 2$ disconnections).

We note that there could be multiple vertices, which have more than k disconnections among $S \cup R$, and for each of them, we can define an ordering as above. We call these vertices *candidate pivots* and the vertex that we pick for defining an ordering the *pivot*. An immediate question is: *which one should we select as the pivot among the candidate pivots?* To answer this question, we quantify the benefits of specifying the ordering based on a specific candidate pivot \hat{v} . There are two benefits (for simplicity, we discuss the case of $\hat{v} \in S_L$ only, and the other case is similar and thus omitted). **Benefit 1:** $(|C| - a)$ branches, namely $B_{a+2}, B_{a+3}, \dots, B_{|C|+1}$, are pruned for \hat{v} . Therefore, the smaller a is, the larger the Benefit 1 is. **Benefit 2:** For Branch B_{a+1} , we have $|C_{a+1}| \leq |C| - b$ since (1) C_{a+1} is updated to be $C - \{u_1, u_2, \dots, u_{a+1}\}$ (please refer to Equation (4)) and (2) the vertices $u_{a+2}, u_{a+3}, \dots, u_b$ can be further excluded from C_{a+1} since including each of these vertices to S_{a+1} would violate the k -biplex definition. Therefore, the larger b is, the larger the Benefit 2 is. In summary, for a vertex with a *smaller* a and/or a *larger* b , the overall benefits would be more significant. Therefore, we select the candidate pivot \hat{v} with the largest $(b - a) = \bar{\delta}(\hat{v}, S_R \cup C_R) - k$ as the pivot. Equivalently, it would select the candidate pivot with the most disconnections within $S \cup C$. Furthermore, to achieve a better worst-case time complexity (details will be introduced in Section 3.4), we first select the pivot among the pivot candidates in S if possible; otherwise, we select one among those in C .

To illustrate, we consider again the example in Figure 2. For a branch B in Figure 2(b), v_0 would be selected as the pivot since v_0 in S has the number of disconnections more than k and the greatest among other vertices in S . For another branch B_3 in Figure 2(c), u_3 would be selected as the pivot since (1) every vertex in S disconnects less than k vertices and (2) u_3 has the number of disconnections more than k and the greatest among other vertices in C .

Remarks. We remark that the branch-and-bound method in [51] selects the pivot with the smallest degree among all candidate pivots in $S \cup C$, from which our strategy differs in two aspects. First, we

select the pivot with the most disconnections as discussed above. Note that the pivot with the most disconnections could be different from that with the smallest degree in bipartite graphs. Second, we prioritize S for selecting a pivot. The intuition behind is that (1) the worst-case running time of Case 1 (i.e., branching based on a pivot from S) is strictly smaller than that of Case 2 and (2) it would help to further improve the time complexity of Case 2 (details are in the proof of Theorem 1 in Section 3.4).

3.4 A Sym-BK Branching based Branch-and-Bound Algorithm: FastBB

Based on the Sym-BK branching strategy and the aforementioned pruning techniques (details will be presented in this section), we design a branch-and-bound algorithm, called FastBB. The pseudo-code of FastBB is presented in Algorithm 2. FastBB differs from BasicBB only in the branching step (i.e., Lines 7 - 10 of Algorithm 2). Next, we elaborate on the pruning conditions (Line 6 in Algorithm 2) in detail.

Pruning conditions. Let $B = (S, C, D)$ be a branch. We first define $\tau_L = \min_{u \in S_R} \delta(u, S_L \cup C_L) + k$ and $\tau_R = \min_{v \in S_L} \delta(v, S_R \cup C_R) + k$, which can be verified to be the upper bound of the number of vertices at the left side and that at the right side of a MBP covered by the branch B , respectively. We can prune the branch B if any of the following four conditions is satisfied.

- (1) S is not a k -biplex.
- (2) $\tau_L < \theta_L$ or $\tau_R < \theta_R$.
- (3) $|E(G[S \cup C])| \leq |E(H^*)|$ or $\tau_L \times \tau_R \leq |E(H^*)|$, where H^* is the MaxBP found so far.
- (4) There exists a vertex $v \in D_L$ such that $\bar{\delta}(v, S_R \cup C_R) \leq k$ and $\{w \in S_R \cup C_R \mid \bar{\delta}(w, S_L \cup C_L) \geq k\} \subseteq \Gamma(v, R)$ or symmetrically there exists such a vertex $u \in D_R$.

Condition (1) holds because of the hereditary property of k -biplex, Condition (2) is based on the size constraints of the two sides of MaxBP to be found, Condition (3) is based on the objective of the problem (i.e., to maximize the number of edges in a MBP), and Condition (4) holds because all k -biplexes covered by this branch (if any) would not be maximal (since an additional vertex v or u can be included in each of them without violating the k -biplex definition).

Worst-case time complexity. The worst-case time complexity of FastBB is strictly better than that of BasicBB, which we show in the following theorem.

THEOREM 1. *Given a bipartite graph G , FastBB finds the MaxBP in time $O(|V| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+4} - 2x^{k+3} + x^2 - x + 1 = 0$. For example, when $k = 1, 2$ and 3 , $\gamma_k = 1.754, 1.888$ and 1.947 , respectively.*

PROOF. We give a sketch of the proof and put the details in the appendix. We recursively maintain two arrays to record the degree of each vertex v within $G[S]$ or $G[S \cup C]$, i.e., $\delta(v, S)$ or $\delta(v, S \cup C)$. Then, the recursion of FastBB-Rec runs in polynomial time $O(|V| \cdot d)$. Specifically, the time cost is dominated by the part of checking pruning condition (4) in line 6. This part has two steps, namely finding all those vertices with at least k disconnections from $S \cup C$ in $O(|S \cup C|)$ time and checking the pruning condition

Algorithm 2: The branch-and-bound algorithm based on Sym-BK branching: FastBB

Input: A graph $G(L \cup R, E)$, k , θ_L and θ_R
Output: The maximal k -biplex H^* with the most edges

```

1  $H^* \leftarrow G[\emptyset]$ ; // Global variable
2 FastBB-Rec( $\emptyset, L \cup R, \emptyset$ ); return  $H^*$ ;
3 Procedure FastBB-Rec( $S, C, D$ )
   /* Termination */
4   if  $G[S \cup C]$  is a  $k$ -biplex then
5      $H^* \leftarrow G[S \cup C]$  if  $|E(G[S \cup C])| > |E(H^*)|$  and return
   /* Pruning */
6   if any of pruning conditions is satisfied then return ;
   /* Sym-BK Branching */
7   Select a pivot vertex  $\hat{v}$  and determine an ordering based on  $\hat{v}$ 
   (Section 3.3);
8   if  $\hat{v} \in S$  then Create  $a + 1$  branches  $\{B_1, B_2, \dots, B_{a+1}\}$ 
   (Equation (2) and (4));
9   else if  $\hat{v} \in C$  then
10    Create  $a + 2$  branches  $\{B_1, B_2, \dots, B_{a+2}\}$  (Equation (2) and
   (5));
11  for each created branch  $B_i$  do
12    FastBB-Rec( $S_i, C_i, D_i$ )

```

(4) for each vertex in D in $O(|D| \cdot d)$ time, where $|S \cup C|$ and $|D|$ are both bounded by $O(|V|)$.

Next, we analyze the number of recursions. Let $T(n)$ be the largest number of recursions where $n = |C|$. We have two cases.

Case 1 ($\hat{v} \in S$). We remove i vertices from C in B_i ($1 \leq i \leq a$) and b vertices from C in B_{a+1} in the worst-case. Hence, we have

$$T_1(n) \leq \sum_{i=1}^a T_1(n-i) + T_1(n-b). \quad (6)$$

As discussed earlier, we have $a \leq k$ and $a < b$. It is easy to verify that we reach the maximum of $T_1(n)$ when $a = k$ and $b = k + 1$. We thus have $T_1(n) \leq \sum_{i=1}^k T_1(n-i) + T_1(n-k-1)$. By solving this linear recurrence, the worst-case running time is $O(|v| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+2} - 2x^{k+1} + 1 = 0$. For example, $\gamma_k = 1.618, 1.839$ and 1.928 when $k = 1, 2$ and 3 , respectively.

Case 2 ($\hat{v} \in C$). We remove i vertices from C in B_i ($1 \leq i \leq a+1$) and $b+1$ vertices from C in B_{a+2} in the worst-case. Hence, we have

$$T_2(n) \leq \sum_{i=1}^{a+1} T_2(n-i) + T_2(n-b-1). \quad (7)$$

Assume $\hat{v} \in C_L$, we consider two scenarios, i.e., $\bar{\delta}(\hat{v}, S_R \cup C_R) \geq k+2$ and $\bar{\delta}(\hat{v}, S_R \cup C_R) = k+1$.

- For Scenario 1, we can imply $a \leq k$ and $b > a+1$. When $a = k$ and $b = k+2$, $T_2(n)$ reaches the maximum. Hence, the worst-case running time is $O(|V| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+4} - 2x^{k+3} + x^2 - x + 1 = 0$. For example, when $k = 1, 2$ and 3 , we have $\gamma_k = 1.754, 1.888$ and 1.947 , respectively.
- For Scenario 2, we can imply $a \leq k$ and $b > a$. When $a = k$ and $b = k+1$, $T_2(n)$ reaches the maximum. Thus the worst-case running time is $O(|V| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive

real root of $x^{k+3} - 2x^{k+2} + 1 = 0$. For example, when $k = 1, 2$ and 3 , we have $\gamma_k = 1.839, 1.928$ and 1.966 , respectively.

The idea of remaining proof is to show that the analysis of Scenario 2 can be further improved based on our pivot selection strategy. Specifically, Scenario 2 would have the worst-case time complexity smaller than that of Scenario 1, and thus the worst-case time complexity of FastBB would be bounded by Scenario 1. \square

Space complexity. The space complexity of FastBB is $O(K \cdot (d+k) + k \cdot |V|)$, which is linear to the size of the input bipartite graph given parameters K and k . The first term $O(K \cdot (d+k))$ is the space cost of storing the returned top- K MaxBPs (each has the size bounded by $2 \times (d+k)$ since otherwise it involves at least $2 \times (d+k) + 1$ vertices and would not be a k -biplex due to the fact that (1) it has the size of one side at least $d+k+1$ and (2) there would be a vertex from the other side which disconnects at least $k+1$ vertices since d is the maximum degree of vertex in G). The second term $O(k \cdot |V|)$ is the space cost of the branch-and-bound search process. Specifically, the space cost of the search process is dominated by that of storing the vertex orderings (Equation (4) or (5)) in all recursions. To reduce the space cost, instead of storing the ordering of all vertices in C , we store that of only those vertices that are before others in the ordering, i.e., the first $a+1$ (resp. $a+2$) vertices of Equation (4) (resp. (5)) depending on the cases. This is because it creates the first $a+1$ or $a+2$ branches only. Considering $a \leq k$, we know that the space cost of storing the vertex ordering in one recursion is bounded by $O(k)$. Furthermore, it can be verified that the number of recursions is bounded by $O(|V|)$ (since whenever the number of recursions increases by one, at least one vertex is removed from C and $C \subseteq V$). In summary, the overall space cost of storing the ordering of vertices is $O(k \cdot |V|)$.

Remark 1. The baseline method FPadp, which adapts the FaPlExen method in [51], has the worst-case time complexity $O(|V| \cdot d \cdot \beta_k^{|V|})$, where β_k is the largest positive real root of $x^{k+3} - 2x^{k+2} + 1 = 0$. We compare between FPadp and FastBB's time complexities: (1) the polynomial in both complexities (i.e., the time cost for each recursion) is $O(|V| \cdot d)$ since both algorithms are branch-and-bound based and (2) the exponential factors in FastBB's time complexity are strictly smaller than those of FPadp, e.g., $\gamma_1 = 1.754$ vs. $\beta_1 = 1.839$, $\gamma_2 = 1.888$ vs. $\beta_2 = 1.928$ and $\gamma_3 = 1.947$ vs. $\beta_3 = 1.966$ for $k = 1, 2$, and 3 , respectively. The differences in the exponential factors are due to the differences in the strategies of branching and ordering vertices (i.e., FastBB uses the Sym-BK branching while FPadp uses a hybrid branching; FastBB prioritizes S over C when selecting a pivot for deciding the ordering of vertices while FPadp does not).

Remark 2. We notice that the delay measure, i.e., the longest running time between any two consecutive outputs, is often used for evaluating enumeration algorithms [45, 48]. In our problem setting, all solutions can only be confirmed and returned in one shot at the end due to its optimization nature. For such optimization-oriented settings [5, 17, 23, 50], the delay measure is not suitable since it would be equal to the total running time trivially.

4 EFFICIENCY AND SCALABILITY BOOSTING TECHNIQUES

In this section, we further introduce three techniques for boosting the efficiency and scalability of the branch-and-bound (BB) algorithms introduced in Section 3, namely *progressive bounding* (PB) in Section 4.1, *inclusion-exclusion* (IE) in Section 4.2, and PBIE, which combines PB and IE, in Section 4.3.

4.1 Progressive Bounding Framework: PB

PB is adapted from an existing study of finding the biclique with the maximum number of edges [23]. The major idea of PB is to run a BB algorithm multiple times, and for each time, it imposes appropriate constraints on the MBP to be found, including lower and upper bounds of the number of vertices on both the left and right sides of the MBP. Then, it returns the MBP with the most edges found at different times. With the constraints captured by the lower and upper bounds, there are two benefits, namely (1) the BB algorithm can be run on a reduced graph instead of the original one and (2) the efficiency of the BB algorithm on the reduced graph can be further boosted with additional pruning techniques based on the upper bounds. Note that the pruning techniques are based on the lower bounds θ_L and θ_R only in the BB algorithms.

Let H^* be the MaxBP with the maximum $E(H^*)$. We have the following prior knowledge about the number of vertices at the left and right sides of H^* , i.e., $|L(H^*)|$ and $|R(H^*)|$.

$$\theta_L \leq |L(H^*)| \leq \delta_{max}^R + k; \quad \theta_R \leq |R(H^*)| \leq \delta_{max}^L + k. \quad (8)$$

where $\delta_{max}^R = \max_{u \in R} \delta(u, L)$ and $\delta_{max}^L = \max_{v \in L} \delta(v, R)$. The lower bounds θ_L and θ_R are inherited from the problem definition. The upper bounds of $\delta_{max}^R + k$ and $\delta_{max}^L + k$ can be verified easily and the proofs for them are thus omitted. We denote by LB_L^i, UB_L^i, LB_R^i , and UB_R^i the lower bound and upper bound of the number of vertices at the left and right sides, respectively, which the PB would use to capture the constraints at the i^{th} time. Then, PB would set these bounds *progressively* as follows.

$$LB_L^i = \max\{LB_L^{i-1}/2, \theta_L\}; \quad UB_L^i = LB_L^{i-1}; \quad (9)$$

$$LB_R^i = \max\{|E(H_{i-1}^*)|/UB_L^i, \theta_R\}; \quad UB_R^i = \delta_{max}^L + k; \quad (10)$$

where $LB_L^0 = \delta_{max}^R + k$ and H_{i-1}^* is the MBP found at the $(i-1)^{th}$ time and H_0^* can be set as $G[\emptyset]$. Essentially, it (1) splits the range of possible values of $|L(H^*)|$, namely $[\theta_L, \delta_{max}^R + k]$, into intervals with lengths decreasing *logarithmically*, (2) uses the boundaries of the intervals as lower and upper bounds of $|L(H^*)|$, and (3) then uses the upper bound of $|L(H^*)|$ and the MBP with the most edges found so far to further tighten the lower bound of $|R(H^*)|$. Note that it would generate $O(\log(\delta_{max}^R + k))$ sets of constraints.

It would then run the BB algorithm for each set of constraints captured by $LB_L^i, UB_L^i, LB_R^i, UB_R^i$ in the order of $i = 1, 2, \dots$. At the i^{th} time, it utilizes the lower and upper bounds as follows. First, it reduces the graph by computing the $(|LB_R^i - k|, |LB_L^i - k)$ -core of G since according to [44, 45], any MBP with at least $|LB_L^i|$ vertices at the left side and $|LB_R^i|$ vertices at the right side must reside in the $(|LB_R^i - k|, |LB_L^i - k)$ -core of G . Second, when it runs a BB algorithm, it prunes a branch $B = (S, C, D)$ if $|S_L| > |UB_L^i|$ or $|S_R| > UB_R^i$.

In summary, PB runs a BB algorithm multiple times, each time on a reduced graph. Hence, PB would boost the practical performance of a BB algorithm.

4.2 Inclusion-Exclusion based Framework: IE

IE is adapted from the *decomposition* technique, which has been widely used for enumerating and finding subgraph structures [5, 7, 40, 51]. The major idea of IE is to partition the graph into multiple ones (which may overlap) and run a BB algorithm on each of the subgraphs. Finally, it returns among the found MBPs the one with the most edges. Specifically, it partitions the graph G to $|L|$ subgraphs, namely $G_i = (L_i, R_i, E_i)$ for $1 \leq i \leq |L|$, as follows.

$$L_i = \Gamma_2(v_i, L) - \{v_1, \dots, v_{i-1}\}; \quad (11)$$

$$R_i = \bigcup_{v \in L_i} \Gamma(v, R); \quad (12)$$

$$E_i = \{(v, u) | v \in L_i, u \in R_i, (v, u) \in E\} \quad (13)$$

where $L = \{v_1, v_2, \dots, v_{|L|}\}$ and $\Gamma_2(v_i, L)$ denotes the set of 2-hop neighbors of v_i in L . We note that the number of vertices in G_i , i.e., $|L_i| + |R_i|$, is bounded by d^3 , where d is maximum degree of the graph G . We verify that the MaxBP H^* must reside in one of the subgraphs formed as above (for which the proof could be found in the appendix). Furthermore, the MBP found in G_i would *include* v_i and *exclude* v_1, v_2, \dots, v_{i-1} .

Moreover, it prunes the following vertices from a subgraph G_i .

- $v \in L_i$ with $\delta(v, R_i) < \theta_R - k$ or $|\Gamma(v, R_i) \cap \Gamma(v_i, R)| < \theta_R - 2k$;
- $u \in R_i$ with $\delta(u, L_i) < \theta_L - k$.

The correctness of pruning the vertices as above can be verified by contradiction with the size constraints based on θ_L and θ_R (the detailed proof can be found in the appendix).

Finally, it runs the BB algorithm on each graph G_i with some vertices pruned by starting from the branch $B_i = (S_i, C_i, D_i)$ with $S_i = \{v_i\}$, $D_i = \{v_1, v_2, \dots, v_{i-1}\}$, and $C_i = V(G_i) - \{v_1, v_2, \dots, v_i\}$. It then returns the MBP with the most edges among all MBPs found.

With the IE framework, the time complexity of a BB algorithm can be improved in certain cases. For example, it improves the time complexity of FastBB from $O(|V| \cdot d \cdot \gamma_k^{|V|})$ to $O(|L| \cdot d^3 \cdot \gamma_k^{d^3})$ for certain sparse graphs (e.g., those with $d^3 < |V|$).

4.3 Combining PB and IE: PBIE

We observe that the PB and IE frameworks can be naturally combined for our problem of finding the MaxBP. Specifically, we can first use PB to construct multiple reduced graphs G_i with corresponding constraints of lower and upper bounds of the number of vertices at the left and right sides of a graph. Then, when for each reduced graph G_i , we further use IE to construct $|L(G_i)|$ subgraphs $G_i^{v_j}$ for $v_j \in L(G_i)$ with some vertices pruned if possible. Finally, we invoke a BB algorithm (e.g., FastBB) on each subgraph $G_i^{v_j}$ with the constraints and return the MBP with the most edges among all found MBPs. The pseudo-code of PBIE is presented in Algorithm 3. **Time complexity.** The time cost is dominated by part of invoking FastBB (line 3-11). There are at most $O(\log(\delta_{max}^R + k))$ iterations (line 3-11). For each iteration, it constructs at most $O(|L|)$ subgraphs. Hence, FastBB is invoked by at most $O(\log(\delta_{max}^R + k) \cdot |L|)$ times.

Algorithm 3: Combine PB and IE: PBIE (for FastBB)

Input: A graph $G(L, R, E)$, k , θ_L and θ_R
Output: The maximal k -biplex H^* with the most edges

- 1 $H_0^* \leftarrow \emptyset$; $LB_L^0 \leftarrow \delta_{max}^R + k$; $i \leftarrow 1$;
- 2 **while** true **do**
- 3 Set $LB_L^i, UB_L^i, LB_R^i, UB_R^i$ according to Equations (9) and (10);
- 4 **if** $UB_L^i \leq \theta_L$ **then**
- 5 **return** H_{i-1}^* ;
- 6 Compute a reduced graph G_i as the
 $(|LB_R^i - k|, |LB_L^i - k|)$ -core of G ;
- 7 $H_i^* \leftarrow H_{i-1}^*$;
- 8 **for** $v_j \in L(G_i)$ **do**
- 9 Construct a subgraph $G_i^{v_j}$ based on Equations (11 - 13)
 with some vertices further pruned;
- 10 $H_i^* \leftarrow$ invoke a FastBB algorithm with $G_i^{v_j}$, k , the
 lower/upper bounds, and H_i^* ;
- 11 $i \leftarrow i + 1$;
- 12 **return** H_{i-1}^* ;

Besides, the number of vertices in $G_{i+1}^{v_j}$ is bounded by $O(d^3)$. Therefore, the time complexity of PBIE (when used for boosting FastBB) is $O(\log(\delta_{max}^R + k) \cdot |L| \cdot d^4 \cdot \gamma_k^{d^3})$ where γ_k is a real number strictly smaller than 2 (refer to Theorem 1 for details). We remark that the large graphs in real applications are usually sparse and have d far smaller than the total number of vertices.

5 EXPERIMENTS

Datasets. We use both real and synthetic datasets in our experiments. The real datasets and their statistics are summarized in Table 1 (<http://konect.cc/>) where the edge density of a bipartite graph $G(L \cup R, E)$ is defined as $2 \times |E| / (|L| + |R|)$ and d denotes the maximum degree. Note that these real graphs are usually sparse overall yet with some dense subgraphs (e.g., the maximum degree at least 16 and up to 12,189) since they follow the power-low distribution. The Erdős-Rényi (ER) synthetic datasets are generated by first creating a certain number of vertices and then randomly adding a certain number of edges between pairs of vertices. We set the number of vertices and edge density as 100k and 20 for synthetic datasets, respectively, by default.

Baselines. We compare our algorithm PBIE+FastBB with three baselines, namely iMBadp [44], FPadp [51] and iTradp [45]. Specifically, PBIE+FastBB adopts the combined framework PBIE and employs the improved algorithm FastBB within the framework. iMBadp and iTradp correspond to the adaptations of existing algorithms designed for enumerating *large MBPs*, namely iMB [44] and iTraversal [45]. Specifically, iMBadp adopts the BK branching for each side of a bipartite graph and is equipped with the pruning techniques developed in this paper. iTradp follows a *reverse search* [6] method and cannot be equipped with the pruning techniques developed in Section 3.4 (since they are not compatible with the reverse search framework). FPadp corresponds to a branch-and-bound algorithm with the hybrid branching strategy [51] and the pruning techniques developed in this paper. Furthermore, FPadp inherits FaPlexen [51] and employs the IE framework.

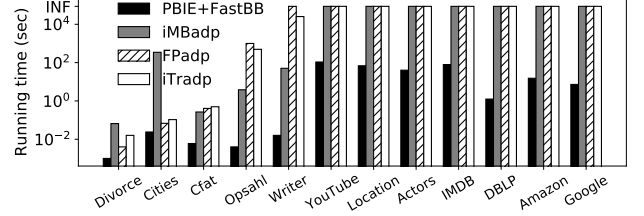


Figure 3: Comparison on all real datasets ($k = 1$)

We remark that all baseline methods are partial enumeration based methods, i.e., they do not enumerate all MBPs for finding the MaxBPs. This is because (1) their original versions were designed to enumerate only those MBPs with sizes at least some threshold but not all MBPs (as specified in the introduction section) and (2) they are enhanced by the four pruning conditions introduced in this paper as much as possible for pruning MBPs that are not MaxBPs (as specified above).

Settings. All algorithms were written in C++ and run on a machine with a 2.66GHz CPU and 32GB main memory running CentOS. We set the time limit (INF) as 24 hours and use 4 representative datasets as default ones, i.e., Writer, Location, DBLP and Google, which cover various graph scales. We set both θ_L and θ_R as $2k + 1$ and both K and k as 1 by default. Our code, data and additional experimental results are available at <https://github.com/KaiqiangYu/SIGMOD23-MaxBP>.

5.1 Comparison among algorithms

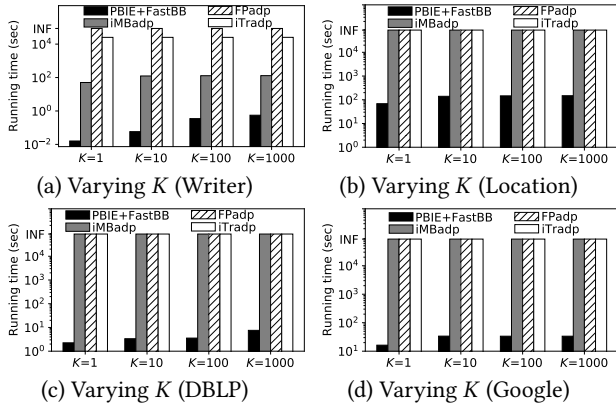
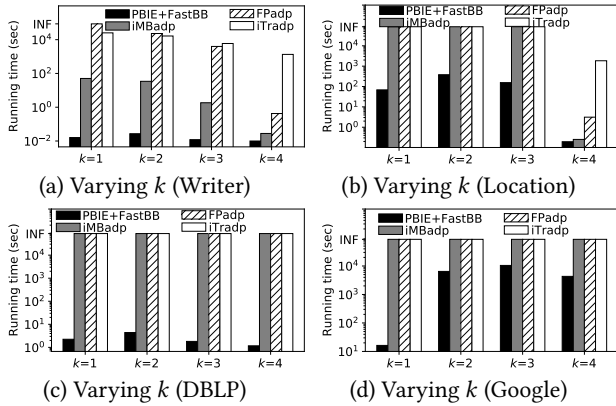
All datasets. We compare all algorithms on various datasets and show the running time in Figure 3. Besides, we report the number of vertices and the number of edges, denoted by $V(H)$ and $|E(H)|$, respectively, within the returned MaxBP in Table 1. We have the following observations. First, the found MaxBPs are usually sufficiently large to be meaningful (e.g., with at least 41 and up to 11,248 edges). Second, PBIE+FastBB outperforms all other algorithms on all datasets. This is consistent with our theoretical analysis that the worst-case running time of PBIE+FastBB is smaller than that of other algorithms. Third, PBIE+FastBB can handle all datasets within INF while others cannot finish on large datasets, e.g., Amazon and Google, which demonstrates its scalability. This is mainly because the framework PBIE would quickly locate the MaxBP at several smaller subgraphs while dramatically pruning many unfruitful vertices so as to reduce the search space.

Varying K . The results of finding K MaxBPs are shown in Figure 4. PBIE+FastBB outperforms other algorithms by around 2-5 orders of magnitude. Besides, it has the running time clearly rise as K increases compared to other algorithms. Possible reasons include (1) FastBB would explore more MBPs to find the K MaxBPs as K grows and (2) PBIE becomes less effective as K grows. In addition, iTradp has the running time almost not changed with K since it needs to explore almost all MBPs without any powerful pruning.

Varying k . The results are shown in Figure 5. PBIE+FastBB significantly outperforms other algorithms by up to five orders of magnitude. In addition, it has the running time first increase and then decrease as k grows. Possible reasons include: (1) the number of k -biplexes increases exponentially with k , which causes the increase of the running time; (2) the thresholds θ_L and θ_R (i.e.,

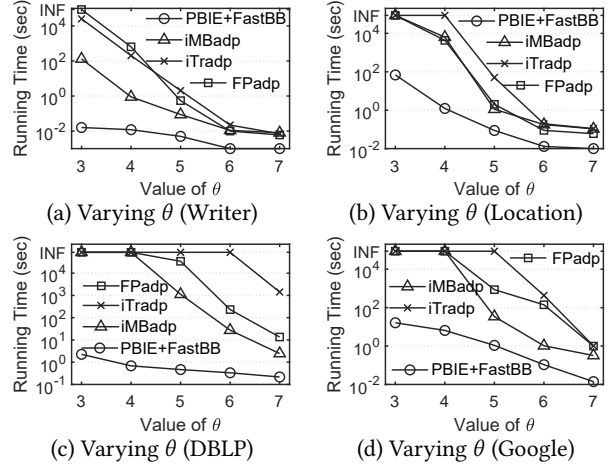
Table 1: Real datasets

Dataset	Category	$ L $	$ R $	$ E $	Density	d	$ V(H) $	$ E(H) $
Divorce	Feature	9	50	225	7.62	37	23	87
Cities	Feature	46	55	1342	26.4	54	43	437
Cfat	Biology	200	200	1537	7.66	16	13	41
Opsahl	Authorship	2,865	4,558	16,910	4.58	116	19	87
Writer	Authorship	89,356	46,213	144,340	2.12	246	38	136
YouTube	Affiliation	94,238	30,087	293,360	4.72	7,591	319	945
Location	Feature	172,091	53,407	293,697	2.60	12,189	287	849
Actors	Affiliation	127,823	383,640	1,470,404	5.64	646	117	341
IMDB	Affiliation	303,617	896,302	3,782,463	5.72	1,590	192	564
DBLP	Authorship	1,425,813	4,000,150	8,649,016	3.18	951	127	369
Amazon	Rating	6,703,391	957,764	12,980,837	3.28	9,990	2,255	1,1248
Google	Hyperlink	17,091,929	3,108,141	14,693,125	1.46	2,318	44	120

Figure 4: Comparison by varying K ($k = 1$)Figure 5: Comparison by varying k

$2k + 1$ by default) increase with k and correspondingly the pruning techniques that are based on θ_L 's and θ_R become more effective.

Varying θ_L and θ_R thresholds. The results are shown in Figure 6. PBIE+FastBB outperforms all other algorithms by achieving up to around 1000 \times speedup. Besides, the running time of all algorithms decreases as θ_L and θ_R grow. The reason is two-fold: (1) the search space (e.g., the number of large k -biplexes with the size of each side at least θ_L and θ_R) decreases as θ_L and θ_R grow and (2) the pruning rules are more effective for larger θ_L 's and θ_R 's.

Figure 6: Comparison by varying $\theta = \theta_L = \theta_R$ ($k = 1$)

Varying # of vertices (synthetic datasets). The results are shown in Figure 7(a). PBIE+FastBB outperforms other algorithms by achieving at least 10 \times speedup and can handle the largest datasets with 1 million vertices and 10 million edges within INF while others cannot. Besides, the speedup increases as the graph become larger. This is mainly because PBIE would prune more unfruitful vertices when locating the MaxBP in several smaller subgraphs. In addition, the results are well aligned with the theoretical results, i.e., the worst-case time complexity of PBIE+FastBB is exponential wrt d^3 while that of others is exponential wrt $|V|$. Hence, PBIE+FastBB has larger speed-ups as the graph scale becomes larger (where d remains almost the same due to the fixed edge density, i.e., 20).

Varying edge density (synthetic datasets). The results are shown in Figure 7(b). PBIE+FastBB achieves at least 10 \times speedup compared with other algorithms. In addition, the speedup decreases as the graph becomes denser. The reason is that the maximum degree of the bipartite graph, i.e., d , increases as the graph becomes denser.

5.2 Performance study

Comparison among various enumeration schemes. We study the effect of various enumeration schemes, namely FastBB, BasicBB, FPAdp, iMBadp, and iTradp. We note that all of them

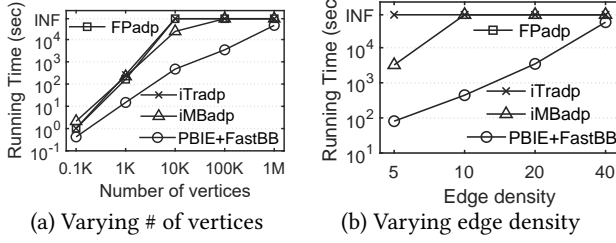
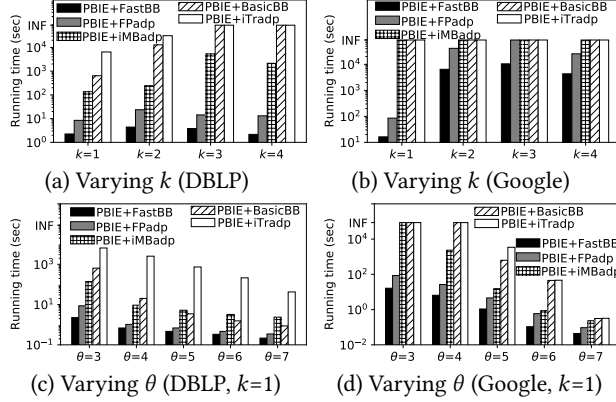
Figure 7: Comparison on synthetic datasets ($k = 1$)

Figure 8: Comparison among various enumeration schemes

are run with the framework PBIE for fair comparison. In particular, BasicBB adopts the classic BK branching and uses the non-decreasing vertex ordering. The results are shown in Figure 8(a) and (b) for varying k , and (c) and (d) for varying $\theta = \theta_L = \theta_R$. First, FastBB outperforms other algorithms, which demonstrates the superiority of proposed Sym-BK branching scheme. Besides, the achieved speedup decreases with θ since the search space (e.g., the number of large MBPs with the size of each side at least θ_L and θ_R) decreases as θ_L and θ_R grow. Second, the algorithms following the BK branching perform better than iTradp that follows a reverse search method. The reason is that the later one cannot be enhanced with effective pruning rules as branch-and-bound algorithms.

Comparison among frameworks. We study the effect of different frameworks and compare four different versions, namely (1) FastBB: without any framework, (2) IE+FastBB: with adapted framework IE, (3) PB+FastBB: with adapted progressive bounding framework PB, and (4) PBIE+FastBB: with proposed combined framework. We note that all frameworks adopt FastBB for fair comparison. The results are shown in Figure 9(a) and (b) for varying k , and (c) and (d) for varying $\theta = \theta_L = \theta_R$. First, both IE+FastBB and PB+FastBB outperform FastBB, which demonstrates the efficiency and scalability of adapted frameworks. Moreover, PBIE+FastBB performs the best and can handle all datasets and settings within INF. This is because PBIE can significantly reduce the size of the original graph (details can be found in the appendix). Second, PB+FastBB performs better than IE+FastBB. This is because PB can quickly locate the MaxBP at a much smaller subgraph. Third, the speedup decreases as θ grows. As discussed earlier, the search space gets smaller with θ and thus the frameworks would have less effects on the running time.

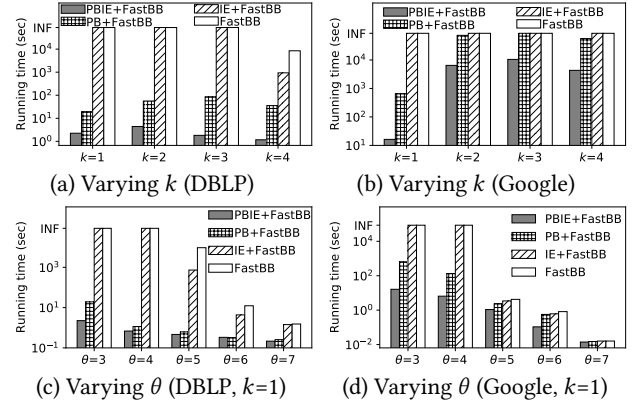


Figure 9: Comparison among frameworks

5.3 Case study: Fraud Detection

We investigate two cohesive models, namely maximal k -biplex and maximum k -biplex, for a fraud detection task [15] on the Software dataset [27]. The dataset contains 459,436 reviews on 21,663 softwares by 375,147 users. We consider the random camouflage attack scenario [15] where a fraud block with 1K fake users (“K” means a thousand), 1K fake softwares, 50K fake comments, and 50K camouflage comments, is injected to the dataset. Specifically, we randomly generate the fake comments (resp. camouflage comments) between pairs of fake users and fake products (resp. real products). We note that this attack can be easily conducted in reality to help fake users evade the detection, e.g., fake users are coordinated to deliberately post comments on some real products [15]. We then find MaxBPs and MBPs from the bipartite graph, and classify all users and products involved in the found subgraphs as fake items and others as real ones.

We measure the running time and F1 score, and show the results in Figure 10 where θ_L is fixed at 4.

- **Varying θ_R .** We find the 2000 MaxBPs (denoted by “MaxBP (2000)”), first-2000 MBPs (denoted by “MBP (2000)”), i.e., the first 2000 MBPs yielded by the algorithm, and all MBPs (denoted by “MBP (All)”) with θ_R varying from 3 to 6, and show the results in Figure 10(a). We have the following observations. (1) MaxBP (2000) achieves the best F1 score (0.99) when $\theta_R = 5$ among all methods. (2) Both MBP (2000) and MBP (All) achieve their best F1 scores, 0.80 and 0.87, respectively, when $\theta_R = 5$. (3) Under the setting with the best F1 scores, i.e., $\theta_R = 5$, MaxBP (2000) and MBP (2000) run comparably fast and both of them run faster than MBP (All). For (1) and (2), the reason could be that the fraud blocks tend to reside in large MBPs with more edges, and for (3), the reason could be that MaxBP (2000) and MBP (2000) need to explore a set of *some* but not all MBPs.
- **Varying K .** We find the K MaxBPs (denoted by “MaxBP (K)”) and the first- K MBPs (denoted by “MBP (K)”) with $\theta_R = 5$ since it give the best F1 scores, and show the results in Figure 10(b). We have the following observations. (1) MaxBP (K) has the F1 score higher than MBP (K) on all settings, and achieves the best F1 score at $K = 2000$. (2) MaxBP (K) provides a better trade-off between F1 score and running time than MBP (K), e.g., MaxBP (K) provides a F1 score 0.94 with the running time 512 seconds

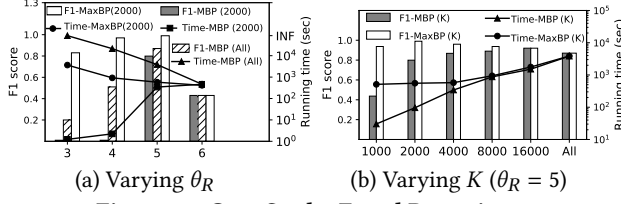


Figure 10: Case Study: Fraud Detection

at $K = 1000$ while MBP (K) provides a similar F1 score 0.91 with the running time 1512 seconds at $K = 16000$.

In summary, MaxBP outperforms MBP for fraud detection in terms of F1 score and the running time, as shown in Figure 10.

We have also explored methods of enumerating maximal bicliques and maximum bicliques for this case study. We found that the best F1-score achieved by enumerating maximal bicliques (with $\theta_R = 4$) and that of finding the maximum bicliques (with $\theta_R = 4$) are 0.48 and 0.54, respectively. In particular, when $\theta_R \geq 5$, there are few large bicliques, and thus both methods have the recall close to 0. This is mainly because biclique requires full connections among vertices, which is usually too strict.

6 RELATED WORK

We find that the maximum k -biplex search problem was recently studied in [22]. Some summary of the similarities and differences between [22] and our paper include: (1) [22] and our paper propose the maximum k -biplex search problem concurrently and independently; (2) both [22] and our paper prove the NP-hardness of the problem, yet they use different NP complete problems; (3) the solution in [22] has the time complexity of $O^*(2^{|V|})$ while our FastBB algorithm has the time complexity of $O^*(\gamma_k^{|V|})$ with $\gamma_k < 2$; and (4) the solutions in this paper run significantly faster than that in [22]. Some more details of the comparison can be found in the appendix. We review other related work as follows.

Cohesive bipartite structures. Recently, many studies have been conducted on finding cohesive subgraphs of bipartite graphs, including bicliques [1, 5, 18, 23, 26, 42, 48], (α, β) -cores [19], quasi-bicliques [16, 17, 21, 37], k -biplexes [30, 44, 45], k -bitrusses [35, 52], etc. Biclique is a bipartite graph where any vertex at one side connects all vertices at the other side. Recent works on biclique focus on enumerating maximal bicliques [1, 18, 48]. Given a bipartite graph, (α, β) -core is the maximal subgraph where any vertex at one side connects a certain number of vertices (i.e., α or β) at the other side. It has many applications, including recommendation systems [9] and community search [14, 36]. Existing works of k -biplexes focus on enumerating large maximal k -biplex [30, 44]. A k -bitruss [35, 52] is a bipartite graph where each edge is contained in at least k butterflies, where a butterfly corresponds to a complete 2×2 biclique [34]. In the literature, there are two types of quasi-biclique, i.e., (1) δ -quasi-biclique [21] is a bipartite graph $G(L, R, E)$ where each vertex in L (resp. R) misses at most $\delta \cdot |R|$ (resp. $\delta \cdot |L|$) edges with $\delta \in [0, 1)$ and (2) γ -quasi-biclique [17, 41] is a bipartite graph $G(L, R, E)$ that can miss at most $\gamma \cdot |L| \cdot |R|$ edges with $\gamma \in [0, 1)$. Existing works of quasi-bicliques focus on finding subgraphs with a certain density and degree [20, 25]. In this paper, we focus on k -biplex since it (1) imposes strict enough requirements on

connections within a subgraph and tolerates some disconnections and (2) satisfies the hereditary property, which facilitates efficient solutions. In [45], a case study of fraud detection on e-commerce platforms is conducted, which shows that k -biplex works better than some other cohesive subgraph structures including biclique, (α, β) -core, and δ -quasi-biclique for the application.

Maximum biclique search. The maximum biclique search problem has attracted much attention in recent years [5, 8, 12, 23, 24, 29, 31, 38, 46, 49, 50]. In general, there are three lines of works, namely maximum edge biclique search (MEBS) [8, 23, 29, 31] which finds a biclique H^* such that $E(H^*)$ is maximized, maximum vertex biclique search (MVBS) [12] which finds a biclique H^* such that $V(H^*)$ is maximized and maximum balanced biclique search (MBBS) [5, 24, 38, 46, 49, 50] which finds a biclique H^* such that $V(H^*)$ is maximized and $L(H^*) = R(H^*)$. First, the MEBS problem is NP-hard, for which many techniques have been proposed. Authors in [8, 31] adopt the integer linear programming techniques to find a MEB, which is not scalable for large bipartite graphs. A recent study [23] proposes a progressive bounding framework to deal with large bipartite graphs. Besides, a Monte Carlo algorithm is proposed in [29], which finds a MEB with a fixed probability. Second, the MVBS problem can be solved in polynomial time by finding a maximum matching [12]. Third, the MBBS problem is NP-hard, for which both exact methods and approximate methods have been developed. To be specific, exact methods proposed in [5, 24, 50] are branch-and-bound algorithms which use the widely-used Bron-Kerbosch (BK) branching [4]. Besides, approximate methods include [38, 46] which introduce a local search framework to find an approximate MBB, [2, 32] which convert MBBS into a maximum independent set problem and adopt approximate algorithms for the independent set problem, and [49] which proposes a heuristic algorithm with tabu search and graph reduction. In summary, there are two types of solutions, namely exact methods and approximate methods. For exact algorithms, most of them follow the branch-and-bound framework and use the BK branching strategy. However, based on our experimental results, BK branching strategy performs worse than our proposed Sym-BK branching strategy for the problem of finding MaxBP. For approximate methods, they cannot be adapted to find the MaxBP exactly.

Maximum quasi-biclique search. The maximum quasi-biclique search problem aims to find a γ -quasi-biclique or δ -quasi-biclique H^* such that $V(H^*)$ is maximized, which is a NP-hard problem [16, 21]. [16, 17] use mixed integer programming to find a maximum γ -quasi-biclique exactly, which cannot handle large datasets. [21, 37] propose greedy algorithms to find an approximate maximum δ -quasi-biclique, which cannot be adapted to find the MaxBP exactly.

7 CONCLUSION

In this paper, we study the *maximum k -biplex search* problem, which is to find K maximal k -biplexes with the most edges. We propose two branch-and-bound algorithms, among which the better one FastBB is based on a novel Sym-BK branching strategy and achieves better worst-case time complexity than adaptations of existing algorithms. We further develop frameworks to boost the efficiency and scalability of the branch-and-bound algorithms including FastBB. Extensive experiments are conducted on real and synthetic datasets

to demonstrate the efficiency of our algorithms and the effectiveness of proposed techniques. In the future, we plan to develop efficient parallel algorithms for the maximum k -biplex search problem and explore the possibility of adapting our algorithms to find other types of maximum cohesive subgraphs in bipartite graphs.

8 PROOF OF LEMMA 2

We prove by showing a polynomial reduction from a well-known NP-complete problem, namely *maximum clique search*, to the maximum k -biplex search problem with $\theta_L = \theta_R = 0$, $k = 1$, and $K = 1$. We define their decision problems as follows.

- **CLIQUE**: given a general graph $G = (V, E)$ and a positive integer α , does G contain a clique with at least α vertices?
- **BIPLEX**: given a bipartite graph $\mathcal{G} = (\mathcal{L} \cup \mathcal{R}, \mathcal{E})$ and two positive integers k and α' , does \mathcal{G} contain a k -biplex with at least α' edges?

Let $G = (V, E)$ and α be the inputs of an instance of **CLIQUE**. W.l.o.g., we assume that $\alpha = \frac{1}{2}|V|$ is a positive integer. This can be achieved with some inflation tricks. Specifically, (1) if the original input α is smaller than $\frac{1}{2}|V|$ (which can be a fractional number), we add to G a new vertex v and $|V|$ edges between v and other vertices. Obviously, every clique would include v , increasing α by 1 but $\frac{1}{2}|V|$ by 0.5. (2) If $\alpha > \frac{1}{2}|V|$, we add to G a new vertex v with no edges. Clearly, every clique would not include v , increasing $\frac{1}{2}|V|$ by 0.5 only. By repeating above two steps, we can make $\alpha = \frac{1}{2}|V|$ finally.

Now, we construct an instance of **BIPLEX** (with $k = 1$) with $\mathcal{G} = (\mathcal{L} \cup \mathcal{R}, \mathcal{E})$ and α' . To be specific,

$$\mathcal{L} = V \text{ and } \mathcal{R} = E \cup W \cup U, |W| = \frac{1}{2}\alpha(\alpha - 5), |U| = 2\alpha$$

where $W \cup U$ is a set of new elements and $|V| = 2\alpha$. Assume $V = \{v_1, \dots, v_{2\alpha}\}$ and $U = \{u_1, \dots, u_{2\alpha}\}$. We have

$$\begin{aligned} \mathcal{E} &= \mathcal{E}(\mathcal{G}[V \cup E]) \cup \mathcal{E}(\mathcal{G}[V \cup W]) \cup \mathcal{E}(\mathcal{G}[V \cup U]) \\ &= \{(v, e) \mid v \in V, e \in E, v \notin e\} \cup \{(v, w) \mid v \in V, w \in W\} \\ &\quad \cup \{(v_i, u_j) \mid v_i \in V, u_j \in U, i \neq j\}, \\ \alpha' &= \frac{1}{2}\alpha^3 + \frac{3}{2}\alpha^2 - \alpha. \end{aligned}$$

To guarantee $|W| \geq 0$, we assume $\alpha \geq 5$ which can be achieved with the inflation tricks. The above construction can be finished in polynomial time.

We then show that G has a clique with at least α vertices iff \mathcal{G} has a 1-biplex with at least α' edges. We consider the following two cases.

Case 1: G has a clique $G[C]$ with α vertices, i.e., $C \subseteq V$ and $|C| = \alpha$. Consider $X = V \setminus C$ and $Y = E(G[C]) \cup W \cup U$. We prove this case by showing that $\mathcal{G}[X \cup Y]$ is a 1-biplex with α' edges. To be specific, for each vertex $v \in X$, we know: (1) vertex v connects all vertices from $E(G[C])$ since v is not an endpoint of any edge in $E(G[C])$, thereby yielding $|X| \times |E(G[C])|$ edges in total; (2) vertex v connects all vertices from W based on the construction, yielding $|X| \times |W|$ edges in total; and (3) vertex v disconnects exactly one vertex in U based on the construction, yielding $|X| \times (|U| - 1)$ edges in total. For each vertex $u \in Y$, we can similarly verify that vertex u disconnects no more than one vertex from X . In summary, we have a 1-biplex $\mathcal{G}[X \cup Y]$ with

$$|\mathcal{E}(\mathcal{G}[X \cup Y])| = |X| \times |E(G[C])| + |X| \times |W| + |X| \times (|U| - 1),$$

where $|X| = \alpha$, $|E(G[C])| = \frac{1}{2}\alpha(\alpha - 1)$, $|W| = \frac{1}{2}\alpha(\alpha - 5)$ and $|U| = 2\alpha$. Therefore, $|\mathcal{E}(\mathcal{G}[X \cup Y])|$ is exactly α' .

Case 2: G does not have a clique with at least α vertices. If no 1-biplex is found in \mathcal{G} , the proof is finished. Otherwise, let $\mathcal{G}[X \cup Y]$ be an arbitrary 1-biplex in \mathcal{G} such that $X \subseteq \mathcal{L}$ and $Y \subseteq \mathcal{R}$. We finish the proof by showing that $|\mathcal{E}(\mathcal{G}[X \cup Y])| < \alpha'$. To estimate $|\mathcal{E}(\mathcal{G}[X \cup Y])|$, we divide Y into two disjoint parts $Y_0 \cup Y_1$, i.e., vertices in Y_0 connect all vertices from X and vertices in Y_1 disconnect exactly one vertex from X . For Y_0 , we have: (1) it includes all vertices from $E(G[V \setminus X]) \subseteq E$ since every edge in $E(G[V \setminus X])$ has no endpoint in X ; (2) it includes all vertices from W based on our construction; and (3) it includes $|V \setminus X|$ vertices from U (with $X = \{v_1, \dots, v_{|X|}\}$, vertices in $\{u_{|X|+1}, \dots, u_{2\alpha}\}$ would connect all vertices from X based on our construction). For Y_1 , it includes at most $|X|$ vertices since otherwise there exists at least a vertex in X that disconnects at least 2 vertices based on the pigeonhole principle. This contradicts to the definition of 1-biplex $\mathcal{G}[X \cup Y]$. In summary, we have

$$\begin{aligned} |\mathcal{E}(\mathcal{G}[X \cup Y])| &= |X| \times |Y_0| + (|X| - 1) \times |Y_1| \\ &\leq |X| \times (|E(G[V \setminus X])| + |W| + |V \setminus X|) + (|X| - 1) \times |X| \end{aligned} \quad (14)$$

Let $x = |X|$. We have $0 \leq x \leq 2\alpha$ and consider two cases.

Case 2.1: $\alpha < x \leq 2\alpha$. Let $y = x - \alpha$ ($0 < y \leq \alpha$). We have $|X| = \alpha + y$ and $|V \setminus X| = \alpha - y$. Moreover, we can obtain $|E(G[V \setminus X])| \leq \frac{1}{2}|V \setminus X|(|V \setminus X| - 1) = \frac{1}{2}(\alpha - y)(\alpha - y - 1)$ where the equality holds iff $G[V \setminus X]$ is a clique. According to equation (14), we have $|\mathcal{E}(\mathcal{G}[X \cup Y])| \leq (\alpha + y)[\frac{1}{2}(\alpha - y)(\alpha - y - 1) + \frac{1}{2}\alpha^2 - \frac{5}{2}\alpha + 2\alpha - (\alpha + y)] + (\alpha + y - 1)(\alpha + y)$ which reduces to

$$|\mathcal{E}(\mathcal{G}[X \cup Y])| - \alpha' \leq \frac{1}{2}y[y^2 - (\alpha - 1)y - \alpha - 2].$$

It is easy to verify that $y^2 - (\alpha - 1)y - \alpha - 2$ is negative for $0 \leq y \leq \alpha$. Therefore, we have $|\mathcal{E}(\mathcal{G}[X \cup Y])| < \alpha'$.

Case 2.2: $0 \leq x \leq \alpha$. Let $y = \alpha - x$ ($0 \leq y \leq \alpha$). We have $|X| = \alpha - y$ and $|V \setminus X| = \alpha + y$. Since $|V \setminus X| = 2\alpha - x \geq \alpha$ and G does not have a clique with at least α vertices, it is easy to verify that $|E(G[V \setminus X])| < \frac{1}{2}|V \setminus X|(|V \setminus X| - 1) - y$ since otherwise there exists a clique with at least α vertices in G (note that the right term can be regarded as a process of iteratively removing y edges from a clique with $|V \setminus X|$ vertices and after removing an edge, the maximum clique in the remaining graph has its size decrease by at most 1, which yields a clique with $|V \setminus X| - y = \alpha + y - y = \alpha$ vertices). According to equation (14), we have $|\mathcal{E}(\mathcal{G}[X \cup Y])| < (\alpha - y)[\frac{1}{2}(\alpha + y)(\alpha + y - 1) - y + \frac{1}{2}\alpha^2 - \frac{5}{2}\alpha + 2\alpha - (\alpha - y)] + (\alpha - y - 1)(\alpha - y)$ which reduces to

$$|\mathcal{E}(\mathcal{G}[X \cup Y])| - \alpha' < \frac{1}{2}y[-y^2 - (\alpha - 3)y - \alpha + 2].$$

It is easy to verify that $y[-y^2 - (\alpha - 3)y - \alpha + 2] \leq 0$ for $0 \leq y \leq \alpha$ and $\alpha \geq 5$. We thus have $|\mathcal{E}(\mathcal{G}[X \cup Y])| < \alpha'$.

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REFERENCES

- [1] Aman Abidi, Rui Zhou, Lu Chen, and Chengfei Liu. 2020. Pivot-based Maximal Biclique Enumeration. In *Proc. Int. Joint Conf. Artif. Intell., IJCAI 2020*. 3558–3564.
- [2] Ahmad A. Al-Yamani, S. Ramsundar, and Dhiraj K. Pradhan. 2007. A Defect Tolerance Scheme for Nanotechnology Circuits. *IEEE Trans. Circuits Syst. I Regul. Pap.* 54-1, 11 (2007), 2402–2409.
- [3] Alex Beutel, Wanhong Xu, Venkatesan Guruswami, Christopher Palow, and Christos Faloutsos. 2013. Copycatch: stopping group attacks by spotting lockstep behavior in social networks. In *Proceedings of the 22nd international conference on World Wide Web*. 119–130.
- [4] Coen Bron and Joep Kerbosch. 1973. Algorithm 457: finding all cliques of an undirected graph. *Commun. ACM* 16, 9 (1973), 575–577.
- [5] Lu Chen, Chengfei Liu, Rui Zhou, Jiajie Xu, and Jianxin Li. 2021. Efficient Exact Algorithms for Maximum Balanced Biclique Search in Bipartite Graphs. In *Proc. ACM SIGMOD Int. Conf. Manage. Data*. 248–260.
- [6] Sara Cohen, Benny Kimpelfeld, and Yehoshua Sagiv. 2008. Generating all maximal induced subgraphs for hereditary and connected-hereditary graph properties. *J. Comput. Syst. Sci.* 74, 7 (2008), 1147–1159.
- [7] Alessio Conte, Tiziano De Matteis, Daniele De Sensi, Roberto Grossi, Andrea Marino, and Luca Versari. 2018. D2K: Scalable Community Detection in Massive Networks via Small-Diameter k-Plexes. In *Proc. 24th ACM SIGKDD Int. Conf. Knowl. Discovery Data Mining*. 1272–1281.
- [8] Milind Dawande, Pinar Keskinocak, Jayashankar M Swaminathan, and Sridhar Tayur. 2001. On bipartite and multipartite clique problems. *Journal of Algorithms* 41, 2 (2001), 388–403.
- [9] Danhao Ding, Hui Li, Zhipeng Huang, and Nikos Mamoulis. 2017. Efficient fault-tolerant group recommendation using alpha-beta-core. In *Proceedings of the 2017 ACM Conference on Information and Knowledge Management*. 2047–2050.
- [10] Fedor V Fomin and Petteri Kaski. 2013. Exact exponential algorithms. *Commun. ACM* 56, 3 (2013), 80–88.
- [11] Siva Charan Reddy Gangireddy, Cheng Long, and Tanmoy Chakraborty. 2020. Unsupervised Fake News Detection: A Graph-based Approach. In *Proceedings of the 31st ACM Conference on Hypertext and Social Media*. 75–83.
- [12] Michael R Garey and David S Johnson. 1979. *Computers and intractability*. Vol. 174. freeman San Francisco.
- [13] Stephan Günnemann, Emmanuel Müller, Sebastian Raubach, and Thomas Seidl. 2011. Flexible Fault Tolerant Subspace Clustering for Data with Missing Values. In *11th IEEE International Conference on Data Mining, ICDM 2011*. 231–240.
- [14] Yang Hao, Mengqi Zhang, Xiaoyang Wang, and Chen Chen. 2020. Cohesive Subgraph Detection in Large Bipartite Networks. In *SSDBM 2020: 32nd International Conference on Scientific and Statistical Database Management*. ACM, 22:1–22:4.
- [15] Bryan Hooi, Hyun Ah Song, Alex Beutel, Neil Shah, Kijung Shin, and Christos Faloutsos. 2016. FRAUDAR: Bounding Graph Fraud in the Face of Camouflage. In *Proc. ACM SIGKDD Int. Conf. Knowl. Discov. Data Mining*. ACM, 895–904.
- [16] Dmitry I. Ignatov. 2019. Preliminary Results on Mixed Integer Programming for Searching Maximum Quasi-Bicliques and Large Dense Biclusters. In *Supplementary Proceedings of ICFA 2019 Conference and Workshops (CEUR Workshop Proceedings, Vol. 2378)*. CEUR-WS.org, 28–32.
- [17] Dmitry I Ignatov, Polina Ivanova, and Albina Zamaletdinova. 2018. Mixed Integer Programming for Searching Maximum Quasi-Bicliques. In *International Conference on Network Analysis*. Springer, 19–35.
- [18] Kyle Kloster, Blair D Sullivan, and Andrew van der Poel. 2019. Mining maximal induced bicliques using odd cycle transversals. In *Proceedings of the 2019 SIAM International Conference on Data Mining*. SIAM, 324–332.
- [19] Boge Liu, Long Yuan, Xuemin Lin, Lu Qin, Wenjie Zhang, and Jingren Zhou. 2020. Efficient (α, β) -core computation in bipartite graphs. *The VLDB Journal* 29, 5 (2020), 1075–1099.
- [20] Hsiao-Fei Liu, Chung-Tsai Su, and An-Chiang Chu. 2013. Fast Quasi-biclique Mining with Giraph. In *IEEE International Congress on Big Data, BigData Congress 2013*. IEEE Computer Society, 347–354.
- [21] Xiaowen Liu, Jinyan Li, and Lusheng Wang. 2008. Quasi-bicliques: Complexity and Binding Pairs. In *Computing and Combinatorics, 14th Annual International Conference, COCOON 2008*, Vol. 5092. Springer, 255–264.
- [22] Wensheng Luo, Kenli Li, Xu Zhou, Yunjun Gao, and Keqin Li. 2022. Maximum Biplex Search over Bipartite Graphs. In *2022 IEEE 38th International Conference on Data Engineering (ICDE)*. IEEE, 898–910.
- [23] Bingqing Lyu, Lu Qin, Xuemin Lin, Ying Zhang, Zhengping Qian, and Jingren Zhou. 2020. Maximum biclique search at billion scale. *Proceedings of the VLDB Endowment* (2020).
- [24] Ciaran McCreesh and Patrick Prosser. 2014. An Exact Branch and Bound Algorithm with Symmetry Breaking for the Maximum Balanced Induced Biclique Problem. In *Integration of AI and OR Techniques in Constraint Programming International Conference, Helmut Simonis (Ed.)*, Vol. 8451. Springer, 226–234.
- [25] Michael Mitzenmacher, Jakub Pachocki, Richard Peng, Charalampos E. Tsourakakis, and Shen Chen Xu. 2015. Scalable Large Near-Clique Detection in Large-Scale Networks via Sampling. In *Proc. ACM SIGKDD Int. Conf. Knowl. Discov. Data Mining*. ACM, 815–824.
- [26] Arko Provo Mukherjee and Srikanta Tirthapura. 2016. Enumerating maximal bicliques from a large graph using mapreduce. *IEEE Transactions on Services Computing* 10, 5 (2016), 771–784.
- [27] Jianmo Ni. 2018. *Amazon Review Data*. <https://nijianmo.github.io/amazon/index.html>
- [28] Ardian Kristanto Poernomo and Vivekanand Gopalkrishnan. 2009. Towards efficient mining of proportional fault-tolerant frequent itemsets. In *Proc. ACM SIGKDD Int. Conf. Knowl. Discov. Data Mining*. 697–706.
- [29] Eran Shaham, Honghai Yu, and Xiao-Li Li. 2016. On finding the maximum edge biclique in a bipartite graph: a subspace clustering approach. In *Proceedings of the 2016 SIAM International Conference on Data Mining*. SIAM, 315–323.
- [30] Kelvin Sim, Jinyan Li, Vivekanand Gopalkrishnan, and Guimei Liu. 2009. Mining maximal quasi-bicliques: Novel algorithm and applications in the stock market and protein networks. *Statistical Analysis and Data Mining* 2, 4 (2009), 255–273.
- [31] Melih Sözdinler and Can Özturan. 2018. Finding Maximum Edge Biclique in Bipartite Networks by Integer Programming. In *2018 IEEE International Conference on Computational Science and Engineering (CSE)*. IEEE, 132–137.
- [32] Mehdi Baradaran Tahoori. 2006. Application-independent defect tolerance of reconfigurable nanoarchitectures. *ACM J. Emerg. Technol. Comput. Syst.* 2, 3 (2006), 197–218.
- [33] Jun Wang, Arjen P De Vries, and Marcel JT Reinders. 2006. Unifying user-based and item-based collaborative filtering approaches by similarity fusion. In *Proceedings of the 29th annual international ACM SIGIR conference on Research and development in information retrieval*. 501–508.
- [34] Kai Wang, Xuemin Lin, Lu Qin, Wenjie Zhang, and Ying Zhang. 2019. Vertex Priority Based Butterfly Counting for Large-scale Bipartite Networks. *Proc. VLDB Endow.* 12, 10 (2019), 1139–1152.
- [35] Kai Wang, Xuemin Lin, Lu Qin, Wenjie Zhang, and Ying Zhang. 2020. Efficient Bitruss Decomposition for Large-scale Bipartite Graphs. In *Proc. 36th IEEE Int. Conf. Data Eng., ICDE 2020*. IEEE, 661–672.
- [36] Kai Wang, Wenjie Zhang, Xuemin Lin, Ying Zhang, Lu Qin, and Yuting Zhang. 2021. Efficient and Effective Community Search on Large-scale Bipartite Graphs. In *Proc. 36th IEEE Int. Conf. Data Eng., ICDE 2021*. IEEE, 85–96.
- [37] Lusheng Wang. 2013. Near optimal solutions for maximum quasi-bicliques. *Journal of Combinatorial Optimization* 25, 3 (2013), 481–497.
- [38] Yiyuan Wang, Shaowei Cai, and Minghao Yin. 2018. New heuristic approaches for maximum balanced biclique problem. *Information Sciences* 432 (2018), 362–375.
- [39] Zhuo Wang, Qun Chen, Boyi Hou, Bo Suo, Zhanhuai Li, Wei Pan, and Zachary G Ives. 2017. Parallelizing maximal clique and k-plex enumeration over graph data. *J. Parallel and Distrib. Comput.* 106 (2017), 79–91.
- [40] Zhengren Wang, Yi Zhou, Mingyu Xiao, and Bakhdadyr Khousainov. 2022. Listing Maximal k-Plexes in Large Real-World Graphs. (2022), 1517–1527.
- [41] Changhui Yan, J Gordon Burleigh, and Oliver Eulenstein. 2005. Identifying optimal incomplete phylogenetic data sets from sequence databases. *Molecular phylogenetics and evolution* 35, 3 (2005), 528–535.
- [42] Jianye Yang, Yun Peng, and Wenjie Zhang. 2022. (p,q)-biclique counting and enumeration for large sparse bipartite graphs. *Proceedings of the VLDB Endowment* (2022).
- [43] Kaiqiang Yu and Cheng Long. 2021. Graph Mining Meets Fake News Detection. In *Data Science for Fake News*. Springer, 169–189.
- [44] Kaiqiang Yu, Cheng Long, P Deepak, and Tanmoy Chakraborty. 2021. On Efficient Large Maximal Biplex Discovery. *IEEE Transactions on Knowledge and Data Engineering* (2021).
- [45] Kaiqiang Yu, Cheng Long, Shengxin Liu, and Da Yan. 2022. Efficient Algorithms for Maximal k-Biplex Enumeration. In *Proc. ACM SIGMOD Int. Conf. Manage. Data*.
- [46] Bo Yuan, Bin Li, Huanhuan Chen, and Xin Yao. 2015. A New Evolutionary Algorithm with Structure Mutation for the Maximum Balanced Biclique Problem. *IEEE Trans. Cybern.* 45, 5 (2015), 1040–1053.
- [47] Xichen Zhang and Ali A Ghorbani. 2020. An overview of online fake news: Characterization, detection, and discussion. *Information Processing & Management* 57, 2 (2020), 102025.
- [48] Yun Zhang, Charles A Phillips, Gary L Rogers, Erich J Baker, Elissa J Chesler, and Michael A Langston. 2014. On finding bicliques in bipartite graphs: a novel algorithm and its application to the integration of diverse biological data types. *BMC bioinformatics* 15, 1 (2014), 110.
- [49] Yi Zhou and Jin-Kao Hao. 2017. Combining tabu search and graph reduction to solve the maximum balanced biclique problem. *CoRR* abs/1705.07339 (2017).
- [50] Yi Zhou, André Rossi, and Jin-Kao Hao. 2018. Towards effective exact methods for the Maximum Balanced Biclique Problem in bipartite graphs. *European Journal of Operational Research* 269, 3 (2018), 834–843.
- [51] Yi Zhou, Jingwei Xu, Zhenyu Guo, Mingyu Xiao, and Yan Jin. 2020. Enumerating Maximal k-Plexes with Worst-Case Time Guarantee. In *The Thirty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2020*. 2442–2449.
- [52] Zhaonian Zou. 2016. Bitruss Decomposition of Bipartite Graphs. In *Database Systems for Advanced Applications - 21st International Conference, DASFAA 2016*, Vol. 9643. Springer, 218–233.

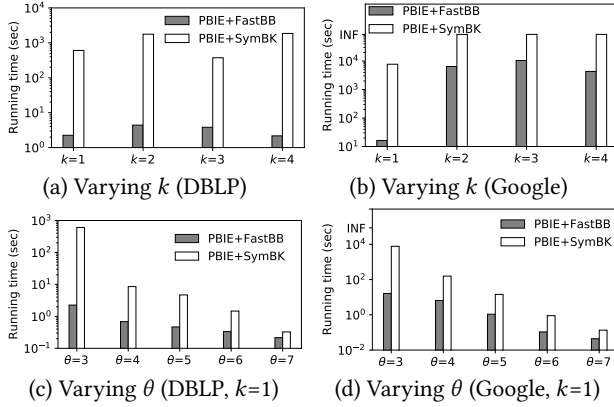


Figure 11: Comparison among various orderings

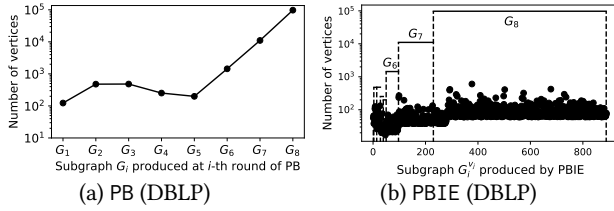


Figure 12: Performance of PBIE on reducing graph size

A ADDITIONAL EXPERIMENTAL RESULTS

Comparison among various orderings of vertices for Sym-BK branching. We consider the Sym-BK branching and study the effect of various orderings of the vertices in C . We compare two vertex orderings for the Sym-BK branching, (1) FastBB: the Sym-BK branching with the proposed ordering and (2) SymBK: the Sym-BK branching with the non-decreasing vertex ordering (where vertices are ranked in an ascending order of their degrees in $S \cup C$, i.e., $\delta(v_i, S \cup C) \leq \delta(v_j, S \cup C)$ for any $i < j$). We note that the non-decreasing vertex ordering would help with effective pruning as shown empirically [1, 48]. The results are shown in Figure 11. PBIE+FastBB outperforms PBIE+SymBK on various settings and datasets, which demonstrates the superiority of our proposed ordering for the Sym-BK branching.

The performance of PBIE on reducing graph size. Recall that PBIE partitions the graph to multiple smaller ones. We study the effect of PBIE on reducing the scale of graph. To this end, we report the size of G_i (which is produced at the i -th round of PB at line 6 of Algorithm 3) in Figure 12(a), and the size of $G_i^{v_j}$ (which is a subgraph of G_i and formed as a partition of G_i by IE based on vertex v_j at line 9 of Algorithm 3) in Figure 12(b). Note that for PBIE, we finally run FastBB on each subgraph $G_i^{v_j}$. We observe that the reduced graph $G_i^{v_j}$ is significantly smaller than (around 0.1% of) G_i and (around 0.01% of) the original graph (i.e., DBLP), which has more than 5 million vertices.

B WORST-CASE TIME COMPLEXITY ANALYSIS

The worst-case time complexity of FastBB is strictly better than that of BasicBB based on Theorem 1 (in Section 3.4). Below we provide a detailed analysis.

Theorem 1. *Given a bipartite graph G , FastBB finds the MaxBP in time $O(|V| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+4} - 2x^{k+3} + x^2 - x + 1 = 0$. For example, when $k = 1, 2$ and 3 , $\gamma_k = 1.754, 1.888$ and 1.947 , respectively.*

PROOF. We recursively maintain two arrays to record the degree of each vertex v within $G[S]$ and $G[S \cup C]$, i.e., $\delta(v, S)$ and $\delta(v, S \cup C)$. Then, each recursion of FastBB-Rec runs in polynomial time $O(|V| \cdot d)$. Specifically, the time cost is dominated by the part of checking pruning condition (4) in line 6 of Algorithm 2. This part has two steps, namely finding all those vertices with at least k disconnections from $S \cup C$ in $O(|S \cup C|)$ time and checking the pruning condition (4) for each vertex in D in $O(|D| \cdot d)$ time, where $|S \cup C|$ and $|D|$ are both bounded by $O(|V|)$.

Next, we analyze the number of recursions. Let $T(n)$ be the largest number of recursions where $n = |C|$. We have two cases.

Case 1 ($\hat{v} \in S$). We remove i vertices from C in B_i ($1 \leq i \leq a$) and b vertices from C in B_{a+1} in the worst-case. Hence, we have

$$T_1(n) \leq \sum_{i=1}^a T_1(n-i) + T_1(n-b). \quad (15)$$

As discussed earlier, we have $a \leq k$ and $a < b$. It is easy to verify that we reach the maximum of $T_1(n)$ when $a = k$ and $b = k+1$. We thus have $T_1(n) \leq \sum_{i=1}^k T_1(n-i) + T_1(n-k-1)$. By solving this linear recurrence, the worst-case running time is $O(|v| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+2} - 2x^{k+1} + 1 = 0$. For example, $\gamma_k = 1.618, 1.839$ and 1.928 when $k = 1, 2$ and 3 , respectively.

Case 2 ($\hat{v} \in C$). We remove i vertices from C in B_i ($1 \leq i \leq a+1$) and $b+1$ vertices from C in B_{a+2} in the worst-case. Hence, we have

$$T_2(n) \leq \sum_{i=1}^{a+1} T_2(n-i) + T_2(n-b-1). \quad (16)$$

Assuming $\hat{v} \in C_L$, we consider two scenarios, namely, $\bar{\delta}(\hat{v}, S_R \cup C_R) \geq k+2$ and $\bar{\delta}(\hat{v}, S_R \cup C_R) = k+1$.

- For Scenario 1, we can imply $a \leq k$ and $b > a+1$. When $a = k$ and $b = k+2$, $T_2(n)$ reaches the maximum. Hence, the worst-case running time is $O(|V| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+4} - 2x^{k+3} + x^2 - x + 1 = 0$. For example, when $k = 1, 2$ and 3 , we have $\gamma_k = 1.754, 1.888$ and 1.947 , respectively.
- For Scenario 2, we can imply $a \leq k$ and $b > a$. When $a = k$ and $b = k+1$, $T_2(n)$ reaches the maximum. Thus the worst-case running time is $O(|V| \cdot d \cdot \gamma_k^{|V|})$ where γ_k is the largest positive real root of $x^{k+3} - 2x^{k+2} + 1 = 0$. For example, when $k = 1, 2$ and 3 , we have $\gamma_k = 1.839, 1.928$ and 1.966 , respectively.

The idea of remaining proof is to show that the analysis of Scenario 2 can be further improved based on our pivot selection strategy. Consequently, Scenario 2 would have the worst-case time complexity smaller than Scenario 1, and thus the worst-case time complexity of FastBB would be bounded by that of Scenario 1.

We first show two pruning rules designed based on our pivot selection strategy, which would help to improve the analysis of Scenario 2. Intuitively, these pruning rules are built upon branch B_1 (formed based on Equation (5)) and try to move from C_1 to S_1

more vertices so as to reduce the number of potential branches to be created with the branching at B_1 . The rationale behind is that (1) after excluding pivot \hat{v} at B_1 , vertices in C that disconnect to \hat{v} , i.e., $\bar{\Gamma}(\hat{v}, C_R)$, would disconnect one vertex fewer at the other side of $S \cup C - \{\hat{v}\}$ and (2) these vertices are thus more likely to appear in the MaxBP to be found in B_1 . To be specific, we have following two pruning rules (wlog, assume $\hat{v} \in C_L$).

Pruning 1. Move from C_1 to S_1 all vertices in $\bar{\Gamma}(\hat{v}, C_R)$, if the following two conditions are satisfied.

- (i) Pivot \hat{v} disconnects exactly $k + 1$ vertices in $S_R \cup C_R$ (i.e., $\bar{\delta}(\hat{v}, S_R \cup C_R) = k + 1$), and
- (ii) For every vertex u in C_R that disconnects to \hat{v} , vertex u disconnects no more than k vertices from $S_L \cup C_L$ (i.e., $\forall u \in \bar{\Gamma}(\hat{v}, C_R), \bar{\delta}(u, S_L \cup C_L) \leq k$).

Proof of Pruning 1. This can be proved by contradiction. Assume that the MaxBP H to be found in B_1 does not include a vertex $w \in \bar{\Gamma}(\hat{v}, C_R)$. We derive the contradiction by showing that $G[V(H) \cup \{\hat{v}\}]$ is a k -biplex which has more edges than H (note that the MaxBP found in this branch cannot include \hat{v} since \hat{v} has been excluded). Specifically, (1) for a vertex $u \in L(H)$, we have $\bar{\delta}(u, R(H)) \leq k$ since H is a k -biplex; (2) for \hat{v} , we have $\bar{\delta}(\hat{v}, R(H)) \leq \bar{\delta}(\hat{v}, S_R \cup C_R \setminus \{w\}) \leq \bar{\delta}(\hat{v}, S_R \cup C_R) - 1 \leq (k+1) - 1 = k$ since $R(H) \subseteq S_R \cup C_R \setminus \{w\}$ (note that H is a subgraph of $G[C \cup S]$ and H does not include w by assumption), vertex w disconnects vertex \hat{v} and $\bar{\delta}(\hat{v}, S_R \cup C_R)$ is exactly $k + 1$ based on the pruning condition (i); (3) for a vertex $u \in S_R \cap L(H)$, we have $\bar{\delta}(u, L(H) \cup \{\hat{v}\}) \leq \bar{\delta}(u, S_L \cup C_L) \leq k$ since $L(H) \cup \{\hat{v}\} \subseteq S_L \cup C_L$ and every vertex in S_R disconnects no more than k vertices from $S_L \cup C_L$ based on the pivot selection; (4) for a vertex $u \in C_R \cap R(H)$, if u connects \hat{v} , we have $\bar{\delta}(u, L(H) \cup \{\hat{v}\}) \leq \bar{\delta}(u, L(H)) \leq k$ since H is a k -biplex. If u disconnects \hat{v} , we have $\bar{\delta}(u, L(H) \cup \{\hat{v}\}) \leq \bar{\delta}(u, S_L \cup C_L) \leq k$ based on $L(H) \cup \{\hat{v}\} \subseteq S_L \cup C_L$ and pruning condition (ii).

EXAMPLE 1. To illustrate, we consider again the example in Figure 2. We consider a branch B'_3 with $S'_3 = \{v_0, u_0, u_1, u_2\}$, $C'_3 = \{u_3, u_6, v_1, v_2, v_3, v_4\}$ and $D'_3 = \{u_4, u_5\}$ shown in Figure 2(c). We then select u_3 as the pivot based on our pivot selection strategy. Consider the branch to be created by excluding u_3 based on Equation (5), we observe that pruning 1 is triggered since (1) pivot u_3 disconnects 3 vertices from the other side of $S'_3 \cup C'_3$, i.e., $\{v_1, v_2, v_3\}$ and each vertex in $\{v_1, v_2, v_3\}$ disconnects 2 vertices from the other side of $S'_3 \cup C'_3$, i.e., $\{u_3, u_6\}$. Therefore, $\{v_1, v_2, v_3\}$ can be directly included to S'_3 .

Pruning 2. Move from $\bar{\Gamma}(\hat{v}, C_R)$ to S_1 those vertices each of which u satisfies the following two conditions.

- (a) u disconnects no more than $k + 1$ vertices from $S_L \cup C_L$ (i.e., $\bar{\delta}(u, S_L \cup C_L) \leq k + 1$), and
- (b) For every vertex v in $C_L \setminus \{\hat{v}\}$ that disconnects to u , vertex v disconnects no more than k vertices from $S_R \cup C_R$ (i.e., $\forall v \in \bar{\Gamma}(u, C_L \setminus \{\hat{v}\}), \bar{\delta}(v, S_R \cup C_R) \leq k$).

We define \mathcal{I} to be the set of those vertices such that $\mathcal{I} \subseteq \bar{\Gamma}(\hat{v}, C_R)$.

Proof of Pruning 2. We prove it by contradiction. Assume that the MaxBP H to be found in B_1 does not include a vertex $w \in \mathcal{I}$. We derive the contradiction by showing that $G[V(H) \cup \{w\}]$ is a k -biplex. Specifically, (1) for a vertex $u \in R(H)$, we have $\bar{\delta}(u, L(H)) \leq$

k since H is a k -biplex; (2) for vertex w , we have $\bar{\delta}(w, L(H)) \leq \bar{\delta}(w, S_L \cup C_L \setminus \{\hat{v}\}) \leq \bar{\delta}(w, S_L \cup C_L) - 1 \leq (k + 1) - 1 = k$ since $L(H) \subseteq S_L \cup C_L \setminus \{\hat{v}\}$ (note that H is a subgraph of $G[S \cup C]$ and \hat{v} has been excluded), vertex w disconnects vertex \hat{v} and $\bar{\delta}(w, S_L \cup C_L) \leq k + 1$ based on the pruning condition (a); (3) for a vertex $u \in S_L \cap L(H)$, we have $\bar{\delta}(u, R(H) \cup \{w\}) \leq \bar{\delta}(u, S_R \cup C_R) \leq k$ since $R(H) \cup \{w\} \subseteq S_R \cup C_R$ and every vertex in S_L disconnects no more than k vertices from $S_R \cup C_R$ based on the pivot selection; and (4) for a vertex $u \in C_L \cap L(H)$, if u connects w , we have $\bar{\delta}(u, R(H) \cup \{w\}) = \bar{\delta}(u, R(H)) \leq k$ since H is a k -biplex. If u disconnects w , we have $\bar{\delta}(u, R(H) \cup \{w\}) \leq \bar{\delta}(u, S_R \cup C_R) \leq k$ since $R(H) \cup \{w\} \subseteq S_R \cup C_R$ and pruning condition (b).

We then improve the analysis of Scenario 2 by using measure-and-conquer techniques [10]. Specifically, we propose a novel function $T_3(\rho)$ for bounding the maximum number of recursions, which assigns different weights to the vertices in C :

$$T_3(\rho), \rho = n_{\leq k} \times w_1 + n_{k+1} \times w_2, 0 < w_1 < w_2 \leq 1, \quad (17)$$

where $n_{\leq k}$ (resp. n_{k+1}) is the number of vertices with no more than k (resp. exactly $k + 1$) disconnections in C . We remark that the previous function $T_2(n)$ assigns the equal weight, i.e., 1, to all vertices in C . The new proposed function is motivated by following observations/intuitions, namely (1) $G[S \cup C]$ in Scenario 2 is a $(k + 1)$ -biplex and would become a k -biplex if no vertex in $S \cup C$ disconnects $k + 1$ vertices at the other side, and (2) those vertices with $k + 1$ disconnections have higher effects on the analysis while vertices with no more than k disconnections is of less importance.

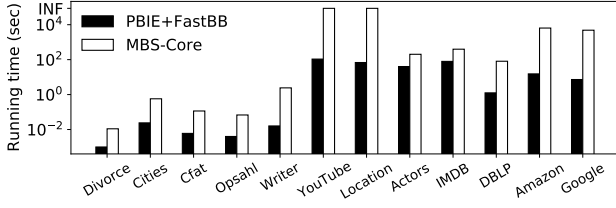
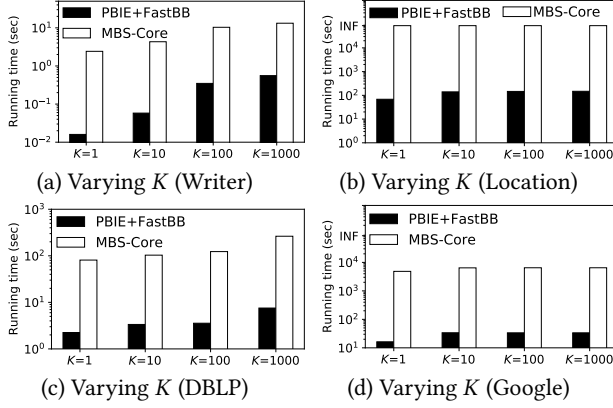
We call a vertex with exactly $k + 1$ disconnections a *boundary* vertex. Wlog, assume that $\hat{v} \in C_L$ and there are x boundary vertices in $\bar{\Gamma}(\hat{v}, C_R)$. We derive $x \geq 1$ in the worst-case (since otherwise Pruning 1 is triggered). Wlog, assume the first $k+1-x$ (resp. the last x) vertices in $\bar{\Gamma}(\hat{v}, C_R)$ are non-boundary vertices (resp. boundary vertices). Consider the worst case where $a = k$, $b = k + 1$, and neither pruning 1 nor pruning 2 would be triggered, we analyze $k + 2$ branches by $T_3(\rho)$ below.

- For B_1 , we exclude only a boundary vertex \hat{v} in the worst case. After branching, those x boundary vertices in $\bar{\Gamma}(\hat{v}, C_R)$ would disconnect k vertices from $S_L \cup C_L \setminus \{\hat{v}\}$ (since they all disconnect \hat{v}). Therefore, the number of recursions of B_1 is

$$T_3(\rho - (w_2 - w_1)x - w_2). \quad (18)$$

- For the remaining branches, i.e., $\{B_2, B_3, \dots, B_{k+2}\}$, recall that B_i ($i \geq 2$) includes vertices $\{\hat{v}, u_1, \dots, u_{i-2}\}$ and excludes vertex u_{i-1} . For each branch B_i , we have (1) a boundary vertex \hat{v} is removed, (2) except \hat{v} , there exists at least a boundary vertex disconnected to u_{i-1} (since otherwise Pruning 2 would be triggered) and it would disconnect to k vertices after excluding u_{i-1} , and (3) for each branch B_i in $\{B_2, \dots, B_{k+2-x}\}$, it removes $i - 1$ vertices, i.e., $\{u_1, \dots, u_{i-1}\}$, from C to S , each of which has the weight w_1 . For each branch B_i in $\{B_{k+3-x}, \dots, B_{k+2}\}$, it removes $k + 2 - x$ vertices with weight w_1 and $i - (k + 2 - x)$ vertices with weight w_2 from C to S . In summary, for each branch B_i in $\{B_2, \dots, B_{k+2-x}\}$, we have

$$T_3(\rho - w_2 - (w_2 - w_1) - w_1) = T_3(\rho - w_1 i - (2w_2 - w_1)). \quad (19)$$

Figure 13: Comparison on all real datasets ($k = 1$)Figure 14: Comparison by varying K ($k = 1$)

For each branch B_i in $\{B_{k+3-x}, \dots, B_{k+2}\}$, we have

$$T_3(\rho - w_2 - (w_2 - w_1) - w_1(k+2-x) - w_2(i - (k+2-x))) \\ = T_3(\rho - w_1(k+1-x) - w_2i - (2w_2 - w_1)). \quad (20)$$

Therefore, for the number of recursions in Scenario 2, we have

$$T_3(\rho) \leq T_3(\rho - (w_2 - w_1)x - w_2) + \sum_{i=1}^{k+1-x} T_3(\rho - w_1i - (2w_2 - w_1)) \\ + \sum_{i=1}^x T_3(\rho - w_1(k+1-x) - w_2i - (2w_2 - w_1)). \quad (21)$$

It is easy to verify that we reach the maximum of $T_3(\rho)$ when $x = 1$. In the following analysis, we set $w_1 = 0.36$ and $w_2 = 1$. Hence, the worst-case running time is $O(\gamma_k^\rho)$ where $\rho \leq |V|$ and γ_k is the largest positive real root of $x^{0.36k+3.64} - x^{0.36k+2.64} - x^{0.36k+2} - x^{0.36k+1.64} + x^{0.36k+1.28} + x^{0.36k+1} + x^{1.36} + 1 = 0$. For example, when $k = 1, 2$ and 3 , we have $\gamma_k = 1.753, 1.866$ and 1.945 . Hence, the time complexity in Scenario 2 is better than that in Scenario 1. \square

C ADDITIONAL PROOFS IN SECTION 4

We prove the correctness of claims mentioned in Section 4.2.

Claim 1. Given $|L|$ subgraphs, namely $G_i = (L_i, R_i, E_i)$ for $1 \leq i \leq |L|$ formed based on Equation (11)-(13), the MaxBP H^* must reside in one of the subgraphs.

PROOF. Assume that v_i is the vertex with the smallest index in H^* . We show that H^* would be found in G_i formed as above, which can be proved by contradiction. Suppose that such a H^* is not in G_i , which means $L(H^*) - L_i \neq \emptyset$ or $R(H^*) - R_i \neq \emptyset$. We thus consider following two cases.

- **Case 1** $L(H^*) - L_i \neq \emptyset$. Assume that a vertex $v \in L(H^*) - L_i$. We derive the contradiction by showing that H^* is not a k -biplex since vertex v disconnects more than k vertices, i.e., $\bar{\delta}(v, R(H^*)) \geq \delta(v_i, R(H^*)) = |R(H^*)| - \bar{\delta}(v_i, R(H^*)) \geq \theta_R - k \geq (2k+1) - k = k+1$ (note that (1) $\bar{\delta}(v, R(H^*)) \geq \delta(v_i, R(H^*))$ since v is not a 2-hop neighbor of v_i by assumption and thus disconnects v_i 's neighbors in $R(H^*)$, (2) $|R(H^*)| \geq \theta_R \geq 2k+1$ by the definition of problem and (3) $\bar{\delta}(v_i, R(H^*)) \leq k$ since H^* is a k -biplex and includes vertex v_i).
- **Case 2** $R(H^*) - R_i \neq \emptyset$. Assume that a vertex $u \in R(H^*) - R_i$. We derive the contradiction by showing that H^* is not a k -biplex since u disconnects more than k vertices, i.e., $\bar{\delta}(u, L(H^*)) = |L(H^*)| \geq \theta_L \geq 2k+1$ (note that $\bar{\delta}(u, L(H^*)) = |L(H^*)|$ since (1) $L(H^*) \subseteq L_i$ otherwise we can finish the proof based on Case 1 and (2) u disconnects all vertices in $L(H^*)$ based on the definition of R_i).

Based on the above two cases, we finish the proof. \square

Claim 2. Given subgraph G_i ($1 \leq i \leq |L|$) formed based on Equation (11)-(13), the following vertices from a subgraph can be pruned from G_i without missing the MaxBP to be found in G_i .

- $v \in L_i$ with $\delta(v, R_i) < \theta_R - k$ or $|\Gamma(v, R_i) \cap \Gamma(v_i, R)| < \theta_R - 2k$;
- $u \in R_i$ with $\delta(u, L_i) < \theta_L - k$.

PROOF. The idea is to show that these vertices cannot appear in the MaxBP H to be found in G_i . This can be proved by contradiction. In general, there are three cases.

- **Case 1.** Assume that H includes a vertex $v \in L_i$ such that $\delta(v, R_i) < \theta_R - k$. We derive the contradiction by showing that H is not a k -biplex since vertex v disconnects more than k vertices from $R(H)$, i.e., $\bar{\delta}(v, R(H)) = |R(H)| - \delta(v, R(H)) > \theta_R - (\theta_R - k) = k$.
- **Case 2.** Assume that H includes a vertex $v \in L_i$ such that $|\Gamma(v, R_i) \cap \Gamma(v_i, R)| < \theta_R - 2k$. We derive the contradiction by showing that H is not a k -biplex since vertex v disconnects more than k vertices from $R(H)$, i.e., $\bar{\delta}(v, R(H)) > \delta(v_i, R(H)) - (\theta_R - 2k) \geq (|R(H)| - k) - (\theta_R - 2k) \geq (\theta_R - k) - (\theta_R - 2k) = k$ (note that (1) $\bar{\delta}(v, R(H)) > \delta(v_i, R(H)) - (\theta_R - 2k)$ since v has less than $\theta_R - 2k$ common neighbors with v_i in R and thus connects at most $\theta_R - 2k$ vertices at $\Gamma(v_i, R(H))$ and (2) $\delta(v_i, R(H)) \geq |R(H)| - k$ since H is a k -biplex).
- **Case 3.** Assume that H includes a vertex $u \in R_i$ such that $\delta(u, L_i) < \theta_L - k$. We derive the contradiction by showing that H is not a k -biplex since vertex u disconnects more than k vertices from $L(H)$, i.e., $\bar{\delta}(u, L(H)) = |L(H)| - \delta(u, L(H)) > \theta_L - (\theta_L - k) = k$.

Combining the above three cases, we thus finish the proof. \square

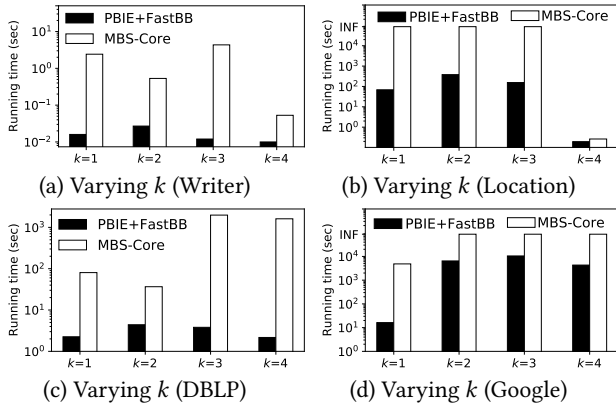


Figure 15: Comparison by varying k

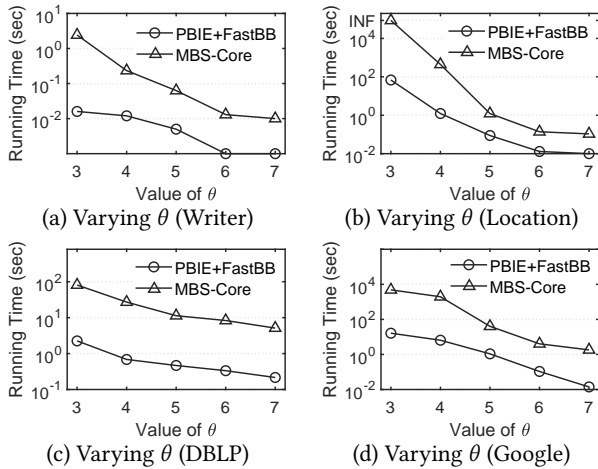


Figure 16: Comparison by varying $\theta = \theta_L = \theta_R$ ($k = 1$)

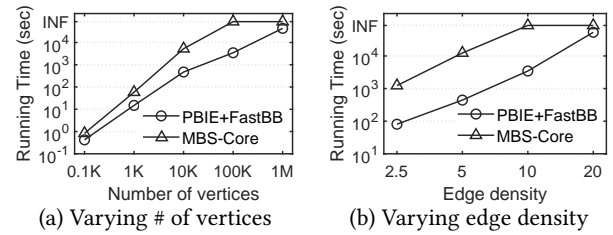


Figure 17: Comparison on synthetic datasets ($k = 1$)

D COMPARISON WITH MBS-CORE [22]

We notice that [22] studies the following maximum k -biplex search problem. Given a positive integer $k > 0$ and the size constraint $S \geq 2k + 1$, it finds the k -biplex with the largest number of edges and the number of vertices at each side at least S . Essentially, this problem corresponds to our problem under the setting of $K = 1$. In [22], the authors prove the NP-hardness of the problem and develop an algorithm called MBS-Core. MBS-Core consists of two sub-procedures, namely a progressive bounding framework called Core and a branch-and-bound enumeration algorithm called MBS. It first partitions the whole graph to multiple smaller ones via Core, and then runs MBS on each of them in a certain order. Specifically, Core is a variant of the progressive bounding framework and adapted from an existing study of finding the biclique with the maximum number of edges [19]. MBS is adapted from the existing BK-based algorithm iMB [44], which is proposed for enumerating large MBPs. We refer to [22] for the details of the MBS-Core algorithm.

We compare our algorithm PBIE+FastBB with MBS-Core on various datasets and settings. The default experimental settings are specified in Section 5. The source code of MBS-Core is obtained from the authors of [22]. In addition, we set $S = \theta$ for MBS-Core. The experimental results of running time are shown in Figure 13 for varying datasets, Figure 14 for varying K , Figure 15 for varying k , Figure 16 for varying θ , and Figure 17 for synthetic datasets.

We have two observations. First, PBIE+FastBB can finish on all datasets and settings while MBS-Core cannot finish within INF on Youtube and Location. Second, PBIE+FastBB outperforms MBS-Core on all datasets and under all settings and achieves up to 100 \times speedup.