Uniform nonsingularity and complementarity problems over symmetric cones

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ABSTRACT. We study the **uniform nonsingularity** property recently proposed by the authors and present its applications to nonlinear complementarity problems over a symmetric cone. In particular, by addressing theoretical issues such as the existence of Newton directions, the boundedness of iterates and the nonsingularity of B-subdifferentials, we show that the non-interior continuation method proposed by Xin Chen and Paul Tseng and the squared smoothing Newton method proposed by Liqun Qi, Defeng Sun and Jie Sun are applicable to a more general class of nonmonotone problems. Interestingly, we also show that the linear complementarity problem is globally uniquely solvable under the assumption of **uniform nonsingularity**.

1. INTRODUCTION

We consider the problem of finding, for a given **continuously differentiable** transformation $f : \mathbb{E} \mapsto \mathbb{E}$ and some given closed convex cone $K \subseteq \mathbb{E}$, an $x \in K$ satisfying

$$(1) \qquad x \geq_K 0, \ f(x) \geq_{K^\sharp} 0 \text{ and } \langle x, f(x) \rangle = 0,$$

where $\mathbb{E}$ is a Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and $K^\sharp$ is the closed dual cone

$$\{s \in \mathbb{E} : \langle s, x \rangle \geq 0 \ \forall x \in K\}.$$

We use $x \geq_K y$ (respectively, $x >_K y$) to mean $x - y \in K$ (respectively, $x - y \in \text{int}(K)$). We shall denote the problem of finding $x \in K$ satisfying (1) by $\text{NCP}_K(f)$. When f is affine, i.e., $f(x) = l(x) + q$ for some linear transformation $l : \mathbb{E} \to \mathbb{E}$ and some vector $q \in \mathbb{E}$, the problem $\text{NCP}_K(f)$ reduces to $\text{LCP}_K(l, q)$ or $\text{LCP}_K(M, q)$, where M is a matrix representation of l. In this case, the problem is called a linear complementarity problem over the cone K. We may also drop the subscript K when the cone is $\mathbb{R}^n_+$.

There recently has been much work on numerical methods for solving $\text{NCP}_K(f)$, including smoothing Newton methods [6, 8, 10, 16, 19, 22, 26], interior-point method [27], and non-interior continuation methods [11, 20]. It is noted that a common feature among these papers is the assumption of the monotonicity when K is different from $\mathbb{R}^n_+$. This assumption is somehow quite natural since it provides a sufficient condition to guarantee the existence of Newton direction as well as the boundedness of iterates encountered in those numerical approaches. A question which is of general theoretical interest is whether one can extend the existing well-developed algorithms for more general problems other than monotone problems.

In the literature of linear complementarity, a class of nonmonotone $\text{LCP}(M, q)$ which has been well solved is the **P**-LCPs, i.e., the matrix M is a **P**-matrix [13]. Numerical approaches for $\text{NCP}(f)$ has also been extensively studied based on the concept of **P**-function. See, e.g., [14]. Accordingly, there has been some considerable effort to extend

the concept of **P**-property to transformations over a general Euclidean space. Generalized from LCPs, a so called **P**-type properties was introduced by Gowda and Song [17] in identifying a class of nonmonotone semidefinite linear complementarity problems (SDLCPs). Later, their **P**-type properties have been extended to transformations over Euclidean Jordan algebras [18]. It is, however, not clear whether any numerical algorithms can be designed based on their **P**-type properties. In a recent paper, Chen and Qi [7] proposed the **Cartesian P**-property and showed that the merit function approach and smoothing method can be applied to a class of nonmonotone SDLCPs, namely **Cartesian P**-SDLCPs. Though they have shown that several impressive results such as the **GUS**-(global unique solvability-)property and the local Lipschitzian property of the solution map can be carried over from LCPs to **Cartesian P**-SDLCPs, the concept of **Cartesian P**-property differs from **strict monotonicity** only when the cone K is a symmetric cone that can be written as a direct sum $K_1 \oplus \cdots \oplus K_\kappa$ of at least two irreducible symmetric cones.

In the current paper, we explore the **uniform nonsingularity** property introduced by the authors in [12] and study its applications to smoothing/continuation methods for $\mathrm{NCP}_K(f)$. We show, in Theorem 3.1, that **uniform nonsingularity** is weaker than the *uniform* **Cartesian P**-property in general. Then, we demonstrate that both the non-interior continuation method [11] and the squared smoothing Newton method [26] are actually applicable to more general problems other than monotone problems. For the non-interior continuation method, we establish the existence of Newton direction and the boundedness of iterates and hence extend Chen-Tseng's algorithm to problems with our **uniform nonsingularity** property. Two key issues for the quadratic convergence of the squared smoothing Newton method are the strong semismoothness and the non-singularity of the B-subdifferential of the squared smoothing function. With **uniform nonsingularity**, we are able to relax the condition on the Jacobian, and yet show that all elements in the B-subdifferential are nonsingular. Moreover, instead of focusing on the squared smoothing function, we extend the result to a class of smooth approximations based on the Chen and Mangasarian smoothing functions. Meanwhile, we show that the $\mathrm{LCP}_K(l, q)$ has the **GUS**-property if l is **uniformly nonsingular**. The **GUS**-property follows from the fact that every solution of the $\mathrm{LCP}_K(l, q)$ is the limit of a convergent trajectory (see Theorem 5.1).

The rest of the paper is organized as follows. In the next section, we briefly review relevant concepts in the theory of Euclidean Jordan algebras. In Section 3, we investigate the relations among **uniform nonsingularity**, **Cartesian P**-property and **strong monotonicity**. In Section 4 we present the applications of **uniform nonsingularity** to two numerical approaches for solving complementarity problems. In Section 5, we show that **uniform nonsingularity** implies the **GUS**-property of the LCP_K. We draw the conclusion in Section 6.

2. Euclidean Jordan algebras

In this section, we review concepts in the theory of Euclidean Jordan algebras that are necessary for the purpose of this paper. Interested readers are referred to Chapters II–IV of [15] for a more comprehensive discussion on the theory of Euclidean Jordan algebras.

Definition 2.1 (Jordan algebra)**.** An algebra $(\mathfrak{J}, \circ)$ over the field $\mathbb{R}$ or $\mathbb{C}$ is said to be a *Jordan algebra* if it is commutative and the endomorphisms $y \mapsto x \circ y$ and $y \mapsto (x \circ x) \circ y$ commute for each $x \in \mathfrak{J}$.

Definition 2.2 (Euclidean Jordan algebra)**.** A finite dimensional Jordan algebra $(\mathfrak{J}, \circ)$ with unit e is said to be *Euclidean* if there exists a positive definite symmetric bilinear form on $\mathfrak{J}$ that is associative; i.e., $\mathfrak{J}$ has an inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle x \circ y, z \rangle = \langle y, x \circ z \rangle \quad \forall x, y, z \in \mathfrak{J}.$$

Henceforth, $(\mathfrak{J}, \circ)$ shall denote a Euclidean Jordan algebra, and e shall denote its unit. We shall identify $\mathfrak{J}$ with a Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$ in the above definition. For each $x \in \mathfrak{J}$, we shall use l_x to denote the *Lyapunov transformation* $y \mapsto x \circ y$, and use P_x to denote $2l_x^2 - l_{x \circ x}$. By the definition of Euclidean Jordan algebra, l_x, whence P_x, is self-adjoint under $\langle \cdot, \cdot \rangle$. The linear transformation P_x is called the *quadratic representation* of x.

Definition 2.3 (Jordan frame)**.** An *idempotent* of $\mathfrak{J}$ is a nonzero element $c \in \mathfrak{J}$ satisfying $c \circ c = c$. An idempotent is said to be *primitive* if it cannot be written as the sum of two idempotents. Two idempotents c and d are said to be *orthogonal* if $c \circ d = 0$. A *complete system of orthogonal idempotents* is a set of idempotents that are pair-wise orthogonal and sum to the unit e. A *Jordan frame* is a complete system of primitive idempotents. The number of elements in any Jordan frame is an invariant called the *rank* of $\mathfrak{J}$ (see paragraph immediately after Theorem III.1.2 of [15]).

Remark 2.1. Orthogonal idempotents are indeed orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ since

$$\langle c, d \rangle = \langle c \circ e, d \rangle = \langle e, c \circ d \rangle.$$

Primitive idempotents have unit norm.

Henceforth, r shall denote the rank of $\mathfrak{J}$.

Theorem 2.1 (Spectral decomposition (of type II))**.** *Each element x of the Euclidean Jordan algebra $(\mathfrak{J}, \circ)$ has a* spectral decomposition (of type II)

$$x = \sum_{i=1}^{r} \lambda_i c_i,$$

where $\lambda_1 \geq \cdots \geq \lambda_r$ (with their multiplicities) are uniquely determined by x, and $\{c_1, \ldots, c_r\} \subset \mathfrak{J}$ forms a Jordan frame.

The coefficients $\lambda_1, \ldots, \lambda_r$ are called the eigenvalues *of x, and they are denoted by $\lambda_1(x), \ldots, \lambda_r(x)$.*

Proof. See Theorem III.1.2 of [15]. □

Remark 2.2. Two elements share the same Jordan frames in their spectral decompositions precisely when they *operator commute* [15, Lemma X.2.2]; i.e., when the Lyapunov transformations l_x and l_y commute.

Theorem 2.2 (Characterization of symmetric cones)**.** *A cone is symmetric if and only if it is linearly isomorphic to the interior of the cone of squares*

$$K(\mathfrak{J}) := \{x \circ x : x \in \mathfrak{J}\}$$

of a Euclidean Jordan algebra $(\mathfrak{J}, \circ)$. Moreover, the interior $\mathrm{int}(K(\mathfrak{J}))$ of the cone of squares coincides with the following equivalent sets:
  (i) *the set $\{x \in \mathfrak{J} : l_x$ is positive definite under $\langle \cdot, \cdot \rangle\}$;*
  (ii) *the set $\{x \in \mathfrak{J} : \lambda_i(x) > 0 \ \forall i\}$.*

Proof. See Theorems III.2.1 and III.3.1 of [15]. □

Henceforth, K shall denote the cone of squares $K(\mathfrak{J})$.

The cone of squares $K(\mathfrak{J})$ can alternatively be described as the set of elements with nonnegative eigenvalues (see proof of Theorem III.2.1 of [15]), whence every symmetric cone can be identified with the set of elements with positive eigenvalues in certain Euclidean Jordan algebra.

For each idempotent $c \in \mathfrak{J}$, the only possible eigenvalues of l_c are 0, $\frac{1}{2}$ and 1; see Theorem III.1.3 of [15]. We shall use $\mathfrak{J}(c, 0)$, $\mathfrak{J}(c, \frac{1}{2})$ and $\mathfrak{J}(c, 1)$ to denote the eigenspaces of l_c corresponding to the eigenvalues 0, $\frac{1}{2}$ and 1, respectively. If μ is not an eigenvalue of l_c, then we use the convention $\mathfrak{J}(c, \mu) = \{0\}$.

Theorem 2.3 (Peirce decomposition). *Given a Jordan frame $\{c_1, \ldots, c_r\}$, the space $\mathfrak{J}$ decomposes into the orthogonal direct sum*

$$\mathfrak{J} = \bigoplus_{1 \le i \le j \le r} \mathfrak{J}_{ij},$$

where $\mathfrak{J}_{ii} := \mathfrak{J}(c_i, 1) = \mathbb{R}c_i$ and $\mathfrak{J}_{ij} := \mathfrak{J}(c_i, \frac{1}{2}) \cap \mathfrak{J}(c_j, \frac{1}{2})$ for $i < j$, such that the orthogonal projector onto $\mathfrak{J}_{ii}$ is P_{c_i}, and that onto $\mathfrak{J}_{ij}$ is $4l_{c_i}l_{c_j}$. Moreover, for any index set $I \subseteq \{1, \ldots, r\}$, the subspaces $\mathfrak{J}_1 = \bigoplus_{i \le j,\ i,j \in I} \mathfrak{J}_{ij}$ and $\mathfrak{J}_0 = \bigoplus_{i \le j,\ i,j \notin I} \mathfrak{J}_{ij}$ are subalgebras of $\mathfrak{J}$, and they are orthogonal in the sense that $\mathfrak{J}_1 \circ \mathfrak{J}_0 = \{0\}$.

The decomposition of $x \in \mathfrak{J}$ into

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{(i,j):1 \le i < j \le r} x_{ij}$$

with $x_i c_i = P_{c_i}(x)$ and $x_{ij} = 4l_{c_i}(l_{c_j}(x))$ is called its Peirce decomposition *with respect to the Jordan frame $\{c_1, \ldots, c_r\}$.*

Proof. See Theorem IV.2.1 and Proposition IV.1.1 of [15]. $\qquad\square$

Remark 2.3. It is straightforward to check that if $\{c_1, \ldots, c_r\}$ is the Jordan frame in a spectral decomposition $x = \sum \lambda_i(x) c_i$ of x, then the Peirce decomposition of x with respect to $\{c_1, \ldots, c_r\}$ coincide with this spectral decomposition.

We end this section with the following note.

Remark 2.4. Every Euclidean Jordan algebra can be written as a direct sum of simple ideals (see, e.g., [15, Proposition III.4.4]), each of which is a Euclidean Jordan algebra under the induced inner product. This means that every symmetric cone can be written as the orthogonal direct sum of *irreducible* symmetric cones.

Henceforth, $\mathfrak{J} = \mathfrak{J}_1 \oplus \cdots \oplus \mathfrak{J}_\kappa$ shall denote the decomposition of $\mathfrak{J}$ into simple ideals, K_v shall denote the irreducible symmetric cone $K(\mathfrak{J}_v)$, and x_v shall denote the component of the element $x \in \mathfrak{J}$ in $\mathfrak{J}_v$.

3. Positive generalized eigenvalue property

In the theory of LCPs [13], the **P**-property of a matrix plays a very important role and it can be defined in a number of ways. Here, we summarize below some known equivalent conditions for a matrix $M \in \mathbb{R}^{n \times n}$ to be a **P**-matrix:

(1) Every principal minor of M is positive.
(2) For every nonzero $x \in \mathbb{R}^n$, there is an index $i \in \{1, 2, \ldots, n\}$ such that

$$x_i(Mx)_i > 0.$$

(3) LCP(M, q) has a unique solution for every $q \in \mathbb{R}^n$ [24].

(4) There exists $\alpha > 0$ such that $\|(M + D)x\| \geq \alpha\|x\|$ for every nonnegative definite diagonal matrix D and any $x \in \mathbb{R}^n$ [12, Lemma 4.1].

The above last characterization of a **P**-matrix as being uniformly nonsingular under the addition of nonnegative D was used by the authors as the basis of definition of a **P**-type property for transformations on Euclidean Jordan algebras in [12]. We recall this definition, and call it the *uniform nonsingularity*.

Definition 3.1. The transformation $f : \mathfrak{J} \to \mathfrak{J}$ is said to be **uniformly nonsingular** if there exists $\alpha > 0$ such that for any $d_1, \ldots, d_r \geq 0$, any $d_{ij} \geq 0$, any Jordan frame $\{c_1, \ldots, c_r\}$ and any $x, y \in \mathfrak{J}$,

$$\left\| f(x) - f(y) + \sum d_i(x_i - y_i)c_i + \sum d_{ij}(x_{ij} - y_{ij}) \right\| \geq \alpha\|x - y\|.$$

In the case when K is polyhedral, **uniform nonsingularity** lies between the concept of **P**-property and uniform **P**-property (see Proposition 4.1 in [12]). For the discussions below, we list several more definitions.

Definition 3.2. A transformation $f : \mathfrak{J} \to \mathfrak{J}$ is said to satisfy

- **strong monotonicity** if there exists $\alpha > 0$ such that for every $x, y \in \mathfrak{J}$,

$$\langle x - y, f(x) - f(y) \rangle \geq \alpha\|x - y\|^2;$$

- the *uniform* **Cartesian P**-property if there exists $\rho > 0$ such that for any $x, y \in \mathfrak{J}$,

$$\max_{v=1,\ldots,\kappa} \langle (x - y)_v, (f(x) - f(y))_v \rangle \geq \rho\|x - y\|^2;$$

- the *uniform* **Jordan P**-property if there exists $\alpha > 0$ such that for any $x, y \in \mathfrak{J}$,

$$\lambda_1((x - y) \circ (f(x) - f(y))) \geq \alpha\|x - y\|^2;$$

- the *uniform* **P**-property if there exists $\alpha > 0$ such that for any $x, y \in \mathfrak{J}$ with $x - y$ operator commuting with $f(x) - f(y)$,

$$\lambda_1((x - y) \circ (f(x) - f(y))) \geq \alpha\|x - y\|^2.$$

Remark 3.1. When f is a strongly monotone *scalar* function we also say that it is *strongly increasing*. When f is a *linear* transformation, **strong monotonicity** reduces to **strict monotonicity**: $\langle x - y, f(x) - f(y) \rangle \leq 0 \implies x = y$.

Remark 3.2. The uniform **Cartesian P**-property is a straightforward extension of the one introduced by Chen and Qi in [7]. When f is linear, the uniform **Cartesian P**-property reduces to the ***Cartesian* P**-*property*: $\max_{v=1,\ldots,\kappa} \langle (x - y)_v, (f(x) - f(y))_v \rangle \leq 0 \implies x = y$.

Remark 3.3. When f is linear, the uniform **Jordan P**-property reduces to the ***Jordan* P**-*property*: $(x - y) \circ (f(x) - f(y)) \leq_K 0 \implies x = y$.
Similarly, when f is linear, the uniform **P**-property reduces to the **P**-*property*:

$$\left.\begin{array}{r} x - y \text{ and } f(x) - f(y) \text{ operator commute} \\ (x - y) \circ (f(x) - f(y)) \leq_K 0 \end{array}\right\} \implies x = y.$$

For a linear transformation $l : \mathfrak{J} \to \mathfrak{J}$, the **Cartesian P**-property implies **uniform nonsingularity** (see Proposition 4.5 in [12]). The next proposition shows an analogous result for nonlinear transformations.

Theorem 3.1. *If f has the uniform **Cartesian** **P***-property, then it is **uniformly nonsingular**.*

Proof. Assume that f has the uniform **Cartesian P**-property. Then for any $x, y \in \mathfrak{J}$ there exists $v \in \{1, ..., \kappa\}$ such that

$$\langle (x-y)_v, (f(x) - f(y))_v \rangle \geq \rho \|x - y\|^2.$$

Let $c_1^v, \ldots, c_{r_v}^v \in \mathfrak{J}_v$ be any Jordan frame and let $x_v = \sum x_i^v c_i^v + \sum x_{ij}^v$, $y_v = \sum y_i^v c_i^v + \sum y_{ij}^v$, $f(x)_v = \sum f_i^v c_i^v + \sum f_{ij}^v$ and $f(y)_v = \sum g_i^v c_i^v + \sum g_{ij}^v$ be Peirce decompositions. Then it follows from the inequality above that

$$\sum_{(x_i^v - y_i^v)(f_i^v - g_i^v) \geq 0} (x_i^v - y_i^v)(f_i^v - g_i^v) + \sum_{\langle x_{ij}^v - y_{ij}^v, f_{ij}^v - g_{ij}^v \rangle \geq 0} \langle x_{ij}^v - y_{ij}^v, f_{ij}^v - g_{ij}^v \rangle$$

$$\geq \sum (x_i^v - y_i^v)(f_i^v - g_i^v) + \sum \langle x_{ij}^v - y_{ij}^v, f_{ij}^v - g_{ij}^v \rangle$$

$$= \langle (x-y)_v, (f(x) - f(y))_v \rangle \geq \rho \|x - y\|^2.$$

Thus either there exists an index $\bar{\imath}$ such that $(x_{\bar{\imath}}^v - y_{\bar{\imath}}^v)(f_{\bar{\imath}}^v - g_{\bar{\imath}}^v) \geq \frac{\rho}{r^2} \|x - y\|^2$, or there exists a pair of indices $(\bar{\imath}, \bar{\jmath})$ such that $\langle x_{\bar{\imath}\bar{\jmath}}^v - y_{\bar{\imath}\bar{\jmath}}^v, f_{\bar{\imath}\bar{\jmath}}^v - g_{\bar{\imath}\bar{\jmath}}^v \rangle \geq \frac{\rho}{r^2} \|x - y\|^2$. In the former case, we have

$$\min_{d_i, d_{ij} \geq 0} \left\| f(x) - f(y) + \sum d_i(x_i - y_i)c_i + \sum d_{ij}(x_{ij} - y_{ij}) \right\|$$

$$\geq \min_{d_i^v, d_{ij}^v \geq 0} \left\| f_v(x) - f_v(y) + \sum d_i^v(x_i^v - y_i^v)c_i^v + \sum d_{ij}^v(x_{ij}^v - y_{ij}^v) \right\|$$

$$= \left(\sum_{(x_i^v - y_i^v)(f_i^v - g_i^v) \geq 0} (f_i^v - g_i^v)^2 + \sum_{\langle x_{ij}^v - y_{ij}^v, f_{ij}^v - g_{ij}^v \rangle \geq 0} \|f_{ij}^v - g_{ij}^v\|^2 \right.$$
$$\left. + \sum_{\langle x_{ij}^v - y_{ij}^v, f_{ij}^v - g_{ij}^v \rangle < 0} \min_{d_{ij}^v \geq 0} \|f_{ij}^v - g_{ij}^v + d_{ij}^v(x_{ij}^v - y_{ij}^v)\|^2 \right)^{1/2}$$

$$\geq |f_{\bar{\imath}}^v - g_{\bar{\imath}}^v| \geq \frac{|f_{\bar{\imath}}^v - g_{\bar{\imath}}^v||x_{\bar{\imath}}^v - y_{\bar{\imath}}^v|}{\|x - y\|} \geq \frac{\rho}{r^2} \|x - y\|.$$

The latter case is similarly established. $\qquad\square$

In general, the converse of Theorem 3.1 is not true; e.g., when K is polyhedral, the uniform **Cartesian P**-property coincides with the uniform **P**-property, which is not implied by **uniform nonsingularity** (see [12, Example 4.1]). On the other hand, the next proposition shows that the converse does hold for *Löwner's operators*. Given a scalar valued function $\phi : \mathbb{R} \to \mathbb{R}$, the corresponding *Löwner's operator* is defined by

$$x \in \mathfrak{J} \mapsto \sum_{i=1}^{r} \phi(\lambda_i(x))c_i,$$

where $x = \sum_{i=1}^{r} \lambda_i(x)c_i$ is any spectral decomposition. We shall denote it by $\phi_{\mathfrak{J}}$.

Theorem 3.2. *For the Löwner's operator $\phi_{\mathfrak{J}}$, the following are equivalent.*

(1) *ϕ is strongly increasing;*
(2) *$\phi_{\mathfrak{J}}$ has **strong monotonicity**.*
(3) *$\phi_{\mathfrak{J}}$ has the uniform **Cartesian P**-property;*
(4) *$\phi_{\mathfrak{J}}$ is **uniformly nonsingular**;*
(5) *$\phi_{\mathfrak{J}}$ has the uniform **Jordan P**-property;*
(6) *$\phi_{\mathfrak{J}}$ has the uniform **P**-property.*

Proof. (1) $\Longrightarrow$ (2): See Theorem 9 in [23].

(2) $\Longrightarrow$ (3): In general, **strong monotonicity** implies the uniform **Cartesian P**-property as

$$\langle x - y, f(x) - f(y) \rangle = \sum_{v=1}^{\kappa} \langle (x-y)_v, (f(x) - f(y))_v \rangle$$
$$\leq \kappa \max_{v=1,\ldots,\kappa} \langle (x-y)_v, (f(x) - f(y))_v \rangle.$$

(3) $\Longrightarrow$ (4): It follows from Theorem 3.1.

(4) $\Longrightarrow$ (1): By using multiples of the unit e and the choices $d_i = d$ and $d_{ij} = 0$ for all i, j in the definition of **uniform nonsingularity**, we deduce that for any $d \geq 0$ and any $x > y$,

$$|\phi(x) - \phi(y) + d(x-y)| \|e\| = \|\phi_{\mathfrak{Z}}(xe) - \phi_{\mathfrak{Z}}(ye) + d(x-y)e\| \geq \alpha(x-y)\|e\|.$$

Hence $\phi(x) \geq \phi(y)$ for any $x > y$. Moreover, with $d = 0$, we get $\phi(x) - \phi(y) = |\phi(x) - \phi(y)| \geq \alpha(x-y)$; i.e., ϕ is strongly increasing.

The implications $(2) \Longrightarrow (5) \Longrightarrow (6)$ follows from [30, Proposition 3.1].

(6) $\Longrightarrow$ (1): By using multiples of the unit e in the definition of the uniform **P**-property, we deduce that for any $x > y$,

$$(\phi(x) - \phi(y))(x - y) = \lambda_1((\phi_{\mathfrak{Z}}(xe) - \phi_{\mathfrak{Z}}(ye)) \circ (x-y)e) \geq \alpha(x-y)^2.$$

$\square$

Next, we consider another special transformation, namely the Lyapunov transformation.

Theorem 3.3. *For the Lyapunov transformation l_a, the following are equivalent:*

(1) $a \in \text{int}(K)$;
(2) l_a *has **strict monotonicity**;*
(3) l_a *has the **Cartesian P**-property;*
(4) l_a *is **uniformly nonsingular**;*
(5) l_a *has the **Jordan P**-property;*
(6) l_a *has the **P**-property.*

Proof. (1) $\Longrightarrow$ (2): It follows from Theorem 2.2.

(2) $\Longrightarrow$ (3): See proof of Theorem 3.2.

(3) $\Longrightarrow$ (4): See Proposition 4.5 in [12].

(4) $\Longrightarrow$ (6): It follows from [12, Proposition 4.4].

(6) $\Longrightarrow$ (1): Let $a = \sum \lambda_i(a)c_i$ be a spectral decomposition. Let I_- be the index set $\{i : \lambda_i(a) \leq 0\}$, which is empty if and only if $a \in \text{int}(K)$; see Theorem 2.2. Consider $x = \sum_{i \in I_-} c_i$, which is zero if and only if I_- is empty. Noting that x and $l_a(x)$ operator commute and

$$x \circ l_a(x) = \sum_{i \in I_-} \lambda_i(a)c_i \leq_K 0,$$

we deduce that if l_a satisfies the **P**-property, then x must be zero, which eventually leads to $a \in \text{int}(K)$.

The implication $(2) \Longrightarrow (5) \Longrightarrow (6)$ follows from [18, Theorem 11], $\square$

4. Applications to nonlinear complementarity problems

4.1. **Smoothing Approximation.** In [5], Chen and Mangasarian introduced a class of smoothing functions $p(\mu, z)$ to approximate the plus function $z^+ := \max\{0, z\}$ by double integrating a probability density function d with parameter $\mu > 0$, so that $p(\mu, z) \to z^+ =: p(0, z)$ as $\mu \downarrow 0$. It turns out that their proposal laid down a basis for many smoothing algorithms for solving complementarity problems; see, e.g., [1, 3, 4, 9, 25] and the references therein. In this section, we focus on a subclass of the Chen-Mangasarian smoothing functions to be used for the smoothing/continuation method mentioned in the introductory section. In particular, we assume that the probability density function has the following properties.

(A1) $d(t)$ is symmetric and piecewise continuous with finite number of pieces.
(A2) $E[|t|]_{d(t)} = \int_{-\infty}^{+\infty} |t| d(t) dt = B < +\infty$.
(A3) $d(t)$ has an infinite support.

Under assumptions (A1) and (A2), the function $p(\mu, z)$ is equivalent to defining

$$(2) \qquad p(\mu, z) = \int_{-\infty}^{z} (z - t) d(\mu, t) dt \quad (\mu > 0),$$

where $d(\mu, t) := \frac{1}{\mu} d(\frac{t}{\mu})$; see proof of Proposition 2.1 of [5].

We summarize below some useful properties of the function $p(\mu, z)$.

Proposition 4.1. *Let $d(t)$ satisfy (A1)–(A3). The following properties hold for the function $p(\mu, z)$ defined in* (2).

(1) *$p(\mu, z)$ is convex and continuously differentiable.*
(2) *$\lim_{z \to -\infty} p(\mu, z) = 0$, $\lim_{z \to \infty} p(\mu, z)/z = 1$, $0 < \frac{\partial}{\partial z} p(\mu, z) < 1$ for all $\mu > 0$.*
(3) *$0 < \frac{\partial}{\partial \mu} p(\mu, z) < B$ for all $z \in \mathbb{R}$.*
(4) *For each $z \in \mathbb{R}$, the function $\mu \in \mathbb{R}_{++} \mapsto p(\mu, z)$ is Lipschitz continuous; moreover, the Lipschitz constant is uniformly bounded above by B over all $z \in \mathbb{R}$.*

Proof. The first two conclusions have been shown in Proposition 1 of [4]. A straightforward calculation shows that

$$\frac{\partial}{\partial \mu} p(\mu, z) = \int_{z/\mu}^{+\infty} t d(t) dt.$$

Thus, the third conclusion follows immediately from assumptions (A1) and (A2), whence the last conclusion follows from the Mean Value Theorem. $\qquad \square$

The following are two well-known smoothing functions derived from probability density functions satisfying assumptions (A1)–(A3).

Example 4.1. *Neural network smoothing function* [5].

$$p(\mu, z) = z + \mu \log(1 + e^{-\frac{z}{\mu}}),$$

where $d(t) = e^{-t}/(1 + e^{-t})^2$.

Example 4.2. *Chen-Harker-Kanzow-Smale (CHKS) function* [2, 21, 28].

$$p(\mu, z) = (z + \sqrt{z^2 + 4\mu})/2,$$

where $d(t) = 2/(t^2 + 4)^{\frac{3}{2}}$.

Henceforth, we shall assume that the smoothing function $p(\mu, z)$ is defined with a density function satisfying (A1)–(A3).

With the help of the Chen and Mangasarian smoothing functions, we can define the smoothing approximation of the Euclidean projector $\mathrm{Proj}_K(z)$ as the Löwner's operator

$$(3) \qquad p_{\mathfrak{J}}(\mu, \cdot) : z \in \mathfrak{J} \mapsto \sum_{i=1}^r p(\mu, \lambda_i(z)) c_i,$$

where $z = \sum_{i=1}^r \lambda_i(z) c_i$ is a spectral decomposition of z. For instance, the Löwner's operator obtained from the CHKS smoothing function is

$$p_{\mathfrak{J}}(\mu, z) = (z + \sqrt{z^2 + 4\mu e})/2,$$

where $\sqrt{x}$ denotes the unique $y \in \mathrm{int}(K)$ with $y^2 = x$.

We formally state several well-known results on the smooth approximation $p_{\mathfrak{J}}(\mu, z)$ in the following proposition.

Proposition 4.2 (Proposition 3.2 of [12]). *The following statements are true.*

(a) $\lim_{\mu \downarrow 0} p_{\mathfrak{J}}(\mu, z) = \mathrm{Proj}_K(z)$.

(b) $p_{\mathfrak{J}}(\mu, z)$ *is continuously differentiable.*

(c) *For each $\mu > 0$, the Jacobian of the map $z \mapsto p_{\mathfrak{J}}(\mu, z)$ is*

$$w \mapsto \sum d_i w_i c_i + \sum d_{ij} w_{ij}$$

where $z = \sum \lambda_i(z) c_i$ is a spectral decomposition, $w = \sum w_i c_i + \sum w_{ij}$ is the Peirce decomposition, $d_i = \frac{\partial}{\partial \mu} p(\mu, \lambda_i(z))$ and

$$d_{ij} = \begin{cases} \frac{p(\mu, \lambda_i(z)) - p(\mu, \lambda_j(z))}{\lambda_i(z) - \lambda_j(z)} & \text{if } \lambda_i(z) \neq \lambda_j(z), \\ \frac{\partial}{\partial \mu} p(\mu, \lambda_i(z)) & \text{if } \lambda_i(z) = \lambda_j(z). \end{cases}$$

Moreover, $d_i, d_{ij} \in (0, 1)$.

(d) *For each $\mu \geq 0$ and each $b >_K 0$, $p_{\mathfrak{J}}(\mu, z) = b$ has a unique solution.*

4.2. Numerical Methods.

In this subsection, we apply **uniform nonsingularity** to two numerical approaches for solving complementarity problems.

4.2.1. *Non-interior continuation method.* In [11], Chen and Tseng developed a non-interior continuation method for solving monotone SDCPs. Later, Chen and Qi [7] showed that Chen-Tseng's algorithm can be applied to the **Cartesian P**-SDLCPs. In this section, we demonstrate that **uniform nonsingularity** ensures the boundedness of neighborhoods as well as the nonsingularity of the Jacobian used in the non-continuation method, and hence extend Chen-Tseng's algorithm to a more general class of nonmonotone and nonlinear problems.

Using the fix-point formulation and the smoothing projection function $p_{\mathfrak{J}}$, the algorithm uses the equation $H_\mu(x, y) = 0$, where

$$H_\mu(x, y) = (\phi_\mu(x, y), f(x) - y)$$

and $\phi_\mu(x, y) = x - p_{\mathfrak{J}}(\mu, x - y)$, to approximate (1).

Lemma 4.1. *Fix any $\mu \in \mathbb{R}_{++}$, any $x, y, u, v \in \mathfrak{J}$. We have that ϕ_μ is continuously differentiable and*

$$J\phi_\mu(x, y)(u, v) = \sum ((1 - d_i) u_i + d_i v_i) c_i + \sum ((1 - d_{ij}) u_{ij} + d_{ij} v_{ij}),$$

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where $x - y = \sum \lambda_i(x - y)c_i$ is a spectral decomposition, $u = \sum u_i c_i + \sum u_{ij}$ and $v = \sum v_i c_i + \sum v_{ij}$ are Peirce decompositions, $d_i = \frac{\partial}{\partial \mu} p(\mu, \lambda_i(x - y))$ and

$$d_{ij} = \begin{cases} \frac{p(\mu, \lambda_i(x-y)) - p(\mu, \lambda_j(x-y))}{\lambda_i(x-y) - \lambda_j(x-y)} & \text{if } \lambda_i(x - y) \neq \lambda_j(x - y), \\ \frac{\partial}{\partial \mu} p(\mu, \lambda_i(x - y)) & \text{if } \lambda_i(x - y) = \lambda_j(x - y). \end{cases}$$

Proof. It follows from Proposition 4.2 and the Chain Rule. $\qquad \square$

For a fixed $\mu > 0$, each step of the Chen-Tseng's algorithm solves the equation $JH_\mu(x, y)(u, v) = (r, s)$ for some $(r, s) \in \mathfrak{J} \times \mathfrak{J}$ to update the Newton direction, and then decreases the parameter. For the convergence analysis, we need the following neighborhood

(4) $$\mathcal{N}_\beta := \{(\mu, x, y) \in \mathbb{R}_{++} \times \mathfrak{J} \times \mathfrak{J} : \|H_\mu(x, y)\| \leq \beta\mu\},$$

where $\beta \in \mathbb{R}_{++}$ is a constant.

With the assumption of monotonicity replaced by the **Cartesian P**-property, Chen and Qi [7] established the nonsingularity of the Jacobian $JH_\mu(x, y)$ and the boundedness of the neighborhood above for the SDLCP. Here, we further relax the **Cartesian P**-property requirement by **uniform nonsingularity**.

First, we address the boundedness of the neighborhood $\mathcal{N}_\beta$. Instead of verifying those conditions in [11], we give a direct proof under the assumption of **uniform nonsingularity**.

Proposition 4.3. *If $f(x)$ is **uniformly nonsingular**, then the set*

$$\{(\mu, x, y) \in \mathcal{N}_\beta : 0 < \mu \leq \mu_0\}$$

is bounded for any $\beta > 0$ and $\mu_0 > 0$.

Proof. Suppose on the contrary that there exist $\beta, \mu_0 > 0$ and a sequence $(\mu_k, x^k, y^k) \in \mathcal{N}_\beta$ with $0 < \mu_k \leq \mu_0$ and $\|x^k\| \to \infty$. By (4), for each k, we have that

(5) $$\|x^k - p_{\mathfrak{J}}(\mu_k, x^k - y^k)\| \leq \beta\mu_k,$$

and

(6) $$\|f(x^k) - y^k\| \leq \beta\mu_k.$$

Then by the **uniform nonsingularity** of f, it must happen that $\|f(x^k) - f(0)\| \geq \alpha\|x^k\| \to \infty$, which together with (6) implies that $\|y^k\| \to \infty$.

Let $x^k - y^k = \sum \lambda_i(x^k - y^k)c_i^k$ be a spectral decomposition, and let $x^k = \sum x_i^k c_i^k + \sum x_{ij}^k$, $y^k = \sum y_i^k c_i^k + \sum y_{ij}^k$, and $f(x^k) = \sum f_i^k c_i^k + \sum f_{ij}^k$ be Peirce decompositions. Since $x^k - y^k = \sum \lambda_i(x^k - y^k)c_i^k = \sum(x_i^k - y_i^k)c_i^k + \sum(x_{ij}^k - y_{ij}^k)$ are both Peirce decompositions with respect to $\{c_1^k, \ldots, c_r^k\}$, it follows that $\lambda_i(x^k - y^k) = x_i^k - y_i^k$ and $x_{ij}^k = y_{ij}^k$. Therefore $p_{\mathfrak{J}}(\mu_k, x^k - y^k) = \sum p(\mu_k, x_i^k - y_i^k)c_i^k$, and thus (5) implies that $\{\|x_{ij}^k\|\}$, whence $\{\|y_{ij}^k\|\}$, is bounded for all i, j, which further implies that $\{\|f_{ij}^k\|\}$ is bounded, and that the index set $I := \{i : \{x_i^k\} \text{ is unbounded}\}$ is nonempty. Observe that $p(\mu_k, \lambda_i(x^k - y^k)) \geq 0$. Hence, by taking a subsequence if necessary, we obtain that that $x_i^k \to +\infty$ for all $i \in I$.

Consider the bounded sequence $\{\tilde{x}^k := \sum_{i \notin I} x_i^k c_i^k + \sum x_{ij}^k\}$. Let $f(x^k) - f(\tilde{x}^k) = \sum \tilde{g}_i^k c_i^k + \sum \tilde{g}_{ij}^k$ be a Peirce decomposition. Consider another bounded sequence

$$\left\{\hat{x}^k := \tilde{x}^k + \sum_{i \notin I} \frac{\varepsilon}{|\tilde{g}_i^k| + 1}\tilde{g}_i^k c_i^k + \sum \frac{\varepsilon}{\|\tilde{g}_{ij}^k\| + 1}\tilde{g}_{ij}^k\right\},$$

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where $\varepsilon = \frac{1}{2}$. Let $f(x^k) - f(\hat{x}^k) = \sum \hat{g}_i^k c_i^k + \sum \hat{g}_{ij}^k$ be a Peirce decomposition. For $i \in I$, let $d_i^k = \max\left\{0, -\frac{\tilde{g}_i^k}{x_i^k}\right\}$; for $i \notin I$, let $d_i^k = \frac{1}{\varepsilon}(|\tilde{g}_i^k| + 1)$. Let $d_{ij}^k = \frac{1}{\varepsilon}(\|\tilde{g}_{ij}^k\| + 1)$ for $1 \leq i < j \leq r$. By construction, $d_i^k \geq 0$ and $d_{ij}^k > 0$. Recall **uniform nonsingularity**:

$$\left\| \begin{array}{c} \sum \hat{g}_i^k c_i^k + \sum \hat{g}_{ij}^k + \sum_{i \in I} d_i^k x_i^k c_i^k \\ -\sum_{i \notin I} d_i^k \frac{\varepsilon}{|\tilde{g}_i^k| + 1} \tilde{g}_i^k c_i^k - \sum d_{ij}^k \frac{\varepsilon}{\|\tilde{g}_{ij}^k\| + 1} \tilde{g}_{ij}^k \end{array} \right\| \geq \alpha \|x^k - \hat{x}^k\|,$$

which simplifies to

$$\left\| f(\tilde{x}^k) - f(\hat{x}^k) + \sum_{i \in I, \tilde{g}_i^k > 0} \tilde{g}_i^k c_i^k \right\| \geq \alpha \|x^k - \hat{x}^k\|.$$

As $k \to +\infty$, the right hand side tends to $+\infty$ while $\{(f(\tilde{x}^k), f(\hat{x}^k))\}$ remains bounded. Thus, by taking a subsequence if necessary, we conclude that there is a index $i \in I$ such that $|\tilde{g}_i^k| \to +\infty$ and $g_i^k > 0$ for all k; i.e., there is some index i with $x_i^k \to +\infty$ and $\tilde{g}_i^k \to +\infty$. For this i, since $f(\tilde{x}^k)$ is bounded, $\tilde{g}_i^k \to +\infty$ implies $f_i^k \to +\infty$. Therefore $y_i^k \to +\infty$ by (6), and hence it follows that

$$x_i^k - p(\mu_k, x_i^k - y_i^k) \geq x_i^k - B\mu_k - (x_i^k - y_i^k)^+ = \min\{x_i^k, y_i^k\} - B\mu_0 \to +\infty,$$

where we have used Proposition 4.1 in the inequality. This contradicts (5). $\qquad \square$

Next, we establish the nonsingularity of the Jacobian over the neighborhood with the aid of the following lemma.

Lemma 4.2. *If f is **uniformly nonsingular** and bounded Jacobian on the open subset $\Omega \subseteq \mathfrak{J}$, then there exists $\lambda > 0$ such that for any $x \in \Omega$, any $u \in \mathfrak{J}$, any Jordan frame $\{c_1, \ldots, c_r\}$, and any $d_i, d_{ij} \in [0, 1]$, it holds*

$$\left\| \sum \left(d_i f_i^u + (1 - d_i)u_i\right) c_i + \sum \left(d_{ij} f_{ij}^u + (1 - d_{ij})u_{ij}\right) \right\| \geq \lambda \|u\|,$$

where $u = \sum u_i c_i + \sum u_{ij}$ and $Jf(x)u = \sum f_i^u c_i + \sum f_{ij}^u$ are Peirce decompositions.

Proof. Let $m \geq 1$ be an upper bound on $\|Jf(x)\|$ over Ω. Fix an arbitrary $x \in \Omega$ and an arbitrary Jordan frame $\{c_1, \ldots, c_r\}$. Since Ω is open, there exists $\delta > 0$ such that any $x + u \in \Omega$ whenever $\|u\| \leq \delta$. The **uniform nonsingularity** of f gives an $\alpha > 0$ such that

$$\min_{d_i, d_{ij} \geq 0} \left\| \sum \left(\frac{g_i - f_i}{t} + d_i u_i\right) c_i + \sum \left(\frac{g_{ij} - f_{ij}}{t} + d_{ij} u_{ij}\right) \right\| \geq \alpha \|u\|$$

for each $u \in \mathfrak{J}$ and each $t \in (0, \delta\|u\|/(\|u\| + 1))$ (so that $x + tu \in \Omega$), where $f(x) = \sum f_i c_i + \sum f_{ij}$, $f(x + tu) = \sum g_i c_i + \sum g_{ij}$ are Peirce decompositions. Without loss of generality, we assume $\alpha \leq 1$. Taking limit as $t \downarrow 0$ then gives

(7) $$\min_{d_i, d_{ij} \geq 0} \left\| \sum \left(f_i^u + d_i u_i\right) c_i + \sum \left(f_{ij}^u + d_{ij} u_{ij}\right) \right\| \geq \alpha \|u\|$$

for each $u \in \mathfrak{J}$. Fix an arbitrary nonzero $u \in \mathfrak{J}$ and let

$$\bar{\lambda} := \frac{1}{\|u\|} \min_{d_i, d_{ij} \in [0, 1]} \left\| \sum (d_i f_i^u + (1 - d_i)u_i) c_i + \sum (d_{ij} f_{ij}^u + (1 - d_{ij})u_{ij}) \right\|.$$

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Evaluating the above minimum value leads to

$$(8) \quad \bar{\lambda}^2 \|u\|^2 = \sum_{u_i f_i^u \geq 0} \min\{|f_i^u|, |u_i|\}^2 + \sum_{\langle u_{ij}, f_{ij}^u \rangle \geq \min\{\|f_{ij}^u\|, \|u_{ij}\|\}^2} \min\{\|f_{ij}^u\|, \|u_{ij}\|\}^2$$

$$+ \sum_{\langle u_{ij}, f_{ij}^u \rangle < \min\{\|f_{ij}^u\|, \|u_{ij}\|\}^2} \|u_{ij} - p_{ij}\|^2.$$

where, for simplicity of notation, p_{ij} denotes the projection $\mathrm{Proj}_{\mathbb{R}_+(u_{ij} - f_{ij}^u)} u_{ij}$. We shall show that $\bar{\lambda} \geq \min\{\frac{1}{4r^2}, \frac{\alpha}{4r(1+mr^2)}\}$, whence proving the proposition. From here onwards, suppose that $\bar{\lambda} < \frac{1}{4r^2}$.

Consider the perturbation $\bar{u} = \sum \bar{u}_i c_i + \sum \bar{u}_{ij}$, where

$$\bar{u}_i = \begin{cases} -\frac{\alpha}{4mr^3} \frac{\|u\| f_i^u}{|f_i^u|+1} & \text{if } |u_i| \leq \bar{\lambda}\|u\|, \\ u_i & \text{otherwise;} \end{cases} \quad \text{and} \quad \bar{u}_{ij} = \begin{cases} -\frac{\alpha}{4mr^3} \frac{\|u\| f_{ij}^u}{\|f_{ij}^u\|+1} & \text{if } \|u_{ij}\| \leq \bar{\lambda}\|u\|, \\ p_{ij} & \text{if } \|u_{ij}\| > \bar{\lambda}\|u\| \text{ and} \\ & \|u_{ij} - p_{ij}\| \leq \bar{\lambda}\|u\|, \\ u_{ij} & \text{otherwise.} \end{cases}$$

Under the assumption $\bar{\lambda} < \frac{1}{4r^2}$ and $\alpha \leq 1$, it holds $\|u - \bar{u}\| \leq (r^2\bar{\lambda} + \frac{\alpha}{4mr})\|u\| < \frac{1}{2}\|u\|$; so that $\bar{y} \in S$, $\|f^u - f^{\bar{u}}\| \leq (mr^2\bar{\lambda} + \frac{\alpha}{4r})\|u\|$ and

$$(9) \qquad \|\bar{u}\| \geq \|u\| - \|u - \bar{u}\| > \frac{1}{2}\|u\|.$$

We now deduce an upper bound on $\min_{d_i, d_{ij} \geq 0} \|f^{\bar{u}} + \sum d_i \bar{u}_i c_i + \sum d_{ij} \bar{u}_{ij}\|$ in terms of $\bar{\lambda}\|u\|$. First, consider the term $\min_{d_i \geq 0} |f_i^{\bar{u}} + d_i \bar{u}_i|$. If $|u_i| \leq \bar{\lambda}\|u\|$, then

$$\min_{d_i \geq 0} |f_i^{\bar{u}} + d_i \bar{u}_i| \leq \min_{d_i \geq 0} |f_i^u + d_i \bar{u}_i| + |f_i^u - f_i^{\bar{u}}|$$

$$(10) \qquad = |f_i^u - f_i^{\bar{u}}| \leq \|f^u - f^{\bar{u}}\| \leq \left(mr^2\bar{\lambda} + \frac{\alpha}{4r}\right)\|u\|,$$

where the equality follows from $\bar{u}_i f_i^u < 0$ when $f_i^u \neq 0$. If $|u_i| > \bar{\lambda}\|u\|$, then we deduce from (8) that either $u_i f_i^u < 0$ or $|f_i^u| \leq \bar{\lambda}\|u\|$. In the former case, the inequality (10) still holds since $\bar{u}_i = u_i$ by definition. In the latter, a similar inequality holds:

$$\min_{d_i \geq 0} |f_i^{\bar{u}} + d_i \bar{u}_i| \leq \min_{d_i \geq 0} |f_i^u + d_i \bar{u}_i| + |f_i^u - f_i^{\bar{u}}| \leq |f_i^u| + |f_i^u - f_i^{\bar{u}}| \leq \left(\bar{\lambda} + mr^2\bar{\lambda} + \frac{\alpha}{4r}\right)\|u\|.$$

Next, consider the term $\min_{d_{ij} \geq 0} |f_{ij}^{\bar{u}} + d_{ij} \bar{u}_{ij}|$. If $|u_{ij}| \leq \bar{\lambda}\|u\|$, then as before, $\min_{d_{ij} \geq 0} \|f_{ij}^{\bar{u}} + d_{ij} \bar{u}_{ij}\| \leq (mr^2\bar{\lambda} + \frac{\alpha}{4r})\|u\|$ by the definition of $\bar{u}_{ij}$. If $|u_{ij}| > \bar{\lambda}\|u\|$, then we deduce that either $\langle u_{ij}, f_{ij}^u \rangle < \min\{\|f_{ij}^u\|, \|u_{ij}\|\}^2$ and $\|u_{ij} - p_{ij}\| \leq \bar{\lambda}\|u\|$, or $\|f_{ij}^u\| \leq \bar{\lambda}\|u\|$. In the former case, $\bar{u}_{ij} = p_{ij} \in \mathbb{R}_+(u_{ij} - f_{ij}^u)$ has negative inner product with f_{ij}^u, whence the minimum of $\|f_{ij}^u + d_{ij} \bar{u}_{ij}\|$ over $d_{ij} \geq 0$ is attained at $d_{ij} = \frac{1}{\|p_{ij}\|}\|\mathrm{Proj}_{\mathbb{R}_+(-p_{ij})} f_{ij}^u\|$ with the value

$$\|f_{ij}^u - \mathrm{Proj}_{\mathbb{R}_+(f_{ij}^u - u_{ij})} f_{ij}^u\|$$

$$= \|f_{ij}^u - \mathrm{Proj}_{\mathbb{R}(f_{ij}^u - u_{ij})} f_{ij}^u\|$$

$$= \|f_{ij}^u - u_{ij} + u_{ij} - \mathrm{Proj}_{\mathbb{R}(f_{ij}^u - u_{ij})}(f_{ij}^u - u_{ij} + u_{ij})\|$$

$$= \|f_{ij}^u - u_{ij} - \mathrm{Proj}_{\mathbb{R}(f_{ij}^u - u_{ij})}(f_{ij}^u - u_{ij}) + u_{ij} - \mathrm{Proj}_{\mathbb{R}(f_{ij}^u - u_{ij})} u_{ij}\|$$

$$= \|f_{ij}^u - u_{ij} - \mathrm{Proj}_{\mathbb{R}(f_{ij}^u - u_{ij})}(f_{ij}^u - u_{ij}) + u_{ij} - \mathrm{Proj}_{\mathbb{R}_+(u_{ij} - f_{ij}^u)} u_{ij}\|$$

$$= \|u_{ij} - p_{ij}\| \leq \bar{\lambda}\|u\|.$$

In the latter, $\min_{d_{ij}\geq0}\|f_{ij}^u + d_{ij}\bar{u}_{ij}\| \leq \|f_{ij}^u\| \leq \bar{\lambda}\|u\|$. Thus in either cases,

$$\min_{d_{ij}\geq0}\|f_{ij}^{\bar{u}} + d_{ij}\bar{u}_{ij}\| \leq \min_{d_{ij}\geq0}\|f_{ij}^u + d_{ij}\bar{u}_{ij}\| + \|f_{ij}^u - f_{ij}^{\bar{u}}\|$$

$$\leq \bar{\lambda}\|u\| + \|f_{ij}^u - f_{ij}^{\bar{u}}\| \leq \left(\bar{\lambda} + mr^2\bar{\lambda} + \frac{\alpha}{4r}\right)\|u\|.$$

In summary, $((1 + mr^2)\bar{\lambda} + \frac{\alpha}{4r})\|u\|$ is an upper bound on all terms, whence leading to

$$\min_{d_i,d_{ij}\geq0}\left\|f^{\bar{u}} + \sum d_i\bar{u}_ic_i + \sum d_{ij}\bar{u}_{ij}\right\| \leq r\left((1 + mr^2)\bar{\lambda} + \frac{\alpha}{4r}\right)\|u\|$$

$$< 2r\left((1 + mr^2)\bar{\lambda} + \frac{\alpha}{4r}\right)\|\bar{u}\|,$$

where the last inequality follows from (9). Finally, it follows from (7) that $2r((1+mr^2)\bar{\lambda}+\frac{\alpha}{4r}) \geq \alpha$, whence $\bar{\lambda} \geq \frac{\alpha}{4r(1+mr^2)}$. $\qquad\square$

Proposition 4.4. *Let C be a compact subset of $\mathfrak{J}$. If f is **uniformly nonsingular**, then for any $\mu \in \mathbb{R}_{++}$, and any $(x,y) \in C \times \mathfrak{J}$ the Jacobian $JH_\mu(x,y)$ is nonsingular. Moreover, we have that*

$$\sup_{(\mu,x,y)\in\mathbb{R}_{++}\times C\times\mathfrak{J}}\|JH_\mu(x,y)^{-1}\| < \infty.$$

Proof. Since f is continuously differentiable, its Jacobian is uniformly bounded on the compact set $\bar{C} := \{x + u : x \in C, \|u\| \leq 1\}$. Thus by Lemma 4.2, the **uniform nonsingularity** of f implies that for any $x \in C \subset \text{int}(\bar{C})$, any $u \in \mathfrak{J}$, any Jordan frame $\{c_1,\ldots,c_r\}$, and any $d_i, d_{ij} \in [0,1]$, it holds

$$(11) \qquad \left\|\sum(d_if_i^u + (1-d_i)u_i)c_i + \sum\left(d_{ij}f_{ij}^u + (1-d_{ij})u_{ij}\right)\right\| \geq \lambda\|u\|,$$

where $Jf(x)u = \sum f_i^uc_i + \sum f_{ij}^u$ is the Peirce decomposition.

We now show that for any $(\mu, x, y) \in \mathbb{R}_{++}\times C\times\mathfrak{J}$, the linear system $JH_\mu(x,y)(u,v) = 0$ only has the trivial solution; i.e., zero is the only solution of the following linear system

$$J\phi_\mu(x,y)(u,v) = 0, \quad Jf(x)u - v = 0.$$

Eliminating v from above equation, we obtain from Lemma 4.1 that

$$\sum(d_if_i^u + (1-d_i)u_i)c_i + \sum(d_{ij}f_{ij}^u + (1-d_{ij})u_{ij}) = 0$$

where $x - y = \sum\lambda_i(x-y)c_i$ is a spectral decomposition, and $u = \sum u_ic_i + \sum u_{ij}$ and $Jf(x)u = \sum f_i^uc_i + \sum f_{ij}^u$ are Peirce decompositions, and $d_i, d_{ij} \in (0,1)$. It is easy to see that $u = 0$ follows immediately from (11), and thus $v = Jf(u) = 0$.

Now we know that for any $(r,s) \in \mathfrak{J}\times\mathfrak{J}$, there is a unique $(u,v) \in \mathfrak{J}\times\mathfrak{J}$ such that

$$\sum(d_iv_i + (1-d_i)u_i)c_i + \sum(d_{ij}v_{ij} + (1-d_{ij})u_{ij}) = r, \quad Jf(u) - v = s.$$

Let $v = \sum v_ic_i + \sum v_{ij}$, $r = \sum r_ic_i + \sum r_{ij}$ and $s = \sum s_ic_i + \sum s_{ij}$ be Peirce decompositions. Eliminating v and rearranging the above equation, we get

$$\sum(d_if_i^u + (1-d_i)u_i)c_i + \sum(d_{ij}f_{ij}^u + (1-d_{ij})u_{ij}) = \sum(r_i + d_is_i)c_i + \sum(r_{ij} + d_{ij}s_{ij}),$$

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Since $0 < d_i, d_{ij} < 1$, it follows that

$$\left\| \sum (r_i + d_i s_i) c_i + \sum (r_{ij} + d_{ij} s_{ij}) \right\|^2 \leq \left(\|r\| + \left\| \sum d_i s_i c_i + \sum d_{ij} s_{ij} \right\| \right)^2$$

$$\leq 2 \left(\|r\|^2 + \left\| \sum d_i s_i c_i + \sum d_{ij} s_{ij} \right\|^2 \right)$$

$$\leq 2(\|r\|^2 + \|s\|^2).$$

Together with (11), we deduce that $\|u\| \leq \frac{\sqrt{2}}{\lambda} \sqrt{\|r\|^2 + \|s\|^2}$. Thus

$$\|v\| \leq m\|u\| + \|s\| \leq \left(\frac{\sqrt{2}}{\lambda} m + 1 \right) \sqrt{\|r\|^2 + \|s\|^2},$$

where m is an upper bound on the continuous transformation $x \mapsto \|Jf(x)\|$ over the compact set C. Consequently $\|JH_\mu(x,y)^{-1}\|$ is uniformly bounded over $\{(\mu, x, y) \in \mathcal{N}_\beta : \mu \in (0, \mu_0]\}$. $\square$

Due to Proposition 4.4 and 4.3, we conclude by Proposition 1 in [11], the following.

Theorem 4.1. *If the transformation $f : \mathfrak{J} \to \mathfrak{J}$ is **uniformly nonsingular**, then the Chen-Tseng's algorithm, when applied to $NCP_{K(\mathfrak{J})}(f)$, is globally convergent with linear convergence rate.*

4.2.2. *Squared smoothing Newton method.* The squared smoothing Newton method designed for solving nonsmooth matrix equations in [26] achieves a quadratic convergence rate applied to SDCPs. Two essential concepts for the convergence analysis are the strong semismoothness and the nonsingularity of B-subdifferential of the squared smoothing function. To establish this, they used the condition that the Jacobian of the problem is positive definite on the affine hull of the critical cone at the solution. In this section, we replace this assumption by **uniform nonsingularity** and extend the convergence result for a class of smoothing functions.

Let $G(\mu, x) := x - p_\mathfrak{J}(\mu, x - f(x))$, then G is continuously differentiable at each $(\mu, x) \in \mathbb{R}_{++} \times \mathfrak{J}$. The squared smoothing Newton method uses the merit function $\phi(\mu, x) := \|E(\mu, x)\|^2$ for the line search, where

$$E(\mu, x) := \begin{bmatrix} \mu \\ G(\mu, x) \end{bmatrix}.$$

First, we establish the nonsingularity of the Jacobian of E.

Proposition 4.5. *Suppose that f is **uniformly nonsingular**. Then the Jacobian $JE(\mu, x)$ is nonsingular for all $(\mu, x) \in \mathbb{R}_{++} \times \mathfrak{J}$. Moreover, if C is a compact subset of $\mathfrak{J}$, then there exists $\gamma > 0$ such that for any $(\mu, x) \in \mathbb{R}_{++} \times C$ and $(\tau, h) \in \mathbb{R} \times \mathfrak{J}$, it holds that*

$$(12) \qquad \|JE(\mu, x)(\tau, h)\| \geq \gamma \|(\tau, h)\|.$$

Proof. Fix any $(\mu, x) \in \mathbb{R}_{++} \times C$. Suppose that there exists $(\tau, h) \in \mathbb{R} \times \mathfrak{J}$ such that $JE(\mu, x)(\tau, h) = 0$, i.e.,

$$(13) \qquad \begin{bmatrix} \tau \\ \tau \nabla_\mu G(\mu, x) + J_x G(\mu, x) h \end{bmatrix} = 0,$$

or equivalently,

$$\tau = 0, \quad J_x G(\mu, x) h = 0.$$

Let $x - f(x) = \sum \lambda_i c_i$ be a spectral decomposition. Let $h = \sum h_i c_i + \sum h_{ij}$ and $Jf(x)h = \sum f_i^h c_i + \sum f_{ij}^h$ be Peirce decompositions. We deduce from Lemma 4.2 that

$$\|J_x G(\mu, x)h\| = \left\| \sum (d_i f_i^h + (1 - d_i)h_i)c_i + \sum (d_{ij} f_{ij}^h + (1 - d_{ij})h_{ij}) \right\| \geq \lambda \|h\|.$$

Thus, $J_x G(\mu, x)h = 0$ implies that $h = 0$, and hence $JE(\mu, x)$ is nonsingular.

"Moreover": Take $m \geq 1$ such that $\|\nabla_\mu G(\mu, x)\| \leq \sqrt{r}B \leq m$, $\lambda \geq \frac{1}{m}$, and $\sqrt{1 - \frac{1}{m^8}} > \frac{1}{m^2}$.

If $\tau \geq \frac{1}{m^4}\|(\tau, h)\|$, then it follows from (13) that

$$\|JE(\mu, x)(\tau, h)\| = \|(\tau, \tau\nabla_\mu G(\mu, x) + J_x G(\mu, x)h)\| \geq \frac{1}{m^4}\|(\tau, h)\|.$$

If $\tau < \frac{1}{m^4}\|(\tau, h)\|$, then we have $\|h\| \geq \sqrt{1 - \frac{1}{m^8}}\|(\tau, h)\|$, whence

$$\|JE(\mu, x)(\tau, h)\| \geq \|\tau\nabla_\mu G(\mu, x) + J_x G(\mu, x)h\| \geq \|J_x G(\mu, x)h\| - \|\tau\nabla_\mu G(\mu, x)\|$$

$$\geq \left(\frac{1}{m}\sqrt{1 - \frac{1}{m^8}} - \frac{1}{m^3} \right) \|(\tau, h)\| \geq \frac{1}{m^3}\|(\tau, h)\|.$$

Hence, (12) is satisfied by taking $\gamma = \frac{1}{m^4}$. $\qquad\square$

For the quadratic convergence of the squared smoothing method, one of the key issues is the nonsingularity of the B-subdifferential of E at $(0, x)$. Since E is locally Lipschitz continuous, it is differentiable almost everywhere by Rademacher's theorem. Let D_E be the set of points at which E is differentiable. The B-subdifferential of E at $(0, x)$, denoted $\partial_B E(\mu, x)$, is defined by

$$\partial_B E(0, x) = \left\{ \lim_{k \to \infty} JE(\mu^k, x^k) : (\mu^k, x^k) \in D_E, \ (\mu^k, x^k) \to (0, x) \right\}.$$

Proposition 4.6. *If f is **uniformly nonsingular**, then every $U \in \partial_B E(0, x)$ is nonsingular.*

Proof. Let U be an element of $\partial_B E(0, x)$. According to the definition of B-subdifferential of E, there is a sequence $\{(\mu^k, x^k)\} \subset D_E$ converging to $(0, x)$ such that

$$U = \lim_{k \to \infty} JE(\mu^k, x^k).$$

Since E is continuously differentiable, we may assume that $\mu^k > 0$ for all k by replacing μ^k with $\mu^k + \frac{1}{k}$. Since x^k converges to x, all x^k will be in some compact set containing x. Thus, by Proposition 4.5,

$$\|U(\tau, h)\| = \lim_{k \to \infty} \|JE(\mu^k, x^k)(\tau, h)\| \geq \gamma\|(\tau, h)\| \quad \forall (\tau, h) \in \mathbb{R} \times \mathfrak{J}$$

implying that U is nonsingular. $\qquad\square$

The other key condition, namely the strong semismoothness of E at $(0, x)$, follows from the strong semismoothness of the projection function over a symmetric cone (see Proposition 3.3 in [29]) and the additional assumption that f has locally Lipschitz Jacobian. Thus, combining Propositions 4.5 and 4.6, we conclude by Theorem 4.2 of [26], the following.

Theorem 4.2. *If the transformation $f : \mathfrak{J} \to \mathfrak{J}$ is **uniformly nonsingular**, then the Qi-Sun-Sun's squared smoothing Newton method, when applied to $NCP_{K(\mathfrak{J})}(f)$ is globally convergent. If, in addition, f has Lipschitz continuous Jacobian around the solution, then the algorithm converges quadratically around the solution.*

We conclude this section by pointing out that the conclusions above is not restricted to the squared smoothing function—it is applicable to a whole class of smoothing functions.

5. GUS-PROPERTY

In this section, we study the unique solvability of $\mathrm{NCP}_K(f)$ under the assumption **uniform nonsingularity** of the nonlinear transformation $f : \mathfrak{J} \to \mathfrak{J}$. This leads to the **GUS**-property of the linear transformation $l : \mathfrak{J} \to \mathfrak{J}$ under **uniform nonsingularity**.

Definition 5.1. A linear transformation $l : \mathfrak{J} \to \mathfrak{J}$ is said to have the *globally unique solvability- (**GUS**-)property* if $\mathrm{LCP}_K(l, q)$ is uniquely solvable for every $q \in \mathfrak{J}$.

To motivate the research, we consider the map $H : \mathbb{R}_+ \times \mathfrak{J} \to \mathfrak{J}$ defined by

$$(14) \qquad (\mu, z) \mapsto (1 - \mu)f(p_{\mathfrak{J}}(\mu, z)) - p_{\mathfrak{J}}(\mu, -z) + \mu b,$$

where $p_{\mathfrak{J}}(\mu, z)$ is the smoothing approximation of $\mathrm{Proj}_K(z)$ defined in (3), and $b \in \mathfrak{J}$ is fixed. This map is a homotopy between $z \mapsto b - p_{\mathfrak{J}}(1, -z)$ and $z \mapsto f(\mathrm{Proj}_K(z)) + \mathrm{Proj}_{-K}(z)$. When $b >_K 0$ and $\mu \in (0, 1]$ is fixed, we refer the equation $H(\mu, z) = 0$ as the smoothing normal map equation (SNME). When $\mu = 1$, the SNME $p_{\mathfrak{J}}(1, -z) = b$ has a unique solution by part (d) of Proposition 4.2. In the limit $\mu \downarrow 0$, the SNME reduces to the normal map equation (NME) $f(\mathrm{Proj}_K(z)) + \mathrm{Proj}_{-K}(z) = 0$, which gives a solution to the $\mathrm{NCP}_K(f)$ via $x = \mathrm{Proj}_K(z)$. This homotopy was used in a recent paper [12] to describe and analyze a continuation method for $\mathrm{NCP}_K(f)$.

The following lemma extracted from [12] will be used in proving our main result for this section.

Lemma 5.1 (Proposition 5.3 of [12]). *If f is **uniformly nonsingular**, then for each fixed $b >_K 0$,*
  (a) *the SNME has a unique solution $z(\mu)$ for each $\mu \in (0, 1]$;*
  (b) *the set $T := \{(\mu, z(\mu)) : \mu \in (0, 1]\}$ is bounded;*
  (c) *every accumulation point $(0, z^*)$ of T gives a solution z^* to the NME.*

Basically, the above result shows that under **uniform nonsingularity**, the set T is bounded and forms a smooth trajectory. Moreover, it has at least one accumulation point as the parameter μ decreases to zero along the trajectory, and every accumulation point is a solution of the NME. However, we have not ruled out the possibility of multiple accumulation points. By exploiting **uniform nonsingularity**, we next show that T actually converges, and that every solution of the NME is the limit of the trajectory with $b = e$, whence establish the unique solvability of f.

We begin with the following lemma, which is a refinement of [12, Proposition 5.2].

Lemma 5.2. *If f is **uniformly nonsingular** and $C \subset \mathfrak{J}$ is compact, then there exists $\sigma > 0$ such that for all $(\mu, z) \in (0, \frac{1}{2}] \times C$ it holds that*

$$\|J_z H(\mu, z)w\| \geq \sigma \|w\|,$$

for any $w \in \mathfrak{J}$.

Proof. The transformation $p_{\mathfrak{J}}$ is continuous on the compact set $[0, \frac{1}{2}] \times C$ and f is continuously differentiable. Hence the set $p_{\mathfrak{J}}([0, \frac{1}{2}] \times C)$ is compact, and the Jacobian Jf is uniformly bounded, say by m, on this compact set; i.e., $\|Jf(p_{\mathfrak{J}}(\mu, z))\| \leq m$ for all $(\mu, z) \in [0, \frac{1}{2}] \times C$.

Fix any $(\mu, z) \in (0, \frac{1}{2}) \times C$ and let $z = \sum_{i=1}^{r} \lambda_i c_i$ be a spectral decomposition. For any $w \in \mathfrak{J}$, we have

$$(15) \qquad J_z H(\mu, z) w = (1 - \mu) J f(p_{\mathfrak{J}}(\mu, z)) \left[\sum d_i w_i c_i + \sum d_{ij} w_{ij} \right]$$
$$+ \sum (1 - d_i) w_i c_i + \sum (1 - d_{ij}) w_{ij}$$

where $w = \sum w_i c_i + \sum w_{ij}$ is a Peirce decomposition, $d_i = \frac{\partial}{\partial \mu} p(\mu, \lambda_i)$ and

$$d_{ij} = \begin{cases} \frac{p(\mu, \lambda_i) - p(\mu, \lambda_j)}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ \frac{\partial}{\partial \mu} p(\mu, \lambda_i) & \text{if } \lambda_i = \lambda_j, \end{cases}$$

with $d_i \in (0, 1)$ and $d_{ij} \in (0, 1)$. Let $y = \sum d_i w_i c_i + \sum d_{ij} w_{ij}$. Then, in the Peirce decomposition $y = \sum y_i c_i + \sum y_{ij}$ we have $y_i = d_i w_i$ and $y_{ij} = d_{ij} w_{ij}$. Rewriting (15) in terms of y gives

$$J_z H(\mu, z) w = (1 - \mu) J f(p_{\mathfrak{J}}(\mu, z)) y + \sum \frac{1 - d_i}{d_i} y_i c_i + \sum \frac{1 - d_{ij}}{d_{ij}} y_{ij}$$

with $\frac{1 - d_i}{d_i}, \frac{1 - d_{ij}}{d_{ij}} > 0$.

Since f is **uniformly nonsingular**, it follows that

$$(1 - \mu) \left\| \frac{f(p_{\mathfrak{J}}(\mu, z) + ty) - f(p_{\mathfrak{J}}(\mu, z))}{t} + \sum \frac{1 - d_i}{d_i (1 - \mu)} y_i c_i + \sum \frac{1 - d_{ij}}{d_{ij} (1 - \mu)} y_{ij} \right\|$$
$$\geq \alpha (1 - \mu) \|y\| \geq \frac{\alpha}{2} \|y\|,$$

which, in the limit as $t \to 0$, becomes

$$\|J_z H(\mu, z) w\| = \left\| (1 - \mu) J f(p_{\mathfrak{J}}(\mu, z)) y + \sum \frac{1 - d_i}{d_i} y_i c_i + \sum \frac{1 - d_{ij}}{d_{ij}} y_{ij} \right\| \geq \frac{\alpha}{2} \|y\|.$$

If $\|y\| \geq \frac{1}{2+m} \|w\|$, then it follows from the above inequality that

$$\|J_z H(\mu, z) w\| \geq \frac{\alpha}{2(2 + m)} \|w\|.$$

Suppose $\|y\| < \frac{1}{2+m} \|w\|$. Observe that (15) can also be rewritten as

$$J_z H(\mu, z) w = (1 - \mu) J f(p_{\mathfrak{J}}(\mu, z)) y + w - y,$$

which implies that

$$\|J_z H(\mu, z) w\| = \|(1 - \mu) J f(p_{\mathfrak{J}}(\mu, z)) y + w - y\|$$
$$\geq \|w\| - \|y\| - \frac{1}{2} \|J f(p_{\mathfrak{J}}(\mu, z)) y\| \geq \frac{1}{2} \|w\|.$$

The proposition follows by taking $\sigma = \min\{\frac{1}{2}, \frac{\alpha}{2(2+m)}\}$. $\qquad \square$

Theorem 5.1. *If f is **uniformly nonsingular**, then $NCP_K(f)$ has a unique solution.*

Proof. Assume that f is **uniformly nonsingular**. The existence of solution to $NCP_K(f)$ is guaranteed by Lemma 5.1. To prove uniqueness, we shall show that the trajectory $T = \{(\mu, z(\mu)) : \mu \in (0, \frac{1}{2}]\}$, where $z(\mu)$ is the unique solution to the SNME with $b = e$, converges as $\mu \downarrow 0$; and that every solution to $NCP_K(f)$ (together with $\mu = 0$) is an accumulation point of T, whence its limit.

In this proof, H refers to the transformation defined by (14) with $b = e$.

Differentiating $H(\mu, z(\mu)) = 0$ with respect to μ gives

$$\nabla_\mu H(\mu, z(\mu)) + J_z H(\mu, z(\mu)) \nabla z(\mu) = 0.$$

By part (b) of Lemma 5.1, the trajectory T is bounded. Together with the continuity of $p_{\mathfrak{J}}$ and f, this implies that $\{\|f(p_{\mathfrak{J}}(\mu, z))\| : (\mu, z) \in T\}$ is bounded, say by d. The transformation f is continuously differentiable, whence its Jacobian Jf is uniformly bounded, say by m, on the compact set $\mathrm{cl}(p_{\mathfrak{J}}(T))$. Observe that $\|\nabla_\mu p_{\mathfrak{J}}(\mu, x)\|^2 = \sum_{i=1}^r \frac{\partial}{\partial \mu} p(\mu, \lambda_i(x))^2 \le rB^2$ by part (4) of Proposition 4.1. Therefore for all $\mu \in (0, \frac{1}{2}]$,

$$
\begin{aligned}
\|\nabla_\mu H(\mu, z(\mu))\| &= \left\| \begin{aligned} &(1-\mu)Jf(p_{\mathfrak{J}}(\mu, z(\mu)))[\nabla_\mu p_{\mathfrak{J}}(\mu, z(\mu))] \\ &- f(p_{\mathfrak{J}}(\mu, z(\mu))) - \nabla_\mu p_{\mathfrak{J}}(\mu, -z(\mu)) + e \end{aligned} \right\| \\
&\le \|(1-\mu)Jf(p_{\mathfrak{J}}(\mu, z(\mu)))[\nabla_\mu p_{\mathfrak{J}}(\mu, z(\mu))]\| \\
&\quad + \|f(p_{\mathfrak{J}}(\mu, z(\mu)))\| + \|\nabla_\mu p_{\mathfrak{J}}(\mu, -z(\mu))\| + \|e\| \\
&\le m\sqrt{r}B + d + \sqrt{r}B + \|e\| =: \bar{d},
\end{aligned}
$$

which together with Lemma 5.2 and the compactness of $\mathrm{cl}(T)$, implies $\|\nabla_\mu z(\mu)\| = \|-J_z H(\mu, z(\mu))^{-1} \nabla_\mu H(\mu, z(\mu))\| \le \frac{\bar{d}}{\sigma}$. For any $\mu_1, \mu_2 \in (0, \frac{1}{2}]$, it follows that

$$\|z(\mu_1) - z(\mu_2)\| = \left\| \int_0^1 (\mu_1 - \mu_2) \nabla_\mu z(\mu_1 + t(\mu_2 - \mu_1)) dt \right\| \le \frac{\bar{d}}{\sigma} |\mu_1 - \mu_2|,$$

which shows that the trajectory T is Cauchy, whence convergent, as $\mu \downarrow 0$.

Now, let z^* be a solution to $\mathrm{NCP}_K(f)$; i.e., z^* satisfies $H(0, z) = 0$. Let $z^* = \sum \lambda_i(z^*)c_i$ be a spectral decomposition, and denote the index sets $\{i : \lambda_i(z^*) > 0\}$ and $\{i : \lambda_i(z^*) \le 0\}$ by I_+ and I_-, respectively. Recall from Theorem 2.3 that under the Jordan frame $\{c_1, \dots, c_r\}$, $\mathfrak{J}$ decomposes into the orthogonal direct sum $\mathfrak{J} = \bigoplus_{1 \le i \le j \le r} \mathfrak{J}_{ij}$ such that the subspaces $\mathfrak{J}_+ := \bigoplus_{i,j \in I_+} \mathfrak{J}_{ij}$ and $\mathfrak{J}_- := \bigoplus_{i,j \in I_-} \mathfrak{J}_{ij}$ are Jordan subalgebras. Denote by l_+ the linear transformation $x \mapsto \mathrm{Proj}_{\mathfrak{J}_+} Jf(z^*)x$ on $\mathfrak{J}_+$. The linear transformation l_+ is invertible. If not, then there exists $0 \ne x \in \mathfrak{J}_+$ such that $l_+(x) = 0$; hence for the perturbation

$$y^\varepsilon := x - \varepsilon \sum_{i \in I_-} \frac{1}{|l_i| + 1} l_i c_i - \varepsilon \sum_{i,j \in I_-} \frac{1}{\|l_{ij}\| + 1} l_{ij} - \varepsilon \sum_{(i,j) \in I_+ \times I_-} \frac{1}{\|l_{ij}\| + 1} l_{ij} \quad (\varepsilon > 0),$$

where $x = \sum_{i \in I_+} x_i c_i + \sum_{i,j \in I_+} x_{ij}$ and $Jf(z^*)x = \sum_{i \in I_-} l_i c_i + \sum_{i,j \in I_-} l_{ij} + \sum_{(i,j) \in I_+ \times I_-} l_{ij}$ are Peirce decompositions, and the nonnegative numbers

$$d_i^\varepsilon := \begin{cases} 0 & \text{if } i \in I_+, \\ (|l_i| + 1)/\varepsilon & \text{otherwise,} \end{cases} \quad \text{and} \quad d_{ij}^\varepsilon := \begin{cases} 0 & \text{if } i, j \in I_+, \\ (\|l_{ij}\| + 1)/\varepsilon & \text{otherwise,} \end{cases}$$

we have

$$
\begin{aligned}
&\left\| Jf(z^*)x + \sum d_i^\varepsilon y_i^\varepsilon c_i + \sum d_{ij}^\varepsilon y_{ij}^\varepsilon \right\| \\
&= \left\| \sum_{i \in I_-} (l_i + d_i^\varepsilon y_i^\varepsilon) c_i + \sum_{i,j \in I_-} (l_{ij} + d_{ij}^\varepsilon y_{ij}^\varepsilon) + \sum_{(i,j) \in I_+ \times I_-} (l_{ij} + d_{ij}^\varepsilon y_{ij}^\varepsilon) \right\| = 0,
\end{aligned}
$$

so that

$$
\begin{aligned}
\left\| Jf(z^*)y^\varepsilon + \sum d_i^\varepsilon y_i^\varepsilon c_i + \sum d_{ij}^\varepsilon y_{ij}^\varepsilon \right\| &= \|Jf(z^*)x - Jf(z^*)y^\varepsilon\| \\
&\le \|Jf(z^*)\| \|x - y^\varepsilon\| < \frac{\alpha}{2} \|x\| < \alpha \|y^\varepsilon\|
\end{aligned}
$$

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for all $\varepsilon > 0$ sufficiently small, contradicting the **uniform nonsingularity** of f. Let $c_+ \in \mathfrak{J}_+$ denote the unit $\sum_{i \in I_+} c_i$ of $\mathfrak{J}_+$, and let $w \in \mathfrak{J}_+$ denote its pre-image under l_+. Similarly denote by $c_- \in \mathfrak{J}_-$ the unit $e - c_+ = \sum_{i \in I_-} c_i$ of $\mathfrak{J}_-$. For $\delta, \bar{\delta} > 0$ with δ sufficiently small,

$$z^* - \delta w - \bar{\delta} c_- = \sum_{i \in I_+} \lambda_i(z^*) c_i - \delta w + \sum_{i \in I_-} (\lambda_i(z^*) - \bar{\delta}) c_i$$

has $\sum_{i \in I_+} \lambda_i(z^*) c_i - \delta w \in \mathrm{int}(K(\mathfrak{J}_+))$ and $\sum_{i \in I_-} (\lambda_i(z^*) - \bar{\delta}) c_i \in -\mathrm{int}(K(\mathfrak{J}_-))$, whence

$$H(0, z^* - \delta w - \bar{\delta} c_-) = f(\mathrm{Proj}_K(z^* - \delta w - \bar{\delta} c_-)) - \mathrm{Proj}_K(-z^* + \delta w + \bar{\delta} c_-)$$

$$= f\left(\sum_{i \in I_+} \lambda_i(z^*) c_i - \delta w\right) + \sum_{i \in I_-} (\lambda_i(z^*) - \bar{\delta}) c_i$$

$$= H(0, z^*) - \delta Jf(z^*) w - \bar{\delta} \sum_{i \in I_-} c_i + o(\delta)$$

$$= -\delta c_+ - \delta(Jf(z^*) w - \mathrm{Proj}_{\mathfrak{J}_+} Jf(z^*) w) - \bar{\delta} c_- + o(\delta).$$

Let $Jf(z^*) w = \sum \bar{l}_i c_i + \sum \bar{l}_{ij}$ be the Peirce decomposition. By [15, Lemma VI.3.1], the image of $H(0, z^* - \delta w - \bar{\delta} c_-)$ under the Frobenius transformation at $-\sum_{(i,j) \in I_+ \times I_-} \bar{l}_{ij}$ is

$$-\delta c_+ - \bar{\delta} c_- - \delta \sum_{i,j \in I_-} \bar{l}_{ij} - \frac{\delta}{4} \mathrm{Proj}_{\mathfrak{J}_-} \left(\sum_{(i,j) \in I_+ \times I_-} \bar{l}_{ij}\right)^2 + o(\delta).$$

Since $(c_+, c_-) \in \mathrm{int}(K(\mathfrak{J}_+)) \times \mathrm{int}(K(\mathfrak{J}_-))$, this image is in $-\mathrm{int}(K)$ when both δ and $\delta/\bar{\delta}$ are sufficiently small. By [15, Proposition VI.3.2] and the fact that the inverse of a Frobenius transformation is a Frobenius transformation, it follows that $H(0, z^* - \delta w - \bar{\delta} c_-) <_K 0$ when $\delta, \bar{\delta} > 0$ has $\delta/\bar{\delta}$ and $\bar{\delta}$ sufficiently small, say when $\bar{\delta} \in (0, \bar{\delta}_0]$ and $\delta \in (0, \rho\bar{\delta}]$. In fact, δ_0 and ρ can be chosen so that

$$H(0, z^* - \rho\bar{\delta} w - \bar{\delta} c_-) \leq_K -\bar{\delta}\bar{x} <_K 0$$

whenever $\bar{\delta} \in (0, \bar{\delta}_0]$, where $\bar{x}$ denotes half of the image of $\rho c_+ + c_- + \rho \sum_{i,j \in I_-} \bar{l}_{ij} + \rho \mathrm{Proj}_{\mathfrak{J}_-} (\sum_{(i,j) \in I_+ \times I_-} \bar{l}_{ij})^2/4$ under the Frobenius transformation at $\sum_{(i,j) \in I_+ \times I_-} \bar{l}_{ij}$. For convenience of notation, we shall denote the sum $z^* - \rho\bar{\delta} w - \bar{\delta} c_-$ by $z^*_{\bar{\delta}}$. It is straightforward to deduce from Proposition 4.2 and the continuous differentiability of f that H is locally Lipschitz at $(0, z^*)$ over $\mathbb{R}_+ \times \mathfrak{J}$, say $\|H(\mu_1, z^* + x) - H(\mu_2, z^* + y)\| \leq l_H(|\mu_1 - \mu_2| + \|x - y\|)$ for all $(\mu_1, x), (\mu_2, y) \in \mathbb{R}_+ \times \mathfrak{J}$ with $\mu_1, \mu_2 \leq \bar{\varepsilon}$ and $\|x\|, \|y\| \leq \bar{\varepsilon}$ with $\bar{\varepsilon} \leq \frac{1}{2}$. Let $\bar{\sigma}$ be the constant given in Lemma 5.2 with the compact set C as the closure of the set of solutions to the SNME over all $\mu \in (0, \frac{1}{2}]$ and all $b >_K 0$ with $\|b - e\| \leq l_H + (2l_H^2 \|\rho w + c_-\|)/\lambda_r(\bar{x})$. Given an arbitrary $\varepsilon > 0$, pick a positive

$$\bar{\delta} < \min\left\{\bar{\delta}_0, \frac{2\bar{\varepsilon} l_H}{\lambda_r(\bar{x})}, \frac{\varepsilon\bar{\sigma}}{2l_H \|\rho w + c_-\| + \lambda_r(\bar{x})}, \frac{\bar{\varepsilon}}{\|\rho w + c_-\|}, \frac{\varepsilon}{2\|\rho w + c_-\|}\right\}.$$

Observe that $\|z^*_{\bar{\delta}} - z^*\| < \bar{\varepsilon}$, so that the Lipschitz bound $\|H(0, z^*_{\bar{\delta}})\| \leq \bar{\delta} l_H \|\rho w + c_-\|$ holds. For $\bar{\mu} := \bar{\delta} \lambda_r(\bar{x})/(2l_H)$, we have $\bar{\mu} \in (0, \bar{\varepsilon}) \subseteq (0, \frac{1}{2})$,

$$H(\bar{\mu}, z^*_{\bar{\delta}}) \leq_K H(0, z^*_{\bar{\delta}}) + \|H(\bar{\mu}, z^*_{\bar{\delta}}) - H(0, z^*_{\bar{\delta}})\| e \leq_K -\bar{\delta}\bar{x} + \frac{l_H \bar{\mu}}{\lambda_r(\bar{x})} \bar{x} = -\frac{\bar{\delta}}{2} \bar{x} <_K 0$$

and
$$\|H(\bar{\mu}, z_{\bar{\delta}}^*)\| \leq \|H(0, z_{\bar{\delta}}^*)\| + l_H \bar{\mu} \leq \bar{\delta}\left(l_H\|\rho w + c_-\| + \frac{\lambda_r(\bar{x})}{2}\right),$$

so that $\Delta b := -\frac{1}{\bar{\mu}} H(\bar{\mu}, z_{\bar{\delta}}^*) >_K 0$ satisfies

$$\|\Delta b\| \leq \frac{\|H(\bar{\mu}, z_{\bar{\delta}}^*) - H(0, z_{\bar{\delta}}^*)\|}{\bar{\mu}} + \frac{\|H(0, z_{\bar{\delta}}^*)\|}{\bar{\mu}}$$

$$\leq l_H + \frac{l_H \bar{\delta}\|\rho w + c_-\|}{\bar{\delta}\lambda_r(\bar{x})/(2l_H)} = l_H + \frac{2l_H^2\|\rho w + c_-\|}{\lambda_r(\bar{x})}$$

and $\bar{\mu}\|\Delta b\| \leq \bar{\delta}(l_H\|\rho w + c_-\| + \lambda_r(\bar{x})/2) < \varepsilon\bar{\sigma}/2$. With a slight abuse of notation, for each fixed $b >_K 0$, let $z(b)$ denote the unique solution to the SNME with $\mu = \bar{\mu}$. Observe that $z(e) = z(\bar{\mu})$ and $z(e + \Delta b) = z_{\bar{\delta}}^*$. Differentiating

$$(1 - \bar{\mu})[f(p_{\mathfrak{J}}(\bar{\mu}, z(b))) + q] - p_{\mathfrak{J}}(\bar{\mu}, -z(b)) + \bar{\mu}b = 0$$

with respect to b then gives

$$J_z H(\bar{\mu}, z(b)) J_b z(b) + \bar{\mu}I = 0,$$

where I is the identity map. Thus, since $z(e + t\Delta b) \in C$ for all $t \in [0, 1]$, we may apply Lemma 5.2 to conclude that

$$\|z_{\bar{\delta}}^* - z(\bar{\mu})\| = \|z(e + \Delta b) - z(e)\|$$

$$= \left\|\int_0^1 J_b z(e + t\Delta b)\Delta b\, dt\right\|$$

$$= \left\|\int_0^1 -\bar{\mu} J_z H(\bar{\mu}, z(e + t\Delta b))^{-1}\Delta b\, dt\right\|$$

$$\leq \frac{\bar{\mu}}{\bar{\sigma}}\|\Delta b\| < \frac{\varepsilon}{2},$$

whence $\|z^* - z(\bar{\mu})\| \leq \|z^* - z_{\bar{\delta}}^*\| + \|z_{\bar{\delta}}^* - z(\bar{\mu})\| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this shows that $(0, z^*)$ is a accumulation point of the trajectory T. $\qquad\square$

Corollary 5.1. *If l is **uniformly nonsingular**, then l has the **GUS**-property.*

6. Conclusion

In this paper, we studied **uniform nonsingularity** and it relation with other existing **P**-type properties for transformations defined on Euclidean Jordan algebras. We demonstrated that the non-interior continuation method and the squared smoothing method converges globally when applied to nonlinear complementarity problems under the assumption of **uniform nonsingularity**. Lastly, we showed that the unique solvability of a nonlinear complementarity problem is implied by the **uniform nonsingularity** of the nonlinear transformation. However, it is still unclear to us if the converse is true.

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