T-algebras and linear optimization over symmetric cones

CHEK BENG CHUA

Abstract. Euclidean Jordan-algebra is a commonly used tool in designing interior-point algorithms for symmetric cone programs. T-algebra, on the other hand, has rarely been used in symmetric cone programming. In this paper, we use both algebraic characterizations of symmetric cones to extend the target-following framework of linear programming to symmetric cone programming. Within this framework, we design an efficient algorithm that finds the analytic centers of convex sets described by linear matrix and convex quadratic constraints.

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1. Introduction

This paper concerns the study of primal-dual interior point algorithms for linear optimization problems over symmetric cones (a.k.a. symmetric cone programming). Primal-dual interior-point algorithms—first designed for linear programming (see, e.g., [27]), and subsequently extended to semidefinite programming (see, e.g., [26, Part II]), symmetric cone programming (see, e.g., [20]) and, recently, homogeneous cone programming [5]—are the most widely used interior-point algorithms in practice. At the same time, they are able to achieve the best iteration complexity bound known to date.

The development of primal-dual algorithms for symmetric cone programming began from two very different perspectives. Yu. Nesterov and M. Todd [20] described their algorithm in the context of *self-concordant barriers* (see the seminal work of Yu. Nesterov and A. Nemirovski [19]) by specializing general logarithmically homogeneous self-concordant barriers to *self-scaled barriers*. L. Faybusovich [9], on the other hand, obtained his algorithm by extending a primal-dual algorithm for semidefinite programming via the theory of Euclidean Jordan algebras. This Jordan-algebraic approach had been so successful that it is now the most common tool in designing interior-point algorithms for symmetric cone programming [1, 2, 7, 21].

In this paper, we present, for the first time, an extension of the target-following framework of linear programming to symmetric cone programming. The target-following framework was first introduced by S. Mizuno [16] for linear complementarity problems, and B. Jansen, C. Roos, T. Terlaky and J.-P. Vial [13] for linear programming as a unifying framework for various primal-dual path-following algorithms and algorithms that find analytic centers of polytopes. The essential ingredient of this framework is the *target map* $(\mathbf{x}, \mathbf{s}) \mapsto (\mathbf{x}_1\mathbf{s}_1, \ldots, \mathbf{x}_n\mathbf{s}_n)$, defined for each pair of positive n-vectors $(\mathbf{x}, \mathbf{s})$. An important feature of the target map is its bijectiveness between the primal-dual strictly feasible region and the cone of positive n-vectors $\mathbb{R}^n_{++}$ [13, 15], whence identifying the primal-dual strictly feasible region with the relatively simple cone $\mathbb{R}^n_{++}$ known as the *target space* (or v-space). Interior-point algorithms based on the target map are known as *target-following* *algorithms*, which are conceptually simple when viewed as following a sequence of targets in the target space. This target map was recently extended to semidefinite programming by the author [6], where the target map was proved to be a bijection between the primal-dual strictly feasible region and the cone of positive definite matrices. See also [17, 18, 22] for other extensions the target map to semidefinite programming, and [12] for an extension to symmetric cone programming. It is noted here that these extensions of the target map do not result in target-following algorithms as they are generally not injective on the whole primal-dual strictly feasible regions.

Our target map is described with the algebraic descriptions of symmetric cone via Euclidean Jordan algebras and T-*algebras*. T-algebras are certain non-associative algebras discovered by È. Vinberg [24] in his attempt to classify *homogeneous cones*. A homogeneous cone K is an open convex cone whose group of automorphisms acts transitively on it. Symmetric cones are precisely those homogeneous cones that are self-dual under suitable choices of inner products. The complete classification of symmetric cones can thus be obtained by classifying self-dual T-algebras [25].

This paper is organized as follows. We begin the next section with the Jordan-algebraic and T-algebraic characterizations of symmetric cones, and the relation between these characterizations. In Section 3, we use both algebraic characterizations to define the notion of weighted analytic centers for symmetric cone programming. This notion allows us to define the target map in Section 4, with which we describe and analyze a target-following algorithm. Finally, in Section 5, we apply the target-following algorithm to the problem of finding analytic centers of sets described by linear matrix inequalities and convex quadratic inequalities.

2. Algebraic characterizations of symmetric cones

2.1. **Jordan algebraic characterization.** Let $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ be a Euclidean space with inner product $\langle\cdot,\cdot\rangle$.

A *homogeneous cone* is an open, convex cone $\mathcal{K} \in \mathbb{E}$ whose *linear automorphism group*

$$\mathrm{Aut}(\mathcal{K}) \stackrel{\text{def}}{=} \{\boldsymbol{\mathcal{A}} \in \mathrm{GL}(\mathbb{E}) : \boldsymbol{\mathcal{A}}(\mathcal{K}) = \mathcal{K}\}$$

acts *transitively* on it; i.e., $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$, $\exists \boldsymbol{\mathcal{A}} \in \mathrm{Aut}(\mathcal{K})$, $\boldsymbol{\mathcal{A}}(\mathbf{x}) = \mathbf{y}$. An open, convex cone $\mathcal{K} \in \mathbb{E}$ is said to be *self-dual* if its *open dual cone*

$$\mathcal{K}^\sharp \stackrel{\text{def}}{=} \{\mathbf{s} \in \mathbb{E} : \langle \mathbf{x}, \mathbf{s}\rangle > 0 \ \forall \mathbf{0} \neq \mathbf{x} \in cl(\mathcal{K})\}$$

coincide with $\mathcal{K}$. A *symmetric cone* $\mathcal{K} \in \mathbb{E}$ is a self-dual homogeneous cone.

Henceforth, $\mathcal{K}$ shall be a symmetric cone.

Every symmetric cone can be associated with a commutative algebra over $\mathbb{R}$ known as a *Jordan algebra*. In short, if $(\mathfrak{J}, \circ)$ is a Euclidean Jordan algebra, then the interior of its cone of squares $\{\mathbf{x} \circ \mathbf{x} : \mathbf{x} \in \mathfrak{J}\}$ is a symmetric cone, and every symmetric cone arises in this manner. For a comprehensive discussion on the relation between symmetric cones and Jordan algebras, we refer the reader to Chapters I–III of [8].

It is known that symmetric cones can be completely characterized as direct sums of five classes of *simple* symmetric cones (see, e.g., [8, Chapter V]):

(1) the second-order cones

$$\mathcal{Q}_n \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}_{n+1} > \sqrt{\mathbf{x}_1^2 + \cdots + \mathbf{x}_n^2}\right\};$$

(2) the cones of real, symmetric, positive definite matrices

$$\mathcal{S}_n \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^{n\times n} : \mathbf{x}^T = \mathbf{x} \text{ and } \mathbf{v}^T\mathbf{x}\mathbf{v} > 0 \ \forall \mathbf{v} \in \mathbb{R}^n\right\};$$

(3) the cones of complex, Hermitian, positive definite matrices

$$\mathcal{C}_n \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{C}^{n\times n} : \mathbf{x}^* = \mathbf{x} \text{ and } \mathbf{v}^*\mathbf{x}\mathbf{v} > 0 \ \forall \mathbf{v} \in \mathbb{C}^n\right\};$$

(4) the cones of Hermitian, positive definite matrices of quaternions

$$\mathcal{H}_n \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{H}^{n\times n} : \mathbf{x}^* = \mathbf{x} \text{ and } \mathbf{v}^*\mathbf{x}\mathbf{v} > 0 \ \forall \mathbf{v} \in \mathbb{H}^n\right\};$$

(5) the cone of 3×3 Hermitian, positive definite matrices of octonions

$$\mathcal{O}_3 \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{O}^{3\times 3} : \mathbf{x}^* = \mathbf{x} \text{ and } \mathbf{v}^*\mathbf{x}\mathbf{v} > 0 \ \forall \mathbf{v} \in \mathbb{O}^3\right\}.$$

This characterization can be deduced from the characterization of formally real Jordan algebras, which was first given by P. Jordan, J. von Neumann and E. Wigner [14]. Formally real Jordan algebras are precisely Euclidean Jordan algebras; see, e.g., [8].

2.2. ***T*-algebraic characterization.** From here on, we shall treat symmetric cones as a subclass of homogeneous cones, and use a different algebraic characterization arising from the study of homogeneous cones.

A *matrix algebra* $\mathfrak{A}$ is a bi-graded algebra $\bigoplus_{i,j=1}^r \mathfrak{A}_{ij}$ over $\mathbb{R}$ satisfying $\mathfrak{A}_{ij}\mathfrak{A}_{kl} \subseteq \delta_{jk}\mathfrak{A}_{il}$, where δ_{jk} is the Kronecker delta symbol. The positive integer r is called the *rank* of the matrix algebra. For each $\mathbf{a} \in \mathfrak{A}$, we denote by $\mathbf{a}_{ij}$ its projection on $\mathfrak{A}_{ij}$. An *involution* $(\cdot)^*$ of a matrix algebra $\mathfrak{A}$ is a linear automorphism on $\mathfrak{A}$ that is involutory (i.e., $(\mathbf{a}^*)^* = \mathbf{a}$), anti-homomorphic (i.e., $(\mathbf{ab})^* = \mathbf{b}^*\mathbf{a}^*$), and further satisfies $\mathfrak{A}_{ij}^* \subseteq \mathfrak{A}_{ji}$. A *$T$-algebra of rank r* is a matrix algebra $\mathfrak{A}$ of rank r with involution $(\cdot)^*$ satisfying the following seven axioms.

I. For each $i \in \{1, \ldots, r\}$, the subalgebra $\mathfrak{A}_{ii}$ is isomorphic to the reals.

For the remaining axioms, we shall use ρ_i to denote the isomorphism from $\mathfrak{A}_{ii}$ to $\mathbb{R}$, and use $\mathbf{e}_i$ to denote the unit of $\mathfrak{A}_{ii}$. Thus $\mathbf{a}_{ii} = \rho_i(\mathbf{a})\mathbf{e}_i$.

II. For each $\mathbf{a} \in \mathfrak{A}$ and each $i, j \in \{1, \ldots, r\}$,

$$\mathbf{a}_{ji}\mathbf{e}_i = \mathbf{a}_{ji} \text{ and } \quad \mathbf{e}_i\mathbf{a}_{ij} = \mathbf{a}_{ij}.$$

III. For each $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$ and each $i, j \in \{1, \ldots, r\}$,

$$\rho_i(\mathbf{a}_{ij}\mathbf{b}_{ji}) = \rho_j(\mathbf{b}_{ji}\mathbf{a}_{ij}).$$

IV. For each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}$ and each $i, j, k \in \{1, \ldots, r\}$,

$$\mathbf{a}_{ij}(\mathbf{b}_{jk}\mathbf{c}_{ki}) = (\mathbf{a}_{ij}\mathbf{b}_{jk})\mathbf{c}_{ki}.$$

V. For each $\mathbf{a} \in \mathfrak{A}$ and each $i, j \in \{1, \ldots, r\}$,

$$\rho_i(\mathbf{a}_{ij}^*\mathbf{a}_{ij}) \geq 0$$

with equality when and only when $\mathbf{a}_{ij} = \mathbf{0}$.

VI. For each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}$ and each $i, j, k, l \in \{1, \ldots, r\}$ with $i \leq j \leq k \leq l$,

$$\mathbf{a}_{ij}(\mathbf{b}_{jk}\mathbf{c}_{kl}) = (\mathbf{a}_{ij}\mathbf{b}_{jk})\mathbf{c}_{kl}.$$

VII. For each $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$ and each $i, j, k, l \in \{1, \ldots, r\}$ with $i \leq j \leq k$ and $l \leq k$,

$$\mathbf{a}_{ij}(\mathbf{b}_{jk}\mathbf{b}_{lk}^*) = (\mathbf{a}_{ij}\mathbf{b}_{jk})\mathbf{b}_{lk}^*.$$

A *change in the grading* of a T-algebra $\mathfrak{A}$ is the replacement of each subspace $\mathfrak{A}_{ij}$ by $\mathfrak{A}_{\pi(i),\pi(j)}$ where π is a permutation of $\{1,\ldots,r\}$. An *inessential change in the grading* of a T-algebra $\mathfrak{A}$ is one that preserves the subspace of upper triangular (or equivalently, lower triangular) elements. In other words, the permutation π in an inessential change satisfies

$$(i<j)\wedge(\pi(i)>\pi(j)) \implies \mathfrak{A}_{ij}=\{\mathbf{0}\}.$$

A T-algebra $\mathfrak{A}$ of rank r is said to be *isomorphic* to another T-algebra of the same rank if, after an inessential change of grading of $\mathfrak{A}$, there is an isomorphism of the bi-graded algebras with involution. The *cone associated with a T-algebra* $\mathfrak{A}$ of rank r is

$$\{\mathbf{t}\mathbf{t}^* : \mathbf{t}\in\mathfrak{A},\ \mathbf{t}_{ij}=\mathbf{0}\ \forall 1\le j<i\le r,\ \rho_i(\mathbf{t})>0\ \forall 1\le i\le r\}.$$

It is denoted by $\mathcal{K}(\mathfrak{A})$. When we write $\mathbf{t}\mathbf{t}^*\in\mathcal{K}(\mathfrak{A})$, we always mean that $\mathbf{t}$ satisfies the conditions in the definition of $\mathcal{K}(\mathfrak{A})$. It was shown by È. Vinberg [24, Proposition III.2] that such $\mathbf{t}$ is uniquely determined by the product $\mathbf{t}\mathbf{t}^*$; see also Propositions 2 and 3. For each $\mathbf{x}=\mathbf{t}\mathbf{t}^*\in\mathcal{K}(\mathfrak{A})$, we shall denote this unique $\mathbf{t}$ by $\mathbf{u}_{\mathbf{x}}$.

Homogeneous cones are completely characterized by T-algebras in the following theorem.

Theorem 1 (Characterization of homogeneous cones, È. Vinberg 1963)**.** *A cone is homogeneous if and only if it is linearly isomorphic to the cone associated with some T-algebra. Moreover the T-algebra is uniquely determined, up to isomorphism, by the homogeneous cone.*

Proof. See Proposition 1 and Theorem 4 of [24]. □

2.2.1. *Notations of T-algebras.* For a T-algebra $\mathfrak{A}$ of rank r with involution $(\cdot)^*$, we shall use the following notations.

(1) $\mathbf{e}_i$ shall denote the unit of the subalgebra $\mathfrak{A}_{ii}$ and ρ_i shall denote the linear function on $\mathfrak{A}$ satisfying $\mathbf{a}_{ii}=\rho_i(\mathbf{a})\mathbf{e}_i$.

(2) $\mathbf{e}$ shall denote the element in $\mathfrak{A}$ satisfying $\mathbf{e}_{ii}=\mathbf{e}_i$ and $\mathbf{e}_{ij}=\mathbf{0}$ for $i\neq j$. Axiom II is equivalent to $\mathbf{e}$ being the unit of the T-algebra $\mathfrak{A}$.

(3) $s(\cdot)$ shall denote the linear function

$$\mathbf{a}\in\mathfrak{A}\mapsto\sum_{i=1}^r\rho_i(\mathbf{a}).$$

(4) $\langle\cdot,\cdot\rangle$ shall denote the bilinear function

$$(\mathbf{a},\mathbf{b})\in\mathfrak{A}\times\mathfrak{A}\mapsto s(\mathbf{a}^*\mathbf{b}).$$

When restricted to $\mathfrak{A}_{ii}$, $(\cdot)^*$ is an involutory, anti-homomorphic, linear automorphism. Hence it must be the identity map. The linear function $s(\cdot)$ is thus invariant under involution. Subsequently $\langle\cdot,\cdot\rangle$ is symmetric. Axiom V is equivalent to $s(\mathbf{a}\mathbf{a}^*)\ge 0$ for all $\mathbf{a}\in\mathfrak{A}$ with equality when and only when $\mathbf{a}=\mathbf{0}$. This is further equivalent to $\langle\cdot,\cdot\rangle$ being positive definite. Thus $\langle\cdot,\cdot\rangle$ is an inner product of $\mathfrak{A}$. We shall denote by $\|\cdot\|$ the induced Euclidean norm $\mathbf{a}\mapsto\sqrt{\langle\mathbf{a},\mathbf{a}\rangle}$.

(5) $\mathfrak{T}$ shall denote the subalgebra

$$\{\mathbf{a}\in\mathfrak{A}:\mathbf{a}_{ij}=\mathbf{0}\ (1\le j<i\le r)\}$$

of upper triangular elements. The associated homogeneous cone $\mathcal{K}(\mathfrak{A})$ can then be described as $\{\mathbf{t}\mathbf{t}^*:\mathbf{t}\in\mathfrak{T},\ \rho_i(\mathbf{t})>0\ \forall i\}$.

(6) $\mathfrak{H}$ shall denote the subspace

$$\{\mathbf{a} \in \mathfrak{A} : \mathbf{a}_{ij} = \mathbf{a}_{ji}^* \ (1 \leq j < i \leq r)\}$$

of "hermitian" elements. We can then view the associated homogeneous cone $\mathcal{K}(\mathfrak{A})$ is as an open cone in the Euclidean space $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$.

(7) $\mathfrak{D}$ shall denote the subalgebra

$$\{\mathbf{a} \in \mathfrak{A} : \mathbf{a}_{ij} = \mathbf{0} \ (i \neq j)\}$$

of diagonal elements (which is also $\mathfrak{T} \cap \mathfrak{H}$). We use the notation $\mathfrak{D}_{++}$ to denote the subset of $\mathfrak{D}$ consisting of those elements with $\rho_i(\mathbf{a}) > 0$, and use the notation $\mathfrak{D}_{\downarrow,++}$ to denote the subset of $\mathfrak{D}_{++}$ consisting of those elements with $\rho_i(\mathbf{a}) \geq \rho_j(\mathbf{a})$ for all $i \leq j$.

(8) $(\cdot)_H$ shall denote the linear map

$$\mathbf{a} \in \mathfrak{A} \mapsto \mathbf{a} + \mathbf{a}^*;$$

i.e., twice the orthogonal projection onto $\mathfrak{H}$ under $\langle \cdot, \cdot \rangle$.

(9) $\langle\!\langle \cdot \rangle\!\rangle$ shall denote the linear map from $\mathfrak{A}$ to $\mathfrak{T}^*$ that takes each $\mathbf{a}$ to the unique lower triangular element $\mathbf{l}$ such that $\mathbf{l}_H$ shares the same lower triangular part as $\mathbf{a}$.

(10) $[\mathfrak{A}]_\ell$ shall denote the sub-algebra $\bigoplus_{i,j=1}^{\ell} \mathfrak{A}_{ij}$, and $[\mathbf{a}]_\ell$ shall denote the projection of $\mathbf{a}$ on $[\mathfrak{A}]_\ell$. Similarly, $[\mathfrak{H}]_\ell$ shall denote the subspace $\mathfrak{H} \cap [\mathfrak{A}]_\ell$ of "hermitian" elements in $[\mathfrak{A}]_\ell$.

2.2.2. *Properties of T-algebras.*

Lemma 1. *For every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}$, $\langle \mathbf{ab}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{a}^*\mathbf{c} \rangle$ and $\langle \mathbf{ba}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{ca}^* \rangle$.*

Proof. Axiom IV is equivalent to $s(\mathbf{a}(\mathbf{bc})) = s((\mathbf{ab})\mathbf{c})$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}$. This is further equivalent to $\langle \mathbf{ab}, \mathbf{c} \rangle = \langle \mathbf{b}, \mathbf{a}^*\mathbf{c} \rangle$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{A}$. Together with Axiom III, we deduce $\langle \mathbf{ba}, \mathbf{c} \rangle = \langle \mathbf{a}^*\mathbf{b}^*, \mathbf{c}^* \rangle = \langle \mathbf{b}^*, \mathbf{ac}^* \rangle = \langle \mathbf{b}, \mathbf{ca}^* \rangle$. ☐

Theorem 2. *The norm $\|\cdot\|$ is sub-multiplicative; i.e., $\forall \mathbf{a}, \mathbf{b} \in \mathfrak{A}$, $\|\mathbf{ab}\| \leq \|\mathbf{a}\| \, \|\mathbf{b}\|$.*

Proof. See proof of Theorem 2 of [5]. ☐

Let $\mathfrak{T}_*$ (resp., $\mathfrak{T}_+$ and $\mathfrak{T}_{++}$) denote the set of elements of $\mathfrak{T}$ with nonzero (resp., nonnegative and positive) diagonal components.

Proposition 1. *The sets $\mathfrak{T}_\star$, $\mathfrak{T}_{++}$, $\mathfrak{T}_\star^*$ and $\mathfrak{T}_{++}^*$ are multiplicative groups.*

Proof. See proof of Proposition 1 of [5]. ☐

We shall denote by $\mathbf{t}^{-1}$ the multiplicative inverse of each element $\mathbf{t} \in \mathfrak{T}_\star$. Similarly for elements $\mathbf{l} \in \mathfrak{T}_\star^*$.

Involution is anti-homomorphic, hence the inverse of the involution of an element $\mathbf{t} \in \mathfrak{T}_\star$ is the involution of its inverse, which we shall denote by $\mathbf{t}^{-*}$. Similarly for elements $\mathbf{l} \in \mathfrak{T}_\star^*$.

2.2.3. *T-algebraic characterization of symmetric cones.* In the special self-dual case of simple symmetric cones, we have the following corresponding T-algebras.

- For each $n \geq 1$, the matrix algebra $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22} = \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$ defined by

$$(\mathbf{ab})_{ii} = \mathbf{a}_{ij}^T \mathbf{b}_{ji} \text{ for } (i, j) \in \{(1, 2), (2, 1)\}$$

is a T-algebra. It is straightforward to check that the associated cone $\mathcal{K}(\mathfrak{A})$ is linearly isomorphic to the second-order cone $\mathcal{Q}_{n+1}$, and that the subspace $\mathfrak{H}$, when equipped with the *Jordan product*[1]

$$\square : (\mathbf{a}, \mathbf{b}) \mapsto \frac{\mathbf{ab} + \mathbf{ba}}{2},$$

is a Euclidean Jordan algebra of rank 2 whose associated symmetric cone is $\mathcal{K}(\mathfrak{A})$.

- Let $\mathfrak{W}$ be a Euclidean Hurwitz algebra (i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$). We shall use $\Re(x)$ and $\bar{x}$ to denote, respectively, the real part and the conjugate of $x \in \mathfrak{W}$.

  Let $\mathfrak{A}_{ij} = \mathfrak{W}$ for each $i, j \in \{1, \ldots, r\}$ with $i \neq j$, $\mathfrak{A}_{ii} = \mathbb{R}$ for $i \in \{1, \ldots, r\}$, and define the matrix algebra $\mathfrak{A} = \bigoplus_{i,j=1}^{r} \mathfrak{A}_{ij}$ by

  $$(\mathbf{ab})_{ij} = \begin{cases} \sum_{k=1}^{r} \Re(\mathbf{a}_{ik}\mathbf{b}_{kj}) & \text{if } i = j, \\ \sum_{k=1}^{r} \mathbf{a}_{ik}\mathbf{b}_{kj} & \text{if } i \neq j, \end{cases}$$

  so that it satisfies Axioms I and II in the definition of a T-algebra. It is straightforward to check that the unary operator $(\cdot)^* : \mathfrak{A} \to \mathfrak{A}$ defined by

  $$(\mathbf{a}^*)_{ij} = \bar{\mathbf{a}_{ji}}$$

  is an involution. Moreover, Proposition V.1.2 of [8] shows that Axioms III–V are also satisfied.

  For $r = 3$, Axiom VI holds since at least one of $\mathbf{a}_{ij}$, $\mathbf{b}_{jk}$ and $\mathbf{c}_{kl}$ is a real number. Finally, Axiom VII holds since Euclidean Hurwitz algebras are alternative (i.e., $x(xy) = (xx)y$ and $(yx)x = y(xx)$, or equivalently, the sub-algebra generated by any two elements is associative), and both x and $\bar{x}$ are in the sub-algebra generated by $x - \Re(x)$ for each $x \in \mathfrak{W}$. Hence $\mathfrak{A}$ is a T-algebra. It is straightforward to check that the associated cone $\mathcal{K}(\mathfrak{A})$ is the cone of 3×3 Hermitian, positive definite matrices of $\mathfrak{W}$; i.e. $\mathcal{K}(\mathfrak{A}) = \mathcal{S}_3$, $\mathcal{C}_3$, $\mathcal{H}_3$ and $\mathcal{O}_3$ when $\mathfrak{W} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively. The algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are associative. Together with Corollary V.2.6 of [8], it follows that the subspace $\mathfrak{H}$, when equipped with the Jordan product $\square$, is a Euclidean Jordan algebra of rank 3 whose associated symmetric cone is $\mathcal{S}_3$, $\mathcal{C}_3$, $\mathcal{H}_3$ and $\mathcal{O}_3$ respectively.

  For $r > 3$, suppose that $\mathfrak{W}$ is associative. Then $\mathfrak{A}$ is clearly a T-algebra. As before, it is straightforward to check that the associated cone $\mathcal{K}(\mathfrak{A})$ is the cone of $r \times r$ Hermitian, positive definite matrices of $\mathfrak{W}$; i.e. $\mathcal{K}(\mathfrak{A}) = \mathcal{S}_r$, $\mathcal{C}_r$ and $\mathcal{H}_r$ when $\mathfrak{W} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively. Once again, since $\mathfrak{W}$ is associative, the subspace $\mathfrak{H}$, when equipped with the Jordan product $\square$, is a Euclidean Jordan algebra of rank r whose associated symmetric cone is $\mathcal{S}_r$, $\mathcal{C}_r$ and $\mathcal{H}_r$ respectively.

Write $\mathcal{K}$ as the direct sum of simple symmetric cones. Henceforth, $\mathfrak{A}$ shall be the direct sum of the T-algebras corresponding to these simple symmetric cones as described above, and $(\mathfrak{J}, \circ)$ shall be the direct sum of the corresponding Euclidean Jordan algebra described above. It is easy to check that $\mathfrak{A}$ is a T-algebra and $\mathfrak{J}$ is a Euclidean Jordan algebra. Let r be the rank of $\mathfrak{A}$.

Each element $\mathbf{x}$ of the Euclidean Jordan algebra $(\mathfrak{J}, \circ)$ has a *spectral decomposition*

$$\mathbf{x} = \sum_{i=1}^{r} \lambda_i \mathbf{c}_i,$$

[1]This gives precisely the *algebra of connectedness*; see, e.g., [24].

where $\lambda_1 \geq \cdots \geq \lambda_r$ are uniquely determined by $\mathbf{x}$, and $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\} \subset \mathfrak{J}$ forms a Jordan frame; see Theorem III.1.2 of [8]. The coefficients $\lambda_1, \ldots, \lambda_r$ are called the *eigenvalues* of $\mathbf{x}$, and they are denoted by $\lambda_1(\mathbf{x}), \ldots, \lambda_r(\mathbf{x})$. A Jordan frame is a set of *primitive idempotents* that are *pair-wise orthogonal* and sum to the unit $\mathbf{e}$. An *idempotent* is a nonzero element $\mathbf{c} \in \mathfrak{J}$ satisfying $\mathbf{c} \circ \mathbf{c} = \mathbf{c}$, and it is said to be *primitive* if it cannot be written as the sum of two idempotents. Two idempotents $\mathbf{c}$ and $\mathbf{d}$ are said to be *orthogonal* if $\mathbf{c} \circ \mathbf{d} = \mathbf{0}$. In terms of $\mathfrak{A}$, idempotents are characterized by $\mathbf{c}^2 = \mathbf{c}$.

The number of elements of any Jordan frame is an invariant called the *rank* of the Jordan algebra; see, e.g., paragraph after Theorem III.1.2 of [8]. By relating this to the Carathéodory number of $\mathcal{K}$, it is immediate that both $\mathfrak{A}$ and $\mathfrak{J}$ have the same rank; see [10] and [23]. Alternatively, one can see this by noting that $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ is a Jordan frame.

In a Euclidean Jordan algebra, we define a linear function $\mathrm{tr}(\cdot)$ by $\mathrm{tr}(\mathbf{x}) = \sum_{i=1}^r \lambda_i(\mathbf{x})$. This function is called the *trace*.

We now show that the function $\mathrm{tr}(\cdot)$ coincides with the restriction of the function $s(\cdot)$ on the subspace $\mathfrak{H}$.

Using the sub-multiplicativity of $\|\cdot\|$, we deduce that if $\mathbf{c}$ is an idempotent, then $\|\mathbf{c}\| = \|\mathbf{c}^2\| \leq \|\mathbf{c}\|^2$, whence $s(\mathbf{c}) = s(\mathbf{c}^2) = \|\mathbf{c}\|^2 \geq 1$. In a Jordan frame $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$, we then have $r \leq s(\mathbf{c}_1 + \cdots + \mathbf{c}_r) = s(\mathbf{e}) = r$, whence $s(\mathbf{c}_1) = \cdots = s(\mathbf{c}_r) = 1$. If $\mathbf{c}$ is a primitive idempotent, then its spectral decomposition shows that $\mathbf{c}$ is a member of some Jordan frame, whence $s(\mathbf{c}) = 1$. This shows that for any $\mathbf{x} \in \mathfrak{H}$ with spectral decomposition $\mathbf{x} = \sum \lambda_i \mathbf{c}_i$, $s(\mathbf{x}) = \sum \lambda_i s(\mathbf{c}) = \sum \lambda_i = \mathrm{tr}(\mathbf{x})$.

2.3. **Quadratic representations.** In the Jordan algebra $(\mathfrak{J}, \circ)$, the quadratic representation of each $\mathbf{x} \in \mathfrak{J}$ is defined to be the linear map

$$\mathcal{Q}_{\mathbf{x}} : \mathbf{y} \in \mathfrak{J} \mapsto 2(\mathbf{x} \circ (\mathbf{x} \circ \mathbf{y})) - (\mathbf{x} \circ \mathbf{x}) \circ \mathbf{y}.$$

In terms of its T-algebra $\mathfrak{A}$, the quadratic representation of each $\mathbf{x} \in \mathfrak{H}$ is

$$\begin{aligned}\mathbf{y} \in \mathfrak{H} \mapsto\ & 2\left(\frac{1}{2}\left(\mathbf{x}\left(\frac{\mathbf{xy}+\mathbf{yx}}{2}\right) + \left(\frac{\mathbf{xy}+\mathbf{yx}}{2}\right)\mathbf{x}\right)\right) - \frac{(\mathbf{xx})\mathbf{y} + \mathbf{y}(\mathbf{xx})}{2} \\ &= \frac{1}{2}\left(\mathbf{x}(\mathbf{xy}) + \mathbf{x}(\mathbf{yx}) - (\mathbf{xx})\mathbf{y}\right)_H .\end{aligned}$$

We can extend the quadratic representations from $\mathfrak{H}$ to $\mathfrak{A}$ by defining, for each $\mathbf{a} \in \mathfrak{A}$,

$$\mathcal{Q}_{\mathbf{a}} : \mathbf{b} \in \mathfrak{H} \mapsto \frac{1}{2}\left(\mathbf{a}(\mathbf{ab}) + \mathbf{a}(\mathbf{ba}^*) - (\mathbf{aa})\mathbf{b}\right)_H .$$

The following proposition and its corollary show that the group of linear automorphisms $\{\mathcal{Q}_{\mathbf{t}} : \mathbf{t} \in \mathfrak{T}_{++}\}$ forms a simply transitive subgroup of $\mathrm{Aut}(\mathcal{K})$.

Proposition 2. *For each $\mathbf{t} \in \mathfrak{T}$, the map $\mathcal{Q}_{\mathbf{t}}$ simplifies to*

$$\mathbf{a} \mapsto \mathbf{t}(\langle\!\langle \mathbf{a} \rangle\!\rangle \mathbf{t}^*) + (\mathbf{t}\langle\!\langle \mathbf{a} \rangle\!\rangle^*)\mathbf{t}^*.$$

If, in addition, $\mathbf{a} = \mathbf{uu}^*$ for some $\mathbf{u} \in \mathfrak{T}$, then

$$\mathcal{Q}_{\mathbf{t}}(\mathbf{uu}^*) = (\mathbf{tu})(\mathbf{tu})^*.$$

Proof. See proof of Proposition 2 of [5]. □

Corollary 1. *The map $\mathbf{t} \mapsto \mathcal{Q}_{\mathbf{t}}$ is a group homomorphism on $\mathfrak{T}_{++}$.*

Proof. See proof of Corollary 1 of [5]. □

2.3.1. *Properties of quadratic representations.* Quadratic representations will be used in the description and analysis of the target-following algorithm in Section 4.1. We now list a few useful properties of quadratic representations.

Proposition 3. *For each $\mathbf{u} \in \mathfrak{T}_{++}$, the map*

$$\mathbf{t} \in \mathfrak{T}_{++} \mapsto \mathfrak{Q}_{\mathbf{t}}(\mathbf{u}\mathbf{u}^*)$$

is a diffeomorphism. Moreover its derivative at $\mathbf{t} \in \mathfrak{T}_{++}$ is

$$\mathbf{w} \in \mathfrak{T} \mapsto (\mathbf{t}\mathbf{u})(\mathbf{w}\mathbf{u})^* + (\mathbf{w}\mathbf{u})(\mathbf{t}\mathbf{u})^*$$

and the derivative of its inverse map at $\mathfrak{Q}_{\mathbf{t}}(\mathbf{u}\mathbf{u}^*) \in \mathcal{K}$ is

$$\mathbf{a} \in \mathfrak{H} \mapsto \mathbf{t}\mathbf{u}\langle\!\langle \mathfrak{Q}_{(\mathbf{t}\mathbf{u})^{-1}}(\mathbf{a})\rangle\!\rangle^* \mathbf{u}^{-1}.$$

Proof. See proof of Proposition 3 of [5]. ☐

Proposition 4. *For each $(\mathbf{x}_1, \ldots, \mathbf{x}_r) \in [\mathfrak{A}]_1 \times \cdots \times [\mathfrak{A}]_r$, and any $\mathbf{t} \in \mathfrak{T}$,*

$$\sum_{\ell=1}^{r} \mathfrak{Q}_{[\mathbf{t}]_\ell}(\mathbf{x}_\ell) = \mathfrak{Q}_{\mathbf{t}}\left(\sum_{\ell=1}^{r} \mathbf{x}_\ell\right). \tag{2.1}$$

Proof. Since $\mathbf{t}_{ik}(\mathbf{x}_\ell)_{kj} = \mathbf{0}$ whenever $i > \ell$ or $k > \ell$, it follows that $[\mathbf{t}]_\ell \mathbf{x}_\ell = \mathbf{t}\mathbf{x}_\ell$, whence

$$\sum_{\ell=1}^{r} [\mathbf{t}]_\ell \mathbf{x}_\ell = \mathbf{t}\left(\sum_{\ell=1}^{r} \mathbf{x}_\ell\right). \tag{2.2}$$

The proposition then follows from Proposition 2. ☐

Proposition 5. *For each $\mathbf{t} \in \mathfrak{T}$, the operator norm of $\mathfrak{Q}_{\mathbf{t}}$ is at most $\sqrt{2}\,\|\mathbf{t}\|^2$.*

Proof. For every $\mathbf{a} \in \mathfrak{H}$, applying the simplification in Proposition 2, the triangle inequality for $\|\cdot\|$ and Axiom III gives $\|\mathfrak{Q}_{\mathbf{t}}\mathbf{a}\| \le 2\,\|\mathbf{t}(\langle\!\langle \mathbf{a}\rangle\!\rangle \mathbf{t}^*)\|$. Together with the submultiplicativity of $\|\cdot\|$, we then get $\|\mathfrak{Q}_{\mathbf{t}}\mathbf{a}\| \le 2\,\|\langle\!\langle \mathbf{a}\rangle\!\rangle\|\,\|\mathbf{t}\|^2$. Now

$$\|\langle\!\langle \mathbf{a}\rangle\!\rangle\|^2 = \sum_{j<i} \mathbf{a}_{ij}\mathbf{a}_{ij}^* + \sum_{i=1}^{r} \frac{1}{4}\rho_i(\mathbf{a})^2 \le \frac{1}{2}\left(\sum_{j\neq i} \mathbf{a}_{ij}\mathbf{a}_{ij}^* + \sum_{i=1}^{r} \rho_i(\mathbf{a})^2\right) = \frac{1}{2}\,\|\mathbf{a}\|^2 \tag{2.3}$$

proves the proposition. ☐

2.4. **Dual cones.** Using Axiom IV and Lemma 1, it is straightforward to check that the adjoint map $\mathfrak{Q}_{\mathbf{a}}^*$ of the quadratic representation of an element $\mathbf{a} \in \mathfrak{A}$ is the quadratic representation $\mathfrak{Q}_{\mathbf{a}^*}$ of its involution. This observation can be used to derive the following description of the cone $\mathcal{K}^\sharp$.

Proposition 6. *The dual cone $\mathcal{K}^\sharp$ is given by $\{\mathbf{l}\mathbf{l}^* : \mathbf{l} \in \mathfrak{T}_{++}^*\}$. The group of automorphisms $\{\mathfrak{Q}_{\mathbf{t}} : \mathbf{t} \in \mathfrak{T}_{++}^*\}$ acts transitively on $\mathcal{K}^\sharp$.*

Proof. See Proposition 4 of [5]. ☐

Proposition 7. *The matrix algebra $\mathfrak{B}$ obtained from $\mathfrak{A}$ by the change in grading*

$$\mathfrak{B}_{ij} = \mathfrak{A}_{r+1-i,r+1-j} \ (1 \le i, j \le r)$$

is a T-algebra whose associated cone $\mathcal{K}(\mathfrak{B})$ is exactly the dual cone $\mathcal{K}^\sharp$.

Proof. See paragraph before Proposition 5 of [5]. ☐

This proposition, together with Proposition III.2 of [24], implies that every element $\mathbf{s} \in \mathcal{K}^\sharp$ can be uniquely written as the product $\mathbf{l}\mathbf{l}^*$ with $\mathbf{l} \in \mathfrak{T}^*_{++}$. We denote this unique $\mathbf{l}$ by $\mathbf{l}_\mathbf{s}$.

Since $\mathcal{K}$ is self-dual, the above argument applies to $\mathcal{K}$. In other words, every element $\mathbf{x}$ in the symmetric cone $\mathcal{K}$ has a Cholesky factorization, as well as an inverse Cholesky factorization.

3. Weighted analytic centers

We consider the following pair of primal-dual symmetric cone programming problems:

$$\inf\{\langle \mathbf{c}, \mathbf{x}\rangle : \ \langle \mathbf{a}_k, \mathbf{x}\rangle = b_k, \ 1 \le k \le m, \ \mathbf{x} \in cl(\mathcal{K})\} \tag{3.1a}$$

and

$$\sup\left\{ \sum_{k=1}^m b_k y_k : \ \sum_{k=1}^m y_k \mathbf{a}_k + \mathbf{s} = \mathbf{c}, \ \mathbf{s} \in cl(\mathcal{K}^\sharp) \right\}, \tag{3.1b}$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{c} \in \mathfrak{A}$ and $b_1, \ldots, b_m \in \mathbb{R}$ are given. Let $\mathcal{F}_p$ (resp., $\mathcal{F}_p^\circ$) denotes the primal feasible (resp., strictly feasible) region, and let $\mathcal{F}_d$ (resp., $\mathcal{F}_d^\circ$) denotes the dual counterpart.

Without any loss of generality, we may assume that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are linearly independent. With this assumption, $(y_1, \ldots, y_m)$ is uniquely determined by each feasible $\mathbf{s}$. Henceforth, we shall use only the $\mathbf{s}$-component when referring to a feasible solution in $\mathcal{F}_d$.

A necessary assumption in any primal-dual interior-point algorithm is the existence of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$; i.e., $\mathcal{F}_p^\circ \times \mathcal{F}_d^\circ \supset \{(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})\}$ is nonempty.

In the special case of semidefinite programming, where $\mathcal{K} = \mathcal{S}_n$, the author [4] defined primal-dual weighted analytic centers of (3.1) to be pairs of solutions to the primal-dual weighted barrier problem

$$\inf_{(\mathbf{x},\mathbf{s}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ} -\sum_{i=1}^n w_i \log(\mathbf{u}_{\boldsymbol{\mathcal{Q}}\mathbf{x}})_{ii}^2 - \sum_{i=1}^n w_i \log(\mathbf{l}_{\boldsymbol{\mathcal{Q}}\mathbf{s}})_{ii}^2 + \langle \mathbf{x}, \mathbf{s}\rangle$$

over all orthonormal similarity transformations $\boldsymbol{\mathcal{Q}} : \mathbb{S}^n \mapsto \mathbb{S}^n$ and all n-tuples $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{++}$. We generalize this to symmetric cone programming by replacing the orthonormal similarity transformation $\boldsymbol{\mathcal{Q}}$ with maps $\boldsymbol{\mathcal{A}}$ from the orthogonal subgroup of $\mathrm{Aut}(\mathcal{K})$. Theorem A.2 states that such maps are precisely *automorphisms of* $\mathfrak{J}$; i.e., bijective homomorphisms of the algebra $\mathfrak{J}$ to itself. We shall denote by $\mathrm{Aut}(\mathfrak{J})$ the group of automorphisms of $\mathfrak{J}$. This gives the primal-dual weighted barrier problem

$$\inf_{(\mathbf{x},\mathbf{s}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ} -\sum_{i=1}^r w_i \log \rho_i(\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}})^2 - \sum_{i=1}^r w_i \log \rho_i(\mathbf{l}_{\boldsymbol{\mathcal{A}}\mathbf{s}})^2 + \langle \mathbf{x}, \mathbf{s}\rangle . \tag{$BP_{\boldsymbol{\mathcal{A}},w}$}$$

The barrier problem ($BP_{\boldsymbol{\mathcal{A}},w}$) has optimal solutions as the objective is continuous, and the set of feasible solutions with values no more than that of $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ is compact. Indeed, at least one of the logarithmic terms explode to ∞ when $\mathbf{x}$ (resp., $\mathbf{s}$) approaches the boundary of $\mathcal{K}$ (resp., $\mathcal{K}^\sharp$), while the linear term explode to ∞ faster than all logarithmic terms in magnitude when either $\mathbf{x}$ or $\mathbf{s}$ (or both) grows without bound.

Using Proposition 3, we deduce that the derivative of $\mathbf{x} \in \mathcal{K} \mapsto \mathbf{u}_\mathbf{x}$ is $\mathbf{a} \in \mathfrak{H} \mapsto \mathbf{u}_\mathbf{x}\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_\mathbf{x}} \mathbf{a}\rangle\!\rangle^*$. Hence the primal weighted barrier $\mathbf{x} \mapsto -\sum w_i \log \rho_i(\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}})^2$ has derivative $\mathbf{a} \in \mathfrak{H} \mapsto -\sum w_i \rho_i(\boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{a})$. Its gradient is thus $-\boldsymbol{\mathcal{A}}^* \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-*}} \mathbf{d}$, where $\mathbf{d}$ is the diagonal element with $\rho_i(\mathbf{d}) = w_i$. Similarly, the gradient of the dual weighted barrier $\mathbf{s} \mapsto -\sum w_i \log \rho_i(\mathbf{l}_{\boldsymbol{\mathcal{A}}\mathbf{s}})^2$ is $-\boldsymbol{\mathcal{A}}^{-1} \boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\boldsymbol{\mathcal{A}}\mathbf{s}}^{-*}} \mathbf{d}$. Thus the necessary and sufficient optimality conditions for $(BP_{\boldsymbol{\mathcal{A}},w})$ are

$$(CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}) \qquad \begin{aligned} \langle \mathbf{a}_k, \mathbf{x}\rangle = b_k \ (k = 1, \ldots, m), &\quad \mathbf{x} \in \mathcal{K}, \\ \sum_{k=1}^m y_k \mathbf{a}_k + \mathbf{s} = \mathbf{c}, &\quad \mathbf{s} \in \mathcal{K}^\sharp, \\ \boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\boldsymbol{\mathcal{A}}\mathbf{s}}^*} \boldsymbol{\mathcal{A}}\mathbf{x} = \mathbf{d}. & \end{aligned}$$

The next theorem shows that $(BP_{\boldsymbol{\mathcal{A}},w})$ has a unique pair of optimal solutions[2], whence so does $(CP_{\boldsymbol{\mathcal{A}},\mathbf{d}})$. We call them the *weighted analytic centers* determined by $(\boldsymbol{\mathcal{A}}, \mathbf{d}) \in \operatorname{Aut}(\mathfrak{J}) \times \mathfrak{D}_{++}$.

Theorem 3 (cf. Proposition 2.9 of [3]). *The primal and dual weighted barriers*

$$\mathbf{x} \mapsto -\sum w_i \log \rho_i(\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}})^2 \quad \textit{and} \quad \mathbf{s} \mapsto -\sum w_i \log \rho_i(\mathbf{l}_{\boldsymbol{\mathcal{A}}\mathbf{s}})^2,$$

where $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ *and* $(w_1, \ldots, w_n) \in \mathbb{R}^n_{++}$, *are strictly convex.*

Proof. We prove the theorem for the primal weighted barrier, and remark that the dual weighted barrier is proved similarly.

From Proposition 2, the derivative of $\mathbf{t} \in \mathfrak{T}_{++} \mapsto \boldsymbol{\mathcal{Q}}_\mathbf{t}\mathbf{a}$ for each $\mathbf{a} \in \mathfrak{H}$ is $\mathbf{u} \in \mathfrak{T} \mapsto (\mathbf{t}(\langle\!\langle \mathbf{a}\rangle\!\rangle \mathbf{u}^*) + \mathbf{u}(\langle\!\langle \mathbf{a}\rangle\!\rangle \mathbf{t}^*))_H = (\mathbf{t}(\langle\!\langle \mathbf{a}\rangle\!\rangle \mathbf{u}^*) + (\mathbf{t}\langle\!\langle \mathbf{a}\rangle\!\rangle^*)\mathbf{u}^*)_H$. By differentiating $\mathbf{t}\mathbf{t}^{-1} = \mathbf{e}$ implicitly, we find that the derivative of $\mathbf{t} \in \mathfrak{T}_{++} \mapsto \mathbf{t}^{-1}$ is $\mathbf{u} \in \mathfrak{T} \mapsto -\mathbf{t}^{-1}\mathbf{u}\mathbf{t}^{-1}$. Recall that the derivative of $\mathbf{x} \in \mathcal{K} \mapsto \mathbf{u}_\mathbf{x}$ is $\mathbf{a} \in \mathfrak{H} \mapsto \mathbf{u}_\mathbf{x}\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_\mathbf{x}}\mathbf{a}\rangle\!\rangle^*$. Thus the second derivative of the primal weighted barrier is

$$(\mathbf{a}, \mathbf{b}) \in \mathfrak{H}^2 \mapsto \sum_{i=1}^r w_i \rho_i \begin{pmatrix} \mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}(\langle\!\langle \boldsymbol{\mathcal{A}}\mathbf{a}\rangle\!\rangle [\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}(\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle^*)\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}]^*) \\ + (\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}\langle\!\langle \boldsymbol{\mathcal{A}}\mathbf{a}\rangle\!\rangle^*)[\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}(\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle^*)\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}]^* \end{pmatrix}_H$$

Using Axioms VI and IV, we can simply the expression in the parentheses to

$$((\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}(\langle\!\langle \boldsymbol{\mathcal{A}}\mathbf{a}\rangle\!\rangle \mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-*}))\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle + ((\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}\langle\!\langle \boldsymbol{\mathcal{A}}\mathbf{a}\rangle\!\rangle^*)\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-*})\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle)_H.$$

By Proposition 2, this is the same as $((\boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{a})\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle)_H$. Therefore the second derivative is

$$(\mathbf{a}, \mathbf{b}) \mapsto \left\langle \mathbf{d}, (\boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{a})\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle + \langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle^*(\boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{a}) \right\rangle,$$

which, by Lemma 1, can be simplified to

$$(\mathbf{a}, \mathbf{b}) \mapsto \left\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{a}, (\mathbf{d}\langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{u}_{\boldsymbol{\mathcal{A}}\mathbf{x}}^{-1}} \boldsymbol{\mathcal{A}}\mathbf{b}\rangle\!\rangle^*)_H \right\rangle.$$

For each $\mathbf{0} \neq \mathbf{a} \in \mathfrak{H}$, $\langle \mathbf{a}, (\mathbf{d}\langle\!\langle \mathbf{a}\rangle\!\rangle^*)_H\rangle = \sum_{i=1}^r w_i \rho_i(\mathbf{a})^2 + 2\sum_{1 \le i < j \le r} w_i \|\mathbf{a}_{ij}\|^2 > 0$. Hence the primal weighted barrier is strictly convex. □

[2]This also follows from the fact that the weighted logarithmic barriers are self-concordant barriers whenever the weights are at least 1; see Corollary 2.2 of [3].

Having defined weighted analytic centers for the symmetric cone programs (3.1), we now show that every pair of primal-dual strictly feasible solutions to (3.1) is a pair of weighted analytic centers. The following lemma will be useful.

Lemma 2. *For every* $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ *and every* $\mathbf{t} \in \mathfrak{T}_{++}$*, there exist a unique* $\widetilde{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathfrak{J})$ *and a unique* $\widetilde{\mathbf{t}} \in \mathfrak{T}_{++}$ *such that* $\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{t}} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}\widetilde{\boldsymbol{\mathcal{A}}}$*. Moreover, for each* $\mathbf{d} \in \mathfrak{D}_{\downarrow,++}$*,* $\boldsymbol{\mathcal{A}}$ *stabilizes* $\mathbf{d}$ *if and only if* $\widetilde{\boldsymbol{\mathcal{A}}}$ *does.*

Proof. *(Existence)* Let $\widetilde{\mathbf{t}} = \mathbf{u}_{\boldsymbol{\mathcal{A}}(\mathbf{tt}^*)} \in \mathfrak{T}_{++}$, and let $\widetilde{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}^{-1}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{t}}$. Since $\widetilde{\boldsymbol{\mathcal{A}}}\mathbf{e} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}^{-1}\boldsymbol{\mathcal{A}}(\mathbf{tt}^*) = \boldsymbol{\mathcal{Q}}_{\mathbf{u}^{-1}_{\boldsymbol{\mathcal{A}}(\mathbf{tt}^*)}}\boldsymbol{\mathcal{A}}(\mathbf{tt}^*) = \mathbf{e}$, we have $\widetilde{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$. By continuity, we further have $\widetilde{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$, whence $\widetilde{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathfrak{J})$ by Theorem A.1.

(Uniqueness) By Theorem A.1, $\operatorname{Aut}(\mathfrak{J}) \cap \{\boldsymbol{\mathcal{Q}}_{\mathbf{u}} : \mathbf{u} \in \mathfrak{T}_{++}\} = \operatorname{Aut}(\mathcal{K})_{\mathbf{e}} \cap \{\boldsymbol{\mathcal{Q}}_{\mathbf{u}} : \mathbf{u} \in \mathfrak{T}_{++}\}$. By Proposition 2, $\boldsymbol{\mathcal{Q}}_{\mathbf{u}}\mathbf{e} = \mathbf{uu}^*$ for every $\mathbf{u} \in \mathfrak{T}_{++}$. We then deduce from Proposition 3 that $\operatorname{Aut}(\mathfrak{J}) \cap \{\boldsymbol{\mathcal{Q}}_{\mathbf{u}} : \mathbf{u} \in \mathfrak{T}_{++}\} = \{\boldsymbol{\mathcal{Q}}_{\mathbf{e}}\}$, whence uniqueness follows.

(Moreover) Recall that $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ is a Jordan frame. Under this Jordan frame, We have the *Pierce decomposition* $\mathfrak{J} = \bigoplus_{1 \le i \le j \le r} \mathfrak{J}_{ij}$, where $\mathfrak{J}_{ii}$ denotes the 1-eigenspace $\mathfrak{A}_{ii}$ of $\mathbf{e}_i$, and $\mathfrak{J}_{ij}$ denotes the common $\frac{1}{2}$-eigenspace $\mathfrak{A}_{ij} \oplus \mathfrak{A}_{ji}$ of $\mathbf{e}_i$ and $\mathbf{e}_j$; see Theorem IV.2.1 of [8]. Let $\mathfrak{J}_{\mathcal{I}}$ denote the sub-algebra $\bigoplus_{i,j \in \mathcal{I}: i \le j} \mathfrak{J}_{ij} = \bigoplus_{i,j \in \mathcal{I}} \mathfrak{A}_{ij}$, and let $\mathbf{e}_{\mathcal{I}}$ denote the unit $\sum_{i \in \mathcal{I}} \mathbf{e}_i$ in the sub-algebra $\mathfrak{J}_{\mathcal{I}}$ for each subset $I \subseteq \{1, \dots, r\}$. Define the equivalence relation $iRj : \rho_i(\mathbf{d}) = \rho_j(\mathbf{d})$. Let $\mathcal{L}_1, \dots, \mathcal{L}_p$ denote its equivalence classes with $\mathcal{L}_1 < \cdots < \mathcal{L}_p$. The sub-algebra generated by $\mathbf{d}$ is then $\bigoplus_{k=1}^{p} \mathbb{R}\mathbf{e}_{\mathcal{L}_k}$. An automorphism stabilizes $\mathbf{d}$ if and only if it stabilizes every element of this sub-algebra. In other words, an automorphism stabilizes $\mathbf{d}$ if and only if it stabilizes every $\mathbf{e}_{\mathcal{L}_k}$. If an automorphism $\boldsymbol{\mathcal{A}}$ stabilizes every $\mathbf{e}_{\mathcal{L}_k}$, then it is an endomorphism on the 1-eigenspace $\mathfrak{J}_{\mathcal{L}_{\le k}}$ of every $\mathbf{e}_{\mathcal{L}_{\le k}}$, where $\mathcal{L}_{\le k}$ denotes the index set $\bigcup_{\ell=1}^{k} \mathcal{L}_\ell$, as $\boldsymbol{\mathcal{A}}\mathbf{x} \circ \mathbf{e}_{\mathcal{L}_k} = \boldsymbol{\mathcal{A}}\mathbf{x} \circ \boldsymbol{\mathcal{A}}\mathbf{e}_{\mathcal{L}_k} = \boldsymbol{\mathcal{A}}(\mathbf{x} \circ \mathbf{e}_{\mathcal{L}_k})$. Conversely, any automorphism that is an endomorphism on the sub-algebras $\mathfrak{J}_{\mathcal{L}_{\le k}}$ must stabilizes its unit $\mathbf{e}_{\mathcal{L}_{\le k}}$, whence every $\mathbf{e}_{\mathcal{L}_k} = \mathbf{e}_{\mathcal{L}_{\le k}} - \mathbf{e}_{\mathcal{L} \le k-1}$. Hence an automorphism stabilizes $\mathbf{d}$ if and only if it is an endomorphism on every $\mathfrak{J}_{\mathcal{L}_{\le k}}$. It is straightforward to deduce from the definition of quadratic representations that each map in $\{\boldsymbol{\mathcal{Q}}_{\mathbf{u}} : \mathbf{u} \in \mathfrak{T}\}$ is an endomorphism on $\mathfrak{J}_{\mathcal{L}_{\le k}}$. Thus the map $\widetilde{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}^{-1}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{t}}$ is an endomorphism on $\mathfrak{J}_{\mathcal{L}_{\le k}}$ if and only if $\boldsymbol{\mathcal{A}}$ is. Consequently $\widetilde{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}^{-1}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{t}}$ stabilizes $\mathbf{d}$ if and only if $\boldsymbol{\mathcal{A}}$ does. □

Theorem 4. *Given any pair of primal-dual strictly feasible solutions* $(\mathbf{x}, \mathbf{s})$ *to* (3.1)*, there exists an automorphism* $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ *and weights* $(w_1, \dots, w_n) \in \mathbb{R}^n_{++}$ *such that* $(\mathbf{x}, \mathbf{s})$ *is the unique solution to* ($BP_{\boldsymbol{\mathcal{A}},w}$)*.*

Proof. Uniqueness have been established. It remains to find automorphism $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ and weights $(w_1, \dots, w_n) \in \mathbb{R}^n_{++}$ such that $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$ solves ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$) with $\mathbf{d} \in \mathfrak{D}$ such that $\rho_i(\mathbf{d}) = w_i$.

Consider a spectral decomposition $\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*}\mathbf{x} = \sum_{i=1}^{r} \lambda_i(\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*}\mathbf{x})\mathbf{c}_i$. Let $\mathbf{d}$ denote the element $\sum_{i=1}^{r} \lambda_i(\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*}\mathbf{x})\mathbf{e}_i \in \mathfrak{D}_{\downarrow,++}$. According to Theorem IV.2.5 of [8], there exists an (not necessarily unique) automorphism $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ such that $\boldsymbol{\mathcal{A}}\mathbf{c}_i = \mathbf{e}_i$. Then $\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*}\mathbf{x} = \mathbf{d}$. Lemma 2 shows that there exists a unique automorphism $\widetilde{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathfrak{J})$ satisfying $\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}\widetilde{\boldsymbol{\mathcal{A}}}$ for some $\widetilde{\mathbf{t}} \in \mathfrak{T}_{++}$.

Observe that $\widetilde{\boldsymbol{\mathcal{A}}}\mathbf{s} = \widetilde{\boldsymbol{\mathcal{A}}}^{-*}\mathbf{s} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}^{*}\boldsymbol{\mathcal{A}}^{-*}\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*}^{-*}\mathbf{s} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}^*}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^{-1}}\mathbf{s} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}^*}\boldsymbol{\mathcal{A}}\mathbf{e} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}^*}\mathbf{e}$ and $\widetilde{\boldsymbol{\mathcal{A}}}\mathbf{x} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}^{-1}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*}\mathbf{x} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}^{-1}}\mathbf{d}$, where we have used the fact that $\boldsymbol{\mathcal{A}}$ is orthogonal; see Theorem A.1. Subsequently $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widetilde{\boldsymbol{\mathcal{A}}}\mathbf{s}}}\widetilde{\boldsymbol{\mathcal{A}}}\mathbf{x} = \boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}}\boldsymbol{\mathcal{Q}}_{\widetilde{\mathbf{t}}^{-1}}\mathbf{d} = \mathbf{d}$ as required. □

3.1. **Approximating weighted analytic centers.** Given $\boldsymbol{\mathcal{A}} \in \mathrm{Aut}(\mathfrak{J})$ and $\mathbf{d} \in \mathfrak{D}_{\downarrow,++}$, consider the problem of approximating the weighted analytic centers determined by ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$). We shall employ Newton's method to approximate the weighted centers.

Suppose we begin with a pair of solutions $(\mathbf{x}_+, \mathbf{s}_+) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$. Linearizing ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$) at $(\mathbf{x}_+, \mathbf{s}_+)$ using Proposition 3 gives

$$\langle \mathbf{a}_k, \boldsymbol{\Delta}_{\mathbf{x}} \rangle = 0 \ (k = 1, \dots, m), \tag{3.2a}$$

$$\sum_{k=1}^{m} \Delta_{y_k} \mathbf{a}_k + \boldsymbol{\Delta}_{\mathbf{s}} = \mathbf{0}, \tag{3.2b}$$

$$\boldsymbol{\mathcal{Q}}_{\mathbf{1}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \boldsymbol{\Delta}_{\mathbf{x}} + \Big(\mathbf{v} \langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{1}^{-1}_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \boldsymbol{\Delta}_{\mathbf{s}} \rangle\!\rangle \Big)_H = \mathbf{d} - \mathbf{v}, \tag{3.2c}$$

where $\mathbf{v}$ denotes $\boldsymbol{\mathcal{Q}}_{\mathbf{1}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \mathbf{x}_+$. Solving this linear system for $(\boldsymbol{\Delta}_{\mathbf{x}}, \boldsymbol{\Delta}_{\mathbf{s}})$ gives the next pair of iterates $(\mathbf{x}_+ + \boldsymbol{\Delta}_{\mathbf{x}}, \mathbf{s}_+ + \boldsymbol{\Delta}_{\mathbf{s}})$.

We shall briefly show that this linearization has a unique pair of solutions $(\boldsymbol{\Delta}_{\mathbf{x}}, \boldsymbol{\Delta}_{\mathbf{s}})$ under some assumption on the proximity of $\boldsymbol{\mathcal{Q}}_{\mathbf{1}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \mathbf{x}_+$ to $\mathbf{d}$.

We shall use $\boldsymbol{\Delta}_{\widetilde{\mathbf{x}}}$ and $\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}$ to denote $\boldsymbol{\mathcal{Q}}_{\mathbf{1}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \boldsymbol{\Delta}_{\mathbf{x}}$ and $\boldsymbol{\mathcal{Q}}_{\mathbf{1}^{-1}_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \boldsymbol{\Delta}_{\mathbf{s}}$, respectively. This simplifies the above linear system to

$$\langle \widetilde{\mathbf{a}}_k, \boldsymbol{\Delta}_{\widetilde{\mathbf{x}}} \rangle = 0 \ (k = 1, \dots, m), \tag{3.3a}$$

$$\sum_{k=1}^{m} \Delta_{y_k} \widetilde{\mathbf{a}}_k + \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}} = \mathbf{0}, \tag{3.3b}$$

$$\boldsymbol{\Delta}_{\widetilde{\mathbf{x}}} + (\mathbf{v} \langle\!\langle \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}} \rangle\!\rangle)_H = \mathbf{d} - \mathbf{v}, \tag{3.3c}$$

where $\widetilde{\mathbf{a}}_k$ denotes $\boldsymbol{\mathcal{Q}}_{\mathbf{1}^{-1}_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \mathbf{a}_k$.

For each $\ell \in \{1, \dots, r\}$, let π_ℓ denote the difference $\rho_\ell(\mathbf{d}) - \rho_{\ell+1}(\mathbf{d})$, with the convention $\rho_{r+1} \equiv 0$. Let $\mathcal{L}$ denote the set $\{\ell : \pi_\ell > 0\}$ of indices of distinct weights. Now consider the linear system

$$\sum_{\ell=1}^{n} \pi_\ell \, \langle [\widetilde{\mathbf{a}}_k]_\ell, \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} \rangle = 0 \ (k = 1, \dots, m), \tag{3.4a}$$

$$\sum_{k=1}^{m} \Delta_{y_k} \widetilde{\mathbf{a}}_k + \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}} = \mathbf{0}, \tag{3.4b}$$

$$\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} + ([\mathbf{z}]_\ell \langle\!\langle [\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell \rangle\!\rangle)_H = [\mathbf{e} - \mathbf{z}]_\ell \ (\ell \in \mathcal{L}), \tag{3.4c}$$

where $\mathbf{z}$ denotes $(\mathbf{d}^{-1} \langle\!\langle \mathbf{v} \rangle\!\rangle)_H$. By (2.2), the weighted sum of the equations in (3.4c), with weights $\{\pi_\ell\}_{\ell \in \mathcal{L}}$, is precisely (3.3c) with $\boldsymbol{\Delta}_{\widetilde{\mathbf{x}}} = \sum_{\ell=1}^{r} \pi_\ell \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}$. Hence

(1) if $(\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}, \ell \in \mathcal{L}; \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}})$ is a solution of the system (3.4), then the pair $(\boldsymbol{\Delta}_{\widetilde{\mathbf{x}}}, \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}})$ with $\boldsymbol{\Delta}_{\widetilde{\mathbf{x}}} = \sum_{\ell=1}^{r} \pi_\ell \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}$ is a solution of (3.3);
(2) conversely, if $(\boldsymbol{\Delta}_{\widetilde{\mathbf{x}}}, \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}})$ is a solution of (3.3), then the tuple $(\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}, \ell \in \mathcal{L}; \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}})$ with $\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} = [\mathbf{e} - \mathbf{z}]_l - ([\mathbf{z}]_l \langle\!\langle [\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_l \rangle\!\rangle)_H$ is a solution of (3.4).

Thus, the existence and uniqueness of solutions to (3.3) (whence (3.2)) is equivalent to that of (3.4). The linear system (3.4) is square. Hence it has unique solutions if and only if its Jacobian is nonsingular. The following proposition shows that this holds when $\left\| (\mathbf{d}^{-1} \langle\!\langle \boldsymbol{\mathcal{Q}}_{\mathbf{1}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}} \mathbf{x}_+ \rangle\!\rangle)_H - \mu \mathbf{e} \right\| < \mu / \sqrt{2}$ for some $\mu > 0$.

Proposition 8 (cf. Proposition 9 of [5]). *Suppose $\{\pi_\ell\}_{\ell\in\mathcal{L}}$ is a finite sequence of positive real numbers, where the index set $\mathcal{L} \subseteq \{1,\dots,r\}$ is nonempty. Suppose further $\mathbf{z} \in \mathcal{K}$ satisfies*

$$\|\mathbf{z} - \mu\mathbf{e}\| \le \gamma\mu \tag{3.5}$$

for some $\gamma \in (0, 1/\sqrt{2})$ and some $\mu > 0$. If $\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} \in [\mathfrak{H}]_\ell$, $(\ell \in \mathcal{L})$ and $\mathbf{\Delta}_{\tilde{\mathbf{s}}} \in \mathfrak{H}$ satisfy

$$\sum_{\ell\in\mathcal{L}} \pi_\ell \langle \mathbf{\Delta}_{\tilde{\mathbf{x}},\ell}, [\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\rangle \ge 0 \tag{3.6}$$

and

$$\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + ([\mathbf{z}]_\ell \langle\!\langle [\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell \rangle\!\rangle)_H = \mathbf{h}_\ell \ (\ell \in \mathcal{L}) \tag{3.7}$$

for some $\mathbf{h}_\ell \in [\mathfrak{H}]_\ell$, $(\ell \in \mathcal{L})$, then

$$\max\left\{ \sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell}\|^2, \mu^2 \sum_{\ell\in\mathcal{L}} \pi_\ell \|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2 \right\} \le \frac{\sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{h}_\ell\|^2}{(1-\sqrt{2}\gamma)^2}. \tag{3.8}$$

Proof. If (3.6) holds, then each summand in $\sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + \mu[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2$ is no less than the corresponding summand in $\sum_{\ell\in\mathcal{L}} \pi_\ell \left(\|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell}\|^2 + \mu^2 \|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2\right)$, which in turn upper bounds the left hand side of (3.8). If the hypothesis (3.7) holds, then

$$\begin{aligned}\|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + \mu[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\| &\le \|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + ([\mathbf{z}]_\ell\langle\!\langle[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\rangle\!\rangle)_H\| + \|(([\mathbf{z}]_\ell - \mu[\mathbf{e}]_\ell)\langle\!\langle[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\rangle\!\rangle)_H\| \\ &\le \|\mathbf{h}_\ell\| + 2\,\|[\mathbf{z}-\mu\mathbf{e}]_\ell\langle\!\langle[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\rangle\!\rangle\|\,,\end{aligned}$$

where we used the triangle inequality on $\|\cdot\|$. Applying the sub-multiplicativity of the norm $\|\cdot\|$ and (2.3) then gives $\|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + \mu[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\| \le \|\mathbf{h}_\ell\| + \sqrt{2}\,\|[\mathbf{z}-\mu\mathbf{e}]_\ell\|\,\|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|$. For each $\ell \in \mathcal{L}$, it is clear that $\|[\mathbf{z}-\mu\mathbf{e}]_\ell\| \le \|\mathbf{z}-\mu\mathbf{e}\|$. Hence using (3.5), we further deduce that $\|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + \mu[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\| \le \|\mathbf{h}_\ell\| + \sqrt{2}\gamma\mu\,\|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|$, and subsequently $\sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell} + \mu[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2$ is bounded from above by

$$\sum_{\ell\in\mathcal{L}} \pi_\ell \left(\|\mathbf{h}_\ell\| + \sqrt{2}\gamma\mu\,\|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|\right)^2 \le \left(\sqrt{\sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{h}_\ell\|^2} + \sqrt{2}\gamma\mu\sqrt{\sum_{\ell\in\mathcal{L}} \pi_\ell \|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2}\right)^2$$

where we have used the triangle inequality on the 2-norm of $\mathbb{R}^{\mathcal{L}}$. Consequently under the hypotheses of the proposition,

$$\begin{aligned}&\max\left\{ \sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell}\|^2, \mu^2 \sum_{\ell\in\mathcal{L}} \pi_\ell \|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2 \right\}^{\frac{1}{2}} \\ &\le \sqrt{\sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{h}_\ell\|^2} + \sqrt{2}\gamma\mu\sqrt{\sum_{\ell\in\mathcal{L}} \pi_\ell \|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2} \\ &\le \sqrt{\sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{h}_\ell\|^2} + \sqrt{2}\gamma \max\left\{ \sum_{\ell\in\mathcal{L}} \pi_\ell \|\mathbf{\Delta}_{\tilde{\mathbf{x}},\ell}\|^2, \mu^2 \sum_{\ell\in\mathcal{L}} \pi_\ell \|[\mathbf{\Delta}_{\tilde{\mathbf{s}}}]_\ell\|^2 \right\}^{\frac{1}{2}},\end{aligned}$$

which proves the proposition. ☐

Corollary 2. *If $\left\|(\mathbf{d}^{-1}\langle\!\langle\mathbf{v}\rangle\!\rangle)_H - \mu\mathbf{e}\right\| < \mu/\sqrt{2}$ for some $\mu > 0$, then the linear system (3.2) with $\mathbf{v} = \mathcal{Q}_{\mathbf{1}^*_{\mathcal{A}\mathbf{s}_+}} \mathcal{A}\mathbf{x}_+$ has a non-singular Jacobian. Consequently it has a unique pair of solutions $(\mathbf{\Delta}_\mathbf{x}, \mathbf{\Delta}_\mathbf{s})$.*

For the analysis of Newton's method, we shall use the function

$$d : (\mathbf{x}, \mathbf{s}) \in \mathfrak{H} \times \mathcal{K}^\sharp \mapsto \left(\sum_{i,j=1}^{r} \frac{\rho_{i\vee j}(\mathbf{d})}{\rho_r(\mathbf{d})} \left\| \rho_{i\vee j}(\mathbf{d})^{-1} (\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}}} \boldsymbol{\mathcal{A}}\mathbf{x})_{ij} - \mathbf{e}_{ij} \right\|^2 \right)^{\frac{1}{2}}, \tag{3.9}$$

where $i \vee j$ denotes $\max\{i, j\}$, to measure the proximity of the pair $(\mathbf{x}, \mathbf{s})$ to the solutions of ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$). Note that

$$d(\mathbf{x}_+, \mathbf{s}_+) = \rho_r(\mathbf{d})^{-\frac{1}{2}} \left(\sum_{\ell \in \mathcal{L}} \pi_\ell \left\| [(\mathbf{d}^{-1} \langle\!\langle \mathbf{v} \rangle\!\rangle)_H - \mathbf{e}]_\ell \right\|^2 \right)^{\frac{1}{2}}, \tag{3.10}$$

where, as before, $\mathbf{v} = \boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_+}} \boldsymbol{\mathcal{A}}\mathbf{x}_+$, $\pi_\ell = \rho_\ell(\mathbf{d}) - \rho_{\ell+1}(\mathbf{d})$ and $\mathcal{L} = \{\ell : \pi_\ell > 0\}$.

Since $\rho_{i\vee j}(\mathbf{d}) \geq \rho_r(\mathbf{d})$, the measure $d(\mathbf{x}_+, \mathbf{s}_+)$ dominates $\left\| (\mathbf{d}^{-1} \langle\!\langle \mathbf{v} \rangle\!\rangle)_H - \mathbf{e} \right\|$. Hence Corollary 2 concludes that (3.2) has unique solutions if $d(\mathbf{x}_+, \mathbf{s}_+) \leq \gamma$ for some $\gamma \in (0, 1/\sqrt{2})$.

For each $\alpha \in \mathbb{R}$, let $\widetilde{\mathbf{x}}_\alpha$ and $\widetilde{\mathbf{s}}_\alpha$ denote, respectively, the sums $\mathbf{v} + \alpha \boldsymbol{\Delta}_{\widetilde{\mathbf{x}}}$ and $\mathbf{e} + \alpha \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}$. Then $(\widetilde{\mathbf{x}}_\alpha, \widetilde{\mathbf{s}}_\alpha) \in \mathcal{K} \times \mathcal{K}^\sharp$ if and only if $(\mathbf{x}_+ + \alpha \boldsymbol{\Delta}_{\mathbf{x}}, \mathbf{s}_+ + \alpha \boldsymbol{\Delta}_{\mathbf{s}}) \in \mathcal{K} \times \mathcal{K}^\sharp$. Moreover, in this case it holds $d(\mathbf{x}_+ + \alpha \boldsymbol{\Delta}_{\mathbf{x}}, \mathbf{s}_+ + \alpha \boldsymbol{\Delta}_{\mathbf{s}}) = d(\widetilde{\mathbf{x}}_\alpha, \widetilde{\mathbf{s}}_\alpha)$. Thus we consider $d(\widetilde{\mathbf{x}}_\alpha, \widetilde{\mathbf{s}}_\alpha)$ instead.

We shall use the following lemma to give an upper bound on $d(\widetilde{\mathbf{x}}_\alpha, \widetilde{\mathbf{s}}_\alpha)$. This lemma generalizes the local Lipschitz constant of Cholesky factorization of real symmetric matrices near the identity matrix.

Lemma 3. *If $\mathbf{h} \in \mathfrak{H}$ satisfies $\|\mathbf{h}\| \leq 1/2$, then*

$$\|\mathbf{l}_{\mathbf{e}+\mathbf{h}} - \mathbf{e}\| \leq \sqrt{2} \|\mathbf{h}\|.$$

Proof. See proof of Lemma 1 of [5]. □

Lemma 4. *Suppose $d(\mathbf{x}_+, \mathbf{s}_+) \leq \gamma$ for some $\gamma \in (0, 1/\sqrt{2})$. Then $\widetilde{\mathbf{s}}_\alpha \in \mathcal{K}^\sharp$ and*

$$d(\widetilde{\mathbf{x}}_\alpha, \widetilde{\mathbf{s}}_\alpha) \leq (1-\alpha)\gamma + \alpha^2 \frac{\gamma^2(4 + 2\sqrt{2} + 4\sqrt{2}\gamma)}{(1-\sqrt{2}\gamma)^2} + \alpha^3 \frac{2\sqrt{2}\gamma^3}{(1-\sqrt{2}\gamma)^3} \tag{3.11}$$

whenever $0 \leq \alpha \leq \min\{1, (1-\sqrt{2}\gamma)/(2\gamma)\}$.

Proof. Let $\{\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}\}_{\ell \in \mathcal{L}}$ be the vectors that satisfy $\sum_{\ell \in \mathcal{L}} \pi_\ell \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} = \boldsymbol{\Delta}_{\widetilde{\mathbf{x}}}$, and, together with $\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}$, solve (3.4). According to Proposition 8, if $\gamma < 1/\sqrt{2}$, then the sums $\sum_{\ell \in \mathcal{L}} \pi_\ell \|\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}\|^2$ and $\sum_{\ell \in \mathcal{L}} \pi_\ell \|[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell\|^2$ are both bounded from above by $\sum_{\ell \in \mathcal{L}} \pi_\ell \|[\mathbf{e} - \mathbf{z}]_\ell\|^2 / (1-\sqrt{2}\gamma)^2$. Recall from (3.10) that the sum in this upper bound is precisely $\rho_r(\mathbf{d}) d(\mathbf{x}_+, \mathbf{s}_+)^2$, which is no more than $\rho_r(\mathbf{d})\gamma^2$. We have thus established that

$$\sum_{\ell \in \mathcal{L}} \pi_\ell \|\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}\|^2, \sum_{\ell \in \mathcal{L}} \pi_\ell \|[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell\|^2 \leq \frac{\rho_r(\mathbf{d})\gamma^2}{(1-\sqrt{2}\gamma)^2}. \tag{3.12}$$

Moreover, since $\sum_{\ell \in \mathcal{L}} \pi_\ell \|[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell\|^2 \geq \pi_r \|[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_r\|^2 = \rho_r(\mathbf{d}) \|\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}\|^2$, it holds

$$\|[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell\| \leq \|\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}\| \leq \frac{\gamma}{1-\sqrt{2}\gamma} \quad \forall \ell \in \mathcal{L}. \tag{3.13}$$

Subsequently, for every $\mathbf{u} \in \mathfrak{T}_+ \setminus \{\mathbf{0}\}$, by Cauchy-Schwarz inequality, $|\langle \boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}, \mathbf{u}\mathbf{u}^* \rangle| \leq \|\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}\| \|\mathbf{u}\|^2 \leq \|\mathbf{u}\|^2 \gamma/(1-\sqrt{2}\gamma)$, which imply, by definition of $\mathcal{K}^\sharp$, that $\widetilde{\mathbf{s}}_\alpha \in \mathcal{K}^\sharp$ for every $\alpha \in [0, (1-\sqrt{2}\gamma)/(2\gamma)]$.

For each $\ell \in \mathcal{L}$, let $\widetilde{\mathbf{x}}_{\alpha,\ell}$ denote the sum $[\mathbf{z}]_\ell + \alpha \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}$. Then

$$\sum_{\ell\in\mathcal{L}} \pi_\ell \widetilde{\mathbf{x}}_{\alpha,\ell} = \widetilde{\mathbf{x}}_\alpha. \tag{3.14}$$

It suffices to demonstrate the same upper bound in (3.11) on

$$\left(\rho_r(\mathbf{d})^{-1} \sum_{\ell\in\mathcal{L}} \pi_\ell \left\| \boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell} - [\mathbf{e}]_\ell \right\|^2 \right)^{\frac{1}{2}}.$$

Indeed, since

$$\begin{aligned}
\sum_{\ell\in\mathcal{L}} \pi_\ell \left\| \boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell} - [\mathbf{e}]_\ell \right\|^2 &= \sum_{\ell\in\mathcal{L}} \pi_\ell \sum_{i,j=1}^{\ell} \left\| (\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell})_{ij} - \mathbf{e}_{ij} \right\|^2 \\
&= \sum_{i,j=1}^{r} \sum_{\substack{\ell\in\mathcal{L} \\ \ell\geq i\vee j}} \pi_\ell \left\| (\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell})_{ij} - \mathbf{e}_{ij} \right\|^2,
\end{aligned}$$

we have, by Cauchy's inequality, that $\sum_{\ell\in\mathcal{L}} \pi_\ell \left\| \boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell} - [\mathbf{e}]_\ell \right\|^2$ upper bounds

$$\sum_{i,j=1}^{r} \left(\sum_{\substack{\ell\in\mathcal{L} \\ \ell\geq i\vee j}} \pi_\ell \right)^{-1} \left(\sum_{\substack{\ell\in\mathcal{L} \\ \ell\geq i\vee j}} \pi_\ell \left\| (\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell})_{ij} - \mathbf{e}_{ij} \right\| \right)^2,$$

which, by (2.1), (3.14) and the triangle inequality on $\|\cdot\|$, is the same as

$$\sum_{i,j=1}^{r} \rho_{i\vee j}(\mathbf{d})^{-1} \left\| \left(\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_\alpha \right)_{ij} - \rho_{i\vee j}(\mathbf{d}) \mathbf{e}_{ij} \right\|^2 = \rho_r(\mathbf{d}) d(\widetilde{\mathbf{x}}_\alpha, \widetilde{\mathbf{s}}_\alpha)^2.$$

For each $\ell \in \mathcal{L}$, let $\mathbf{l}_{\alpha,\ell}$ denote the lower triangular element $[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha} - \mathbf{e}]_\ell = \mathbf{l}_{[\mathbf{e}]_\ell + \alpha[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell} - [\mathbf{e}]_\ell$, which is well-defined when $\alpha \in [0, (1-\sqrt{2}\gamma)/(2\gamma)]$. Similar to the proof of [5, Lemma 2], we can use Proposition 2, (3.3c) and the triangle inequality on $\|\cdot\|$ to show that

$$\begin{aligned}
&\left\| [\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell} - [\mathbf{e}]_\ell \right\| \\
&\leq (1-\alpha) \left\| [\mathbf{z} - \mathbf{e}]_\ell \right\| + \left\| \mathbf{l}_{\alpha,\ell} \mathbf{l}_{\alpha,\ell}^* \right\| + 2 \left\| ([\mathbf{z} - \mathbf{e}]_\ell) \langle\!\langle \mathbf{l}_{\alpha,\ell} \mathbf{l}_{\alpha,\ell}^* \rangle\!\rangle \right\| \\
&\quad + 2\alpha \left\| \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} \mathbf{l}_{\alpha,\ell} \right\| + \left\| \mathbf{l}_{\alpha,\ell}^* \mathbf{l}_{\alpha,\ell} \right\| + \left\| \boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\alpha,\ell}^*} ([\mathbf{z} - \mathbf{e}]_\ell) \right\| + \alpha \left\| \boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\alpha,\ell}^*} (\boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell}) \right\|.
\end{aligned}$$

Lemma 3 states that $\|\mathbf{l}_{\alpha,\ell}\|$ is no more than $\sqrt{2}\alpha \|[\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell\|$ whenever $0 \leq \alpha \leq (1-\sqrt{2}\gamma)/(2\gamma)$. This upper bound, together with the sub-multiplicativity of $\|\cdot\|$, the sub-multiplicativity of $\|\cdot\|$, the inequality (2.3) and Proposition 5, implies that for $0 \leq \alpha \leq (1-\sqrt{2}\gamma)/(2\gamma)$, $\left\| [\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\widetilde{\mathbf{s}}_\alpha}]_\ell^*} \widetilde{\mathbf{x}}_{\alpha,\ell} - \mathbf{e}]_\ell \right\|$ is bounded from above by

$$\begin{aligned}
&(1-\alpha) \left\| [\mathbf{z} - \mathbf{e}]_\ell \right\| + 4\alpha^2 \left\| [\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell \right\|^2 + 4\sqrt{2}\alpha^2 \left\| [\mathbf{z} - \mathbf{e}]_\ell \right\| \left\| [\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell \right\|^2 \\
&+ 2\sqrt{2}\alpha^2 \left\| \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} \right\| \left\| [\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell \right\| + 2\sqrt{2}\alpha^3 \left\| \boldsymbol{\Delta}_{\widetilde{\mathbf{x}},\ell} \right\| \left\| [\boldsymbol{\Delta}_{\widetilde{\mathbf{s}}}]_\ell \right\|^2.
\end{aligned}$$

Using the triangle inequality on the 2-norm of $\mathbb{R}^{\mathcal{L}}$, we can then bound the expression $\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|[\boldsymbol{\mathcal{Q}}_{[\mathbf{l}_{\tilde{\mathbf{s}}_\alpha}]^*_\ell}\widetilde{\mathbf{x}}_{\alpha,\ell} - \mathbf{e}]_\ell\right\|^2\right)^{\frac{1}{2}}$ from above by the sum

$$(1-\alpha)\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|[\mathbf{z}-\mathbf{e}]_\ell\right\|^2\right)^{\frac{1}{2}} + 4\alpha^2\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|[\boldsymbol{\Delta}_{\tilde{\mathbf{s}}}]_\ell\right\|^4\right)^{\frac{1}{2}}$$
$$+ 4\sqrt{2}\alpha^2\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|[\mathbf{z}-\mathbf{e}]_\ell\right\|^2 \left\|[\boldsymbol{\Delta}_{\tilde{\mathbf{s}}}]_\ell\right\|^4\right)^{\frac{1}{2}} + 2\sqrt{2}\alpha^2\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|\boldsymbol{\Delta}_{\tilde{\mathbf{x}},\ell}\right\|^2 \left\|[\boldsymbol{\Delta}_{\tilde{\mathbf{s}}}]_\ell\right\|^2\right)^{\frac{1}{2}}$$
$$+ 2\sqrt{2}\alpha^3\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|\boldsymbol{\Delta}_{\tilde{\mathbf{x}},\ell}\right\|^2 \left\|[\boldsymbol{\Delta}_{\tilde{\mathbf{s}}}]_\ell\right\|^4\right)^{\frac{1}{2}}.$$

It remains to bound each of these five summands.

By (3.10), the first summand is $(1-\alpha)\rho_r(\mathbf{d})^{\frac{1}{2}}d(\mathbf{x}_+,\mathbf{s}_+)$. Using the upper bound (3.13), we can bound the remaining four summands from above by

$$\frac{4\alpha^2\gamma}{1-\sqrt{2}\gamma}\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|[\boldsymbol{\Delta}_{\tilde{\mathbf{s}}}]_\ell\right\|^2\right)^{\frac{1}{2}}, \frac{4\sqrt{2}\alpha^2\gamma^2}{(1-\sqrt{2}\gamma)^2}\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|[\mathbf{z}-\mathbf{e}]_\ell\right\|^2\right)^{\frac{1}{2}},$$
$$\frac{2\sqrt{2}\alpha^2\gamma}{1-\sqrt{2}\gamma}\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|\boldsymbol{\Delta}_{\tilde{\mathbf{x}},\ell}\right\|^2\right)^{\frac{1}{2}} \quad\text{and}\quad \frac{2\sqrt{2}\alpha^3\gamma^2}{(1-\sqrt{2}\gamma)^2}\left(\sum_{\ell\in\mathcal{L}} \pi_\ell \left\|\boldsymbol{\Delta}_{\tilde{\mathbf{x}},\ell}\right\|^2\right)^{\frac{1}{2}},$$

respectively. Finally, the lemma follows from the hypothesis $d(\mathbf{x}_+,\mathbf{s}_+) \le \gamma$ and the inequality (3.12). $\square$

The next theorem states that the region of quadratic convergence for Newton's method contains all pairs of solutions $(\mathbf{x},\mathbf{s}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$ satisfying $d(\mathbf{x},\mathbf{s}) \le \gamma$, when γ is no more than, say, $\frac{1}{11}$.

Theorem 5. *If Newton's method is applied to solve the nonlinear system ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$) starting from initial iterates $(\mathbf{x},\mathbf{s}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$ satisfying $d(\mathbf{x},\mathbf{s}) \le \gamma$ for some $\gamma \in (0, 1/\sqrt{2})$ with*

$$\gamma\kappa \le 1, \tag{3.15}$$

where the function d is the proximity measure defined in (3.9) and κ denotes the sum $(4+2\sqrt{2}+4\sqrt{2}\gamma)/(1-\sqrt{2}\gamma)^2 + 2\sqrt{2}\gamma/(1-\sqrt{2}\gamma)^3$, then an infinite sequence of iterates $\{(\mathbf{x}_k,\mathbf{s}_k)\}$ is generated, and the iterates satisfy $d(\mathbf{x}_{k+1},\mathbf{s}_{k+1}) \le \kappa d(\mathbf{x}_k,\mathbf{s}_k)^2 \le \gamma$.

Proof. We shall prove the theorem by induction on k.

Let $\gamma_k = d(\mathbf{x}_k,\mathbf{s}_k)$. If (3.15) holds with $\gamma \in (0,1/\sqrt{2})$, then $(1-\sqrt{2}\gamma_k)/(2\gamma_k) > 1$ as $\gamma_k \le \gamma$. Hence we deduce from Corollary 2 that the search directions $(\boldsymbol{\Delta}_{\mathbf{x}},\boldsymbol{\Delta}_{\mathbf{s}})$ are well defined, and from Lemma 4 that $\mathbf{s}_k + \alpha\boldsymbol{\Delta}_{\mathbf{s}} \in \mathcal{K}^\sharp$ and

$$d(\mathbf{x}_k+\alpha\boldsymbol{\Delta}_{\mathbf{x}},\mathbf{s}_k+\alpha\boldsymbol{\Delta}_{\mathbf{s}}) \le (1-\alpha)\gamma_k + \alpha^2\frac{\gamma_k^2(4+2\sqrt{2}+4\sqrt{2}\gamma_k)}{(1-\sqrt{2}\gamma_k)^2} + \alpha^3\frac{2\sqrt{2}\gamma_k^3}{(1-\sqrt{2}\gamma_k)^3}$$

for all $\alpha \in [0,1]$. In particular, $\mathbf{s}_{k+1} \in \mathcal{K}^\sharp$ and $d(\mathbf{x}_{k+1},\mathbf{s}_{k+1}) \le \kappa\gamma_k^2 \le \gamma$ under the hypothesis of the theorem. Subsequently, $\|\mathbf{z}-\mathbf{e}\| \le d(\mathbf{x}_{k+1},\mathbf{s}_{k+1}) < 1$, where $\mathbf{z}$ denotes $(\mathbf{d}^{-1}\langle\!\langle\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_{k+1}}}\boldsymbol{\mathcal{A}}\mathbf{x}_{k+1}\rangle\!\rangle)_H$, shows that $\mathbf{z} \in \mathcal{K}$. This means $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}_{k+1}}}\boldsymbol{\mathcal{A}}\mathbf{x}_{k+1} = \sum_{\ell=1}^r(\rho_\ell(\mathbf{d}) - \rho_{\ell+1}(\mathbf{d}))[\mathbf{z}]_\ell \in \mathcal{K}$ as $\mathcal{K}$ is a convex cone, whence $(\mathbf{x}_{k+1},\mathbf{s}_{k+1}) \in \mathcal{K}\times\mathcal{K}^\sharp$. $\square$

4. Target Map

In linear programming (LP), target-following algorithms are based on the fact that the map $(\mathbf{x}, \mathbf{s}) \mapsto \mathbf{xs}$ sending each pair of strictly feasible primal-dual solutions to the positive vector of their component-wise product is bijective; see [15]. This map is commonly known as the *target map*. The target map was extended to semidefinite programming (SDP) by the author [4] and proved to be a bijection between the set of strictly feasible primal-dual solutions and the cone of symmetric positive definite matrices. In this section, we further extend the target map to symmetric cone programming, and show that it is a bijection.

For each pair $(\mathbf{x}, \mathbf{s}) \in \mathcal{K} \times \mathcal{K}^\sharp$, let $\mathcal{A}(\mathbf{x}, \mathbf{s})$ denote the set of automorphisms $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ satisfying $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}}} \boldsymbol{\mathcal{A}}\mathbf{x} \in \mathfrak{D}_{\downarrow,++}$. Theorem 4 implies that $\mathcal{A}(\mathbf{x}, \mathbf{s})$ is nonempty. With this set $\mathcal{A}(\mathbf{x}, \mathbf{s})$, we define the target map $\boldsymbol{\mathcal{T}} : \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ \to \mathcal{K}$ by

$$\boldsymbol{\mathcal{T}} : (\mathbf{x}, \mathbf{s}) \mapsto \boldsymbol{\mathcal{A}}^* \boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\boldsymbol{\mathcal{A}}\mathbf{s}}} \boldsymbol{\mathcal{A}}\mathbf{x}, \tag{4.1}$$

where $\boldsymbol{\mathcal{A}} \in \mathcal{A}(\mathbf{x}, \mathbf{s})$ is arbitrary. We shall briefly show that $\boldsymbol{\mathcal{T}}$ is well-defined via the following characterization of $\mathcal{A}(\mathbf{x}, \mathbf{s})$.

Lemma 5. *Suppose $(\mathbf{x}, \mathbf{s}) \in \mathcal{K} \times \mathcal{K}^\sharp$ and $\widehat{\boldsymbol{\mathcal{A}}} \in \mathcal{A}(\mathbf{x}, \mathbf{s})$. Denote $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widehat{\boldsymbol{\mathcal{A}}}\mathbf{x}$ by $\mathbf{d}$. Then $\mathcal{A}(\mathbf{x}, \mathbf{s}) = \{\widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathfrak{J}) : \widetilde{\boldsymbol{\mathcal{A}}}\mathbf{d} = \mathbf{d}\}$. Moreover, $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}}} \boldsymbol{\mathcal{A}}\mathbf{x} = \mathbf{d}$ for all $\boldsymbol{\mathcal{A}} \in \mathcal{A}(\mathbf{x}, \mathbf{s})$.*

Proof. Let $\widehat{\mathbf{t}}$ denote $\mathbf{l}^*_{\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}$, and let $\widetilde{\boldsymbol{\mathcal{A}}} \in \operatorname{Aut}(\mathfrak{J})$ be arbitrary. By Lemma 2, there exists an automorphism $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ that satisfies $\widetilde{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{Q}}_{\widehat{\mathbf{t}}^{-1}} = \boldsymbol{\mathcal{Q}}_{\mathbf{t}^{-1}}\boldsymbol{\mathcal{A}}^{-1}$ for some $\mathbf{t} \in \mathfrak{T}_{++}$. By Theorem A.2, $\widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s} = \widetilde{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{Q}}_{\widehat{\mathbf{t}}^*}\mathbf{e} = \boldsymbol{\mathcal{Q}}_{\mathbf{t}^*}\boldsymbol{\mathcal{A}}^{-1}\mathbf{e} = \boldsymbol{\mathcal{Q}}_{\mathbf{t}^*}\mathbf{e}$ and $\widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} = \widetilde{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{Q}}_{\widehat{\mathbf{t}}^{-1}}\boldsymbol{\mathcal{Q}}_{\widehat{\mathbf{t}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} = \boldsymbol{\mathcal{Q}}_{\mathbf{t}^{-1}}\boldsymbol{\mathcal{A}}^{-1}\mathbf{d}$, whence $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} = \boldsymbol{\mathcal{Q}}_{\mathbf{t}}\boldsymbol{\mathcal{Q}}_{\mathbf{t}^{-1}}\boldsymbol{\mathcal{A}}^{-1}\mathbf{d} = \boldsymbol{\mathcal{A}}^{-1}\mathbf{d}$. Consequently $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widetilde{\boldsymbol{\mathcal{A}}}\widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} \in \mathfrak{D}_{\downarrow,++}$ if and only if $\boldsymbol{\mathcal{A}}$ stabilizes $\mathbf{d}$. By Lemma 2, this holds if and only if $\widetilde{\boldsymbol{\mathcal{A}}}$ stabilizes $\mathbf{d}$. $\square$

If $\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2 \in \mathcal{A}(\mathbf{x}, \mathbf{s})$, then the previous lemma implies that $\boldsymbol{\mathcal{A}}_2\boldsymbol{\mathcal{A}}_1^{-1}$ stabilizes $\mathbf{d}$. By Theorem A.2, $\boldsymbol{\mathcal{A}}_1^*\mathbf{d} = \boldsymbol{\mathcal{A}}_2^*\boldsymbol{\mathcal{A}}_2\boldsymbol{\mathcal{A}}_1^{-1}\mathbf{d} = \boldsymbol{\mathcal{A}}_2^*\mathbf{d}$ shows that the target map is well-defined.

Theorem 6. *For the primal-dual symmetric cone programming problems (3.1), where $\mathcal{K}$ is the symmetric cone associated with the simple Euclidean Jordan algebra $\mathfrak{J}$, the target map $\boldsymbol{\mathcal{T}}$ in (4.1) is a bijection between $\mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$ and $\mathcal{K}$.*

Proof. Given $\mathbf{w} \in \mathcal{K}$, let $\mathbf{w} = \sum_{i=1}^r \lambda_i \mathbf{c}_i$ be a spectral decomposition with $\lambda_1 \geq \cdots \geq \lambda_r > 0$. Let $\mathbf{d}$ denote the diagonal element in $\mathfrak{D}_{\downarrow,++}$ with $\rho_i(\mathbf{d}) = \lambda_i$. Pick any automorphism $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ such that $\mathbf{c}_i = \boldsymbol{\mathcal{A}}^*\mathbf{e}_i$. Then $\boldsymbol{\mathcal{T}}(\mathbf{x}, \mathbf{s}) = \mathbf{w}$ if and only if there exists $\widehat{\boldsymbol{\mathcal{A}}} \in \mathcal{A}(\mathbf{x}, \mathbf{s})$ and $\widehat{\boldsymbol{\mathcal{A}}}^* \boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} = \boldsymbol{\mathcal{A}}^*\mathbf{d}$. Since $\boldsymbol{\mathcal{A}}^*\mathbf{d}$ is the spectral decomposition of $\mathbf{w}$, it follows that $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} \in \mathfrak{D}_{\downarrow,++}$ and $\widehat{\boldsymbol{\mathcal{A}}}^* \boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} = \boldsymbol{\mathcal{A}}^*\mathbf{d}$ if and only if $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\boldsymbol{\mathcal{A}}}\mathbf{s}}} \widehat{\boldsymbol{\mathcal{A}}}\mathbf{x} = \mathbf{d}$ and $\widehat{\boldsymbol{\mathcal{A}}}^{-*}\boldsymbol{\mathcal{A}}^*$ stabilizes $\mathbf{d}$. By Theorem A.2 and Lemma 5, such automorphism $\widehat{\boldsymbol{\mathcal{A}}}$ exists if and only if $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\boldsymbol{\mathcal{A}}\mathbf{s}}} \boldsymbol{\mathcal{A}}\mathbf{x} = \mathbf{d}$. This holds precisely when, and only when, $(\mathbf{x}, \mathbf{s})$ is the unique pair of solutions to ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$). $\square$

4.1. **Target-following algorithm for symmetric cone programming.** We shall use the following local norm on the target space $\mathcal{K}$:

$$\|\cdot\|_{\mathbf{w}} : \mathbf{h} \in \mathfrak{J} \mapsto \left(\lambda_r(\mathbf{w})^{-1} \sum_{i,j=1}^r \lambda_{i\vee j}(\mathbf{w})^{-1} \left\| (\boldsymbol{\mathcal{A}}\mathbf{h})_{ij} \right\|^2 \right)^{\frac{1}{2}},$$

where $\boldsymbol{\mathcal{A}}$ is any automorphism in $\mathrm{Aut}(\mathfrak{J})$ such that $\boldsymbol{\mathcal{A}}\mathbf{w} \in \mathfrak{D}_{\downarrow,++}$. According to Theorem IV.2.5 of [8], we can always find an automorphism in $\mathrm{Aut}(\mathfrak{J})$ satisfying $\boldsymbol{\mathcal{A}}\mathbf{w} \in \mathfrak{D}_{\downarrow,++}$ with $\rho_i(\boldsymbol{\mathcal{A}}\mathbf{w}) = \lambda_i(\mathbf{w})$. Since there exist, in general, more than one such automorphism, we need to show that $\|\cdot\|_{\mathbf{w}}$ is well-defined.

Let $\mathbf{d} = \boldsymbol{\mathcal{A}}\mathbf{w} \in \mathfrak{D}_{\downarrow,++}$. Using Lemma 5 with $(\mathbf{x},\mathbf{s}) = (\mathbf{w},\mathbf{e})$ we deduce that if $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_2$ are any two such automorphisms, then $\boldsymbol{\mathcal{A}}_1\boldsymbol{\mathcal{A}}_2^{-1}$ stabilizes $\boldsymbol{\mathcal{A}}_1\mathbf{w} = \boldsymbol{\mathcal{A}}_2\mathbf{w} = \mathbf{d}$. Let $\mu_1 > \cdots > \mu_p > 0$ be the distinct eigenvalues of $\mathbf{w}$. In proving Lemma 2, we argued that if an automorphism stabilizes $\mathbf{d}$, then it is an endomorphism on the eigenspaces of each $\mathbf{e}_{\mathcal{L}_k} := \sum_{i\in\mathcal{L}_k} \mathbf{e}_i$, where $\mathcal{L}_k = \{i : \lambda_i(\mathbf{w}) = \mu_k\}$. In particular, it is an endomorphism on the 1-eigenspace $\bigoplus_{i,j\in\mathcal{L}_k} \mathfrak{A}_{ij}$ of $\mathbf{e}_{\mathcal{L}_k}$, and on the common $\frac{1}{2}$-eigenspace $\bigoplus_{i\in\mathcal{L}_k,j\in\mathcal{L}_\ell} \mathfrak{A}_{ij}$ of $\mathbf{e}_{\mathcal{L}_k}$ and $\mathbf{e}_{\mathcal{L}_\ell}$ $(k \neq \ell)$. With the convention $\lambda_{r+1} \equiv 0$, we can write

$$\begin{aligned}\lambda_r(\mathbf{w})\,\|\mathbf{x}\|_{\mathbf{w}}^2 &= \sum_{k,\ell=1}^{p} \sum_{i\in\mathcal{L}_k,j\in\mathcal{L}_\ell} \lambda_{i\vee j}(\mathbf{w})^{-1}\,\|(\boldsymbol{\mathcal{A}}\mathbf{x})_{ij}\|^2 \\ &= \sum_{k,\ell=1}^{p} \sum_{i\in\mathcal{L}_k,j\in\mathcal{L}_\ell} \mu_{k\vee\ell}^{-1}\,\|(\boldsymbol{\mathcal{A}}\mathbf{x})_{ij}\|^2 = \sum_{k,\ell=1}^{p} \mu_{k\vee\ell}^{-1}\,\|(\boldsymbol{\mathcal{A}}\mathbf{x})_{\mathcal{L}_k\mathcal{L}_\ell}\|^2 .\end{aligned}$$

Since

$$\|(\boldsymbol{\mathcal{A}}_1\mathbf{x})_{\mathcal{L}_k\mathcal{L}_\ell}\| = \left\|(\boldsymbol{\mathcal{A}}_1\boldsymbol{\mathcal{A}}_2^{-1}\boldsymbol{\mathcal{A}}_2\mathbf{x})_{\mathcal{L}_k\mathcal{L}_\ell}\right\| = \left\|\boldsymbol{\mathcal{A}}_1\boldsymbol{\mathcal{A}}_2^{-1}(\boldsymbol{\mathcal{A}}_2\mathbf{x})_{\mathcal{L}_k\mathcal{L}_\ell}\right\| = \|(\boldsymbol{\mathcal{A}}_2\mathbf{x})_{\mathcal{L}_k\mathcal{L}_\ell}\| ,$$

the local norm is well-defined.

We propose the following target-following framework.

Algorithm 1. (Target-following framework for symmetric cone programming) *Given* $(\mathbf{x}_{\mathrm{in}}, \mathbf{s}_{\mathrm{in}}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$ *and a target* $\mathbf{w}_{\mathrm{out}} \in \mathcal{K}$.

(1) *Pick some* $\delta \in (0,1)$ *and a sequence of targets* $\{\mathbf{w}_k\}_{k=0}^N \subset \mathcal{K}$ *such that* $\mathbf{w}_0 = \boldsymbol{\mathcal{T}}(\mathbf{x}_{\mathrm{in}}, \mathbf{s}_{\mathrm{in}})$, $\mathbf{w}_N = \mathbf{w}_{\mathrm{out}}$, *and* $\|\mathbf{w}_{k+1} - \mathbf{w}_k\|_{\mathbf{w}_k} \leq \delta$ *for* $k = 1, \ldots, N$.
(2) *Set* $(\mathbf{x}_+, \mathbf{s}_+) = (\mathbf{x}_{\mathrm{in}}, \mathbf{s}_{\mathrm{in}})$.
(3) *For* $k = 1, \ldots, N$,
  (a) *Solve* (3.2) *with any* $\boldsymbol{\mathcal{A}} \in \mathrm{Aut}(\mathfrak{J})$ *satisfying* $\boldsymbol{\mathcal{A}}\mathbf{w}_k \in \mathfrak{D}_{\downarrow,++}$, *and* $\mathbf{d} = \boldsymbol{\mathcal{A}}\mathbf{w}_k$, *and set* $(\mathbf{x}_{++}, \mathbf{s}_{++}) = (\mathbf{x}_+ + \boldsymbol{\Delta}_{\mathbf{x}}, \mathbf{s}_+ + \boldsymbol{\Delta}_{\mathbf{s}})$.
  (b) *Update* $(\mathbf{x}_+, \mathbf{s}_+) \leftarrow (\mathbf{x}_{++}, \mathbf{s}_{++})$.
(4) *Output* $(\mathbf{x}_{\mathrm{out}}, \mathbf{s}_{\mathrm{out}}) = (\mathbf{x}_+, \mathbf{s}_+)$.

Consider the following scenarios in which the target-following framework can be applied.

(1) If the objective is to approximate the solutions to the symmetric cone programming problems (3.1), the target $\mathbf{w}_{\mathrm{out}}$ can be chosen to be a very small positive multiple of the unit. Of course, if the sequence of targets are multiples of the unit, then Algorithm 1 reduces to a path-following algorithm.
(2) If the objective is to approximate the analytic center of a compact set described by linear matrix inequalities and convex quadratic inequalities, then we can describe this set with the dual problem (3.1b) with null objective, and choose the target $\mathbf{w}_{\mathrm{out}} = \mathbf{e}$.

In these scenarios, we can always choose the targets $\mathbf{w}_k$ to share the same Jordan frame in their spectral decompositions. The following lemma justifies this choice as it shows that $\|\mathbf{w}_{k+1} - \mathbf{w}_k\|_{\mathbf{w}_k}$ is minimized over all $\mathbf{w}_{k+1}$ with prescribed eigenvalues when $\mathbf{w}_{k+1}$ shares the same Jordan frame with $\mathbf{w}_k$.

Lemma 6 (cf. Lemma 3 of [6]). *If $u_1 \geq \cdots \geq u_n > 0$, then it holds, for every $\mathbf{x} \in \mathfrak{J}$,*

$$\sum_{i,j=1}^{r} u_{i\vee j} \left\| u_{i\vee j}^{-1}\mathbf{x}_{ij} - \mathbf{e}_{ij} \right\|^2 \geq \sum_{i=1}^{r} u_i^{-1} \left\| \lambda_i(\mathbf{x}) - u_i \right\|^2 .$$

Proof. By expanding both sides of the desired inequality, it is clear that it suffices to bound the sum $\sum_{i,j=1}^{r} u_{i\vee j}^{-1} \|\mathbf{x}_{ij}\|^2$ from below by $\sum_{i=1}^{r} u_i^{-1}\lambda_i^2$. Let $\mathbf{x} = \sum \lambda_i \mathbf{c}_i$ be a spectral decomposition. Then $\mathbf{x}^2 = \sum \lambda_i^2 \mathbf{c}_i$, whence

$$\begin{bmatrix} \rho_1(\mathbf{x}^2) \\ \vdots \\ \rho_r(\mathbf{x}^2) \end{bmatrix} = \begin{bmatrix} \rho_1(\mathbf{c}_1) & \cdots & \rho_1(\mathbf{c}_r) \\ \vdots & \ddots & \vdots \\ \rho_r(\mathbf{c}_1) & \cdots & \rho_r(\mathbf{c}_r) \end{bmatrix} \begin{bmatrix} \lambda_1^2 \\ \vdots \\ \lambda_n^2 \end{bmatrix},$$

where the matrix on the right side of the equation is doubly-stochastic. By the Hardy-Littlewood-Pólya Theorem [11], we have $\sum_{i=1}^{k} \rho_i(\mathbf{x}^2) \leq \sum_{i=1}^{k} \lambda_i^2$ for all $k \in \{1,\ldots,r\}$. Consequently

$$\begin{aligned} \sum_{i,j=1}^{n} u_{i\vee j}^{-1} \|\mathbf{x}_{ij}\|^2 &= u_n^{-1} \sum_{i,j=1}^{r} \|\mathbf{x}_{ij}\|^2 - \sum_{k=1}^{r-1} (u_{k+1}^{-1} - u_k^{-1}) \sum_{i,j=1}^{k} \|\mathbf{x}_{ij}\|^2 \\ &\geq u_n^{-1} \sum_{i=1}^{r} \rho_i(\mathbf{x}^2) - \sum_{k=1}^{r-1} (u_{k+1}^{-1} - u_k^{-1}) \sum_{i=1}^{k} \rho_i(\mathbf{x}^2) \\ &\geq u_n^{-1} \sum_{i=1}^{r} \lambda_i^2 - \sum_{k=1}^{r-1} (u_{k+1}^{-1} - u_k^{-1}) \sum_{i=1}^{k} \lambda_i^2 = \sum_{i=1}^{r} u_i^{-1} \lambda_i^2 \end{aligned}$$

proves the theorem. □

By applying the transformation $(\mathbf{x},\mathbf{s}) \mapsto (\boldsymbol{\mathcal{A}}\mathbf{x}, \boldsymbol{\mathcal{A}}\mathbf{s})$ to (3.1) for some suitable $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$, we may assume, without any loss of generality, that $\boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\mathbf{s}}^*}\mathbf{x} \in \mathfrak{D}_{\downarrow,++}$, or equivalently, $\boldsymbol{\mathcal{T}}(\mathbf{x},\mathbf{s}) \in \mathfrak{D}_{\downarrow,++}$. Henceforth, we assume that $\boldsymbol{\mathcal{T}}(\mathbf{x},\mathbf{s}) \in \mathfrak{D}_{\downarrow,++}$ and the sequence of targets $\{\mathbf{w}_k\}_{k=0}^{N} \subset \mathfrak{D}_{\downarrow,++}$. This has the advantage that the proximity measure used in approximating $\mathbf{w}_k$ using (3.2) can be simplified to

$$d(\mathbf{x},\mathbf{s}) = \left(\rho_r(\mathbf{d})^{-1} \sum_{i,j=1}^{r} \rho_{i\vee j}(\mathbf{d})^{-1} \left\| (\boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\mathbf{s}}^*}\mathbf{x})_{ij} - \mathbf{d}_{ij} \right\|^2 \right)^{\frac{1}{2}} = \left\| \boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\mathbf{s}}^*}\mathbf{x} - \mathbf{d} \right\|_{\mathbf{d}} .$$

Moreover, this specializes the framework to the following algorithm.

Algorithm 2. (Target-following algorithm for symmetric cone programming)
Given $(\mathbf{x}_{\text{in}}, \mathbf{s}_{\text{in}}) \in \mathcal{F}_p^\circ \times \mathcal{F}_d^\circ$ with $\boldsymbol{\mathcal{Q}}_{\mathbf{l}_{\mathbf{s}_{\text{in}}}^*}\mathbf{x}_{\text{in}} \in \mathfrak{D}_{\downarrow,++}$ and a target $\mathbf{w}_{\text{out}} \in \mathfrak{D}_{\downarrow,++}$.

(1) *Pick some $\delta \in (0,1)$ and a sequence of targets $\{\mathbf{w}_k\}_{k=0}^{N} \subset \mathfrak{D}_{\downarrow,++}$ such that $\mathbf{w}_0 = \boldsymbol{\mathcal{T}}(\mathbf{x}_{\text{in}}, \mathbf{s}_{\text{in}})$, $\mathbf{w}_N = \mathbf{w}_{\text{out}}$, and $\|\mathbf{w}_{k+1} - \mathbf{w}_k\|_{\mathbf{w}_k} \leq \delta$ for $k = 1,\ldots,N$.*
(2) *Set $(\mathbf{x}_+, \mathbf{s}_+) = (\mathbf{x}_{\text{in}}, \mathbf{s}_{\text{in}})$.*
(3) *For $k = 1,\ldots,N$,*
   (a) *Solve (3.2) with $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{I}}$, $\mathbf{d} = \mathbf{w}_k$, and set $(\mathbf{x}_{++}, \mathbf{s}_{++}) = (\mathbf{x}_+ + \boldsymbol{\Delta}_{\mathbf{x}}, \mathbf{s}_+ + \boldsymbol{\Delta}_{\mathbf{s}})$.*
   (b) *Update $(\mathbf{x}_+, \mathbf{s}_+) \leftarrow (\mathbf{x}_{++}, \mathbf{s}_{++})$.*
(4) *Output $(\mathbf{x}_{\text{out}}, \mathbf{s}_{\text{out}}) = (\mathbf{x}_+, \mathbf{s}_+)$.*

4.2. **Analysis of algorithm.** Consider one iteration of Algorithm 2. As before, let π_ℓ denote the difference $\rho_\ell(\mathbf{d}) - \rho_{\ell+1}(\mathbf{d})$, with the convention $\rho_{r+1} \equiv 0$, and let $\mathcal{L}$ denote the set $\{\ell : \pi_\ell > 0\}$.

Recall from Corollary 2 and the paragraphs following it that (3.2) has a unique pair of solutions if $d(\mathbf{x}_+, \mathbf{s}_+) \le \gamma$ for some $\gamma \in (0, 1/\sqrt{2})$. This can be enforced via the following lemma.

Lemma 7. *If* $\left\|\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\mathbf{s}}}\mathbf{x} - \mathbf{w}_{k-1}\right\|_{\mathbf{w}_{k-1}} \le \beta$ *and* $\|\mathbf{d} - \mathbf{w}_{k-1}\|_{\mathbf{w}_{k-1}} \le \delta$ *for some* $\beta, \delta \in (0,1)$, *then* $d(\mathbf{x}_+, \mathbf{s}_+) \le \frac{\beta+\delta}{1-\delta}$.

Proof. We have

$$\begin{aligned} d(\mathbf{x}_+, \mathbf{s}_+) \le &\left(\rho_r(\mathbf{d})^{-1} \sum_{i,j=1}^r \rho_{i\vee j}(\mathbf{d})^{-1} \left\|(\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\mathbf{s}_+}}\mathbf{x}_+)_{ij} - (\mathbf{w}_{k-1})_{ij}\right\|^2\right)^{\frac{1}{2}} \\ &+ \left(\rho_r(\mathbf{d})^{-1} \sum_{i,j=1}^r \rho_{i\vee j}(\mathbf{d})^{-1} \|(\mathbf{w}_{k-1} - \mathbf{d})_{ij}\|^2\right)^{\frac{1}{2}} \\ \le &\ d(\mathbf{x}_+, \mathbf{s}_+; \mathbf{w}_{k-1}) \max_i \frac{\rho_i(\mathbf{w}_{k-1})}{\rho_i(\mathbf{d})} + \|\mathbf{d} - \mathbf{w}_{k-1}\|_{\mathbf{w}_{k-1}} \max_i \frac{\rho_i(\mathbf{w}_{k-1})}{\rho_i(\mathbf{d})}. \end{aligned}$$

If $\|\mathbf{d} - \mathbf{w}_{k-1}\|_{\mathbf{w}_{k-1}} \le \delta$, then

$$\delta^2 \ge \rho_r(\mathbf{w}_{k-1})^{-1} \sum_{i=1}^r \rho_i(\mathbf{w}_{k-1})^{-1}(\rho_i(\mathbf{d}) - \rho_i(\mathbf{w}_{k-1}))^2 \ge \sum_{i=1}^r \left(\frac{\rho_i(\mathbf{d})}{\rho_i(\mathbf{w}_{k-1})} - 1\right)^2,$$

whence $\rho_i(\mathbf{d})/\rho_i(\mathbf{w}_{k-1}) \ge 1 - \delta$. □

We now give the main theorem of this section.

Theorem 7. *In Algorithm 2, if* $\delta \in (0,1)$ *is such that there exists some* $\beta \in (0,1)$ *satisfying*

$$\frac{\gamma^2(4 + 2\sqrt{2} + 4\sqrt{2}\gamma)}{(1-\sqrt{2}\gamma)^2} + \frac{2\sqrt{2}\gamma^3}{(1-\sqrt{2}\gamma)^3} \le \beta, \tag{4.2}$$

where $\gamma = (\beta+\delta)/(1-\delta) < 1/\sqrt{2}$, *then* $(\mathbf{x}_{++}, \mathbf{s}_{++})$ *is well-defined and strictly feasible in each iteration, and the algorithm terminates with* $\left\|\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\mathbf{s}_{\text{out}}}}\mathbf{x}_{\text{out}} - \mathbf{w}_{\text{out}}\right\|_{\mathbf{w}_{\text{out}}} \le \beta$.

Proof. Let d_k denote the proximity measure defined in (3.9) for ($CP_{\boldsymbol{\mathcal{A}},\mathbf{d}}$) with $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{I}}$ and $\mathbf{d} = \mathbf{w}_k$; i.e., $d_k(\mathbf{x}, \mathbf{s}) = \left\|\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\mathbf{s}}}\mathbf{x} - \mathbf{w}_k\right\|_{\mathbf{w}_k}$. We shall prove the theorem by induction that the iterates $(\mathbf{x}_+, \mathbf{s}_+)$ are strictly feasible and $d_{k-1}(\mathbf{x}_+, \mathbf{s}_+) \le \beta$ at the beginning of each iteration. This is certainly true for the first iteration. By Lemma 7, we have $d_k(\mathbf{x}_+, \mathbf{s}_+) \le \gamma$. If (4.2) holds, then we may apply Theorem 5 to deduce that the iterates $(\mathbf{x}_{++}, \mathbf{s}_{++})$ are strictly feasible with $d_k(\mathbf{x}_{++}, \mathbf{s}_{++}) \le \beta$. This completes the induction. □

5. Finding Analytic Centers

In this section, we consider an algorithm that finds the analytic center $\boldsymbol{\mathcal{T}}^{-1}(\widehat{\mu}\mathbf{e})$ for any given $\widehat{\mu} > 0$. This algorithm can be used to find analytic centers of compact sets described by linear matrix inequalities and convex quadratic constraints; see Section 4.1.

It can also be combined with a path-following algorithm to solve the symmetric cone program (3.1).

Given a pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ with $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\mathbf{s}}}}\widehat{\mathbf{x}} \in \mathfrak{D}_{\downarrow,++}$, we shall construct a finite sequence of targets $\{\mathbf{w}_k\}_{k=0}^N \in \mathfrak{D}_{\downarrow,++}$ such that $\mathbf{w}_0 = \boldsymbol{\mathcal{T}}(\widehat{\mathbf{x}}, \widehat{\mathbf{s}}) = \boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\mathbf{s}}}}\widehat{\mathbf{x}}$,

$$\|\mathbf{w}_k - \mathbf{w}_{k-1}\|_{\mathbf{w}_{k-1}} \le \delta \quad \text{for } 1 \le k \le N, \tag{5.1}$$

and $\mathbf{w}_N = \widehat{\mu}\mathbf{e}$, with β and δ satisfying the hypothesis of Theorem 7, thus allowing us to apply Algorithm 2 to approximate $\boldsymbol{\mathcal{T}}^{-1}(\widehat{\mu}\mathbf{e})$.

Of course, if $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\mathbf{s}}}}\widehat{\mathbf{x}} \in \mathfrak{D}_{\downarrow,++}$ is a positive multiple of $\mathbf{e}$, then we need simply to follow the central path to approximate $\boldsymbol{\mathcal{T}}^{-1}(\widehat{\mu}\mathbf{e})$; i.e., pick the targets to be positive multiples of $\mathbf{e}$. Henceforth, we assume that $\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\widehat{\mathbf{s}}}}\widehat{\mathbf{x}} \in \mathfrak{D}_{\downarrow,++}$ is not a positive multiple of $\mathbf{e}$.

Since the targets are diagonal matrices $\mathbf{w}_k \in \mathfrak{D}_{\downarrow,++}$, we may restrict our attention to the vector of diagonal entries $\mathbf{x}^k = (\rho_1(\mathbf{w}_k), \ldots, \rho_r(\mathbf{w}_k))$ and work in $\mathbb{R}^r_{\downarrow,++}$ instead. Under this restriction, the condition (5.1) becomes

$$\sqrt{\frac{1}{\mathbf{x}_n^{k-1}} \sum_{i=1}^n \frac{\left(\mathbf{x}_i^k - \mathbf{x}_i^{k-1}\right)^2}{\mathbf{x}_i^{k-1}}} \le \delta \quad (1 \le k \le N).$$

Such sequence $\{\mathbf{x}^k\}_{k=0}^N$ is called a *δ-sequence*, and N is called its *length*; see [16]. In [6], the author gave an upper bound on the length of a shortest δ-sequence from any $\mathbf{x}^0 \in \mathbb{R}^n_{\downarrow,++}$ to the ray $\{(\mu, \ldots, \mu) \in \mathbb{R}^n : \mu > 0\}$. This is summarized in the following theorem.

Theorem 8. *For every $\mathbf{x}^0 \in \mathbb{R}^r_{\downarrow,++}$ and every $\delta \in (0,1)$, there exists a δ-sequence $\{\mathbf{x}^k\}_{k=0}^N$ with $\mathbf{x}^N = (\mu, \ldots, \mu)$, where $\mu = \sum_{i=1}^r \mathbf{x}_i^0/r$, and length*

$$N \le \left\lceil \frac{\sqrt{r}}{\delta - \frac{1}{2}\delta^2} \log\left(\frac{4\mu}{\mathbf{x}_r^0}\right) \right\rceil.$$

Proof. Follows from Lemmas 13, 14 and 15 of [6]. □

Combining Theorem 8 with a δ-sequence on the central path, we have the following theorem.

Theorem 9. *Suppose $\beta \in (0,1)$ is fixed. Given any pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ for the primal-dual symmetric cone programming problems (3.1), and any positive real number $\widehat{\mu}$, there is a sequence of at most*

$$O\left(\sqrt{r}\left(\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\lambda_r(\widehat{\mathbf{x}} \circ \widehat{\mathbf{s}})} + \left|\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\widehat{\mu}}\right|\right)\right)$$

targets such that Algorithm 2 finds a pair of primal-dual feasible solutions $(\mathbf{x}, \mathbf{s})$ satisfying $\left\|\boldsymbol{\mathcal{Q}}_{\mathbf{l}^*_{\mathbf{s}}}\mathbf{x} - \widehat{\mu}\mathbf{e}\right\| \le \beta\widehat{\mu}$.

As an immediate corollary, we have the following worst-case iteration bound on solving symmetric cone problems using Algorithm 2.

Corollary 3. *Given any pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ and any $\varepsilon > 0$, there is a sequence of at most*

$$O\left(\sqrt{r}\left(\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\lambda_r(\widehat{\mathbf{x}} \circ \widehat{\mathbf{s}})} + \left|\log \varepsilon^{-1}\right|\right)\right)$$

targets such that Algorithm 2 find a pair of primal-dual feasible solutions $(\mathbf{x}, \mathbf{s})$ *satisfying* $\langle \mathbf{x}, \mathbf{s} \rangle \leq \varepsilon \langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle$.

Proof. If $(\mathbf{x}, \mathbf{s}) \in \mathcal{K} \times \mathcal{K}^\sharp$ satisfies $\left\| \boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*} \mathbf{x} - \mu \mathbf{e} \right\| \leq \beta\mu$ for some $\beta \in (0, 1)$ and some $\mu > 0$, then $\langle \mathbf{x}, \mathbf{s} \rangle - r\mu = \left\langle \mathbf{e}, \boldsymbol{\mathcal{Q}}_{\mathbf{l}_\mathbf{s}^*} \mathbf{x} - \mu \mathbf{e} \right\rangle \leq \sqrt{r}\beta\mu$. Apply the preceding theorem with $\widehat{\mu} = \varepsilon \langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle / (\beta\sqrt{r} + r)$. □

6. Conclusion

In this paper, we consider the T-algebraic characterization of symmetric cones by viewing them as homogeneous cones, and relate it to the Jordan-algebraic characterization. The two different but related characterizations are used to define primal-dual weighted analytic centers and a target map for linear optimization problems over symmetric cones. This opens the door to target-following algorithms for symmetric cone programming. An application of target-following algorithm is approximating the analytic centers of sets described by linear matrix and convex quadratic constraints. Such sets can be modelled as the feasible region of a symmetric cone program by introducing the zero objective function. The analytic center of the set is then given by the weighted analytic center corresponding to unit weights. Thus the analytic center can be efficiently estimated by following a sequence of targets ending at said weighted analytic center.

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Appendix A. Automorphisms of Euclidean Jordan algebras

In Section II.1 of [25], it was stated without proof that if $(\mathfrak{J}, \cdot)$ is a Euclidean Jordan algebra and $\mathcal{K}$ is its associated symmetric cone, then the stabilizer subgroup $\mathrm{Aut}(\mathcal{K})_{\mathbf{e}}$ of the unit $\mathbf{e}$ in $\mathrm{Aut}(\mathcal{K})$ coincide with the group of automorphisms $\mathrm{Aut}(\mathfrak{J})$ of $\mathfrak{J}$. Here we give a proof of this fact.

Theorem A.1. *Given a Euclidean Jordan algebra $(\mathfrak{J}, \cdot)$ with unit $\mathbf{e}$ and associated symmetric cone $\mathcal{K}$, the stabilizer subgroup $\mathrm{Aut}(\mathcal{K})_{\mathbf{e}}$ of the unit $\mathbf{e}$ in $\mathrm{Aut}(\mathcal{K})$ coincide with the group of automorphisms $\mathrm{Aut}(\mathfrak{J})$ of $\mathfrak{J}$.*

Proof. Consider the inner product $\langle \cdot, \cdot \rangle : (\mathbf{x}, \mathbf{y}) \mapsto \operatorname{trace} \boldsymbol{\mathcal{L}}_{\mathbf{x}\cdot\mathbf{y}}$, where $\boldsymbol{\mathcal{L}}_{\mathbf{x}}$ denotes the linear map $\mathbf{y} \mapsto \mathbf{x} \cdot \mathbf{y}$. Let $\mathrm{O}(\mathfrak{J})$ denote the orthogonal group of the Euclidean space $(\mathfrak{J}, \langle \cdot, \cdot \rangle)$; i.e., $\mathrm{O}(\mathfrak{J}) = \{\boldsymbol{\mathcal{A}} \in \mathrm{GL}(\mathfrak{J}) : \langle \boldsymbol{\mathcal{A}}\mathbf{x}, \boldsymbol{\mathcal{A}}\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \, \forall \mathbf{x}, \mathbf{y} \in \mathfrak{J}\}$.

Let φ be the characteristic function of $\mathcal{K}$; i.e.,

$$\varphi : \mathbf{x} \in \mathcal{K} \mapsto \int_{\mathcal{K}^\sharp} e^{-\langle \mathbf{x}, \mathbf{y} \rangle} d\mathbf{y},$$

where $d\mathbf{y}$ denotes the Euclidean measure on $(\mathfrak{J}, \langle \cdot, \cdot \rangle)$. Let $\mathbf{x}^\sharp$ denote the negative gradient of the logarithmic derivative of φ at $\mathbf{x}$. We deduce from Propositions II.3.4 and III.2.2 of [8] that $\exp \boldsymbol{\mathcal{L}}_{\mathbf{x}} \in \mathrm{Aut}(\mathcal{K})$. Thus by Proposition I.3.1 of [8], we have $\log \varphi(\exp \boldsymbol{\mathcal{L}}_{\mathbf{x}} \cdot \mathbf{e}) = \log \varphi(\mathbf{e}) - \log \det \exp \boldsymbol{\mathcal{L}}_{\mathbf{x}} = \log \varphi(\mathbf{e}) - \operatorname{trace} \boldsymbol{\mathcal{L}}_{\mathbf{x}}$. Differentiating this at $\mathbf{0}$ gives $\operatorname{trace} \boldsymbol{\mathcal{L}}_{\mathbf{h}} = -D \log \varphi(\mathbf{e})[\mathbf{h}]$. Since $\operatorname{trace} \boldsymbol{\mathcal{L}}_{\mathbf{h}} = \langle \mathbf{h}, \mathbf{e} \rangle$ and $-D \log \varphi(\mathbf{e})[\mathbf{h}] = \langle \mathbf{e}^\sharp, \mathbf{h} \rangle$, it follows that $\mathbf{e}$ is a fixed point of the map $\mathbf{x} \in \mathcal{K} \mapsto \mathbf{x}^\sharp$. Proposition I.4.3 of [8] then states that $\operatorname{Aut}(\mathcal{K}) \cap \mathrm{O}(\mathfrak{J}) = \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$.

We now show that $\operatorname{Aut}(\mathfrak{J})$ coincides with $\operatorname{Aut}(\mathcal{K}) \cap \mathrm{O}(\mathfrak{J}) = \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$. It is straightforward to check that every automorphism of $\mathfrak{J}$ is an automorphism of $\mathcal{K}$ (which is the interior of the cone of squares) that stabilizes the unit $\mathbf{e}$. For the other direction, it suffices to show that every linear map $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathcal{K}) \cap \mathrm{O}(\mathfrak{J}) = \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$ preserves orthogonality of idempotents and maps every primitive idempotent to some primitive idempotent, for if $\mathbf{x} = \sum \lambda_i \mathbf{c}_i$ is the spectral decomposition, then we have

$$\boldsymbol{\mathcal{A}}(\mathbf{x}^2) = \boldsymbol{\mathcal{A}}\left(\sum \lambda_i^2 \mathbf{c}_i\right) = \sum \lambda_i^2 \boldsymbol{\mathcal{A}}(\mathbf{c}_i) = \left(\sum \lambda_i \boldsymbol{\mathcal{A}}(\mathbf{c}_i)\right)^2 = (\boldsymbol{\mathcal{A}}\mathbf{x})^2,$$

whence $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$ by polarization. Suppose $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathcal{K}) \cap \mathrm{O}(\mathfrak{J}) = \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$. Two idempotents are $\mathbf{c}$ and $\mathbf{d}$ are orthogonal if and only if $\langle \mathbf{c}, \mathbf{d} \rangle = 0$. One direction of this statement follows from the definition of $\langle \cdot, \cdot \rangle$. For the other direction, suppose that $\mathbf{c}$ and $\mathbf{d}$ are two idempotents satisfying $\langle \mathbf{c}, \mathbf{d} \rangle = 0$. Since the inner product $\langle \cdot, \cdot \rangle$ is associative (see Proposition II.4.3 of [8]), $\boldsymbol{\mathcal{L}}_{\mathbf{c}}$ is self-adjoint. Proposition III.1.3 of [8] then implies that $\boldsymbol{\mathcal{L}}_{\mathbf{c}}$ is positive semidefinite. Thus it has a self-adjoint, positive semidefinite square root $\boldsymbol{\mathcal{L}}_{\mathbf{c}}^{\frac{1}{2}}$. Hence

$$0 = \langle \mathbf{c}, \mathbf{d} \rangle = \left\langle \mathbf{c}, \mathbf{d}^2 \right\rangle = \langle \mathbf{c} \cdot \mathbf{d}, \mathbf{d} \rangle = \langle \boldsymbol{\mathcal{L}}_{\mathbf{c}} \mathbf{d}, \mathbf{d} \rangle = \left\langle \boldsymbol{\mathcal{L}}_{\mathbf{c}}^{\frac{1}{2}} \mathbf{d}, \boldsymbol{\mathcal{L}}_{\mathbf{c}}^{\frac{1}{2}} \mathbf{d} \right\rangle$$

shows that $\boldsymbol{\mathcal{L}}_{\mathbf{c}}^{\frac{1}{2}} \mathbf{d} = \mathbf{0}$, whence $\mathbf{c} \cdot \mathbf{d} = \boldsymbol{\mathcal{L}}_{\mathbf{c}}^{\frac{1}{2}} \boldsymbol{\mathcal{L}}_{\mathbf{c}}^{\frac{1}{2}} \mathbf{d} = \mathbf{0}$; i.e., $\mathbf{c}$ and $\mathbf{d}$ are orthogonal. Since $\boldsymbol{\mathcal{A}}$ is orthogonal, it follows that orthogonal idempotents remain orthogonal under $\boldsymbol{\mathcal{A}}$. Proposition IV.3.2 of [8] states $\mathbf{c}$ is a primitive idempotent if and only if $\{\lambda \mathbf{c} : \lambda \geq 0\}$ is an extreme ray of $\mathcal{K}$. Since $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathcal{K})$, it maps each extreme ray to some extreme ray of $\mathcal{K}$. Thus it maps each primitive idempotent $\mathbf{c}$ to a positive multiple $\lambda \mathbf{d}$ of some primitive idempotent $\mathbf{d}$. In fact, λ must be unit since

$$\begin{aligned} 0 < \langle \mathbf{d}, \mathbf{d} \rangle &= \left\langle \mathbf{d}^2, \mathbf{e} \right\rangle = \langle \mathbf{d}, \mathbf{e} \rangle = \langle \mathbf{d}, \boldsymbol{\mathcal{A}}\mathbf{e} \rangle = \lambda^{-1} \langle \boldsymbol{\mathcal{A}}\mathbf{c}, \boldsymbol{\mathcal{A}}\mathbf{e} \rangle \\ &= \lambda^{-1} \langle \mathbf{c}, \mathbf{e} \rangle = \lambda^{-1} \left\langle \mathbf{c}^2, \mathbf{e} \right\rangle = \lambda^{-1} \langle \mathbf{c}, \mathbf{c} \rangle = \lambda^{-1} \langle \boldsymbol{\mathcal{A}}\mathbf{c}, \boldsymbol{\mathcal{A}}\mathbf{c} \rangle = \lambda \langle \mathbf{d}, \mathbf{d} \rangle \end{aligned}$$

Hence $\boldsymbol{\mathcal{A}}$ maps each primitive idempotent to some primitive idempotent. □

The proof of the theorem shows that both $\operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$ and $\operatorname{Aut}(\mathfrak{J})$ coincide with certain orthogonal subgroup of $\operatorname{Aut}(\mathcal{K})$. The next theorem gives a similar result.

Theorem A.2. *Given a Euclidean Jordan algebra* $(\mathfrak{J}, \cdot)$ *with unit* $\mathbf{e}$ *and associated symmetric cone* $\mathcal{K}$*, the groups* $\operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$ *and* $\operatorname{Aut}(\mathfrak{J})$ *are both equivalent to the orthogonal subgroup of* $\operatorname{Aut}(\mathcal{K})$ *under the inner product* $\langle \cdot, \cdot \rangle : (\mathbf{x}, \mathbf{y}) \mapsto \operatorname{tr}(\mathbf{x} \cdot \mathbf{y})$.

Proof. Let $\mathrm{O}(\mathcal{K})$ denote the orthogonal subgroup of $\operatorname{Aut}(\mathcal{K})$ under $\langle \cdot, \cdot \rangle$. By Proposition II.4.2 of [8], if $\boldsymbol{\mathcal{A}} \in \operatorname{Aut}(\mathfrak{J})$, then $\operatorname{tr}(\boldsymbol{\mathcal{A}}\mathbf{x} \cdot \boldsymbol{\mathcal{A}}\mathbf{y}) = \operatorname{tr} \boldsymbol{\mathcal{A}}(\mathbf{x} \cdot \mathbf{y}) = \operatorname{tr}(\mathbf{x} \cdot \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{J}$, whence $\boldsymbol{\mathcal{A}}$ is orthogonal. Therefore $\operatorname{Aut}(\mathfrak{J}) = \operatorname{Aut}(\mathcal{K})_{\mathbf{e}} \subseteq \mathrm{O}(\mathcal{K})$. According to Proposition I.1.8 of [8] and the paragraph following it, $\operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$ is a maximal compact subgroup of $\operatorname{Aut}(\mathcal{K})$. Hence $\mathrm{O}(\mathcal{K}) \subseteq \operatorname{Aut}(\mathcal{K})_{\mathbf{e}}$. □

Division of Mathematical Sciences, School of Physical & Mathematical Sciences, Nanyang Technological University, Singapore 637371, Singapore

E-mail address: cbchua@ntu.edu.sg