Target-following framework for symmetric cone programming

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ABSTRACT. We extend the target map, together with the weighted barriers and the notions of weighted analytic centers, from linear programming to general convex conic programming. This extension is obtained from a novel geometrical perspective of the weighted barriers, that views a weighted barrier as a weighted sum of barriers for a strictly decreasing sequence of faces. Using the Euclidean Jordan-algebraic structure of symmetric cones, we give an algebraic characterization of a strictly decreasing sequence of its faces, and specialize this target map to produce a computationally-tractable target-following algorithm for symmetric cone programming. The analysis is made possible with the use of *triangular* automorphisms of the cone, a new tool in the study of symmetric cone programming. As an application of this algorithm, we demonstrate that starting from any given any pair of primal-dual strictly feasible solutions, the primal-dual central path of a symmetric cone program can be efficiently approximated.

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1. INTRODUCTION

In this paper, we consider primal-dual interior point algorithms for linear optimization problems over symmetric cones (a.k.a. symmetric cone programming):

$$\sup\left\{\sum_{j=1}^{m} b_j y_j : \sum_{j=1}^{m} y_j \mathbf{a}_j + \mathbf{x} = \mathbf{c},\ \mathbf{x} \in cl(\Omega)\right\}, \tag{1.1a}$$

where Ω is a given (open) symmetric cone in a finite-dimensional real vector space with inner product $\langle\cdot,\cdot\rangle$, $\mathbf{a}_1,\ldots,\mathbf{a}_m,\mathbf{c}$ are given vectors, and $b_1,\ldots,b_m$ are given real numbers. Its dual problem is the symmetric cone program

$$\inf\{\langle\mathbf{c},\mathbf{x}\rangle : \langle\mathbf{a}_j,\mathbf{s}\rangle = b_j,\ 1 \le j \le m,\ \mathbf{s} \in cl(\Omega)\}. \tag{1.1b}$$

Without any loss of generality, we assume that the vectors $\mathbf{a}_1,\ldots,\mathbf{a}_m$ are linearly independent. With this assumption, $(y_1,\ldots,y_m)$ is uniquely determined by each $\mathbf{x}$ satisfying the equality constraints, and we thus view $(y_1,\ldots,y_m)$ as a function of $\mathbf{x}$. Henceforth, we shall use only the $\mathbf{x}$-component when referring to a feasible solution. For the purpose of studying interior point algorithms, we also assume that the primal-dual symmetric cone programs have *strictly feasible solutions*; i.e., there exists $\mathbf{x},\mathbf{s} \in \Omega$ satisfying the equality constraints in their respective problems.

Primal-dual interior-point algorithms—first designed for linear programming (see, e.g., [27]), and subsequently extended to semidefinite programming (see, e.g., [26, Part II]), symmetric cone programming (see, e.g., [19]) and, recently, homogeneous cone programming [6]—are the most widely used interior-point algorithms in practice. At the same time, they are able to achieve the best iteration complexity bound known to date.

The development of primal-dual algorithms for symmetric cone programming began from two very different perspectives. Yu. Nesterov and M. Todd [19] described their algorithm in the context of *self-concordant barriers* (see the seminal work of Yu. Nesterov and A. Nemirovski [18]) by specializing general logarithmically homogeneous self-concordant barriers to *self-scaled barriers*. L. Faybusovich [8], on the other hand, obtained his algorithm by extending a primal-dual algorithm for semidefinite programming via the theory of Euclidean Jordan algebras. This Jordan-algebraic approach had been so successful that it is now the most commonly used tool in designing interior-point algorithms for symmetric cone programming [1, 2, 4, 21].

In the special case of linear programming, various primal-dual path-following algorithms were simultaneously analyzed under the *target-following framework* by B. Jansen, C. Roos, T. Terlaky and J.-Ph. Vial [12]. The target-following framework was first introduced by S. Mizuno [15] for linear complementarity problems. It was subsequently used by Jansen et al. as a unifying framework for various primal-dual path-following algorithms for linear programming and algorithms that find analytic centers of polyhedral sets. The essential ingredient of this framework is the *target map* $(\mathbf{x},\mathbf{s}) \mapsto (\mathbf{x}_1\mathbf{s}_1,\ldots,\mathbf{x}_n\mathbf{s}_n)$, defined for each pair of positive n-vectors $(\mathbf{x},\mathbf{s})$. An important feature of the target map is its bijectivity between the primal-dual strictly feasible region and the cone of positive n-vectors $\mathbb{R}^n_{++}$ [12, 14], whence identifying the primal-dual strictly feasible region with the relatively simple cone $\mathbb{R}^n_{++}$ known as the *target space* (or v-space). Interior-point algorithms based on the target map are known as *target-following algorithms*, which are conceptually elegant when viewed as following a sequence of targets in the target space.

Various attempts were made to generalize the concept of target maps to semidefinite programming [16, 17, 22], symmetric cone programming [11, 23] and general convex conic programming [24]. It is noted here that these extensions of the target map do not result in target-following algorithms as they are generally not injective on the whole primal-dual strictly feasible region; see Section 1.4.

In this paper, we present an extension of the target-following framework to symmetric cone programming. This extension is obtained from a novel geometrical perspective of the weighted barriers, that views a weighted barrier as a weighted sum of barriers for a strictly decreasing sequence of faces. Using the Euclidean Jordan-algebraic structure of symmetric cones, we give an algebraic characterization of a strictly decreasing sequence of its faces, and specialize this target map to produce a computationally-tractable target-following algorithm for symmetric cone programming. The analysis is made possible with the use of *triangular* automorphisms of the cone, a new tool in the study of symmetric cone programming. As an application of this algorithm, we demonstrate that starting from any given any pair of primal-dual strictly feasible solutions, the primal-dual central path of a symmetric cone program can be efficiently approximated.

1.1. **Linear programming revisited.** Let us begin by revisiting the special case of linear programming, where $\Omega = \mathbb{R}^r_{++} = (0,\infty)^r$. In this case, the minimizer of the *weighted barrier problem*—which is the problem of minimizing the function

$$\mathbf{x} \in \Omega \mapsto -\sum_{i=1}^{r} \omega_i \log \mathbf{x}_i + \sum_{j=1}^{m} b_j y_j$$

over the primal strictly feasible region—and the unique set of Lagrange multipliers satisfying the Lagrange optimality conditions form the pair of primal-dual *weighted analytic centers* associated with the weights $\omega_1,\ldots,\omega_r$. Under the target map, a pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ is mapped to $(\omega_1, \ldots, \omega_r)$ if and only if it is the pair of primal-dual weighted analytic centers associated with the weights $\omega_1, \ldots, \omega_r$. The weighted sum of logarithmic barriers

$$\mathbf{x} \in \Omega \mapsto -\sum_{i=1}^{r} \omega_i \log \mathbf{x}_i$$

is called a *weighted logarithmic barrier for* $\mathbb{R}^r_{++}$ associated with $(\omega_1, \ldots, \omega_r)$ (or simply a *weighted barrier for* $\mathbb{R}^r_{++}$).

We now describe a generalization of the weighted barriers, and the notion of weighted analytic centers, to symmetric cone programming, and generally to convex conic programming—where Ω is an open convex cone.

First we select a sequence of faces of $\mathbb{R}^r_+$ satisfying $\mathbb{R}^r_+ = \mathsf{F}_1 \rhd \cdots \rhd \mathsf{F}_r \rhd \mathsf{F}_{r+1} = \{\mathbf{0}\}$, where $\mathsf{F} \rhd \widetilde{\mathsf{F}}$ means that $\widetilde{\mathsf{F}}$ is a proper face of F. This choice of faces determines a permutation π on the index set $\{1, \ldots, r\}$ such that $\mathsf{F}_i = \{\mathbf{x} \in \mathbb{R}^r_+ : \mathbf{x}_{\pi(1)} = \cdots = \mathbf{x}_{\pi(i-1)} = 0\}$ for $i = 2, \ldots, r, r+1$. In analogy to flag manifolds, we use the term *flag* when referring to such sequence of faces.

Definition 1 (Flag of faces)**.** A *flag of faces* (or simply *flag*) of a convex cone K is a strictly decreasing sequence of faces

$$cl(\mathsf{K}) = \mathsf{F}_1 \rhd \cdots \rhd \mathsf{F}_p \rhd \mathsf{F}_{p+1} = \{\mathbf{0}\}.$$

A flag is said to be *complete* if it is not a subsequence of another flag of the same cone.

Next, we consider the weighted sum of logarithmic barriers:

$$f_{(\omega,\mathsf{f})} : \mathbf{x} \mapsto \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) f_{\mathsf{F}_i}(\mathbf{x}), \tag{1.2}$$

where $\omega = (\omega_0, \omega_1, \ldots, \omega_r)$ is a nondecreasing sequence $0 = \omega_0 < \omega_1 \leq \cdots \leq \omega_r$ of real numbers, $\mathsf{f} = (\mathsf{F}_1, \ldots, \mathsf{F}_r, \mathsf{F}_{r+1})$ is a complete flag of $\mathbb{R}^r_+$, and f_{F_i} denotes the (modified) Legendre-Fenchel conjugate[1] of the logarithmic barrier of the face F_i; i.e., f_{F_i} is the barrier $\mathbf{x} \mapsto -\log \mathbf{x}_{\pi(i)} - \cdots - \log \mathbf{x}_{\pi(r)}$ for the open dual cone $int(\mathsf{F}_i)^\sharp = \{\mathbf{x} \in \mathbb{R}^r : \mathbf{x}_{\pi(i)}, \ldots, \mathbf{x}_{\pi(r)} > 0\}$ of the interior $int(\mathsf{F}_i)$ of the face F_i. When the complete flag f is paired with the nondecreasing sequence ω, we call this pair a *weighted complete flag*.

Definition 2 (Weighted complete flag)**.** A *weighted complete flag* of a convex cone K is a pair (ω, f), with $\omega = (\omega_0, \omega_1, \ldots, \omega_r)$ a nondecreasing sequence $0 = \omega_0 < \omega_1 \leq \cdots \leq \omega_r$ of real numbers, and $\mathsf{f} = (\mathsf{F}_1, \ldots, \mathsf{F}_r, \mathsf{F}_{r+1})$ a complete flag of K. The sequence ω is called a *weight sequence*, and the numbers $\omega_1, \ldots, \omega_r$ are called its *weights*.

Using partial summation, we can write this weighted sum logarithmic barriers as

$$f_{(\omega,\mathsf{f})}(\mathbf{x}) = \sum_{i=1}^{r} \omega_i (f_{\mathsf{F}_i}(\mathbf{x}) - f_{\mathsf{F}_{i+1}}(\mathbf{x})) = -\sum_{i=1}^{r} \omega_i \log \mathbf{x}_{\pi(i)} = -\sum_{i=1}^{r} \omega_{\pi^{-1}(i)} \log \mathbf{x}_i.$$

This is precisely a weighted logarithmic barrier for $\mathbb{R}^r_{++}$ associated with the weights $(\omega_{\pi^{-1}(1)}, \ldots, \omega_{\pi^{-1}(r)})$. Conversely, every weighted logarithmic barrier for $\mathbb{R}^r_{++}$ can

[1]The *(modified) Legendre-Fenchel conjugate* of a function $f : \mathsf{S} \to \mathbb{R}$ on a (nonempty) convex set S in a Euclidean space with inner product $\langle \cdot, \cdot \rangle$ is the function $f^\sharp : \mathbf{s} \mapsto \sup\{-\langle \mathbf{s}, \mathbf{x} \rangle - f(\mathbf{x}) : \mathbf{x} \in \mathsf{S}\}$ with domain $\{\mathbf{s} : f^\sharp(\mathbf{s}) < +\infty\}$. When f is closed (e.g., continuous f on open domain S), we have $f^{\sharp\sharp} = f$.

be written as a weighted sum of the form (1.2) once the reordering π of the indices that puts the weights in nondecreasing order is determined.

If we replace each logarithmic barrier f_{F_i} in this weighted sum by the image $\mathbf{e}_{\mathsf{F}_i}$ of the vector of all ones $\mathbf{1}$ under the *duality map* $-\nabla f_{\mathsf{F}_i}(\cdot)$, we recover the image of the primal-dual weighted analytic centers $(\mathbf{x}, \mathbf{s})$ under the target map: the image $\mathbf{e}_{\mathsf{F}_i}$ is the 0-1 vector with nonzero entries precisely at positions $\pi(i), \ldots, \pi(r)$, whence $\mathbf{e}_{\mathsf{F}_i} - \mathbf{e}_{\mathsf{F}_{i+1}}$ is the $\pi(i)$'th unit vector, and subsequently

$$(\omega_{\pi^{-1}(1)}, \ldots, \omega_{\pi^{-1}(r)}) = \sum_{i=1}^{r} \omega_i(\mathbf{e}_{\mathsf{F}_i} - \mathbf{e}_{\mathsf{F}_{i+1}}) = \sum_{i=1}^{r} (\omega_i - \omega_{i-1})\mathbf{e}_{\mathsf{F}_i}.$$

In summary, for a weighted complete flag (ω, f) of $\mathbb{R}^r_+$,

(1) the nonnegative sum of barriers (1.2) is a weighted logarithmic barrier for $\mathbb{R}^r_{++}$, which we call the *weighted barrier associated with* (ω, f);
(2) the pair of primal-dual solutions $(\mathbf{x}, \mathbf{s})$ to the weighted barrier problem determined by this weighted logarithmic barrier is a pair of weighted analytic centers, which we call the pair of *weighted centers associated with* (ω, f); and
(3) the weighted sum $-\sum_{i=1}^{r}(\omega_i - \omega_{i-1})\nabla f_{\mathsf{F}_i}(\mathbf{1})$ is the image of the weighted analytic centers $(\mathbf{x}, \mathbf{s})$ under the target map, which we call the *target vector associated with* (ω, f).

When the weights are not pair-wise distinct, a weighted barrier for $\mathbb{R}^r_{++}$ is associated with more than one weighted complete flags since the permutation π is not uniquely determined. Thus we group the weighted complete flags into equivalence classes according on the weighted barriers with which they associate.

Definition 3 (Equivalence of weighted complete flags)**.** Two weighted complete flags (ω, f) and $(\widetilde{\omega}, \widetilde{\mathsf{f}})$ of an open convex cone K are said to be *equivalent* if they have the same weights $\omega_1, \ldots, \omega_r$, and

$$\omega_i > \omega_{i-1} \implies \mathsf{F}_i = \widetilde{\mathsf{F}}_i$$

for $i = 1, \ldots, r$.

It is straightforward to verify that two weighted complete flags of $\mathbb{R}^r_+$ are equivalent if and only if they associate with the same target vector. Hence we have an alternative definition for the target map $(\mathbf{x}, \mathbf{s}) \mapsto (\mathbf{x}_1\mathbf{s}_1, \ldots, \mathbf{x}_r\mathbf{s}_r)$:

$$(\mathbf{x}, \mathbf{s}) \mapsto -\sum_{i=1}^{r}(\omega_i - \omega_{i-1})\nabla f_{\mathsf{F}_i}(\mathbf{1}),$$

where (ω, f) is a weighted complete flag of $\mathbb{R}^r_+$ such that $(\mathbf{x}, \mathbf{s})$ is the pair of weighted centers associated with (ω, f).

1.2. **Extension to convex conic programming.** We now extend this idea to linear optimization over a general closed convex cone $cl(\mathsf{K})$. In order to associate a weighted complete flag (ω, f) of the open dual cone $\mathsf{K}^\sharp$ with a weighted barrier, we would need to fix, for each and every face F of $cl(\mathsf{K}^\sharp)$, an a priori logarithmically homogeneous self-concordant barrier $f^\sharp_{\mathsf{F}}$. Since the Legendre-Fenchel conjugate $f_{cl(\mathsf{K}^\sharp)}$ is strictly convex, the weighted barrier problem it determines has a unique solution. With these barriers, we then define the weighted barrier, pair of weighted centers, and target vector associated with (ω, f), respectively, as

(1) the nonnegative sum of barriers in (1.2),
(2) the pair of primal-dual solutions $(\mathbf{x}, \mathbf{s})$ to the weighted barrier problem determined by the weighted barrier (1.2), and

(3) the weighted sum $-\sum_{i=1}^{r}(\omega_i - \omega_{i-1})\nabla f_{\mathsf{F}_i}(\mathbf{e})$, where $\mathbf{e} \in \mathsf{K}$ is the fixed point of the duality map $-\nabla f_{cl(\mathsf{K}^\sharp)}(\cdot)$.

Definition 4 (Target map)**.** The *target map* for a linear optimization problem over the closure $cl(\mathsf{K})$ of an open convex cone is the map defined over its primal-dual strictly feasible region by

$$(\mathbf{x}, \mathbf{s}) \mapsto \sum_{i=1}^{r} -(\omega_i - \omega_{i-1})\nabla f_{\mathsf{F}_i}(\mathbf{e}),$$

where (ω, f) is a weighted complete flag of $\mathsf{K}^\sharp$ such that $(\mathbf{x}, \mathbf{s})$ is the pair of weighted centers associated with (ω, f), and $\mathbf{e} \in \mathsf{K}$ is the fixed point of the duality map $-\nabla f_{cl(\mathsf{K}^\sharp)}(\cdot)$.

One of our main results generalizes the bijectivity of the target map for linear programming over the set of primal-dual strictly feasible solutions.

Theorem 1 (Bijectivity of target map)**.** *The target map is a bijection between the primal-dual strictly feasible region and the open dual cone* $\mathsf{K}^\sharp$*.*

Proof. We first demonstrate that the association of each pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ to a weighted barrier $f_{(\omega,\mathsf{f})}$ is a bijection. It suffices to show that each pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ solves the weighted barrier problem determined by a unique weighted barrier $f_{(\omega,\mathsf{f})}$.

To this end, we note that the pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ solves the weighted barrier problem determined by $f_{(\omega,\mathsf{f})}$ if and only if

$$\mathbf{s} = -\sum_{i=1}^{r}(\omega_i - \omega_{i-1})\nabla f_{\mathsf{F}_i}(\mathbf{x}).$$

Since $\mathbf{s} \in \mathsf{K}^\sharp$ and $-\nabla f_{\mathsf{F}_1}(\mathbf{x}) = -\nabla f_{cl(\mathsf{K}^\sharp)}(\mathbf{x}) \in \mathsf{K}^\sharp \subset cl(\mathsf{K}^\sharp)$, we can find some positive δ_1 such that the difference $\mathbf{s} - \delta_1(-\nabla f_{\mathsf{F}_1}(\mathbf{x}))$ is on the boundary of $\mathsf{K}^\sharp$. Let $\mathsf{F}_2 \lhd cl(\mathsf{K}^\sharp)$ be the minimal face containing the difference $\mathbf{s} - \delta_1(-\nabla f_{\mathsf{F}_1}(\mathbf{x}))$. If the minimal face F_2 is not the trivial cone $\{\mathbf{0}\}$, we repeat this process with $\mathbf{s}$ replaced by the difference $\mathbf{s} - \delta_1(-\nabla f_{\mathsf{F}_1}(\mathbf{x})) \in int(\mathsf{F}_2)$, $\mathbf{x}$ replaced by the projection $\mathrm{Proj}_{\mathsf{F}_2 - \mathsf{F}_2}\,\mathbf{x} \in int(\mathsf{F}_2)^\sharp$, and K replaced by the cone $int(\mathsf{F}_2)$.

After a finite number (at most the dimension of K) of iterations of this process, we have a flag and a corresponding strictly increasing sequence of weights $\{\delta_1 + \cdots + \delta_i\}_{i=1}^{p}$ satisfying $\mathbf{s} = -\sum_{i=1}^{p}\delta_i\nabla f_{\mathsf{F}_i}(\mathbf{x})$. The weighted complete flag (ω, f), obtained by extending this flag and sequence of weights to a weighted complete flag, then defines a weighted barrier $f_{(\omega,\mathsf{f})}$ associated with $(\mathbf{x}, \mathbf{s})$.

In the above argument, we have in fact demonstrated a bijection between elements $\mathbf{w} \in \mathsf{K}^\sharp$ and weighted logarithmic barriers via

$$\mathbf{w} = -\sum_{i=1}^{r}(\omega_i - \omega_{i-1})f_{\mathsf{F}_i}(\mathbf{e}),$$

by taking $(\mathbf{x}, \mathbf{s}) = (\mathbf{e}, \mathbf{w})$. Composing these two bijections proves the theorem. □

1.3. **Specialization to symmetric cone programming.** In the special case of symmetric cone programming, where Ω is a symmetric cone, we consider the Euclidean Jordan algebra $\mathfrak{J}$ of rank r associated with Ω, and use the standard log-determinant barriers

$$\mathbf{x} \mapsto -\log\det(\mathbf{x})$$

for all faces of $cl(\Omega)$; see Section A.1. This results in the following weighted barrier and target map associated with the weighted complete flag (ω, f):

$$f_{(\omega,\mathsf{f})} : \mathbf{x} \mapsto -\sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \log\det \mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}),$$

and

$$(\mathbf{x}, \mathbf{s}) \mapsto \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathbf{e}_{\mathsf{F}_i},$$

where $\mathbf{e}_{\mathsf{F}_i}$ is the identity element in the Euclidean Jordan subalgebra $\mathfrak{J}_{\mathsf{F}_i}$ of $\mathfrak{J}$ associated with the symmetric cone $int(\mathsf{F}_i)$.

The association between weighted complete flags (ω, f) and targeted pairs of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ is given by

$$\mathbf{s} = \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) (\mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}))^{-1},$$

where the inverse is taken as an element of the Euclidean Jordan subalgebra $\mathfrak{J}_{\mathsf{F}_i}$. In the unweighted case where $\omega = \mathbf{1}$, we get the familiar expression $\mathbf{s} = \mathbf{x}^{-1}$, which we readily express in terms of the Jordan product as the perturbed complementarity condition

$$\mathbf{x} \circ \mathbf{s} = \mathbf{e}$$

with $\mathbf{e}$ the identity element of the Euclidean Jordan algebra $\mathfrak{J}$. In the general weighted case, however, we would introduce a partition of $\mathbf{s}$ into $(\omega_1 - \omega_0)\mathbf{s}_1 + \cdots + (\omega_r - \omega_{r-1})\mathbf{s}_r$ with $\mathbf{s}_i = (\mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}))^{-1}$; i.e.,

$$\mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}) \circ \mathbf{s}_i = \mathbf{e}_{\mathsf{F}_i}.$$

This is done so that we can enjoy the benefit of applying well studied numerical solution methods for the perturbed complementarity condition. In particular, we choose to use the Nesterov-Todd method [19] when computing the search direction, and measure progress via the function

$$(\mathbf{x}, \{\mathbf{s}_i\}) \mapsto \frac{1}{\sqrt{\omega_1}} \sqrt{\sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \| \mathcal{P}_{\mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x})^{1/2}} \mathbf{s}_i - \mathbf{e}_{\mathsf{F}_i} \|^2},$$

where $\mathcal{P}_{\mathbf{x}}$ denotes the *quadratic representation* of $\mathbf{x}$; see Section A. A weighted sum is used here as the computed search directions are orthogonal under the induced weighted inner product. The multiplicative factor of $\frac{1}{\sqrt{\omega_1}}$ scales the induced unit ball centered at the targeted primal-dual solutions so that it just sits within the primal-dual feasible region.

The Nesterov-Todd method applies scalings to the primal-dual variables to get

$$\mathcal{P}_{\mathbf{p}_i} \mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}) \circ \mathcal{P}_{\mathbf{p}_i^{-1}} \mathbf{s}_i = \mathbf{e}_{\mathsf{F}_i},$$

where $\mathbf{p}_i \in int(\mathsf{F}_i)$ is commonly known as a *scaling point*: it is chosen so that after scaling, $\mathcal{P}_{\mathbf{p}_i} \mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}) = \mathcal{P}_{\mathbf{p}_i^{-1}} \mathbf{s}_i$. This method is chosen for its simplicity in the analysis of algorithm, as shall be evident in Section 3.1. One drawback of partitioning $\mathbf{s}$ is that we now have a sequence of r dual variables $(\mathbf{s}_1, \ldots, \mathbf{s}_r)$ to solve for; i.e., there is an increased in the size of the Newton system. This can be circumvented by using *triangular* transformations $\mathcal{A}_i$ (see Definition 7) instead of quadratic representations, together with an appropriate choice of complete flag f; see Section 3.1 for details.

We thus apply Newton's method to solve

$$\mathcal{A}_i^t \mathrm{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}) \circ \mathcal{A}_i^{-1} \mathbf{s}_i = \mathbf{e}_{\mathsf{F}_i},$$

where $\mathcal{A}_i$ is a triangular automorphism of $int(\mathsf{F}_i)$ satisfying $\mathcal{A}_i^t \operatorname{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x}) = \mathcal{A}_i^{-1}\mathbf{s}_i$, and measure progress via the *proximity measure*

$$d_F(\mathbf{x}, \mathbf{s}; \omega) = \sqrt{\frac{1}{\omega_1} \sum_{i=1}^{r} \frac{(\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s}) - \omega_i)^2}{\omega_i}},$$

which is obtained naturally from the measure for $(\mathbf{x}, \{\mathbf{s}_i\})$. Here, $\lambda_i(\cdot)$ denotes the i'th smallest eigenvalue; see Section A.1. In short, we prove the following quadratic convergence result.

Proposition 1. *If the target* $\sum_{i=1}^{r}(\omega_i - \omega_{i-1})\mathbf{e}_{\mathsf{F}_i}$ *is selected in such a way that the current primal-dual iterates* $(\mathbf{x}, \mathbf{s})$ *satisfy*

$$\mathbf{s} = \sum_{i=1}^{r} \delta_i \operatorname{Proj}_{\mathsf{F}_i - \mathsf{F}_i}(\mathbf{x})^{-1}$$

for some $\delta_1, \ldots, \delta_r > 0$*, and* $d_F(\mathbf{x}, \mathbf{s}; \omega) < \frac{\sqrt{5}-1}{2} < 1$*, then taking a full step along the search directions* $(\mathbf{\Delta_x}, \mathbf{\Delta_s})$ *determined by the Nesterov-Todd method using a suitable triangular automorphism keeps the iterates within the primal-dual strictly feasible region, and satisfies*

$$d_F(\mathbf{x} + \mathbf{\Delta_x}, \mathbf{s} + \mathbf{\Delta_s}; \omega) \leq \frac{d_F(\mathbf{x}, \mathbf{s}; \omega)^2}{1 - d_F(\mathbf{x}, \mathbf{s}; \omega)}.$$

The above inequality allows us to design a globally convergence target-following algorithm (see Algorithm 2), with which we show that points on the primal-dual central path can be efficiently approximated when given any primal-dual strictly feasible solutions within a prescribed wide-neighborhood of the central path; i.e., in the set $\{(\mathbf{x}, \mathbf{s}) \in \Omega^2 : \lambda_1(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s}) \geq \gamma\frac{\langle \mathbf{x}, \mathbf{s}\rangle}{r}\}$ for some $\gamma \in (0, 1)$. This is summarized in the following theorem.

Theorem 2. *Suppose* $\beta \in (0, 1)$ *is fixed. Given any pair of primal-dual strictly feasible solutions* $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ *for the primal-dual symmetric cone programming problems* (1.1)*, and any positive real number* $\widehat{\mu}$*, there is a sequence of at most*

$$O\left(\sqrt{r}\left(\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}}\rangle}{r\lambda_1(\mathcal{P}_{\widehat{\mathbf{x}}^{1/2}}\widehat{\mathbf{s}})} + \left|\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}}\rangle}{r\widehat{\mu}}\right|\right)\right)$$

weights such that Algorithm 2 finds a pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ *satisfying* $\|\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} - \widehat{\mu}\mathbf{e}\| \leq \beta\widehat{\mu}$*.*

1.4. **Comparison with existing notions target maps.** As earlier mentioned, there were other attempts at extending the concept of target maps to semidefinite programming [16, 17, 22], symmetric cone programming [11, 23] and general convex conic programming [24].

In the works [16, 17], the authors consider various notions of target map, and demonstrate that each of these target maps is injective on some neighborhood of the primal-dual central path. However, it is not known if any of these target maps are injective on the whole strictly feasible region. Thus, unlike our target map and the target-following algorithms derived from it, any target-following algorithm based on the target maps in [16, 17] requires all targets to stay within some neighborhood of the central path.

In the work [22], the authors consider the mapping of primal-dual strictly feasible solutions to the diagonal matrix of eigenvalues of their product as the target map. This target map is only injective along the primal-dual central path, hence the target-following algorithm based on it can only follow the central path.

The target maps considered in [11, 23] and [24] are all generalizations of the target map induced by the Nesterov-Todd method [17] to symmetric cone programming and general convex conic programming. Hence, we expect that any target-following algorithm based on these will again require all targets to stay within some neighborhood of the central path. While our algorithm is also based on the Nesterov-Todd approach, we do not use self-adjoint automorphisms $\mathcal{P}_{\mathbf{p}}$ for the primal-dual scalings, but instead employ triangular scalings; this new tool enables our algorithm to work beyond neighborhoods of the central path.

1.5. **Organization of paper.** This paper is organized as follows. In Section 2, we use the Euclidean Jordan-algebraic characterization of symmetric cones to define the notion of weighted analytic centers for symmetric cone programming. This notion allows us to define the target map, with which we describe and analyze a target-following algorithm in Section 3. Finally in Section 4, we apply the target-following algorithm to the problem of finding the primal-dual central path of a symmetric cone program.

2. Target map for symmetric cone programming

Throughout this paper, Ω denotes a symmetric cone, and $(\mathfrak{J}, \circ)$ denotes a Euclidean Jordan algebra of rank r with identity element $\mathbf{e}$ such that the associated symmetric cone $\Omega(\mathfrak{J})$ coincides with Ω; i.e., the interior of the cone of squares of $\mathfrak{J}$ is the symmetric cone Ω. Here, and throughout, we equip $\mathfrak{J}$ with the inner product $\langle\cdot,\cdot\rangle : (\mathbf{x}, \mathbf{y}) \mapsto \operatorname{tr}(\mathbf{x} \circ \mathbf{y})$. We refer the reader to the appendix for more details on Euclidean Jordan algebras, including various notations used in this paper.

We shall denote the automorphism group of Ω by $\mathsf{G}(\Omega)$ and its connected component containing the identity by G. We note that since Ω is self-adjoint, so is its automorphism group $\mathsf{G}(\Omega)$; i.e. for each automorphism $\mathcal{A} \in \mathsf{G}(\Omega)$, its adjoint $\mathcal{A}^t$ is also an automorphism of Ω.

2.1. **Weighted barriers and target map.** In order to have a Jordan-algebraic description of the target map, we would first need a description of flags of faces of $cl(\Omega)$.

2.1.1. *Weighted flags and Jordan frames.* We begin with an algebraic characterization of faces of Ω given by L. Faybusovich [9, Theorem 2]: each face $\mathsf{F} \lhd cl(\Omega)$ is the cone of squares $cl(\Omega(\mathfrak{J}_{\mathbf{c}_2+\cdots+\mathbf{c}_k}))$ of the subalgebra $\mathfrak{J}_{\mathbf{c}_2+\cdots+\mathbf{c}_k}$ in the Peirce decomposition $\mathfrak{J} = \mathfrak{J}_{\mathbf{c}_2+\cdots+\mathbf{c}_k} \oplus \mathfrak{J}_{\mathbf{c}_2+\cdots+\mathbf{c}_k,\mathbf{c}_1} \oplus \mathfrak{J}_{\mathbf{c}_1}$ with respect to the idempotent $\mathbf{c}_2 + \cdots + \mathbf{c}_k$, where $\lambda_1\mathbf{c}_1 + \lambda_2\mathbf{c}_2 + \cdots + \lambda_k\mathbf{c}_k$ is the type I spectral decomposition of an arbitrary $\mathbf{x} \in relint(\mathsf{F})$ with $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_k$.

We now extend this algebraic characterization to one for flag of faces of Ω.

Theorem 3. *Given a flag $\mathsf{f} = (\mathsf{F}_1, \ldots, \mathsf{F}_p, \mathsf{F}_{p+1})$ of Ω, there exists a unique complete system of orthogonal idempotents $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_p)$ such that*

$$\mathsf{F}_i = cl(\Omega(\mathfrak{J}_{\mathbf{c}_i+\cdots+\mathbf{c}_p})) \quad \textit{for } i = 1, \ldots, p.$$

Proof. We shall prove by induction on the rank r of the Euclidean Jordan algebra $\mathfrak{J}$. When $r = 1$, the theorem trivially holds with $\mathbf{c}_1 = \mathbf{e}$.

Suppose that the theorem holds for every Euclidean Jordan algebra of rank no more than some $r \geq 2$. Consider a Euclidean Jordan algebra $\mathfrak{J}$ of rank $r+1$. If $p = 1$, then the theorem trivially holds with $\mathbf{c}_1 = \mathbf{e}$. Otherwise, by the preceding facial characterization, there is an idempotent $\widetilde{\mathbf{c}} \neq \mathbf{e}$ such that F_2 is the cone of squares of $\mathfrak{J}_{\widetilde{\mathbf{c}}}$. This idempotent is the identity element in the subalgebra $\mathfrak{J}_{\widetilde{\mathbf{c}}}$, and is thus unique. The Euclidean Jordan algebra $\mathfrak{J}_{\widetilde{\mathbf{c}}}$ has rank at most r. By the inductive hypothesis, there is a unique system of orthogonal idempotents $(\mathbf{c}_2, \ldots, \mathbf{c}_p)$ with $\mathbf{c}_2+\cdots+\mathbf{c}_p = \widetilde{\mathbf{c}}$

such that F_i is the cone of squares of the subalgebra $\mathfrak{J}_{\mathbf{c}_i+\cdots+\mathbf{c}_p}$ for $i = 2, \ldots, p$. With $\mathbf{c}_1 = \mathbf{e} - \widetilde{\mathbf{c}}$, $\mathsf{F}_1 = cl(\Omega)$ is the cone of squares of $\mathfrak{J}_{\mathbf{c}_1+\cdots+\mathbf{c}_p} = \mathfrak{J}_{\mathbf{e}} = \mathfrak{J}$. □

The above description of flags leads to the following definition of *weighted Jordan frame* and its equivalent classes: there is a natural correspondence between (equivalent) weighted Jordan frames and (equivalent) weighted complete flags of Ω.

Definition 5 (Weighted Jordan frame)**.** A *weighted Jordan frame* of a Euclidean Jordan algebra $\mathfrak{J}$ is a pair (ω, C), with $\omega = (\omega_0, \omega_1, \ldots, \omega_r)$ is a nondecreasing sequence $0 = \omega_0 < \omega_1 \leq \cdots \leq \omega_r$ of real numbers, and C a Jordan frame of $\mathfrak{J}$. The sequence ω is called a *weight sequence*, and the real numbers $\omega_1, \ldots, \omega_r$ are called its *weights*.

Definition 6 (Equivalence of weighted Jordan frame)**.** Two weighted Jordan frames (ω, C) and $(\widetilde{\omega}, \widetilde{\mathsf{C}})$ of a Euclidean Jordan algebra $\mathfrak{J}$ are said to be *equivalent* if they have the same weights $(\omega_0, \omega_1, \ldots, \omega_r)$, and

$$\omega_i > \omega_{i-1} \implies \mathbf{c}_i + \cdots + \mathbf{c}_r = \widetilde{\mathbf{c}}_i + \cdots + \widetilde{\mathbf{c}}_r$$

for $i = 1, \ldots, r$. In other words, two weighted Jordan frames are equivalent if the weighted complete flags they determined are equivalent.

2.1.2. *Triangular automorphisms.* In our target-following algorithm, we will employ special automorphisms of Ω that respect the structure of flags of faces. These are called triangular automorphisms.

Definition 7 (Triangular transformation)**.** Given a Jordan frame $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$, a linear transformation $\mathcal{A} \in L[\mathfrak{J}, \mathfrak{J}]$ is said to be C-*triangular* if, for each $i \in \{1, \ldots, r\}$, the subalgebra $\mathfrak{J}_{\mathbf{c}_i+\cdots+\mathbf{c}_r}$ is an invariant subspace of $\mathcal{A}$, and the restriction of $\mathcal{A}$ to each subspace in the Peirce decomposition of $\mathfrak{J}$ with respect to C is some multiple of the identity transformation.

In matrix theory, the *Gauss decomposition* of a square matrix $\mathbf{A}$ is its decomposition into the product $\mathbf{LU}$ of a lower triangular matrix with an upper triangular matrix. This decomposition is obtained as a consequence of the Gaussian elimination process. For a symmetric positive definite matrix, we often further require that the two triangular matrices have positive diagonal entries, and are transposes of each other. This symmetric Gauss decomposition $\mathbf{A} = \mathbf{LL}^T$ is commonly known as the *Cholesky decomposition*. The Cholesky decomposition produces the linear transformation $\mathbf{X} \mapsto \mathbf{LXL}^T$ that is representable by a triangular matrix under a suitable choice of basis. This triangular transformation is in fact an automorphism of the cone of positive definite matrices, and we recover the original matrix $\mathbf{X}$ by applying this triangular automorphism to the identity matrix.

We shall briefly see that this C-triangular automorphisms of the identity component of $\mathsf{G}(\Omega)$ generalizes the triangular automorphisms to the setting of Euclidean Jordan algebra. Moreover, these triangular automorphisms can be used in place of quadratic representation for the primal-dual scalings in the Nesterov-Todd method.

Proposition 2 (Symmetric Gauss decomposition, cf. Theorem VI.3.6 of [7])**.** *For each Jordan frame* C *of* $\mathfrak{J}$*,*

(1) *each* $\mathbf{x} \in \Omega$ *can be uniquely expressed as* $\mathbf{x} = \mathcal{A}\mathbf{e} = \widetilde{\mathcal{A}}^t\mathbf{e}$*, where* $\mathcal{A}, \widetilde{\mathcal{A}} \in \mathsf{G}$ *are* C*-triangular;*

(2) *each* $\mathcal{A} \in \mathsf{G}$ *can be uniquely decomposed into* $\mathcal{A} = \mathcal{B}\mathcal{Q} = \widetilde{\mathcal{Q}}\widetilde{\mathcal{B}}$*, where* $\mathcal{B}, \widetilde{\mathcal{B}} \in \mathsf{G}$ *are* C*-triangular and* $\mathcal{Q}, \widetilde{\mathcal{Q}} \in \mathsf{G}$ *are orthogonal;*

Proof. All statements, except the last expression in each item, are proved in Theorem VI.3.6 of [7]. To prove these last expressions, we reverse the ordering of primitive idempotents in C before applying Theorem VI.3.6 of [7]. □

Example 1. *For the Jordan algebra of $r \times r$ real symmetric matrices, a Jordan frame C is the r-tuple $(\mathbf{q}_1\mathbf{q}_1^T, \ldots, \mathbf{q}_r\mathbf{q}_r^T)$ with the vectors $\mathbf{q}_1, \ldots, \mathbf{q}_r$ taken from the columns of an orthogonal matrix $\mathbf{Q}$, and a C-triangular automorphism $\mathcal{A} \in \mathsf{G}$ takes the form $\mathbf{X} \mapsto \mathbf{QLQ}^T\mathbf{XQL}^T\mathbf{Q}^T$ for some lower triangular matrix $\mathbf{L}$ with positive diagonal entries, and an orthogonal automorphism $\mathcal{Q} \in \mathsf{G}$ takes the form $\mathbf{X} \mapsto \mathbf{PXP}^T$ for some orthogonal matrix $\mathbf{P}$. Thus the first item in the above corollary gives the Cholesky and inverse Cholesky decompositions, and the second item is the QR-decomposition.*

Theorem 4 (Triangular Nesterov-Todd scaling)**.** *For each pair $(\mathbf{x}, \mathbf{s}) \in \Omega^2$ and each Jordan frame C of $\mathfrak{J}$, there exists a unique C-triangular automorphism $\mathcal{A} \in \mathsf{G}$ satisfying $\mathcal{A}^t\mathbf{x} = \mathcal{A}^{-1}\mathbf{s}$.*

Proof. By the preceding proposition, there exists a unique C-triangular automorphism $\widetilde{\mathcal{A}} \in \mathsf{G}$ satisfying $\widetilde{\mathcal{A}}^t\mathbf{e} = \mathbf{x}$, and a unique C-triangular automorphism $\widetilde{\mathcal{B}} \in \mathsf{G}$ satisfying $\widetilde{\mathcal{B}}\mathbf{e} = (\widetilde{\mathcal{A}}\mathbf{s})^{1/2}$. The theorem follows from

$$\begin{aligned}
\mathcal{A}^t\mathbf{x} = \mathcal{A}^{-1}\mathbf{s} &\iff (\widetilde{\mathcal{A}}\mathcal{A})^t\mathbf{e} = (\widetilde{\mathcal{A}}\mathcal{A})^{-1}(\widetilde{\mathcal{A}}\mathbf{s}) = (\widetilde{\mathcal{A}}\mathcal{A})^{-1}\mathcal{P}_{(\widetilde{\mathcal{A}}\mathbf{s})^{1/2}}\mathbf{e} \\
&\iff (\widetilde{\mathcal{A}}\mathcal{A})(\widetilde{\mathcal{A}}\mathcal{A})^t\mathbf{e} = \mathcal{P}_{\widetilde{\mathcal{B}}\mathbf{e}}\mathbf{e} \overset{\text{(Lemma A.2)}}{=} \widetilde{\mathcal{B}}\mathcal{P}_{\mathbf{e}}\widetilde{\mathcal{B}}^t\mathbf{e} = \widetilde{\mathcal{B}}\widetilde{\mathcal{B}}^t\mathbf{e} \\
&\overset{\text{(Lemma A.4)}}{\iff} (\widetilde{\mathcal{A}}\mathcal{A}\mathbf{e})^2 = (\widetilde{\mathcal{B}}\mathbf{e})^2 \iff \widetilde{\mathcal{A}}\mathcal{A}\mathbf{e} = \widetilde{\mathcal{B}}\mathbf{e} \overset{\text{(Proposition 2)}}{\iff} \widetilde{\mathcal{A}}\mathcal{A} = \widetilde{\mathcal{B}}. \qquad \square
\end{aligned}$$

Finally, we show that we can always find a Jordan frame $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$ for which the unique C-triangular automorphism in the above proposition scales a given primal-dual pair in Ω^2 to a "diagonal" element; i.e., an element $\mathbf{d}$ with $\mathbf{d}_{\mathbf{c}_i,\mathbf{c}_j} = \mathbf{0}$ for all $i < j$ in its Peirce decomposition $\mathbf{d} = \sum_{i=1}^r \mathbf{d}_{\mathbf{c}_i} + \sum_{i<j} \mathbf{d}_{\mathbf{c}_i,\mathbf{c}_j}$. This will be subsequently used to give an algebraic proof of bijectivity of the target map.

Theorem 5. *For each pair $(\mathbf{x}, \mathbf{s}) \in \Omega^2$, there exists a Jordan frame $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$ of $\mathfrak{J}$ and a unique C-triangular automorphism $\mathcal{A} \in \mathsf{G}$ satisfying*

$$\mathcal{A}^t\mathbf{x} = \mathcal{A}^{-1}\mathbf{s} = \sum_{i=1}^r \sqrt{\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})}\mathbf{c}_i.$$

Proof. Take any spectral decomposition $\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} = \sum_{i=1}^r \lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})\widetilde{\mathbf{c}}_i$. By Proposition 2, there is a $\widetilde{\mathsf{C}}$-triangular automorphism $\mathcal{B} \in \mathsf{G}$ and an orthogonal automorphism $\mathcal{Q} \in \mathsf{G}$ such that $\mathcal{P}_{\mathbf{x}^{1/2}} = \mathcal{B}\mathcal{Q}$. We take $\mathcal{A}$ to be the automorphism $\mathcal{P}_{\mathbf{x}^{1/2}}^{-1}\mathcal{P}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}\mathcal{Q} = \mathcal{Q}^t\mathcal{B}^{-1}\mathcal{P}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}\mathcal{Q} \in \mathsf{G}$, and $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$ to be the r-tuple $(\mathcal{Q}^t\widetilde{\mathbf{c}}_1, \ldots, \mathcal{Q}^t\widetilde{\mathbf{c}}_r)$; so that $\mathcal{A}^{-1}\mathbf{s} = \mathcal{Q}^t\mathcal{P}^{-1}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} = \mathcal{Q}^t(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/2}$ and $\mathcal{A}^t\mathbf{x} = \mathcal{Q}^t\mathcal{P}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}\mathcal{P}^{-1}_{\mathbf{x}^{1/2}}\mathbf{x} = \mathcal{Q}^t\mathcal{P}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}\mathbf{e} = \mathcal{Q}^t(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/2}$, with

$$\mathcal{Q}^t(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/2} = \mathcal{Q}^t\sum_{i=1}^r \sqrt{\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})}\widetilde{\mathbf{c}}_i = \sum_{i=1}^r \sqrt{\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})}\mathbf{c}_i. \qquad \square$$

It remains to check that C is a Jordan frame and $\mathcal{A}$ is a C-triangular transformation. The former holds since orthogonal automorphisms in G are automorphisms of $\mathfrak{J}$ that stabilizes the identity element $\mathbf{e}$; see Theorem A.7. The latter follows from $\mathcal{A} = \mathcal{Q}^t\mathcal{B}^{-1}\mathcal{P}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}\mathcal{Q}$ and the fact that both $\mathcal{B}^{-1}$ and $\mathcal{P}_{(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})^{1/4}}$ are $\widetilde{\mathsf{C}}$-triangular.

2.1.3. *Weighted analytic centers.* In defining weighted barriers for Ω, we use the log-determinant barriers for faces F of $cl(\Omega)$: $\mathbf{s} \in int(\mathsf{F}) \mapsto -\log\det_{\mathsf{F}}(\mathbf{s})$, where $\det_{\mathsf{F}}(\mathbf{s})$ denotes the determinant of $\mathbf{s}$ as an element of the Jordan subalgebra $\mathfrak{J}_{\mathsf{F}}$ that is the linear span of the face F (so that $int(\mathsf{F}) = \Omega(\mathfrak{J}_{\mathsf{F}})$). Its Legendre-Fenchel conjugate is then the composition of the orthogonal projection onto this subalgebra, and the log-determinant barrier of the associated symmetric cone.

Remark 1. While the choice of barriers is irrelevant to the bijectivity of the target map, this choice is taken here for the convenience of designing and analyzing the target-following algorithm based on it. In fact, it gives an algebraic means of finding a weighted Jordan frame that associates with a given pair of weighted centers; see proof of Theorem 6.

The one-to-one correspondence between weighted complete flags and weighted Jordan frames results in the following definition of weighted barriers.

Definition 8 (Weighted log-determinant barrier for symmetric cone)**.** The *weighted log-determinant barrier* for Ω associated with the weighted Jordan frame (ω, C) of the Euclidean Jordan algebra $\mathfrak{J}$ (or simply *weighted barrier*) is the function

$$f_{(\omega,\mathsf{C})} : \mathbf{x} \in \Omega \mapsto -\sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \log \det_{\mathbf{c}_{i:r}}(\mathbf{x}_{\mathbf{c}_{i:r}}),$$

where $\mathbf{c}_{i:r}$ denotes the idempotent $\mathbf{c}_i + \cdots + \mathbf{c}_r$, and $\det_{\mathbf{c}_{i:r}}(\mathbf{x})$ denotes the determinant of $\mathbf{x}$ as an element of the Jordan subalgebra $\mathfrak{J}_{\mathbf{c}_{i:r}}$.

Up to equivalence of weighted Jordan frame, there is a one-to-one correspondence between the weighted barriers for Ω and the weighted Jordan frames of $\mathfrak{J}$.

The log-determinant barrier $\mathbf{x} \mapsto -\log\det(\mathbf{x})$ for the symmetric cone Ω is strictly convex, and all barriers $\mathbf{x} \mapsto -\log\det_{\mathbf{c}_{i:r}}(\mathbf{x}_{\mathbf{c}_{i:r}})$ are convex. Therefore the weighted barrier $f_{(\omega,\mathsf{C})}$ is strictly convex, and hence the weighted barrier problem

$$\inf \left\{ f_{(\omega,\mathsf{C})}(\mathbf{x}) + \sum_{j=1}^{p} b_j y_j : \sum_{j=1}^{m} y_j \mathbf{a}_j + \mathbf{x} = \mathbf{c},\ \mathbf{x} \in \Omega \right\}$$

has a unique solution. We call this the *primal weighted analytic center associated with the weighted Jordan frame* (ω, C) for the symmetric cone program.

We now consider the Lagrange optimality conditions for this weighted barrier problem. The gradient of the log-determinant barrier at the element $\mathbf{x}$ is the negation of its inverse $\mathbf{x}^{-1}$. Therefore the gradient of the weighted barrier at $\mathbf{x}$ is $-\sum_{i=1}^{r}(\omega_i - \omega_{i-1})\mathbf{x}_{\mathbf{c}_{i:r}}^{-1}$, and the Lagrange optimality conditions are

$$\begin{aligned} &\sum_{j=1}^{m} y_j \mathbf{a}_j + \mathbf{x} = \mathbf{c}, \quad \mathbf{x} \in \Omega, \\ (\mathrm{WCE}_{(\omega,\mathsf{C})}) \qquad &\langle \mathbf{a}_j, \mathbf{s} \rangle = b_j, \quad 1 \le j \le m, \\ &\mathbf{s} = \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathbf{x}_{\mathbf{c}_{i:r}}^{-1}. \end{aligned}$$

We shall call these conditions the *weighted center equations* given by the weighted Jordan frame (ω, C), and the unique $\mathbf{s}$ satisfying the above conditions the *dual weighted analytic center associated with the weighted Jordan frame* (ω, C).

By following the proof of Thereom 1, we can show that every pair of primal-dual strictly feasible solutions to the symmetric cone programs (1.1) is a pair of weighted analytic centers.

Theorem 6 (Completeness of weighted log-determinant barriers)**.** *Given any pair of primal-dual strictly feasible solutions* $(\mathbf{x}, \mathbf{s})$ *to the symmetric cone programs* (1.1)*, there exist a weighted Jordan frame* (ω, C) *such that* $(\mathbf{x}, \mathbf{s})$ *is the unique solution to the weighed central equations* ($\mathrm{WCE}_{(\omega,\mathsf{C})}$)*. Moreover, up to equivalence, the weighted Jordan frame* (ω, C) *is uniquely determined by the pair* $(\mathbf{x}, \mathbf{s})$*.*

Proof. While this follows from the constructive proof of Theorem 1 as a special case, we can instead use the proof Theorem 5 to find the weighted Jordan frame (ω, C). Indeed in the proof of Theorem 5, we construct a Jordan frame $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$ such that there is unique a C-triangular automorphism $\mathcal{A} \in \mathsf{G}$ satisfying $\mathcal{A}^t\mathbf{x} = \mathcal{A}^{-1}\mathbf{s} = \sum_{i=1}^r \sqrt{\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})}\mathbf{c}_i$.[2] From Lemma 1, we deduce $(\mathbf{x}_{\mathbf{c}_{i:r}})^{-1} = (\mathcal{A}_i^{-t}(\mathcal{A}^t\mathbf{x})_{\mathbf{c}_{i:r}})^{-1}$, where $\mathcal{A}_i$ denotes the restriction of $\mathcal{A}$ to the subalgebra $\mathfrak{J}_{\mathbf{c}_{i:r}}$. From Lemma A.2, we see that this expression is equivalent to $\mathcal{A}_i((\mathcal{A}^t\mathbf{x})_{\mathbf{c}_{i:r}})^{-1}$. Therefore, for any weight sequence ω, we have

$$\begin{aligned} \sum_{i=1}^r (\omega_i - \omega_{i-1})(\mathbf{x}_{\mathbf{c}_{i:r}})^{-1} &= \sum_{i=1}^r (\omega_i - \omega_{i-1})\mathcal{A}_i((\mathcal{A}^t\mathbf{x})_{\mathbf{c}_{i:r}})^{-1} \\ \text{(Lemma 1)} \qquad &= \mathcal{A}\sum_{i=1}^r (\omega_i - \omega_{i-1})((\mathcal{A}^t\mathbf{x})_{\mathbf{c}_{i:r}})^{-1} \\ &= \mathcal{A}\sum_{i=1}^r (\omega_i - \omega_{i-1})\sum_{j=i}^r \frac{1}{\sqrt{\lambda_j(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})}}\mathbf{c}_j. \end{aligned}$$

In particular, for the weight sequence $\omega = (0, \lambda_1(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s}), \ldots, \lambda_r(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s}))$, the above expression simplifies to $\mathcal{A}\sum_{i=1}^r \sqrt{\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s})}\mathbf{c}_i = \mathbf{s}$. ☐

2.1.4. *Target map.* For each pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ of the symmetric programs (1.1), Theorem 6 states that, up to equivalence, there exists a unique weighted Jordan frame (ω, C) satisfying $\mathbf{s} = \sum_{i=1}^r (\omega_i - \omega_{i-1})\mathbf{x}_{\mathbf{c}_{i:r}}^{-1}$, where $\mathbf{c}_{i:r}$ denotes the idempotent $\mathbf{c}_i + \cdots + \mathbf{c}_r$. With this weighted Jordan frame (ω, C), we define the *target map* as

$$\mathcal{T} : (\mathbf{x}, \mathbf{s}) \mapsto \sum_{i=1}^r (\omega_i - \omega_{i-1})\mathbf{c}_{i:r} = \sum_{i=1}^r \omega_i \mathbf{c}_i.$$

We note here that the idempotent $\mathbf{c}_{i:r} = \mathbf{e}_{\mathbf{c}_{i:r}} = \nabla(\log\det_{\mathbf{c}_{i:r}})(\mathbf{e}_{\mathbf{c}_{i:r}})$, where the identity element $\mathbf{e}$ is the fixed point of the duality map $\mathbf{y} \mapsto -\nabla(-\log\det(\mathbf{y})) = \mathbf{y}^{-1}$.

The following theorem is a special case of Theorem 1, and has a constructive algebraic proof, obtained by replacing the geometric argument in the proof of Theorem 1 with the algebraic version in the proof of Theorem 6.

Theorem 7 (Bijectivity of target map for symmetric cone programming)**.** *The target map for the symmetric cone programs* (1.1) *is a bijection between the primal-dual strictly feasible region and the symmetric cone* Ω*.*

3. Target-following algorithms for symmetric cone programming

Using the target map $\mathcal{T}$ defined in the previous section, we propose the following target-following framework.

Algorithm 1. (Target-following framework for symmetric cone programming) *Given a pair of primal-dual strictly feasible solutions* $(\mathbf{x}^{\text{in}}, \mathbf{s}^{\text{in}})$ *and a target* $\mathbf{w}^{\text{out}} \in \Omega$.

(1) *Set* $(\mathbf{x}^+, \mathbf{s}^+) = (\mathbf{x}^{\text{in}}, \mathbf{s}^{\text{in}})$*, and* $\mathbf{w}^+ = \mathcal{T}(\mathbf{x}^{\text{in}}, \mathbf{s}^{\text{in}})$.
(2) *Repeat the following steps until* $\mathbf{w}^+$ *is close to* $\mathbf{w}^{\text{out}}$,
  (a) *Select the next target* $\mathbf{w}^{++}$ *leading towards* $\mathbf{w}^{\text{out}}$.
  (b) *Compute an approximation* $(\mathbf{x}^{++}, \mathbf{s}^{++})$ *of the pre-image* $\mathcal{T}^{-1}(\mathbf{w}^{++})$.
  (c) *Update* $(\mathbf{x}^+, \mathbf{s}^+) \leftarrow (\mathbf{x}^{++}, \mathbf{s}^{++})$ *and* $\mathbf{w}^+ \leftarrow \mathbf{w}^{++}$.
(3) *Output* $(\mathbf{x}^{\text{out}}, \mathbf{s}^{\text{out}}) = (\mathbf{x}^+, \mathbf{s}^+)$.

[2] This involves a QR-decomposition in the case of semidefinite programming.

The two main steps in this framework are the selection of the next target $\mathbf{w}^{++}$ and the computation of the next pair of iterates $(\mathbf{x}^{++}, \mathbf{s}^{++})$. In the next section, we consider the problem of computing the next pair of iterates.

3.1. **Approximating weighted analytic centers.** We consider the problem of approximating the weighted analytic centers determined by the weighted center equations (WCE$_{(\omega,\mathsf{C})}$), given a weighted Jordan frame (ω, C) that defines the next target $\mathbf{w}^{++}$ and a pair of current iterates $(\mathbf{x}^{+}, \mathbf{s}^{+})$. For simplicity of notations, we shall denote by $\mathbf{c}_{i:r}$ the idempotent $\mathbf{c}_i + \cdots + \mathbf{c}_r$, by $\mathfrak{J}_i$ and Ω_i, respectively, the Jordan subalgebra $\mathfrak{J}_{\mathbf{c}_{i:r}}$ and its associated symmetric cone $\Omega(\mathfrak{J}_{\mathbf{c}_{i:r}})$, and by G_i the identity component of $\mathsf{G}(\mathfrak{J}_i)$.

3.1.1. *Nesterov-Todd scaling.* We begin by writing the last equation in the weighted center equations (WCE$_{(\omega,\mathsf{C})}$) as

$$\mathbf{s} - \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathbf{s}_i = \mathbf{0} \tag{3.1a}$$

$$\mathbf{x}_{\mathbf{c}_{i:r}} - \mathbf{x}_i = \mathbf{0}, \qquad i = 1, \ldots, r \tag{3.1b}$$

$$\mathbf{x}_i \circ \mathbf{s}_i = \mathbf{c}_{i:r}, \qquad i = 1, \ldots, r. \tag{3.1c}$$

For the moment, we cast aside the first two equations and consider the application of a primal-dual Nesterov-Todd-type scaling in (3.1c):

$$(\{\mathbf{x}_i\}, \{\mathbf{s}_i\}) \mapsto (\{\mathcal{A}_i^t \mathbf{x}_i\}, \{\mathcal{A}_i^{-1} \mathbf{s}_i\})$$

where $\mathcal{A}_i$ is some automorphism in G_i such that $\mathcal{A}_i^t \mathbf{x}_i = \mathcal{A}_i^{-1} \mathbf{s}_i$ for each i. The bilinear equations (3.1c) are invariant under this transformation since $\mathbf{x}_i \circ \mathbf{s}_i = \mathbf{c}_{i:r}$ if and only if $\mathcal{A}_i^t \mathbf{x}_i \circ \mathcal{A}_i^{-1} \mathbf{s}_i = \mathbf{c}_{i:r}$ for any $\mathcal{A}_i \in \mathsf{G}_i$; see Lemma A.5. The advantage of using the Nesterov-Todd-type scaling is that we can simplify the linearization of $\mathcal{A}_i^t \mathbf{x}_i \circ \mathcal{A}_i^{-1} \mathbf{s}_i = \mathbf{c}_{i:r}$ by scaling with $\mathcal{L}^{-1}_{\mathcal{A}_i^t \mathbf{x}_i} = \mathcal{L}^{-1}_{\mathcal{A}_i^{-1} \mathbf{s}_i}$ to get

$$\mathcal{A}_i^t \mathbf{\Delta}_{\mathbf{x}_i} + \mathcal{A}_i^{-1} \mathbf{\Delta}_{\mathbf{s}_i} = (\mathcal{A}_i^t \mathbf{x}_i)^{-1} - \mathcal{A}_i^t \mathbf{x}_i.$$

As we turn out attention to the first two equations in (3.1), we quickly realize that the automorphisms $\mathcal{A}_i$ used in the primal-dual scalings of the bilinear equations (3.1c) should be chosen so that for some automorphism $\mathcal{A} \in \mathsf{G}$,

$$(\mathcal{A}^t \mathbf{x})_{\mathbf{c}_{i:r}} = \mathcal{A}_i^t \mathbf{x}_{\mathbf{c}_{i:r}} \quad \forall \mathbf{x} \in \mathfrak{J} \quad \forall i \in \{1, \ldots, r\}, \tag{3.2}$$

and

$$\mathcal{A}^{-1} \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathbf{s}_i = \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathcal{A}_i^{-1} \mathbf{s}_i \quad \forall \mathbf{s}_i \in \mathfrak{J}_i. \tag{3.3}$$

The next lemma shows that these conditions do hold if we take $\mathcal{A} \in \mathsf{G}$ to be C-triangular, and take $\mathcal{A}_i$ to be its restriction to the subalgebra $\mathfrak{J}_i$.

Lemma 1. *If $\mathcal{A}_i$ is the restriction of a nonsingular C-triangular transformation $\mathcal{A}$, then both (3.2) and (3.3) hold.*

Proof. By definition of $\mathcal{A}_i$, we have $\mathcal{A}\mathbf{s}_i = \mathcal{A}_i \mathbf{s}_i$ for all $\mathbf{s}_i \in \mathfrak{J}$. Therefore for all $\mathbf{s}_i \in \mathfrak{J}_i$ with $i \in \{1, \ldots, r\}$,

$$\mathcal{A} \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathcal{A}_i^{-1} \mathbf{s}_i = \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathcal{A} \mathcal{A}_i^{-1} \mathbf{s}_i = \sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \mathbf{s}_i.$$

Since $\mathfrak{J}_i$ is invariant under $\mathcal{A}$ and the subspaces in a Peirce decomposition are pairwise orthogonal, we have for all $\mathbf{s}_i \in \mathfrak{J}_i$

$$\left\langle \mathcal{A}_i^t \mathbf{x}_{\mathbf{c}_i}, \mathbf{s}_i \right\rangle = \left\langle \mathbf{x}_{\mathbf{c}_i}, \mathcal{A}_i \mathbf{s}_i \right\rangle = \left\langle \mathbf{x}, \mathcal{A} \mathbf{s}_i \right\rangle = \left\langle \mathcal{A}^t \mathbf{x}, \mathbf{s}_i \right\rangle = \left\langle (\mathcal{A}^t \mathbf{x})_{\mathbf{c}_i}, \mathbf{s}_i \right\rangle. \qquad \square$$

In summary, we will transform the primal-dual variables by $(\mathbf{x},\mathbf{s}) \in \mathfrak{J}^2 \mapsto (\mathcal{A}^t\mathbf{x}, \mathcal{A}^{-1}\mathbf{s})$ and $(\mathbf{x}_i,\mathbf{s}_i) \in \mathfrak{J}_i^2 \mapsto (\mathcal{A}_i^t\mathbf{x}_i, \mathcal{A}^{-1}\mathbf{s}_i)$, where $\mathcal{A} \in \mathsf{G}$ is C-triangular and $\mathcal{A}_i \in \mathsf{G}_i$ is the restriction of $\mathcal{A}$ to $\mathfrak{J}_i$. We further require that $\mathcal{A}_i^t\mathbf{x}_i = \mathcal{A}_i^{-1}\mathbf{s}_i$ for each $i \in \{1,\dots,r\}$ at the current primal-dual iterates $(\mathbf{x}^+,\mathbf{s}^+)$, where $\mathbf{x}_i = (\mathbf{x}^+)_{\mathbf{c}_i}$ and $\mathbf{s}_i \in \mathfrak{J}_i$ is such that $\mathbf{s}^+ = \sum_{i=1}^r(\omega_i - \omega_{i-1})\mathbf{s}_i$.

From $\mathbf{x}_i = (\mathbf{x}^+)_{\mathbf{c}_i}$, $\mathcal{A}_i^t\mathbf{x}_i = \mathcal{A}_i^{-1}\mathbf{s}_i$ and $\mathbf{s}^+ = \sum_{i=1}^r(\omega_i - \omega_{i-1})\mathbf{s}_i$, we arrive at

$$\mathbf{s}^+ = \sum_{i=1}^r(\omega_i-\omega_{i-1})\mathcal{A}_i\mathcal{A}_i^t(\mathbf{x}^+)_{\mathbf{c}_i} = \sum_{i=1}^r(\omega_i-\omega_{i-1})\mathcal{A}(\mathcal{A}^t\mathbf{x}^+)_{\mathbf{c}_i} = \mathcal{A}\mathcal{M}\mathcal{A}^t\mathbf{x}^+,$$

where $\mathcal{M} : \mathbf{x} \mapsto \sum_{i=1}^r(\omega_i-\omega_{i-1})\mathbf{x}_{\mathbf{c}_i}$. Since $\mathcal{M}$ is not an automorphism of Ω, we cannot expect $\mathcal{A}\mathcal{M}\mathcal{A}^t\mathbf{x}^+ \in \Omega$ in general, thence the above equation may not be satisfied with any choice of $\mathcal{A} \in \mathsf{G}$. On the other hand, if $(\mathcal{A}^t\mathbf{x}^+)_{\mathbf{c}_i,\mathbf{c}_j} = 0$ for all $i < j$, then we can replace $\mathcal{M}$ with the automorphism $\mathcal{D} = \mathcal{P}_{\sum_{i=1}^r\sqrt{\omega_i}\mathbf{c}_i} \in \mathsf{G}$. This happens whenever the next target $\mathbf{w}^{++}$ is selected in such a way that the current primal-dual iterates $(\mathbf{x}^+,\mathbf{s}^+)$ satisfy

$$\mathbf{s}^+ = \sum_{i=1}^r(\widetilde{\omega}_i - \widetilde{\omega}_{i-1})(\mathbf{x}^+)^{-1}_{\mathbf{c}_{i:r}} \tag{3.4}$$

for some weight sequence $\widetilde{\omega}$; i.e., we select f to be some complete flag $\widetilde{\mathsf{f}}$ such that $(\mathbf{x}^+,\mathbf{s}^+)$ is the pair of weighted analytic centers associated with the weighted complete flag $(\widetilde{\omega},\widetilde{\mathsf{f}})$. We note that this complete flag $\widetilde{\mathsf{f}}$ can be obtained from the construction in the proof of Theorem 6.

Remark 2. This assumption means that we only need to (and, in fact, only allowed to) choose the weight sequence ω when selecting the next target. Thus the analysis in this paper only allow the algorithm to target at the collection of targets with specific weight sequence, but non-specific complete flag; i.e., targets that share the same set of eigenvalues. This is not an issue if the final target is a multiple of the identity element $\mathbf{e}$; i.e., if the algorithm aims to locate points on the central path. For all other purposes, we would need to resort to another approach described in [5], which unfortunately has a more involved analysis.

Under the above assumption, the following lemma shows that with the choice $\mathcal{D} = \mathcal{P}_{\sum_{i=1}^r\sqrt{\omega_i}\mathbf{c}_i} \in \mathsf{G}$, we are able to find C-triangular automorphism $\mathcal{A} \in \mathsf{G}$ satisfying $\mathbf{s}^+ = \mathcal{A}\mathcal{D}\mathcal{A}^t\mathbf{x}^+$.

Lemma 2. *There exists a unique C-triangular automorphism $\mathcal{A} \in \mathsf{G}$ satisfying $\mathcal{A}^t\mathbf{x}^+ = \mathcal{D}^{-1}\mathcal{A}^{-1}\mathbf{s}^+$, where $\mathcal{D} \in \mathsf{G}$ is the automorphism $\mathcal{P}_{\sum_{i=1}^r\sqrt{\omega_i}\mathbf{c}_i}$. Moreover, if the next target $\mathbf{w}^{++}$ is selected in such a way that the current primal-dual iterates $(\mathbf{x}^+,\mathbf{s}^+)$ satisfy (3.4) for some weight sequence $\widetilde{\omega}$, then $\mathcal{A}^t\mathbf{x}^+ = \mathcal{D}^{-1}\mathcal{A}^{-1}\mathbf{s}^+ = \sum_{i=1}^r\sqrt{\widetilde{\omega}_i/\omega_i}\mathbf{c}_i$*

Proof. By Theorem 4, there is a unique C-triangular automorphism $\widetilde{\mathcal{A}} \in \mathsf{G}$ satisfying $\widetilde{\mathcal{A}}^t\mathbf{x}^+ = \widetilde{\mathcal{A}}^{-1}\mathbf{s}^+$. We can then take $\mathcal{A}$ to be the C-triangular automorphism $\widetilde{\mathcal{A}}\mathcal{D}^{-1/2} \in \mathsf{G}$, and check that $\mathcal{A}^t\mathbf{x}^+ = \mathcal{D}^{-1/2}\widetilde{\mathcal{A}}^t\mathbf{x}^+ = \mathcal{D}^{-1/2}\widetilde{\mathcal{A}}^{-1}\mathbf{s}^+ = \mathcal{D}^{-1}\mathcal{A}^{-1}\mathbf{s}^+$. Uniqueness of $\mathcal{A}$ follows from that of $\widetilde{\mathcal{A}}$. Moreover, if (3.4) holds, we see from the proof of Theorem 6 that $\widetilde{\mathcal{A}}$ can be chosen such that $\widetilde{\mathcal{A}}^t\mathbf{x}^+ = \widetilde{\mathcal{A}}^{-1}\mathbf{s}^+ = \sum_{i=1}^r\sqrt{\widetilde{\omega}}\mathbf{c}_i$; subsequently, $\mathcal{D}^{-1}\mathcal{A}^{-1}\mathbf{s}^+ = \mathcal{D}^{-1/2}\widetilde{\mathcal{A}}^{-1}\mathbf{s}^+ = \mathcal{D}^{-1/2}\sum_{i=1}^r\sqrt{\widetilde{\omega}_i}\mathbf{c}_i = \sum_{i=1}^r\sqrt{\widetilde{\omega}_i/\omega_i}\mathbf{c}_i$. □

Let $\mathcal{A} \in \mathsf{G}$ be the C-triangular automorphism in the lemma, and denote its restriction to $\mathfrak{J}_i$ by $\mathcal{A}_i$. We then choose $\mathbf{s}_i = \mathcal{A}_i\mathcal{A}_i^t\mathbf{x}_i \in \mathfrak{J}_i$, and check that $\mathbf{s}^+ =$

$\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\mathbf{s}_i$: using lemma 1, we deduce $\mathbf{x}_i=(\mathcal{A}^{-t}\mathbf{d})_{\mathbf{c}_{i:r}}=\mathcal{A}_i^{-t}\mathbf{d}_{\mathbf{c}_{i:r}}$, where $\mathbf{d}$ denotes the element $\sum_{i=1}^{r}\sqrt{\widetilde{\omega}_i/\omega_i}\mathbf{c}_i$, whence $\mathbf{s}_i=\mathcal{A}_i\mathbf{d}_{\mathbf{c}_{i:r}}$; we then have

$$\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\mathbf{s}_i=\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\mathcal{A}_i\mathbf{d}_{\mathbf{c}_{i:r}}=\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\mathcal{A}\mathbf{d}_{\mathbf{c}_{i:r}}=\mathcal{A}\mathcal{D}\mathbf{d}=\mathbf{s}^{+}.$$

With the primal-dual scaling $(\mathbf{x},\mathbf{s},\{\mathbf{x}_i\},\{\mathbf{s}_i\})\mapsto(\mathcal{A}^{t}\mathbf{x},\mathcal{A}^{-1}\mathbf{s},\{\mathcal{A}_i^{t}\mathbf{x}_i\},\{\mathcal{A}_i^{-1}\mathbf{s}_i\})$, the linearization of the rewritten last equation (3.1) of the weighted center equations at the current iterate $(\mathbf{x}^{+},\mathbf{s}^{+},\{\mathbf{x}_i=(\mathbf{x}^{+})_{\mathbf{c}_{i:r}}\},\{\mathbf{s}_i=\mathcal{A}_i\mathcal{A}_i^{t}\mathbf{x}_i\})$, after scaling by $\mathcal{L}^{-1}_{\mathbf{d}_{\mathbf{c}_{i:r}}}$, becomes

$$\boldsymbol{\Delta}_{\mathbf{s}}-\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\boldsymbol{\Delta}_{\mathbf{s}_i}=\mathbf{0}, \tag{3.5a}$$

$$(\boldsymbol{\Delta}_{\mathbf{x}})_{\mathbf{c}_i}-\boldsymbol{\Delta}_{\mathbf{x}_i}=\mathbf{0},\quad i=1,\dots,r, \tag{3.5b}$$

$$\mathcal{A}_i^{t}\boldsymbol{\Delta}_{\mathbf{x}_i}+\mathcal{A}^{-1}\boldsymbol{\Delta}_{\mathbf{s}_i}=\mathbf{d}^{-1}_{\mathbf{c}_{i:r}}-\mathbf{d}_{\mathbf{c}_{i:r}}=\sum_{j=i}^{r}\frac{\omega_j-\widetilde{\omega}_j}{\sqrt{\widetilde{\omega}_j\omega_j}}\mathbf{c}_j,\quad i=1,\dots,r. \tag{3.5c}$$

The weighted sum of these equations, with weights $(\omega_i-\omega_{i-1})$, is

$$\sum_{i=1}^{r}(\omega_i-\omega_{i-1})(\mathcal{A}^{t}\boldsymbol{\Delta}_{\mathbf{x}_1})_{\mathbf{c}_{i:r}}+\mathcal{A}^{-1}\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\boldsymbol{\Delta}_{\mathbf{s}_i}=\sum_{j=i}^{r}\sqrt{\frac{\omega_j}{\widetilde{\omega}_j}}(\omega_j-\widetilde{\omega}_j)\mathbf{c}_j.$$

Thus the search directions $(\boldsymbol{\Delta}_{\mathbf{x}},\boldsymbol{\Delta}_{\mathbf{s}})$ are obtained by solving

$$\begin{gathered}\sum_{j=1}^{m}\Delta_{y_j}\mathbf{a}_j+\boldsymbol{\Delta}_{\mathbf{x}}=\mathbf{0},\\ \sum_{i=1}^{r}\langle\mathbf{a}_j,\boldsymbol{\Delta}_{\mathbf{s}}\rangle=0,\quad 1\le j\le m,\\ \sum_{i=1}^{r}(\omega_i-\omega_{i-1})(\mathcal{A}^{t}\boldsymbol{\Delta}_{\mathbf{x}})_{\mathbf{c}_{i:r}}+\mathcal{A}^{-1}\boldsymbol{\Delta}_{\mathbf{s}}=\sum_{j=i}^{r}\sqrt{\frac{\omega_j}{\widetilde{\omega}_j}}(\omega_j-\widetilde{\omega}_j)\mathbf{c}_j.\end{gathered} \tag{3.6}$$

Since the linear operator $\mathbf{x}\in\mathfrak{J}\mapsto\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\mathbf{x}_{\mathbf{c}_{i:r}}$ is positive definite, the search directions are uniquely determined.

3.1.2. *Proximity measure.* Proximity of the iterates to the weighted analytic centers is measured in terms of $(\{\mathbf{x}_i\},\{\mathbf{s}_i\})$ via the backward error

$$d_F(\{\mathbf{y}_i\},\{\mathbf{w}_i\};(\omega,\mathsf{C}))\stackrel{\text{def}}{=}\frac{1}{\sqrt{\omega_1}}\sqrt{\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\|\mathcal{P}_{\mathbf{y}_i^{1/2}}\mathbf{w}_i-\mathbf{c}_{i:r}\|^2} \tag{3.7}$$

defined on $\Omega(\mathfrak{J}_1\times\cdots\times\mathfrak{J}_r)\times(\mathfrak{J}_1\times\cdots\times\mathfrak{J}_r)$. This error is induced by the inner product $\langle\cdot,\cdot\rangle_\omega:(\{\mathbf{u}_i\},\{\mathbf{v}_i\})\mapsto\frac{1}{\omega_1}\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\langle\mathbf{u}_i,\mathbf{v}_i\rangle$ on the Cartesian product $\mathfrak{J}_1\times\cdots\times\mathfrak{J}_r$ of Euclidean Jordan algebras, which is chosen because the search directions $(\{\boldsymbol{\Delta}_{\mathbf{x}_i}\},\{\boldsymbol{\Delta}_{\mathbf{s}_i}\})$ satisfy $\langle\{\boldsymbol{\Delta}_{\mathbf{x}_i}\},\{\boldsymbol{\Delta}_{\mathbf{s}_i}\}\rangle_\omega=0$. The factor $1/\sqrt{\omega_1}$ is the greatest factor so that

$$d_F(\{\mathbf{y}_{\mathbf{c}_{i:r}}\},\{\mathbf{w}_i\};(\omega,\mathsf{C}))<1\implies\sum_{i=1}^{r}(\omega_i-\omega_{i-1})\mathbf{w}_i\in\Omega,$$

which is a consequence of the following lemma.

Lemma 3. *For* $\mathbf{w} = \sum_{i=1}^r (\omega_i - \omega_{i-1})\mathbf{w}_i$,

$$d_F(\{\mathbf{y}_{\mathbf{c}_{i:r}}\}, \{\mathbf{w}_i\}; (\omega, \mathsf{C})) \geq \sqrt{\frac{1}{\omega_1}\sum_{i=1}^r \frac{(\lambda_i(\mathcal{P}_{\mathbf{y}^{1/2}}\mathbf{w}) - \omega_i)^2}{\omega_i}}.$$

Proof. By Lemma A.3, $\mathcal{P}_{\mathbf{y}_i^{1/2}}\mathbf{w}_i$ and $\mathcal{A}_{\mathbf{y}_i}\mathbf{w}_i$ shares the same spectrum for any automorphism $\mathcal{A}_{\mathbf{y}_i} \in \mathsf{G}(\Omega_i)$ satisfying $\mathcal{A}_{\mathbf{y}_i}^t \mathbf{c}_{i:r} = \mathbf{y}_i$. In particular, we may use a C-triangular transformation $\mathcal{A}_{\mathbf{y}}$ satisfying $\mathcal{A}_{\mathbf{y}}^t \mathbf{e} = \mathbf{y}$ (see Proposition 2) and take $\mathcal{A}_{\mathbf{y}_i}$ to be the restriction of $\mathcal{A}_{\mathbf{y}}$ to $\mathfrak{J}_i$; this results in $\|\mathcal{P}_{\mathbf{y}_{\mathbf{c}_{i:r}}^{1/2}}\mathbf{w}_i - \mathbf{c}_{i:r}\| = \|\mathcal{A}_{\mathbf{y}_i}\mathbf{w}_i - \mathbf{c}_{i:r}\| = \|\mathcal{A}_{\mathbf{y}}\mathbf{w}_i - \mathbf{c}_{i:r}\|$, whence $\omega_1 d_F(\{\mathbf{y}_i\}, \{\mathbf{w}_i\}; (\omega, \mathsf{C}))^2 = \sum_{i=1}^r (\omega_i - \omega_{i-1})\|\mathcal{A}_{\mathbf{y}}\mathbf{w}_i - \mathbf{c}_{i:r}\|^2$. In terms of the Pierce decomposition with respect to C,

$$\begin{aligned}
&\omega_1 d_F(\{\mathbf{y}_i\}, \{\mathbf{w}_i\}; (\omega, \mathsf{C}))^2 \\
&= \sum_{i=1}^r (\omega_i - \omega_{i-1}) \sum_{j=i}^r ((\mathcal{A}_{\mathbf{y}}\mathbf{w}_i)_{\mathbf{c}_j} - 1)^2 + \sum_{i=1}^r 2(\omega_i - \omega_{i-1}) \sum_{i \leq j < k} \|(\mathcal{A}_{\mathbf{y}}\mathbf{w}_i)_{\mathbf{c}_j,\mathbf{c}_k}\|^2 \\
&= \sum_{j=1}^r \sum_{i=1}^j (\omega_i - \omega_{i-1})((\mathcal{A}_{\mathbf{y}}\mathbf{w}_i)_{\mathbf{c}_j} - 1)^2 + 2\sum_{j<k}\sum_{i=1}^j (\omega_i - \omega_{i-1})\|(\mathcal{A}_{\mathbf{y}}\mathbf{w}_i)_{\mathbf{c}_j,\mathbf{c}_k}\|^2.
\end{aligned}$$

Using Cauchy's inequality and the triangle inequality, we get

$$\begin{aligned}
\omega_1 d_F(\{\mathbf{y}_i\}, \{\mathbf{w}_i\}; (\omega, \mathsf{C}))^2 \geq\ & \sum_{j=1}^r \frac{1}{\omega_j}\left(\sum_{i=1}^j (\omega_i - \omega_{i-1})(\mathcal{A}_{\mathbf{y}}\mathbf{w}_i)_{\mathbf{c}_j} - \omega_j\right)^2 \\
& + 2\sum_{j<k}\frac{1}{\omega_j}\left\|\sum_{i=1}^j (\omega_i - \omega_{i-1})(\mathcal{A}_{\mathbf{y}}\mathbf{w}_i)_{\mathbf{c}_j,\mathbf{c}_k}\right\|^2 \\
=\ & \sum_{j=1}^r \frac{1}{\omega_j}\left((\mathcal{A}_{\mathbf{y}}\mathbf{w})_{\mathbf{c}_j} - \omega_j\right)^2 + 2\sum_{j<k}\frac{1}{\omega_j}\left\|(\mathcal{A}_{\mathbf{y}}\mathbf{w})_{\mathbf{c}_j,\mathbf{c}_k}\right\|^2,
\end{aligned}$$

where $\mathbf{w} = \sum_{i=1}^r (\omega_i - \omega_{i-1})\mathbf{w}_i$. The lemma then follows from the next theorem. □

Theorem 8 (cf. Lemma 3 of [3]; cf. Hoffman-Wielandt Inequality)**.** *For any* $0 < \omega_1 \leq \cdots \leq \omega_n$, *any* $\mathbf{x} \in \mathfrak{J}$ *and any Jordan frame* C,

$$\sum_{i=1}^r \frac{1}{\omega_i}\left(\mathbf{x}_{\mathbf{c}_i} - \omega_i\right)^2 + 2\sum_{i<j}\frac{1}{\omega_i}\left\|\mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j}\right\|^2 \geq \sum_{i=1}^r \frac{1}{\omega_i}\left\|\lambda_i(\mathbf{x}) - \omega_i\right\|^2.$$

Proof. By expanding both sides of the desired inequality, it is clear that it suffices to bound the sum $\sum_{i=1}^r \frac{1}{\omega_i}\mathbf{x}_{\mathbf{c}_i}^2 + 2\sum_{i<j}\frac{1}{\omega_i}\left\|\mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j}\right\|^2$ from below by $\sum_{i=1}^r \frac{1}{\omega_i}\lambda_i(\mathbf{x})^2$. Let $\mathbf{x} = \sum \lambda_i(\mathbf{x})\widetilde{\mathbf{c}}_i$ be a spectral decomposition. From $\mathbf{x}^2 = \sum \lambda_i(\mathbf{x})^2\widetilde{\mathbf{c}}_i$, we get

$$\begin{bmatrix} (\mathbf{x}^2)_{\mathbf{c}_1} \\ \vdots \\ (\mathbf{x}^2)_{\mathbf{c}_r} \end{bmatrix} = \begin{bmatrix} (\widetilde{\mathbf{c}}_1)_{\mathbf{c}_1} & \cdots & (\widetilde{\mathbf{c}}_r)_{\mathbf{c}_1} \\ \vdots & \ddots & \vdots \\ (\widetilde{\mathbf{c}}_1)_{\mathbf{c}_r} & \cdots & (\widetilde{\mathbf{c}}_r)_{\mathbf{c}_r} \end{bmatrix} \begin{bmatrix} \lambda_1(\mathbf{x})^2 \\ \vdots \\ \lambda_n(\mathbf{x})^2 \end{bmatrix},$$

where the matrix on the right side of the equation is doubly-stochastic. By the Hardy-Littlewood-Pólya Theorem [10], we have $\sum_{i=k}^r (\mathbf{x}^2)_{\mathbf{c}_i} \leq \sum_{i=k}^r \lambda_i(\mathbf{x})^2$ for any

$k \in \{1, \ldots, r\}$. Consequently

$$\begin{aligned}
&\sum_{i=1}^{r} \frac{1}{\omega_i} \mathbf{x}_{\mathbf{c}_i}^2 + 2\sum_{i<j} \frac{1}{\omega_i} \left\| \mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j} \right\|^2 \\
&= \frac{1}{\omega_1} \sum_{i,j=1}^{r} \left\| \mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j} \right\|^2 - \sum_{k=2}^{r} \left(\frac{1}{\omega_{k-1}} - \frac{1}{\omega_k} \right) \sum_{i,j=k}^{r} \left\| \mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j} \right\|^2 \\
&\geq \frac{1}{\omega_1} \sum_{i=1}^{r} (\mathbf{x}^2)_{\mathbf{c}_i} - \sum_{k=2}^{r} \left(\frac{1}{\omega_{k-1}} - \frac{1}{\omega_k} \right) \sum_{i=k}^{r} (\mathbf{x}^2)_{\mathbf{c}_i} \\
&\geq \frac{1}{\omega_1} \sum_{i=1}^{r} \lambda_i(\mathbf{x})^2 - \sum_{k=2}^{r} \left(\frac{1}{\omega_{k-1}} - \frac{1}{\omega_k} \right) \sum_{i=k}^{r} \lambda_i(\mathbf{x})^2 \\
&= \sum_{i=1}^{r} \frac{1}{\omega_i} \lambda_i(\mathbf{x})^2
\end{aligned}$$

proves the lemma. □

We note that for $\mathbf{x}_i = (\mathbf{x}^+)_{\mathbf{c}_{i:r}}$ and $\mathbf{s}_i = \mathcal{A}_i \mathcal{A}_i^t \mathbf{x}_i$,

$$\begin{aligned}
d_F(\{\mathbf{x}_i\}, \{\mathbf{s}_i\}; (\omega, \mathsf{C})) &= \frac{1}{\sqrt{\omega_1}} \sqrt{\sum_{i=1}^{r} (\omega_i - \omega_{i-1}) \sum_{j=i}^{r} \left(\frac{\widetilde{\omega}_j - \omega_j}{\omega_j} \right)^2} \\
&= \sqrt{\frac{1}{\omega_1} \sum_{i=1}^{r} \frac{(\widetilde{\omega}_i - \omega_i)^2}{\omega_i}}.
\end{aligned}$$

This suggest the following proximity measure for $(\mathbf{x}, \mathbf{s})$:

$$d_F(\mathbf{x}, \mathbf{s}; \omega) \stackrel{\text{def}}{=} \sqrt{\frac{1}{\omega_1} \sum_{i=1}^{r} \frac{(\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}} \mathbf{s}) - \omega_i)^2}{\omega_i}}. \tag{3.8}$$

From this definition, it is straightforward to deduce that

$$\frac{\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}} \mathbf{s})}{\omega_i} \in [1 - d_F(\mathbf{x}, \mathbf{s}; \omega), 1 + d_F(\mathbf{x}, \mathbf{s}; \omega)].$$

Lemma 4. *For all* $\alpha \in [0, 1]$,

$$\begin{aligned}
d_F(\{\mathbf{x}_i + \alpha \mathbf{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \alpha \mathbf{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C})) &\leq (1 - \alpha)\gamma + \frac{\alpha^2}{\omega_1} \sum_{i=1}^{r} \frac{(\omega_i - \widetilde{\omega}_i)^2}{\widetilde{\omega}_i} \\
&\leq (1 - \alpha)\gamma + \alpha^2 \frac{\gamma^2}{1 - \gamma},
\end{aligned}$$

where γ *denotes* $d_F(\{\mathbf{x}_i\}, \{\mathbf{s}_i\}; (\omega, \mathsf{C}))$.

Proof. Denote by $\widetilde{\mathbf{x}}_i$, $\widetilde{\mathbf{s}}_i$, $\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}$ and $\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}$ the scaled elements $\mathcal{A}_i^t \mathbf{x}_i$, $\mathcal{A}_i^{-1} \mathbf{s}_i$, $\mathcal{A}_i^t \mathbf{\Delta}_{\mathbf{x}_i}$ and $\mathcal{A}_i^{-1} \mathbf{\Delta}_{\mathbf{s}_i}$ respectively.

We first show that d_F is invariant under the primal-dual scaling $(\{\mathbf{y}_i\}, \{\mathbf{w}_i\}) \mapsto (\{\mathcal{A}_i^t \mathbf{y}_i\}, \{\mathcal{A}_i^{-t} \mathbf{w}_i\})$. From Lemma A.2, we have $\mathcal{P}_{\mathcal{A}_i^t \mathbf{y}_i} = \mathcal{A}_i^t \mathcal{P}_{\mathbf{y}_i} \mathcal{A}_i$. We can then follow the method of F. Alizadeh and S.H. Schmieta [1, Proposition 21] to show that $\mathcal{P}_{\mathbf{y}_i^{1/2}} \mathbf{w}_i$ and $\mathcal{P}_{(\mathcal{A}_i^t \mathbf{y}_i)^{1/2}} \mathcal{A}_i^{-1} \mathbf{w}_i$ share the same set of eigenvalues by demonstrating that their quadratic representations are similar to each other. This shows that each summand in d_F, and hence d_F, is invariant under the primal-dual scaling.

From the first two sets of equations (3.5a) and (3.5b) of the linearization, we see that the search directions $(\{\mathbf{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{\Delta}_{\mathbf{s}_i}\})$ are orthogonal under the inner product $\langle\cdot,\cdot\rangle_\omega$. Thus

$$\begin{aligned}\|\{\mathbf{\Delta}_{\mathbf{x}_i}\}\|_\omega^2 + \|\{\mathbf{\Delta}_{\mathbf{s}_i}\}\|_\omega^2 &= \|\{\mathbf{\Delta}_{\mathbf{x}_i} + \mathbf{\Delta}_{\mathbf{s}_i}\}\|_\omega^2 \\ &= \sum_{i=1}^r (\omega_i - \omega_{i-1}) \sum_{j=i}^r \left(\frac{\omega_j - \widetilde{\omega}_j}{\sqrt{\widetilde{\omega}_j \omega_j}}\right)^2 \\ &= \sum_{i=1}^r \omega_i \left(\frac{\omega_i - \widetilde{\omega}_i}{\sqrt{\widetilde{\omega}_i \omega_i}}\right)^2 = \sum_{i=1}^r \frac{(\omega_i - \widetilde{\omega}_i)^2}{\widetilde{\omega}_i}\end{aligned}$$

where we have used the scaled third set of equations (3.5c) in the second equality. Therefore $\|\{\mathbf{\Delta}_{\mathbf{x}_i}\}\|_\omega, \|\{\mathbf{\Delta}_{\mathbf{s}_i}\}\|_\omega \le \sqrt{\sum_{i=1}^r \frac{(\omega_i - \widetilde{\omega}_i)^2}{\widetilde{\omega}_i}}$. For each $i \in \{1, \ldots, r\}$, we have $\|\mathbf{\Delta}_{\mathbf{x}_i}\| = \|(\mathbf{\Delta}_{\mathbf{x}})_{\mathbf{c}_{i:r}}\| \le \|\mathbf{\Delta}_{\mathbf{x}}\| = \|\mathbf{\Delta}_{\mathbf{x}_1}\|$, and hence $\|\mathbf{\Delta}_{\mathbf{x}_i}\| \le \|\mathbf{\Delta}_{\mathbf{x}_1}\| \le \frac{1}{\sqrt{\omega_1}}\|\{\mathbf{\Delta}_{\mathbf{x}_i}\}\|_\omega \le \frac{1}{\sqrt{\omega_1}}\sqrt{\sum_{i=1}^r \frac{(\omega_i - \widetilde{\omega}_i)^2}{\widetilde{\omega}_i}}$.

From the third set of equations (3.5c) of the linearization, we have

$$\begin{aligned}&(\widetilde{\mathbf{x}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}) \circ (\widetilde{\mathbf{s}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}) - \mathbf{c}_{i:r} \\ &= \widetilde{\mathbf{x}}_i \circ \widetilde{\mathbf{s}}_i + \alpha(\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i} \circ \widetilde{\mathbf{s}}_i + \widetilde{\mathbf{x}}_i \circ \mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}) + \alpha^2 \mathbf{\Delta}_{\widetilde{\mathbf{x}}_i} \circ \mathbf{\Delta}_{\widetilde{\mathbf{s}}_i} - \mathbf{c}_{i:r} \\ &= (1-\alpha)(\widetilde{\mathbf{x}}_i \circ \widetilde{\mathbf{s}}_i - \mathbf{c}_{i:r}) + \alpha(\widetilde{\mathbf{x}}_i \circ \widetilde{\mathbf{s}}_i + \mathbf{\Delta}_{\widetilde{\mathbf{x}}_i} \circ \widetilde{\mathbf{s}}_i + \widetilde{\mathbf{x}}_i \circ \mathbf{\Delta}_{\widetilde{\mathbf{s}}_i} - \mathbf{c}_{i:r}) + \alpha^2 \mathbf{\Delta}_{\widetilde{\mathbf{x}}_i} \circ \mathbf{\Delta}_{\widetilde{\mathbf{s}}_i} \\ &= (1-\alpha)(\widetilde{\mathbf{x}}_i \circ \widetilde{\mathbf{s}}_i - \mathbf{c}_{i:r}) + \alpha^2 \mathbf{\Delta}_{\widetilde{\mathbf{x}}_i} \circ \mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}.\end{aligned}$$

Therefore, by Lemma 30 of [1],

$$\begin{aligned}\|\mathcal{P}_{(\widetilde{\mathbf{x}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i})^{1/2}}(\widetilde{\mathbf{s}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}) - \mathbf{c}_{i:r}\| &\le \|(\widetilde{\mathbf{x}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}) \circ (\widetilde{\mathbf{s}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}) - \mathbf{c}_{i:r}\| \\ &\le (1-\alpha)\|\widetilde{\mathbf{x}}_i \circ \widetilde{\mathbf{s}}_i - \mathbf{c}_{i:r}\| + \alpha^2\|\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}\|\|\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}\| \\ &= (1-\alpha)\|\mathcal{P}_{\widetilde{\mathbf{x}}_i^{1/2}}\widetilde{\mathbf{s}}_i - \mathbf{c}_{i:r}\| + \alpha^2\|\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}\|\|\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}\| \\ &= (1-\alpha)\|\mathcal{P}_{\mathbf{x}_i^{1/2}}\mathbf{s}_i - \mathbf{c}_{i:r}\| + \alpha^2\|\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}\|\|\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}\|,\end{aligned}$$

where the first equality follows from the fact that $\widetilde{\mathbf{x}}_i = \widetilde{\mathbf{s}}_i$. Consequently,

$$\begin{aligned}&d_F(\{\mathbf{x}_i + \alpha\mathbf{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \alpha\mathbf{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C})) \\ &= d_F(\{\widetilde{\mathbf{x}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}\}, \{\widetilde{\mathbf{s}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}\}; (\omega, \mathsf{C})) \\ &= \frac{1}{\sqrt{\omega_1}}\sqrt{\sum_{i=1}^r (\omega_i - \omega_{i-1})\|\mathcal{P}_{(\widetilde{\mathbf{x}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i})^{1/2}}(\widetilde{\mathbf{s}}_i + \alpha\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}) - \mathbf{c}_{i:r}\|^2} \\ &\le \frac{1-\alpha}{\sqrt{\omega_1}}\sqrt{\sum_{i=1}^r (\omega_i - \omega_{i-1})\|\mathcal{P}_{\mathbf{x}_i^{1/2}}\mathbf{s}_i - \mathbf{c}_{i:r}\|^2} + \frac{\alpha^2}{\sqrt{\omega_1}}\sqrt{\sum_{i=1}^r (\omega_i - \omega_{i-1})\|\mathbf{\Delta}_{\widetilde{\mathbf{x}}_i}\|^2\|\mathbf{\Delta}_{\widetilde{\mathbf{s}}_i}\|^2} \\ &\le (1-\alpha)d_F(\{\mathbf{x}_i\}, \{\mathbf{s}_i\}; (\omega, \mathsf{C})) + \frac{\alpha^2}{\omega_1}\sum_{i=1}^r \frac{(\omega_i - \widetilde{\omega}_i)^2}{\widetilde{\omega}_i}\end{aligned}$$

proves the lemma. □

With this lemma, we prove the following quadratic convergence result.

Proposition 3. *If next target* $\mathbf{w}^{++} = \sum_{i=1}^r \omega_i \mathbf{c}_i$ *is selected in such a way that the current primal-dual iterates* $(\mathbf{x}^+, \mathbf{s}^+)$ *satisfy*

$$\mathbf{s}^+ = \sum_{i=1}^r \widehat{\delta}_i (\mathbf{x}^+)^{-1}_{\mathbf{c}_{i:r}}$$

for some $\widehat{\delta}_1, \ldots, \widehat{\delta}_r > 0$*, and* $d_F(\mathbf{x}^+, \mathbf{s}^+; \omega) < \frac{\sqrt{5}-1}{2} < 1$*, then taking a full step along the directions* $(\boldsymbol{\Delta}_\mathbf{x}, \boldsymbol{\Delta}_\mathbf{s})$*, determined by the linear system* (3.6) *with* $\mathcal{A} \in \mathsf{G}$ *a* C*-triangular automorphism satisfying* $\mathcal{P}_{\sum_{i=1}^r \sqrt{\omega_i}\mathbf{c}_i} \mathcal{A}^t \mathbf{x}^+ = \mathcal{A}^{-1}\mathbf{s}^+$*, keeps the iterates within the primal-dual strictly feasible region, and*

$$\begin{aligned} d_F(\mathbf{x}^+ + \boldsymbol{\Delta}_\mathbf{x}, \mathbf{s}^+ + \boldsymbol{\Delta}_\mathbf{s}; \omega) &\le d_F(\{\mathbf{x}_i + \boldsymbol{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \boldsymbol{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C})) \\ &\le \frac{d_F(\mathbf{x}^+, \mathbf{s}^+; \omega)^2}{1 - d_F(\mathbf{x}^+, \mathbf{s}^+; \omega)} < 1, \end{aligned}$$

where $\mathbf{x}_i = (\mathbf{x}^+)_{\mathbf{c}_{i:r}}$*,* $\mathbf{s}_i = \mathcal{A}_i\mathcal{A}_i^t\mathbf{x}_i$*,* $\boldsymbol{\Delta}_{\mathbf{x}_i} = (\boldsymbol{\Delta}_\mathbf{x})_{\mathbf{c}_{i:r}}$*,* $\boldsymbol{\Delta}_{\mathbf{s}_i} = \mathbf{x}_i^{-1} - \mathbf{s}_i - \mathcal{A}_i\mathcal{A}_i^t\boldsymbol{\Delta}_{\mathbf{x}_i}$*, and* $\mathcal{A}_i$ *is the restriction of* $\mathcal{A}$ *to* $\mathfrak{J}_{\mathbf{c}_{i:r}}$*.*

Proof. Recall that $\boldsymbol{\Delta}_{\mathbf{x}_i} = (\boldsymbol{\Delta}_\mathbf{x})_{\mathbf{c}_{i:r}}$ and $\boldsymbol{\Delta}_\mathbf{s} = \sum_{i=1}^r (\omega_i - \omega_{i-1})\boldsymbol{\Delta}_{\mathbf{s}_i}$, whence $d_F(\mathbf{x} + \boldsymbol{\Delta}_\mathbf{x}, \mathbf{s} + \boldsymbol{\Delta}_\mathbf{s}; \omega) \le d_F(\{\mathbf{x}_i + \boldsymbol{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \boldsymbol{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C}))$ by Lemma 3. Thus it suffices to show that $\mathbf{x}_i + \alpha\boldsymbol{\Delta}_{\mathbf{x}_i}, \mathbf{s}_i + \alpha\boldsymbol{\Delta}_{\mathbf{s}_i} \in \Omega_i$ for $\alpha \in [0, 1]$, and that

$$d_F(\{\mathbf{x}_i + \boldsymbol{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \boldsymbol{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C})) \le \frac{d_F(\mathbf{x}, \mathbf{s}; \omega)^2}{1 - d_F(\mathbf{x}, \mathbf{s}; \omega)}.$$

By Lemma 3, if $\mathbf{x}_i + \alpha\boldsymbol{\Delta}_{\mathbf{x}_i} \in cl(\Omega_i)\backslash\Omega_i$ for any $\alpha \in [0, 1]$, then $d_F(\{\mathbf{x}_i + \alpha\boldsymbol{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \alpha\boldsymbol{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C})) \ge 1$. Assuming $\gamma := d_F(\mathbf{x}, \mathbf{s}; \omega) < 1$, the previous lemma states that

$$d_F(\{\mathbf{x}_i + \alpha\boldsymbol{\Delta}_{\mathbf{x}_i}\}, \{\mathbf{s}_i + \alpha\boldsymbol{\Delta}_{\mathbf{s}_i}\}; (\omega, \mathsf{C})) \le (1 - \alpha)\gamma + \alpha^2 \frac{\gamma^2}{1 - \gamma} < 1,$$

for all $\alpha \in [0, 1]$, whence $\mathbf{x}_i + \alpha\boldsymbol{\Delta}_{\mathbf{x}_i} \in \Omega_i$ for all $\alpha \in [0, 1]$ by the continuity of $\alpha \mapsto \mathbf{x}_i + \alpha\boldsymbol{\Delta}_{\mathbf{x}_i}$. We then apply Lemma 3 once again, together with the above inequality, to conclude that $\mathbf{s}_i + \alpha\boldsymbol{\Delta}_{\mathbf{s}_i} \in \Omega_i$ for all $\alpha \in [0, 1]$. □

3.2. **Choice of targets.** The analysis in the preceding section requires the assumption that the next target $\mathbf{w}^{++}$ is selected in such a way that the current primal-dual iterates $(\mathbf{x}^+, \mathbf{s}^+)$ satisfy

$$\mathbf{s}^+ = \sum_{i=1}^r \widehat{\delta}_i (\mathbf{x}^+)^{-1}_{\mathbf{c}_{i:r}}$$

for some $\widehat{\delta}_1, \ldots, \widehat{\delta}_r > 0$. This, in general, decides the choice of the complete flag f. Thus, we now only need to decide the values of the weights ω; see Remark 2. In light of the proximity measure $d_F(\cdot, \cdot; \omega)$, we shall use the following proximity measure on the set of weights:

$$d_F(\widetilde{\omega}; \omega) \stackrel{\text{def}}{=} \sqrt{\frac{1}{\omega_1} \sum_{i=1}^r \frac{(\widetilde{\omega}_i - \omega_i)^2}{\omega_i}}, \tag{3.9}$$

With this choice of targets and proximity measure, the target-following framework is now specialized to the following:

Algorithm 2. (Target-following algorithm for symmetric cone programming)
Given a pair of primal-dual strictly feasible solutions $(\mathbf{x}^{\text{in}}, \mathbf{s}^{\text{in}})$ *and target weights* ω^{out}*.*

(1) *Pick some* $\delta \in (0, 1)$ *and a sequence of weights* $\{\omega^k\}_{k=0}^N$ *such that*

$$\mathbf{s}^+ = \sum_{i=1}^r (\omega_i^{(0)} - \omega_{i-1}^{(0)})(\mathbf{x}^+)^{-1}_{\mathbf{c}_i + \cdots + \mathbf{c}_r}$$

for some Jordan frame $(\mathbf{c}_1, \ldots, \mathbf{c}_r)$*,* $d_F(\omega^k; \omega^{k-1}) \le \delta$ *for* $k = 1, \ldots, N$*, and* $\omega^N = \omega^{\text{out}}$*.*

(2) *Set* $(\mathbf{x}^+, \mathbf{s}^+) = (\mathbf{x}^{\text{in}}, \mathbf{s}^{\text{in}})$*.*
(3) *For* $k = 1, \ldots, N$*,*

(a) *Solve the linear system* (3.6) *with* $(\mathbf{c}_1, \ldots, \mathbf{c}_r)$ *a Jordan frame from Theorem* 6 *for* $(\mathbf{x}^+, \mathbf{s}^+)$*, with* $(\omega_1, \ldots, \omega_r)$ *the weights in* ω^k*, and with* $\mathcal{A} \in \mathsf{G}$ *the automorphism from Lemma* 2*.*

(b) *Update* $(\mathbf{x}^+, \mathbf{s}^+) \leftarrow (\mathbf{x}^+ + \mathbf{\Delta_x}, \mathbf{s}^+ + \mathbf{\Delta_s})$*.*

(4) *Output* $(\mathbf{x}^{\text{out}}, \mathbf{s}^{\text{out}}) = (\mathbf{x}^+, \mathbf{s}^+)$*.*

3.3. **Analysis of algorithm.** Consider one iteration of Algorithm 2. Recall from Proposition 3 that we can take a full step when $d_F(\mathbf{x}^+, \mathbf{s}^+) < (\sqrt{5}-1)/2$. This can be enforced during the update of weights via the following lemma.

Lemma 5. *If* $d_F(\mathbf{x}, \mathbf{s}; \omega) \le \beta$ *and* $d_F(\widetilde{\omega}; \omega) \le \alpha$ *for some* $\alpha, \beta \in (0,1)$*, then*

$$d_F(\mathbf{x}, \mathbf{s}; \widetilde{\omega}) \le \frac{\beta + \alpha}{1 - \alpha}.$$

Proof. We have

$$\begin{aligned} d_F(\mathbf{x}, \mathbf{s}; \widetilde{\omega}) &\le \sqrt{\frac{1}{\widetilde{\omega}_1} \sum_{i=1}^r \frac{(\lambda_i(\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s}) - \omega_i)^2}{\widetilde{\omega}_i}} + \sqrt{\frac{1}{\widetilde{\omega}_1} \sum_{i=1}^r \frac{(\omega_i - \widetilde{\omega}_i)^2}{\widetilde{\omega}_i}} \\ &\le (d_F(\mathbf{x}, \mathbf{s}; \omega) + d_F(\widetilde{\omega}; \omega)) \max_i \frac{\omega_i}{\widetilde{\omega}_i}. \end{aligned}$$

If $d_F(\widetilde{\omega}; \omega) \le \alpha$, then

$$\alpha^2 \ge \frac{1}{\omega_1} \sum_{i=1}^r \frac{(\widetilde{\omega}_i - \omega_i)^2}{\omega_i} \ge \sum_{i=1}^r \left(\frac{\widetilde{\omega}_i}{\omega_i} - 1 \right)^2,$$

whence $\min_i \frac{\widetilde{\omega}_i}{\omega_i} \ge 1 - \alpha$. □

We now give the main theorem of this section, which states that for all $\alpha > 0$ sufficiently small, say $\alpha \le \frac{1}{6}$, then Algorithm 2 terminates with a good approximation of $\mathcal{T}^{-1}(\sum_{i=1}^r \omega_i^{\text{out}} \mathbf{c}_i)$ for some Jordan frame $\mathsf{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_r)$.

Theorem 9. *In Algorithm* 2*, if* $\alpha \in (0,1)$ *is such that there exists some* $\beta \in (0, \frac{\sqrt{5}-1}{2})$ *satisfying*

$$\frac{(\beta + \alpha)^2}{(1-\beta)(1-\alpha)} \le \beta, \tag{3.10}$$

then $(\mathbf{x}^+, \mathbf{s}^+)$ *is well-defined and strictly feasible in each iteration, and the algorithm terminates with* $d_F(\mathbf{x}^{\text{out}}, \mathbf{s}^{\text{out}}; \omega^{\text{out}}) \le \beta_{\min}$*, where* $\beta_{\min}$ *is the least* β *satisfying the inequality.*

Proof. We shall prove the theorem by induction that the iterates $(\mathbf{x}^+, \mathbf{s}^+)$ are strictly feasible and $d_F(\mathbf{x}^+, \mathbf{s}^+; \omega^k) \le \beta_{\min}$ at the beginning of each iteration. This is certainly true for the first iteration. By Lemma 5, we have $d_F(\mathbf{x}^+, \mathbf{s}^+; \omega^{k+1}) \le (\beta_{\min} + \alpha)/(1-\alpha)$. If the hypothesis (3.10) holds, then we may apply Proposition 3 to deduce that the iterates $(\mathbf{x}^+ + \mathbf{\Delta_x}, \mathbf{s}^+ + \mathbf{\Delta_s})$ are strictly feasible with

$$d_F(\mathbf{x}^+ + \mathbf{\Delta_x}, \mathbf{s}^+ + \mathbf{\Delta_s}) \le \frac{((\beta_{\min} + \alpha)/(1-\alpha))^2}{1 - (\beta_{\min} + \alpha)/(1-\alpha)} = \frac{(\beta_{\min} + \alpha)^2}{(1-\beta_{\min})(1-\alpha)} \le \beta_{\min}.$$

This completes the induction. □

4. Finding analytic centers

In this section, we consider an algorithm that finds the analytic center $\mathcal{T}^{-1}(\widehat{\mu}\mathbf{e})$ for any given $\widehat{\mu} > 0$. This algorithm can be used to find analytic centers of compact sets described by linear matrix inequalities and convex quadratic constraints. It can also be combined with a path-following algorithm to solve the symmetric cone program (1.1).

Given a pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$, we shall construct a finite sequence of targets $\{\omega^k\}_{k=0}^N$ such that

$$\mathbf{s}^+ = \sum_{i=1}^{r} (\omega_i^{(0)} - \omega_{i-1}^{(0)}) (\mathbf{x}^+)^{-1}_{\mathbf{c}_i + \cdots + \mathbf{c}_r}$$

for some Jordan frame $(\mathbf{c}_1, \ldots, \mathbf{c}_r)$,

$$d_F(\omega^k; \omega^{k-1}) = \sqrt{\frac{1}{\omega_1^{k-1}} \sum_{i=1}^{r} \frac{(\omega_i^k - \omega_i^{k-1})^2}{\omega_i^{k-1}}} \leq \alpha \quad \text{for } 1 \leq k \leq N, \tag{4.1}$$

and $\omega^N = \widehat{\mu}\mathbf{1}$, with α satisfying the hypothesis of Theorem 9, thus allowing us to apply Algorithm 2 to approximate $\mathcal{T}^{-1}(\widehat{\mu}\mathbf{e})$.

Any sequence $\{\omega^k\}_{k=0}^N$ satisfying (4.1) is called a α-*sequence*, and N is called its *length*; see [15]. In [3], the author gave an upper bound on the length of a shortest α-sequence from any weight sequence ω^0 to the ray $\{\mu\mathbf{1} : \mu > 0\}$. For the sake of completeness, we repeat the argument here.

Consider the local metric defined by the inner product

$$\langle \cdot, \cdot \rangle_\omega : (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^r \times \mathbb{R}^r \mapsto \frac{1}{\omega_n} \sum_{i=1}^{r} \frac{\mathbf{u}_i \mathbf{v}_i}{\omega_i}$$

at each weight sequence ω. We denote by $\|\cdot\|_\omega$ the norm induced by the above inner product. In terms of this local metric, an α-sequence $\{\omega^k\}_{k=0}^N$ is one that satisfies

$$\left\|\omega^k - \omega^{k-1}\right\|_{\omega^{k-1}} \leq \alpha \quad \text{for } 1 \leq k \leq N.$$

The length of a piecewise smooth curve $\xi : [0,1] \to \mathsf{W}$, where $\mathsf{W} \subset \mathbb{R}^r_{++}$ denotes the set of weight sequences, is defined to be

$$\int_0^1 \left\| \frac{d\xi(t)}{dt} \right\|_{\xi(t)} dt = \int_0^1 \sqrt{\frac{1}{\xi_n} \sum_{i=1}^{n} \frac{\dot{\xi}_i^2}{\xi_i}} dt,$$

and denoted by $l(\xi)$. The next lemma gives an upper bound on a shortest α-sequence between any two weight sequences in terms of the length of a piecewise smooth curve joining them. Its proof can be obtained by adapting the proof of a similar result in [20], and is thus omitted here.

Lemma 6 (c.f. Lemma 3.3 of [20])**.** *For every piecewise smooth curve $\xi : [0,1] \to \mathsf{W}$ and every $\alpha \in (0,1)$, there exists an α-sequence $\{\omega^k\}_{k=0}^N$ with $\omega^0 = \xi(0)$, $\omega^1 = \xi(1)$ and length*

$$N \leq \left\lceil \frac{l(\xi)}{\alpha - \frac{1}{2}\alpha^2} \right\rceil.$$

Next we show the existence of a piecewise smooth curve from a given weight sequence ω to the ray $\{\mu\mathbf{1} : \mu > 0\}$ with length $O(\log(\frac{1}{r\omega_1} \sum_{i=1}^r \omega_i))$.

Lemma 7. *For each weight sequence ω, there exists a piecewise smooth curve $\xi : [0,1] \mapsto \mathsf{W}$ with $\xi(0) = \omega$, $\xi(1) = \mu\mathbf{1}$, and length*

$$l(\xi) \leq \sqrt{r} \log\left(\frac{4\mu}{\omega_1}\right),$$

where μ denotes the average weight $\frac{1}{r}\sum_{i=1}^r \omega_i$.

Proof. The lemma is trivially true when $\omega = \mu\mathbf{1}$. Otherwise, $\omega_1 = \cdots = \omega_p < \omega_{p+1} \leq \cdots \leq \omega_r$ for some $p \in \{1, \ldots, r-1\}$.

Consider the straight line segment $\widetilde{\xi} : [0, \bar{t}] \mapsto \mathsf{W}$ starting from ω, along which the weights of least value increases at the same rate, with the other weights decreasing at rates proportional to their values, while maintain the average weight throughout, and ending when the weights of least value coincide with the next higher value; i.e., $\widetilde{\xi}$ is defined by

(4.2)
$$\widetilde{\xi}(t)_1 = \cdots = \widetilde{\xi}(t)_p = \omega_1 + t\frac{r\mu - p\omega_1}{p} \quad \text{and} \quad \widetilde{\xi}(t)_i = \omega_i - t\omega_i \quad \text{for } i = p+1,\ldots,r,$$

where $\bar{t} \in (0,1)$ is such that $\widetilde{\xi}(\bar{t})_1 = \cdots = \widetilde{\xi}(\bar{t})_p = \widetilde{\xi}(\bar{t})_{p+1}$; as required $\widetilde{\xi}(t)_1 + \cdots + \widetilde{\xi}(t)_r = r\mu$ is independent of t. Its length is

$$\begin{aligned}
&\int_0^{\bar{t}} \left(\frac{1}{\omega_1 + t\frac{r\mu-p\omega_1}{p}} \left(\sum_{i=1}^{p} \frac{(\frac{r\mu-p\omega_1}{p})^2}{\omega_1 + t\frac{r\mu-p\omega_1}{p}} + \sum_{i=p+1}^{r} \frac{\omega_i^2}{\omega_i - t\omega_i} \right)\right)^{1/2} dt \\
&= \int_0^{\bar{t}} \left(\frac{\frac{p}{r\mu-p\omega_1}}{\frac{p\omega_1}{r\mu-p\omega_1} + t} \left(\frac{r\mu - p\omega_1}{\frac{p\omega_1}{r\mu-p\omega_1} + t} + \sum_{i=p+1}^{r} \frac{\omega_i}{1-t} \right)\right)^{1/2} dt \\
&= \sqrt{p} \int_0^{\bar{t}} \left(\frac{1}{\frac{p\omega_1}{r\mu-p\omega_1} + t} \left(\frac{1}{\frac{p\omega_1}{r\mu-p\omega_1} + t} + \frac{1}{1-t} \right)\right)^{1/2} dt \\
&= \sqrt{p} \int_0^{\bar{t}} \frac{\sqrt{\frac{r\mu}{r\mu-p\omega_1}}}{(\frac{p\omega_1}{r\mu-p\omega_1} + t)\sqrt{1-t}}\, dt \\
&= \sqrt{p} \log \frac{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} + 1}{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} - 1} - \sqrt{p} \log \frac{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} + \sqrt{1-\bar{t}}}{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} - \sqrt{1-\bar{t}}}.
\end{aligned}$$

From the definition of $\widetilde{\xi}(t)_1$, we can simplify

$$\frac{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} + \sqrt{1-\bar{t}}}{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} - \sqrt{1-\bar{t}}} = \frac{1 + \sqrt{(1-\bar{t})\frac{r\mu-p\omega_1}{r\mu}}}{1 - \sqrt{(1-\bar{t})\frac{r\mu-p\omega_1}{r\mu}}} = \frac{1 + \sqrt{\frac{r\mu-p\widetilde{\xi}(\bar{t})_1}{r\mu}}}{1 - \sqrt{\frac{r\mu-p\widetilde{\xi}(\bar{t})_1}{r\mu}}} = R(\tfrac{p\widetilde{\xi}(\bar{t})_1}{r\mu}),$$

and

$$\frac{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} + 1}{\sqrt{\frac{r\mu}{r\mu-p\omega_1}} - 1} = \frac{1 + \sqrt{\frac{r\mu-p\omega_1}{r\mu}}}{1 - \sqrt{\frac{r\mu-p\omega_1}{r\mu}}} = R(\tfrac{p\omega_1}{r\mu}) = R(\tfrac{p\widetilde{\xi}(0)_1}{r\mu}),$$

where $R : (0,1] \to [1,\infty)$ is the decreasing function $u \mapsto (1+\sqrt{1-u})/(1-\sqrt{1-u})$ satisfying $R(u) \le (1+1)/(1-(1-\frac{1}{2}u)) = 4/u$. This gives

$$l(\widetilde{\xi}) = \sqrt{p} \log R(\tfrac{p\widetilde{\xi}(0)_1}{r\mu}) - \sqrt{p} \log R(\tfrac{p\widetilde{\xi}(\bar{t})_1}{r\mu}).$$

As long as $\widetilde{\xi}(\bar{t}) \ne \mu\mathbf{1}$, we repeat this process to construct another straight line segment starting from $\widetilde{\xi}(\bar{t})$. Eventually, we get a piecewise linear curve ξ joining ω to $\mu\mathbf{1}$ with $q \le r$ straight line segments, and total length

$$l(\xi) = \sum_{i=1}^{q} \left(\sqrt{p_i} \log R(\tfrac{p_i\omega_1^i}{r\mu}) - \sqrt{p_i} \log R(\tfrac{p_i\omega_1^{i+1}}{r\mu}) \right),$$

where ω^i is the weight sequence at the start of the i'th straight segment, ω^{q+1} denotes the weights $\mu\mathbf{1}$, and p_i is the number of weights of least value in ω^i. We claim that for any $a > 1$, the function $u \in (0, 1/a] \mapsto \log R(u) - \log R(au)$ is increasing[3]. Thus, since both $\{\omega_1^i\}_{i=1}^q$ increasing by construction, we have the upper bound

$$\begin{aligned} l(\xi) &< \sum_{i=1}^{q} \sqrt{p_i} \left(\log R(\tfrac{r\omega_1^i}{r\mu}) - \log R(\tfrac{r\omega_1^{i+1}}{r\mu}) \right) \\ &\leq \sqrt{r} \sum_{i=1}^{q} \left(\log R(\tfrac{\omega_1^i}{\mu}) - \log R(\tfrac{\omega_1^{i+1}}{\mu}) \right) \\ &= \sqrt{r}(\log R(\tfrac{\omega_1}{\mu}) - \log R(1)) \leq \sqrt{r} \log \tfrac{4\mu}{\omega_1}. \end{aligned}$$

□

From the above two lemmas, we deduce the following upper bound on the length of a shortest α-sequence from a given weight sequence ω to the ray $\{\mu\mathbf{1} : \mu > 0\}$.

Theorem 10. *For every weight sequence ω^0 and every $\alpha \in (0,1)$, there exists an α-sequence $\{\omega^k\}_{k=0}^N$ with $\omega^N = \mu\mathbf{1}$, where $\mu = \sum_{i=1}^r \omega_i^0/r$, and length*

$$N \leq \left\lceil \frac{\sqrt{r}}{\alpha - \frac{1}{2}\alpha^2} \log \left(\frac{4\mu}{\omega_1^0} \right) \right\rceil .$$

Corollary 1. *Suppose $\beta \in (0, \frac{\sqrt{5}-1}{2})$ is fixed. Let $\alpha \in (0,1)$ be a number satisfying the inequality (3.10) in Theorem 9. Given any pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ for the primal-dual symmetric cone programming problems (1.1), there is a sequence of at most*

$$N \leq \left\lceil \frac{\sqrt{r}}{\alpha - \frac{1}{2}\alpha^2} \log \left(\frac{4 \langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\lambda_1(\mathcal{P}_{\widehat{\mathbf{x}}^{1/2}}\widehat{\mathbf{s}})} \right) \right\rceil .$$

weights such that Algorithm 2 finds a pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ satisfying $\|\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} - \mu\mathbf{e}\| = \mu d_F(\mathbf{x}, \mathbf{s}; \mu\mathbf{1}) \leq \beta\mu$, where $\mu = \frac{1}{r}\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle$.

Combining the corollary with an α-sequence on the central path, we have the following theorem.

Theorem 11. *Suppose $\beta \in (0,1)$ is fixed. Given any pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ for the primal-dual symmetric cone programming problems (1.1), and any positive real number $\widehat{\mu}$, there is a sequence of at most*

$$O\left(\sqrt{r} \left(\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\lambda_1(\mathcal{P}_{\widehat{\mathbf{x}}^{1/2}}\widehat{\mathbf{s}})} + \left| \log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\widehat{\mu}} \right| \right) \right)$$

weights such that Algorithm 2 finds a pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ satisfying $\|\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} - \widehat{\mu}\mathbf{e}\| = \widehat{\mu} d_F(\mathbf{x}, \mathbf{s}; \widehat{\mu}\mathbf{1}) \leq \beta\widehat{\mu}$.

As an immediate corollary, we have the following worst-case iteration bound on solving symmetric cone problems using Algorithm 2.

Corollary 2. *Given any pair of primal-dual strictly feasible solutions $(\widehat{\mathbf{x}}, \widehat{\mathbf{s}})$ and any $\varepsilon > 0$, there is a sequence of at most*

$$O\left(\sqrt{r} \left(\log \frac{\langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle}{r\lambda_r(\mathcal{P}_{\widehat{\mathbf{x}}^{1/2}}\widehat{\mathbf{s}})} + \left| \log \varepsilon^{-1} \right| \right) \right)$$

targets such that Algorithm 2 find a pair of primal-dual strictly feasible solutions $(\mathbf{x}, \mathbf{s})$ satisfying $\langle \mathbf{x}, \mathbf{s} \rangle \leq \varepsilon \langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle$.

Proof. If $(\mathbf{x}, \mathbf{s}) \in \Omega^2$ satisfies $\|\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} - \mu\mathbf{e}\| \leq \beta\mu$ for some $\beta \in (0,1)$ and some $\mu > 0$, then $\langle \mathbf{x}, \mathbf{s} \rangle - r\mu = \langle \mathbf{e}, \mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{s} - \mu\mathbf{e} \rangle \leq \sqrt{r}\beta\mu$. Apply the preceding theorem with $\widehat{\mu} = \varepsilon \langle \widehat{\mathbf{x}}, \widehat{\mathbf{s}} \rangle / (\beta\sqrt{r} + r)$. □

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[3] We have $\frac{R(u)}{R(au)} = a\left(\frac{1+\sqrt{1-u}}{1+\sqrt{1-au}}\right)^2$, and it is straightforward to check that $u \mapsto \frac{1+\sqrt{1-u}}{1+\sqrt{1-au}}$ is increasing when $a > 1$.

5. Conclusion

We extend the target map $(\mathbf{x}, \mathbf{s}) \mapsto (\mathbf{x}_1\mathbf{s}_1, \ldots, \mathbf{x}_n\mathbf{s}_n)$, together with the weighted barriers $\mathbf{x} \mapsto -\sum_{i=1}^n \omega_i \log \mathbf{x}_i$ and the notions of weighted analytic centers, from linear programming to general convex conic programming. This extension is obtained from a geometrical perspective of the weighted barriers, via the facial structure of the nonnegative orthant, that views a weighted barrier as a weighted sum of barriers for a strictly decreasing sequence of faces. When we replace decreasing sequences of faces of the nonnegative orthant with decreasing sequences of faces of an arbitrary closed convex cone, we arrive at weighted barriers for the convex cone; provided that we have made a priori choices of barriers for all faces of the convex cone. This potentially opens the door to efficient target-following algorithms for general convex conic programming, once we know how to design and analyze efficient primal-dual algorithms for general convex conic programming.

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[section]

Appendix A. Euclidean Jordan algebras

In this section, we give a very brief introduction to Euclidean Jordan algebras, stating various known results and proving some new ones that are necessary for the development of this paper. For a comprehensive discussion on symmetric cones and Jordan algebras, we refer the reader to the excellent exposition by J. Faraut and A. Korányi [7].

A *Jordan algebra* $(\mathfrak{J}, \circ, +)$ is a commutative algebra whose multiplication operator $\circ$ satisfies the *Jordan identity* $\mathbf{x} \circ (\mathbf{x}^2 \circ \mathbf{y}) = \mathbf{x}^2 \circ (\mathbf{x} \circ \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{J}$, where $\mathbf{x}^2$ denotes $\mathbf{x} \circ \mathbf{x}$. The multiplication operator $\circ$ is often called the *Jordan product* of the Jordan algebra. We use $\mathcal{L}_{\mathbf{x}}$ to denote the *Lyapunov operator* $\mathbf{y} \in \mathfrak{J} \mapsto \mathbf{x} \circ \mathbf{y}$.

A Jordan algebra $(\mathfrak{J}, \circ)$ is said to be *formally real* if

$$(\mathbf{x}^2 + \mathbf{y}^2 = \mathbf{0} \implies \mathbf{x} = \mathbf{y} = \mathbf{0}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathfrak{J}.$$

A formally real Jordan algebra has a *identity element* $\mathbf{e}$: an element that satisfies $\mathbf{e} \circ \mathbf{x} = \mathbf{x}$; see [13]. It is also noted in [13] that a formally real Jordan algebra is *power associative*[4]: if we recursively define, for each element $\mathbf{x} \in \mathfrak{J}$, the *k'th power* by $\mathbf{x}^0 = \mathbf{e}$ and $\mathbf{x}^k = \mathbf{x}^{k-1} \circ \mathbf{x}$ for $k = 1, 2, \ldots$, then $\mathbf{x}^{k+l} = \mathbf{x}^k \circ \mathbf{x}^l$ for all nonnegative integers k and l (i.e., the collection of all powers of an element $\mathbf{x}$ forms a semigroup). This important fact results in the existence of the *minimal polynomial* for each element $\mathbf{x} \in \mathfrak{J}$: the monic polynomial in $\mathbb{R}[X]$ that generates the principle ideal $\{p \in \mathbb{R}[X] : p(\mathbf{x}) = 0\}$. The maximum degree of all minimal polynomials is called the *rank* of the Jordan algebra. An element $\mathbf{x} \in \mathfrak{J}$ is said to be *regular* if the degree of its minimal polynomial coincides with the rank of $\mathfrak{J}$.

Henceforth, r shall denote the rank of $\mathfrak{J}$.

The minimal polynomials also give us two important functions. For a regular element $\mathbf{x}$, its *trace* is the coefficient of the second highest power in its minimal polynomial, and its *determinant* is the constant term in its minimal polynomial. As the set of regular elements is dense in $\mathfrak{J}$, and both functions are continuously extendable to $\mathfrak{J}$ as polynomials, we can define the trace and determinant of all elements of $\mathfrak{J}$; see [7, Proposition II.2.1].

[4] This is actually true in general for all Jordan algebras; see [7, Proposition II.1.2].

Example A.1. *The space of $r \times r$ real symmetric matrices $\mathbb{S}^r$ equipped with the symmetrized product $\frac{1}{2}(\mathbf{AB} + \mathbf{BA})$ is a formally real Jordan algebra with identity $\mathbf{I}$. The notions of minimal polynomial, trace and determinant are as we commonly defined.*

It is known (see, e.g., [7, Proposition VIII.4.2]) that a Jordan algebra with identity is formally real if and only if it is a *Euclidean Jordan algebra*; i.e., there is a symmetric positive definite bilinear functional $B : \mathfrak{J}^2 \mapsto \mathbb{R}$ that is *associative*; i.e., $B(\mathbf{x} \circ \mathbf{y}, \mathbf{z}) = B(\mathbf{y}, \mathbf{x} \circ \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathfrak{J}$. Equivalently, a Euclidean Jordan algebra is a Jordan algebra with identity such that the bilinear function $(\mathbf{x}, \mathbf{y}) \mapsto \mathrm{tr}(\mathbf{x} \circ \mathbf{y})$ is positive definite; see [7, Proposition III.1.5]. Thus a Euclidean Jordan algebra can be given a Euclidean structure with the inner product $\langle \cdot, \cdot \rangle : (\mathbf{x}, \mathbf{y}) \in \mathfrak{J}^2 \mapsto \mathrm{tr}(\mathbf{x} \circ \mathbf{y})$ in such a way that $\mathcal{L}_{\mathbf{x}}$ is self-adjoint.

It is known that if $(\mathfrak{J}, \circ)$ is a Euclidean Jordan algebra, then the interior of its cone of squares $\{\mathbf{x} \circ \mathbf{x} : \mathbf{x} \in \mathfrak{J}\}$ is a symmetric cone in the Euclidean space $(\mathfrak{J}, \langle \cdot, \cdot \rangle)$. *We denote this interior by* $\Omega(\mathfrak{J})$. Moreover, this symmetric cone coincides with the set of elements with positive definite Lyapunov operators; see [7, Theorem III.2.1]. A key ingredient in showing the homogeneity of the cone of squares is the *quadratic representation* of $\mathfrak{J}$: $\mathcal{P}_{\mathbf{x}} : \mathbf{x} \in \mathfrak{J} \mapsto 2\mathcal{L}_{\mathbf{x}}^2 - \mathcal{L}_{\mathbf{x}}$. The collection of quadratic representations at all $\mathbf{x} \in \Omega(\mathfrak{J})$ gives a transitive subset of automorphisms of $\Omega(\mathfrak{J})$. In particular, to each $\mathbf{x} \in \Omega(\mathfrak{J})$ is associated a unique $\mathbf{y} \in \Omega(\mathfrak{J})$ such that $\mathcal{P}_{\mathbf{y}}\mathbf{e} = \mathbf{x}$. We denote such $\mathbf{y}$ by $\mathbf{x}^{1/2}$, and called it the *square root of* $\mathbf{x}$.

Conversely, given any symmetric cone Ω, there is an Euclidean Jordan-algebraic structure such that the symmetric cone coincides with the interior of the cone of squares. Moreover the closure $cl(\Omega)$ of the symmetric cone coincides with the cone of squares; see [7, Theorem III.3.1].

Alternatively, the symmetric cone $\Omega(\mathfrak{J})$ can be defined as the connected component of the set of invertible elements containing the identity element; see [7, Theorem III.2.1]. An element $\mathbf{x}$ is said to be *invertible* if there exists a linear combination of powers of $\mathbf{x}$ whose Jordan product with $\mathbf{x}$ is the identity element. This linear combination of powers is called the *inverse of* $\mathbf{x}$, and denoted by $\mathbf{x}^{-1}$. It is unique since the subalgebra generated by $\mathbf{x}$ and the identity element $\mathbf{e}$ is associative. An element $\mathbf{x}$ is invertible if and only if its quadratic representation is nonsingular, and in this case, $\mathcal{P}_{\mathbf{x}}^{-1} = \mathcal{P}_{\mathbf{x}^{-1}}$; see [7, Theorem II.3.1].

Example A.2. *For the Jordan algebra of $r \times r$ real symmetric matrices, the cone of squares is the cone of all positive semidefinite matrices. The quadratic representation of a matrix $\mathbf{X}$ is $\mathbf{Y} \mapsto \mathbf{XYX}$. The notions of square root and inverse are as we commonly defined.*

A.1. **Spectral decompositions.** A key ingredient in the study of formally real Jordan algebra by P. Jordan et al. [13] is the set of *idempotents*. An idempotent of $\mathfrak{J}$ is a nonzero element $\mathbf{c} \in \mathfrak{J}$ satisfying $\mathbf{c} \circ \mathbf{c} = \mathbf{c}$. Amongst the idempotents are the *primitive* ones: idempotents that cannot be written as the sum of two idempotents. Two idempotents $\mathbf{c}$ and $\mathbf{d}$ are said to be *orthogonal* if $\mathbf{c} \circ \mathbf{d} = \mathbf{0}$. Orthogonal idempotents are indeed orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ since

$$\langle \mathbf{c}, \mathbf{d} \rangle = \langle \mathbf{c} \circ \mathbf{e}, \mathbf{d} \rangle = \langle \mathbf{e}, \mathbf{c} \circ \mathbf{d} \rangle .$$

From its definition, it is straightforward to check that the sum of two orthogonal idempotents is an idempotent, and that an element $\mathbf{c}$ is an idempotent if and only if the element $\mathbf{e} - \mathbf{c}$ is an idempotent. A *complete system of orthogonal idempotents* is a set of idempotents that are pair-wise orthogonal and sum to the identity element $\mathbf{e}$. A *Jordan frame* is a complete system of orthogonal primitive idempotents. The number of elements in any Jordan frame always coincide with the rank of $\mathfrak{J}$; see paragraph immediately after Theorem III.1.2 of [7].

Example A.3. *For the Jordan algebra of $r \times r$ real symmetric matrices, an idempotent is a product $\mathbf{P}\mathbf{P}^T$ where the matrix $\mathbf{P}$ has orthogonal columns of unit length. It is primitive if and only if it is of rank one. A complete system of orthogonal idempotents is a p-tuple $(\mathbf{P}_1\mathbf{P}_1^T, \ldots, \mathbf{P}_p\mathbf{P}_p^T)$ with the columns of $\mathbf{P}_1, \ldots, \mathbf{P}_p$ taken from the columns of an orthogonal matrix. A Jordan frame C is then a complete system of r orthogonal idempotents with each $\mathbf{P}_i$ a column-matrix.*

For each $\mathbf{x} \in \mathfrak{J}$, there exists numbers $\lambda_1 \leq \cdots \leq \lambda_r$ and a Jordan frame $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ such that $\mathbf{x} = \lambda_1\mathbf{c}_1 + \cdots + \lambda_r\mathbf{c}_r$. This is known as a *spectral decomposition of type II* of $\mathbf{x}$; see Theorem III.1.2 of [7]. Moreover, the set of values of λ_i's (with their multiplicities) remain unchanged over all such Jordan frames.

When the primitive idempotents corresponding to the same eigenvalues are combined, we have the *spectral decomposition of type I*: $\mathbf{x} = \mu_1\widetilde{\mathbf{c}}_1 + \cdots + \mu_k\widetilde{\mathbf{c}}_k$, where $\mu_1 < \cdots < \mu_k$ are the distinct eigenvalues of $\mathbf{x}$, and $\widetilde{\mathbf{c}}_i$ is the sum of the primitive idempotents corresponding to the eigenvalue μ_i; see also Theorems III.1.1 [7]. This spectral decomposition is unique.

The values λ_i in a type II spectral decomposition are called the *eigenvalues* of $\mathbf{x}$, and are denoted by $\lambda_i(\mathbf{x})$, with $\lambda_1(\mathbf{x}) \leq \cdots \leq \lambda_r(\mathbf{x})$. In terms of the spectral decompositions, the inverse of an invertible element $\mathbf{x}$ is the element $\mathbf{x}^{-1} = \lambda_1(\mathbf{x})^{-1}\mathbf{c}_1 + \cdots + \lambda_r(\mathbf{x})^{-1}\mathbf{c}_r$, and the square root of an element $\mathbf{x}$ in the symmetric cone $\Omega(\mathfrak{J})$ is the element $\mathbf{x}^{1/2} = \lambda_1(\mathbf{x})^{1/2}\mathbf{c}_1 + \cdots + \lambda_r(\mathbf{x})^{1/2}\mathbf{c}_r$.

For an element $\mathbf{x}$ with the type I spectral decomposition $\mathbf{x} = \mu_1\mathbf{c}_1 + \cdots + \mu_k\mathbf{c}_k$, the orthogonality of the idempotents implies that a polynomial $p \in \mathbb{R}[X]$ is in the principle ideal generated by the minimal polynomial if and only if $p(\mu_1) = \cdots = p(\mu_k) = 0$. Thus the minimal polynomial of an element $\mathbf{x}$ is $t \mapsto (t - \mu_1)\cdots(t-\mu_k)$. Consequently, an element is regular if and only if it has distinct eigenvalues. Moreover, the trace of $\mathbf{x}$ is the sum $\lambda_1(\mathbf{x}) + \cdots + \lambda_r(\mathbf{x})$, and its determinant is the product $\lambda_1(\mathbf{x}) \times \cdots \times \lambda_r(\mathbf{x})$. The norm of an element $\mathbf{x}$ is then $\sqrt{\lambda_1(\mathbf{x})^2 + \cdots + \lambda_r(\mathbf{x})^2}$. In particular, the square of the norm of an idempotent is the number of pairwise orthogonal primitive idempotents summing up to it.

The logarithm of the determinant plays an important role in interior-point methods for symmetric cone programming: its negation serves as a barrier (called the *log-determinant barrier*) for the symmetric cone Ω. We note that the gradient of this log-determinant barrier at $\mathbf{x} \in \Omega$ is $-\mathbf{x}^{-1}$, and its Hessian is $\mathcal{P}_{\mathbf{x}}^{-1} = \mathcal{P}_{\mathbf{x}^{-1}}$; see [7, Proposition II.3.3 and Proposition III.4.2][5].

A.2. **Peirce decomposition.** For any idempotent $\mathbf{c}$, its Lyapunov operator $\mathcal{L}_{\mathbf{c}}$ can only have eigenvalues 0, 1/2 or 1; see [7, p. 62]. We denote by $\mathfrak{J}(\mathbf{c}, 0)$, $\mathfrak{J}(\mathbf{c}, 1/2)$ and $\mathfrak{J}(\mathbf{c}, 1)$ the (possibly empty) eigenspaces associated with the eigenvalues 0, 1/2 and 1, respectively. Since $\mathcal{L}_{\mathbf{e}}$ is the identity map, the eigenspaces of the orthogonal idempotents $\mathbf{c}$ and $\mathbf{e} - \mathbf{c}$ satisfy $\mathfrak{J}(\mathbf{c}, 0) = \mathfrak{J}(\mathbf{e} - \mathbf{c}, 1)$, $\mathfrak{J}(\mathbf{c}, 1/2) = \mathfrak{J}(\mathbf{e} - \mathbf{c}, 1/2)$ and $\mathfrak{J}(\mathbf{c}, 1) = \mathfrak{J}(\mathbf{e} - \mathbf{c}, 0)$. Recall that $\mathcal{L}_{\mathbf{c}}$ is self-adjoint. Thus we have the orthogonal decomposition

$$\begin{aligned}\mathfrak{J} &= \mathfrak{J}(\mathbf{c}, 1) + \mathfrak{J}(\mathbf{c}, 1/2) + \mathfrak{J}(\mathbf{c}, 0)\\ &= \mathfrak{J}(\mathbf{c}, 1) + \mathfrak{J}(\mathbf{c}, 1/2) \cap \mathfrak{J}(\mathbf{e} - \mathbf{c}, 1/2) + \mathfrak{J}(\mathbf{e} - \mathbf{c}, 1).\end{aligned}$$

This is known as the *Peirce decomposition of* $\mathfrak{J}$ *with respect to* $\mathbf{c}$.

[5]Part (ii) of [7, Proposition III.4.2], while stated only for *simple* Euclidean Jordan algebras, is in fact true for general Euclidean Jordan algebras. This follows from the fact that when a Euclidean Jordan algebra is written as the direct sum $\mathfrak{J}_1 \oplus \cdots \oplus \mathfrak{J}_\kappa$ of simple Euclidean Jordan algebras, the determinant $\det(\mathbf{x}_1, \ldots, \mathbf{x}^{(\kappa)})$ decomposes into the product $\det_1(\mathbf{x}_1)\cdots\det^{(\kappa)}(\mathbf{x}^{(\kappa)})$ of determinants on each component, and the quadratic representation $\mathcal{P}_{(\mathbf{x}_1, \ldots, \mathbf{x}^{(\kappa)})}$ is block diagonal with the quadratic representations $\mathcal{P}_{\mathbf{x}_1}, \ldots, \mathcal{P}_{\mathbf{x}^{(\kappa)}}$ as the diagonal blocks.

For simplicity of notation, we shall use $\mathfrak{J}_{\mathbf{c}}$ to denote the (nonempty) eigenspace $\mathfrak{J}(\mathbf{c}, 1)$, and for each pair of orthogonal idempotents $(\mathbf{c}, \mathbf{c}')$, we shall use $\mathfrak{J}_{\mathbf{c},\mathbf{c}'}$ to denote the (nonempty) common eigenspace $\mathfrak{J}(\mathbf{c}, 1/2) \cap \mathfrak{J}(\mathbf{c}', 1/2)$. The above Peirce decomposition $\mathfrak{J} = \mathfrak{J}_{\mathbf{c}} + \mathfrak{J}_{\mathbf{c},\mathbf{e}-\mathbf{c}} + \mathfrak{J}_{\mathbf{e}-\mathbf{c}}$ with respect to $\mathbf{c}$ can be generalized to one with respect to a collection of pairwise orthogonal idempotents that sums up to the identity element.

Theorem 12 (Peirce decomposition, cf. Theorem IV.2.1 of [7]). *For each complete system of orthogonal idempotents* $\mathsf{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_p\}$*, the space* $\mathfrak{J}$ *decomposes into the orthogonal direct sum*

$$\mathfrak{J} = \bigoplus_{i=1}^{p} \mathfrak{J}_{\mathbf{c}_i} \oplus \bigoplus_{i<j} \mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j}$$

in such a way that

(1) $\mathfrak{J}_{\mathbf{c}_i}$ *is a Jordan subalgebra of* $\mathfrak{J}$ *with identity element* $\mathbf{c}_i$*;*
(2) *the orthogonal projector onto* $\mathfrak{J}_{\mathbf{c}_i}$ *is* $\mathcal{P}_{\mathbf{c}_i}$*, and that onto* $\mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j}$ *is* $4\mathcal{L}_{\mathbf{c}_i}\mathcal{L}_{\mathbf{c}_j}$*; and*
(3) *for each* $1 \le i, j, k, l \le p$ *with* $\{i, j\} \cap \{k, l\} = \emptyset$*,*

$$\mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j} \circ \mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j} \subseteq \mathfrak{J}_{\mathbf{c}_i} + \mathfrak{J}_{\mathbf{c}_j}, \qquad \mathfrak{J}_{\mathbf{c}_i} \circ \mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_k} \subseteq \mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_k},$$
$$\mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j} \circ \mathfrak{J}_{\mathbf{c}_j,\mathbf{c}_k} \subseteq \mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_k}, \qquad \mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j} \circ \mathfrak{J}_{\mathbf{c}_k,\mathbf{c}_l} = \{\mathbf{0}\}.$$

Proof. This theorem follows from Theorems 8 and 9 of [13], and their proofs. □

For each $\mathbf{x} \in \mathfrak{J}$, its decomposition into $\mathbf{x} = \sum_{i=1}^{p} \mathbf{x}_{\mathbf{c}_i} + \sum_{i<j} \mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j}$ with $\mathbf{x}_{\mathbf{c}_i} = 2\mathcal{P}_{\mathbf{c}_i}(\mathbf{x})$ and $\mathbf{x}_{\mathbf{c}_i,\mathbf{c}_j} = 4\mathcal{L}_{\mathbf{c}_i}(\mathcal{L}_{\mathbf{c}_j}(\mathbf{x}))$ is called its *Peirce decomposition* with respect to the complete system of idempotents $\{\mathbf{c}_1, \ldots, \mathbf{c}_p\}$.

Example A.4. *For the Jordan algebra of* $r \times r$ *real symmetric matrices, the Peirce decomposition of a matrix* $\mathbf{X}$ *with respect to a complete system of orthogonal idempotents* $(\mathbf{P}_1\mathbf{P}_1^T, \ldots, \mathbf{P}_p\mathbf{P}_p^T)$ *is* $\mathbf{X} = \sum_{i=1}^{p} \mathbf{P}_i\mathbf{P}_i^T\mathbf{X}\mathbf{P}_i\mathbf{P}_i^T + \sum_{i<j} \mathbf{P}_i\mathbf{P}_i^T\mathbf{X}\mathbf{P}_j\mathbf{P}_j^T + \mathbf{P}_j\mathbf{P}_j^T\mathbf{X}\mathbf{P}_i\mathbf{P}_i^T$*.*

The Peirce decomposition allows us to express the eigenvalues and eigenspaces of the Lyapunov operator $\mathcal{L}_{\mathbf{x}}$ in terms of the spectral decomposition of $\mathbf{x}$: if $\mathbf{x} = \mu_1\mathbf{c}_1 + \cdots + \mu_k\mathbf{c}_k$ is the type I spectral decomposition of $\mathbf{x}$, then the subspace $\mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j}$, if nonempty, is an eigenspace of $\mathcal{L}_{\mathbf{x}}$ associated with the eigenvalue $\frac{1}{2}(\mu_i + \mu_j)$. Subsequently, the eigenvalues and eigenspaces of the quadratic representation $\mathcal{P}_{\mathbf{x}}$ can be similarly obtained: the subspace $\mathfrak{J}_{\mathbf{c}_i,\mathbf{c}_j}$, if nonempty, is an eigenspace of $\mathcal{P}_{\mathbf{x}}$ associated with the eigenvalue $\mu_i\mu_j$. These observations leads to the following lemma.

Lemma A.1 (cf. Lemma 12 of [1]). *If* $\mathcal{L}_{\mathbf{x}}$ *(resp.,* $\mathcal{P}_{\mathbf{x}}$*) and* $\mathcal{L}_{\mathbf{y}}$ *(resp.,* $\mathcal{P}_{\mathbf{y}}$*) are similar to each other, then* $\mathbf{x}$ *and* $\mathbf{y}$ *have the same set of eigenvalues.*

Proof. In the Peirce decomposition with respect to a complete system of orthogonal idempotents $(\mathbf{c}_1, \ldots, \mathbf{c}_p)$, the subspace $\mathfrak{J}_{\mathbf{c}_i}$ is generated by any set of orthogonal idempotents summing up to $\mathbf{c}_i$, and has dimension $\|\mathbf{c}_i\|^2$. Thus if two Lyapunov operators (or quadratic representations) are similar to each other, then the corresponding elements share the same eigenvalues, and each eigenvalue occurs the same number of times in each type II spectral decomposition. □

A.3. **Some new results.**

Lemma A.2. *For each automorphism* $\mathcal{A}$ *in the identity component* G *of the automorphism group* $\mathsf{G}(\Omega)$ *of* Ω *and all* $\mathbf{x} \in \Omega$*,*

$$-\log\det(\mathcal{A}\mathbf{x}) = -\log\det(\mathbf{x}) + c_{\mathcal{A}}, \quad (\mathcal{A}^t\mathbf{x})^{-1} = \mathcal{A}^{-1}\mathbf{x}^{-1} \quad \textit{and} \quad \mathcal{P}_{\mathcal{A}^t\mathbf{x}} = \mathcal{A}^t\mathcal{P}_{\mathbf{x}}\mathcal{A}.$$

Proof. When $\mathcal{A}$ is a quadratic representation, the first equation follows from [7, Proposition III.4.2]. In general, we decompose $\mathcal{A}$ into the product $\mathcal{P}_{\mathbf{p}}\mathcal{Q}$ of the quadratic representation of some $\mathbf{p} \in \Omega$ and some orthogonal automorphism $\mathcal{Q}$ in the identity component of $\mathsf{G}(\Omega)$ (see [7, Theorem III.5.1]), and note that the determinant is invariant under automorphisms of $\mathfrak{J}$ (see [7, Theorem II.4.2]), whence invariant under $\mathcal{Q}$ (see Theorem A.7). Differentiating the first equation twice gives $-\mathcal{A}^t(\mathcal{A}\mathbf{x})^{-1} = -\mathbf{x}^{-1}$ and $\mathcal{A}^t\mathcal{P}_{\mathcal{A}\mathbf{x}}^{-1}\mathcal{A} = \mathcal{P}_{\mathbf{x}}^{-1}$. Since Ω is self-dual implies that $\mathsf{G}(\Omega)^t = \mathsf{G}(\Omega)$, the other two equations follows. □

Lemma A.3. *For each $\mathbf{x} \in \Omega$ and each $\mathcal{A} \in \mathsf{G}$ satisfying $\mathcal{A}\mathbf{x} = \mathbf{e}$, the elements $\mathbf{z} \in \mathfrak{J}$ and $\mathcal{A}\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{z}$ always have the same set of eigenvalues.*

Proof. We shall show that the quadratic representations of $\mathbf{z}$ and $\mathcal{A}\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{z}$ are similar to each other, whence by Lemma A.1, both elements have the same set of eigenvalues. By the choice of $\mathcal{A}$, $\mathcal{A}\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{e} = \mathcal{A}\mathbf{x} = \mathbf{e}$. Therefore $\mathcal{Q} := \mathcal{A}\mathcal{P}_{\mathbf{x}^{1/2}} \in \mathsf{G}$ is orthogonal by Theorem A.7. By Lemma A.2, $\mathcal{P}_{\mathcal{A}\mathcal{P}_{\mathbf{x}^{1/2}}\mathbf{z}} = \mathcal{Q}\mathcal{P}_{\mathbf{z}}\mathcal{Q}^T$ is similar to $\mathcal{P}_{\mathbf{z}}$. □

Lemma A.4. *For each $\mathcal{A} \in \mathsf{G}$, $\mathcal{A}\mathcal{A}^t\mathbf{e} = (\mathcal{A}\mathbf{e})^2$.*

Proof. By Lemma A.2, $\mathcal{A}\mathcal{A}^t\mathbf{e} = \mathcal{A}\mathcal{P}_{\mathbf{e}}\mathcal{A}^t\mathbf{e} = \mathcal{P}_{\mathcal{A}\mathbf{e}}\mathbf{e} = (\mathcal{A}\mathbf{e})^2$. □

Lemma A.5. *For any $\mathcal{A} \in \mathsf{G}$, and any $\mathbf{x}, \mathbf{s} \in \Omega^2$, $\mathbf{x} \circ \mathbf{s} = \mathbf{e}$ if and only if $\mathcal{A}^t\mathbf{x} \circ \mathcal{A}^{-1}\mathbf{s} = \mathbf{e}$.*

Proof. Since every automorphism $\mathcal{A}$ decomposes into $\mathcal{A} = \mathcal{P}_{\mathbf{p}}\mathcal{Q}^t$ for some orthogonal automorphism $\mathcal{Q}$ in the identity component of $\mathsf{G}(\Omega)$ and some $\mathbf{p} \in \Omega$ (see [7, Theorem III.5.1]), we can write $\mathcal{A}^t\mathbf{x} \circ \mathcal{A}^{-1}\mathbf{s} = \mathcal{Q}\mathcal{P}_{\mathbf{p}}\mathbf{x} \circ \mathcal{Q}\mathcal{P}_{\mathbf{p}^{-1}}\mathbf{s}$.

From the fact that the orthogonal subgroup of $\mathsf{G}(\Omega)$ coincides with both the automorphism group $\mathsf{G}(\mathfrak{J})$ and the stabilizer subgroup $\mathsf{G}(\Omega)_{\mathbf{c}_{i:r}}$ of the unit $\mathbf{c}_{i:r}$ in $\mathsf{G}(\Omega)$ (see Theorem A.7), we have $\mathcal{A}^t\mathbf{x}\circ\mathcal{A}^{-1}\mathbf{s} = \mathcal{Q}\mathcal{P}_{\mathbf{p}}\mathbf{x}\circ\mathcal{Q}\mathcal{P}_{\mathbf{p}^{-1}}\mathbf{s} = \mathcal{Q}(\mathcal{P}_{\mathbf{p}}\mathbf{x}\circ\mathcal{P}_{\mathbf{p}^{-1}}\mathbf{s})$, and subsequently, $\mathcal{A}^t\mathbf{x} \circ \mathcal{A}^{-1}\mathbf{s} = \mathbf{c}_{i:r}$ if and only if $\mathcal{P}_{\mathbf{p}}\mathbf{x} \circ \mathcal{P}_{\mathbf{p}^{-1}}\mathbf{s} = \mathbf{e}$. The lemma then follows from Lemma 28 of [1] (cf. Theorem 3.1 (i) of [19]). □

Example A.5. *For the Jordan algebra of $r \times r$ real symmetric matrices, an automorphism of the cone of positive definite matrices takes the form $\mathbf{X} \mapsto \mathbf{P}\mathbf{X}\mathbf{P}^T$ for some invertible matrix $\mathbf{P}$. It is in the identity component G if and only if $\det(\mathbf{P}) > 0$. The first lemma specializes to the well-known facts $-\log\det(\mathbf{P}\mathbf{X}\mathbf{P}^T) = -\log\det(\mathbf{X}) - \log\det(\mathbf{P})^2$, $(\mathbf{P}^T\mathbf{X}\mathbf{P})^{-1} = \mathbf{P}^{-1}\mathbf{X}^{-1}\mathbf{P}^{-T}$, and $(\mathbf{P}^T\mathbf{X}\mathbf{P})\mathbf{Y}(\mathbf{P}^T\mathbf{X}\mathbf{P}) = \mathbf{P}^T(\mathbf{X}(\mathbf{P}\mathbf{Y}\mathbf{P}^T)\mathbf{X})\mathbf{P}$. The second lemma follows easily from $\mathbf{X}\mathbf{S} + \mathbf{S}\mathbf{X} = 2\mathbf{I} \iff \mathbf{X}\mathbf{S} = \mathbf{I}$.*

A.4. **Automorphisms of Euclidean Jordan algebras.** In Section II.1 of [25], it was stated without proof that if $(\mathfrak{J}, \circ)$ is a Euclidean Jordan algebra and Ω is its associated symmetric cone, then the stabilizer subgroup $\mathsf{G}(\Omega)_{\mathbf{e}}$ of the unit $\mathbf{e}$ in $\mathsf{G}(\Omega)$ coincide with the group of automorphisms $\mathsf{G}(\mathfrak{J})$ of $\mathfrak{J}$. Here we give a proof of this fact.

Theorem A.6. *Given a Euclidean Jordan algebra $(\mathfrak{J}, \circ)$ with unit $\mathbf{e}$ and associated symmetric cone Ω, the stabilizer subgroup $\mathsf{G}(\Omega)_{\mathbf{e}}$ of the unit $\mathbf{e}$ in $\mathsf{G}(\Omega)$ coincide with the group of automorphisms $\mathsf{G}(\mathfrak{J})$ of $\mathfrak{J}$.*

Proof. Consider the inner product $\langle\cdot,\cdot\rangle : (\mathbf{x}, \mathbf{y}) \mapsto \operatorname{trace}\mathcal{L}_{\mathbf{x}\circ\mathbf{y}}$, where $\mathcal{L}_{\mathbf{x}}$ denotes the linear map $\mathbf{y} \mapsto \mathbf{x} \circ \mathbf{y}$. Let $\mathsf{O}(\mathfrak{J})$ denote the orthogonal group of the Euclidean space $(\mathfrak{J}, \langle\cdot,\cdot\rangle)$; i.e., $\mathsf{O}(\mathfrak{J}) = \{\mathcal{A} \in GL(\mathfrak{J}) : \langle\mathcal{A}\mathbf{x}, \mathcal{A}\mathbf{y}\rangle = \langle\mathbf{x}, \mathbf{y}\rangle\ \forall\mathbf{x}, \mathbf{y} \in \mathfrak{J}\}$.

Let φ be the characteristic function of Ω; i.e.,

$$\varphi : \mathbf{x} \in \Omega \mapsto \int_{\Omega^\sharp} e^{-\langle\mathbf{x},\mathbf{y}\rangle} d\mathbf{y},$$

where $d\mathbf{y}$ denotes the Euclidean measure on $(\mathfrak{J}, \langle\cdot,\cdot\rangle)$. Let $\mathbf{x}^\sharp$ denote the negative gradient of the logarithmic derivative of φ at $\mathbf{x}$. We deduce from Propositions II.3.4 and III.2.2 of [7] that $\exp \mathcal{L}_\mathbf{x} \in \mathsf{G}(\Omega)$. Thus by Proposition I.3.1 of [7], we have $\log\varphi(\exp\mathcal{L}_\mathbf{x} \circ \mathbf{e}) = \log\varphi(\mathbf{e}) - \log\det\exp\mathcal{L}_\mathbf{x} = \log\varphi(\mathbf{e}) - \operatorname{trace}\mathcal{L}_\mathbf{x}$. Differentiating this at $\mathbf{0}$ gives $\operatorname{trace}\mathcal{L}_\mathbf{h} = -D\log\varphi(\mathbf{e})[\mathbf{h}]$. Since $\operatorname{trace}\mathcal{L}_\mathbf{h} = \langle \mathbf{h}, \mathbf{e}\rangle$ and $-D\log\varphi(\mathbf{e})[\mathbf{h}] = \left\langle \mathbf{e}^\sharp, \mathbf{h}\right\rangle$, it follows that $\mathbf{e}$ is a fixed point of the map $\mathbf{x} \in \Omega \mapsto \mathbf{x}^\sharp$. Proposition I.4.3 of [7] then states that $\mathsf{G}(\Omega) \cap \mathsf{O}(\mathfrak{J}) = \mathsf{G}(\Omega)_\mathbf{e}$.

We now show that $\mathsf{G}(\mathfrak{J})$ coincides with $\mathsf{G}(\Omega) \cap \mathsf{O}(\mathfrak{J}) = \mathsf{G}(\Omega)_\mathbf{e}$. It is straightforward to check that every automorphism of $\mathfrak{J}$ is an automorphism of Ω (which is the interior of the cone of squares) that stabilizes the unit $\mathbf{e}$. For the other direction, it suffices to show that every linear map $\mathcal{A} \in \mathsf{G}(\Omega) \cap \mathsf{O}(\mathfrak{J}) = \mathsf{G}(\Omega)_\mathbf{e}$ preserves orthogonality of idempotents and maps every primitive idempotent to some primitive idempotent, for if $\mathbf{x} = \sum \lambda_i \mathbf{c}_i$ is the spectral decomposition, then we have $\mathcal{A}(\mathbf{x}^2) = \mathcal{A}\left(\sum \lambda_i^2 \mathbf{c}_i\right) = \sum \lambda_i^2 \mathcal{A}(\mathbf{c}_i) = \left(\sum \lambda_i \mathcal{A}(\mathbf{c}_i)\right)^2 = (\mathcal{A}\mathbf{x})^2$, whence $\mathcal{A} \in \mathsf{G}(\mathfrak{J})$ by polarization. Suppose $\mathcal{A} \in \mathsf{G}(\Omega) \cap \mathsf{O}(\mathfrak{J}) = \mathsf{G}(\Omega)_\mathbf{e}$. Two idempotents are $\mathbf{c}$ and $\mathbf{d}$ are orthogonal if and only if $\langle \mathbf{c}, \mathbf{d}\rangle = 0$. One direction of this statement follows from the definition of $\langle\cdot,\cdot\rangle$. For the other direction, suppose that $\mathbf{c}$ and $\mathbf{d}$ are two idempotents satisfying $\langle \mathbf{c}, \mathbf{d}\rangle = 0$. Since the inner product $\langle\cdot,\cdot\rangle$ is associative (see Proposition II.4.3 of [7]), $\mathcal{L}_\mathbf{c}$ is self-adjoint. Proposition III.1.3 of [7] then implies that $\mathcal{L}_\mathbf{c}$ is positive semidefinite. Thus it has a self-adjoint, positive semidefinite square root $\mathcal{L}_\mathbf{c}^{1/2}$. Hence $0 = \langle \mathbf{c}, \mathbf{d}\rangle = \left\langle \mathbf{c}, \mathbf{d}^2\right\rangle = \langle \mathbf{c}\circ\mathbf{d}, \mathbf{d}\rangle = \langle \mathcal{L}_\mathbf{c}\mathbf{d}, \mathbf{d}\rangle = \left\langle \mathcal{L}_\mathbf{c}^{1/2}\mathbf{d}, \mathcal{L}_\mathbf{c}^{1/2}\mathbf{d}\right\rangle$ shows that $\mathcal{L}_\mathbf{c}^{1/2}\mathbf{d} = \mathbf{0}$, whence $\mathbf{c}\circ\mathbf{d} = \mathcal{L}_\mathbf{c}^{1/2}\mathcal{L}_\mathbf{c}^{1/2}\mathbf{d} = \mathbf{0}$; i.e., $\mathbf{c}$ and $\mathbf{d}$ are orthogonal. Since $\mathcal{A}$ is orthogonal, it follows that orthogonal idempotents remain orthogonal under $\mathcal{A}$. Proposition IV.3.2 of [7] states $\mathbf{c}$ is a primitive idempotent if and only if $\{\lambda\mathbf{c} : \lambda \geq 0\}$ is an extreme ray of Ω. Since $\mathcal{A} \in \mathsf{G}(\Omega)$, it maps each extreme ray to some extreme ray of Ω. Thus it maps each primitive idempotent $\mathbf{c}$ to a positive multiple $\lambda\mathbf{d}$ of some primitive idempotent $\mathbf{d}$. In fact, λ must be unit since

$$\begin{aligned} 0 < \langle \mathbf{d}, \mathbf{d}\rangle &= \left\langle \mathbf{d}^2, \mathbf{e}\right\rangle = \langle \mathbf{d}, \mathbf{e}\rangle = \langle \mathbf{d}, \mathcal{A}\mathbf{e}\rangle = \lambda^{-1}\langle \mathcal{A}\mathbf{c}, \mathcal{A}\mathbf{e}\rangle \\ &= \lambda^{-1}\langle \mathbf{c}, \mathbf{e}\rangle = \lambda^{-1}\left\langle \mathbf{c}^2, \mathbf{e}\right\rangle = \lambda^{-1}\langle \mathbf{c}, \mathbf{c}\rangle = \lambda^{-1}\langle \mathcal{A}\mathbf{c}, \mathcal{A}\mathbf{c}\rangle = \lambda\langle \mathbf{d}, \mathbf{d}\rangle \end{aligned}$$

Hence $\mathcal{A}$ maps each primitive idempotent to some primitive idempotent. □

The proof of the theorem shows that both $\mathsf{G}(\Omega)_\mathbf{e}$ and $\mathsf{G}(\mathfrak{J})$ coincide with certain orthogonal subgroup of $\mathsf{G}(\Omega)$. The next theorem gives a similar result.

Theorem A.7. *Given a Euclidean Jordan algebra $(\mathfrak{J}, \circ)$ with unit $\mathbf{e}$ and associated symmetric cone Ω, the groups $\mathsf{G}(\Omega)_\mathbf{e}$ and $\mathsf{G}(\mathfrak{J})$ are both equivalent to the orthogonal subgroup of $\mathsf{G}(\Omega)$ under the inner product $\langle\cdot,\cdot\rangle : (\mathbf{x}, \mathbf{y}) \mapsto \operatorname{tr}(\mathbf{x}\circ\mathbf{y})$.*

Proof. Let $\mathsf{O}(\Omega)$ denote the orthogonal subgroup of $\mathsf{G}(\Omega)$ under $\langle\cdot,\cdot\rangle$. By Proposition II.4.2 of [7], if $\mathcal{A} \in \mathsf{G}(\mathfrak{J})$, then $\operatorname{tr}(\mathcal{A}\mathbf{x}\circ\mathcal{A}\mathbf{y}) = \operatorname{tr}\mathcal{A}(\mathbf{x}\circ\mathbf{y}) = \operatorname{tr}(\mathbf{x}\circ\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{J}$, whence $\mathcal{A}$ is orthogonal. Therefore $\mathsf{G}(\mathfrak{J}) = \mathsf{G}(\Omega)_\mathbf{e} \subseteq \mathsf{O}(\Omega)$. According to Proposition I.1.8 of [7] and the paragraph following it, $\mathsf{G}(\Omega)_\mathbf{e}$ is a maximal compact subgroup of $\mathsf{G}(\Omega)$. Hence $\mathsf{O}(\Omega) \subseteq \mathsf{G}(\Omega)_\mathbf{e}$. □

Example A.6. *For the Jordan algebra of $r \times r$ real symmetric matrices, an automorphism of the Jordan algebra takes the form $\mathbf{X} \mapsto \mathbf{Q}\mathbf{X}\mathbf{Q}^T$ for some orthogonal matrix $\mathbf{Q}$, which clearly stabilizes the identity $\mathbf{I}$. It is also orthogonal under the trace inner product:* $\operatorname{tr}(\mathbf{Q}\mathbf{X}\mathbf{Q}^T)(\mathbf{Q}\mathbf{Y}\mathbf{Q}^T) = \operatorname{tr}\mathbf{X}\mathbf{Y}$.

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