Stabilization of a Class of Switched Systems via Designing Switching Laws

Z. G. Li, C. Y. Wen, and Y. C. Soh

***Abstract*—In this note, one state transformation is used to construct switching laws for a class of switched systems totally composed of unstable subsystems. Some sufficient conditions for determining the switching law, such that the system is asymptotically stable, are derived.**

***Index Terms*—Stabilization, switched systems, switching law.**

I. INTRODUCTION

In this note, we consider the stabilization of a class of switched nonlinear systems described by

$$\dot{x}(t) = f(x(t), m(t)) \tag{1}$$

where

$x(t)$ $\in R^r$ continuous state;
$m(t)$ $\in \bar{M} = \{1, \ldots, n\}$ discrete state;
$m(t)$ left continuous with each i corresponding to a mode $f(x, i)$ with $f(0, i) = 0$.

All $f(x, i)$ $(i = 1, 2, \ldots, n)$ are assumed to be continuously differentiable and $\dot{x}(t) = f(x(t), i)$ are all unstable systems. If $f(x(t), m(t)) = A(m(t))x(t)$, then we can obtain the following switched linear system

$$\dot{x}(t) = A(m(t))x(t). \tag{2}$$

Wicks *et al.* [6] showed that there exists a switching law for the asymptotic stabilization of system (2) if $n = 2$ and there is a stable convex combination of $A(1)$ and $A(2)$. Although it is NP-hard to identify stable convex combinations of two matrices [1], it is still possible to find a stable convex combination for a class of switched systems. However, there is no result available on how to find such a combination. This problem is more of a design problem than a stability analysis problem, and it is a major problem in switched control systems [5].

In this note, we shall present a method to find such a combination. We introduce a linear state transformation to decompose each subsystem into stable and unstable parts. For each stable part, there naturally exists a Lyapunov function. Under some conditions imposed on the original switched system, the sum of these Lyapunov functions is shown to be a Lyapunov function for a convex combination of the whole switched system. This ensures the existence of a switching law of $m(t)$ for a switched nonlinear system (1) or a switched linear system (2) to be asymptotically stable. We shall first derive some sufficient conditions to determine a stable convex combination of switched linear systems (2). The linear approximation method and the obtained results for linear systems are then used to consider switched nonlinear systems (1). For simplicity of presentation, we only consider the case of $n = 2$. However, the results obtained in this note can be easily generalized.

The rest of the note is organized as follows. Some supporting results are given in Section II. Section III contains some sufficient conditions for the stabilization of switched linear systems and Section IV considers the stabilization of switched nonlinear systems. A numerical example is given in Section V to illustrate the application of the results. Finally, the note is concluded in Section VI.

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II. SUPPORTING RESULTS

In this section, we shall introduce some supporting results.

Lemma 1: There exists a switching law for switched nonlinear system (1) such that the system is asymptotically stable if there exist positive numbers α_i $(1 \le i \le n)$ satisfying $\sum_{i=1}^n \alpha_i = 1$ such that $\dot{x}(t) = \sum_{i=1}^n \alpha_i f(x, i)$ is an asymptotically stable system.

Proof: Since there exist positive numbers α_i $(1 \le i \le n)$ such that $\dot{x} = \sum_{i=1}^n \alpha_i f(x, i)$ is asymptotically stable, there exists a Lyapunov function $V(x)$ such that

$$\frac{\partial V}{\partial x} \sum_{i=1}^n \alpha_i f(x, i) < 0 \tag{3}$$

It follows that for any t, there exists an $i \in \{1, 2, \ldots, n\}$ such that

$$\frac{\partial V}{\partial x} f(x, i) < 0 \tag{4}$$

From (4), we know that there exists a common Lyapunov function for the whole system (1). Thus, switched nonlinear system (1) is asymptotically stable.

Remark 1: Condition (4) will be used to design switching laws. Under such switching laws, the switchings of the system may be arbitrarily fast. □

Let $f(x, i) = A(i)x$. Then, we have the following.

Lemma 2: There exists a switching law for switched linear system (2) such that the system is asymptotically stable if there exist positive numbers α_i $(1 \le i \le n)$ satisfying $\sum_{i=1}^n \alpha_i = 1$ such that $\dot{x} = \sum_{i=1}^n \alpha_i A(i)x$ is an asymptotically stable system.

Remark 2: If $n = 2$, then Lemma 2 becomes a result in [6]. □

Proposition 1: Switched linear system (2) is asymptotically stable if and only if the following switched linear system (5) is asymptotically stable.

$$\dot{z}(t) = TA(m(t))T^{-1}z(t) \tag{5}$$

where $z(t) = Tx(t)$ and T is nonsingular.

Proof: The proof can be easily proved by using the condition that T is nonsingular. □

Proposition 2: Switched nonlinear system (1) is asymptotically stable if and only if the following switched nonlinear system (6) is asymptotically stable.

$$\dot{z}(t) = Tf(T^{-1}z(t), m(t)) \tag{6}$$

where $z(t) = Tx(t)$ and T is nonsingular. □

Proof: The proof can be easily shown by using the condition that T is nonsingular.

III. Stabilization of Switched Linear Systems

In this section, we consider the stabilization of switched linear system (2) in the case that $n = 2$.

For a matrix $A(i)$, define

$$\hat{X}_i = \{x \mid x \text{ is an eigenvector of } A(i) \text{ corresponding to an eigenvalue with negative real part.}\} \tag{7}$$

Assumption 1: A basis for R^r can be selected from $\hat{X}_1 \cup \hat{X}_2$. In other words, $\hat{\theta}(i)$ $(i = 1, 2)$ vectors can be chosen respectively from $\hat{X}_i$, denoted as $X_{i,j} \in \hat{X}_i$ $(j = 1, 2, \ldots, \hat{\theta}(i))$, such that

$$\sum_{i=1}^{2} \hat{\theta}(i) = r \tag{8}$$

$$\text{Span}(X_{1,1}, \ldots, X_{1,\hat{\theta}(1)}, X_{2,1}, \ldots, X_{2,\hat{\theta}(2)}) = R^r. \tag{9}$$

Assumption 2: $\text{Span}(X_{1,1}, \ldots, X_{1,\hat{\theta}(1)})$ and $\text{Span}(X_{2,1}, \ldots, X_{2,\hat{\theta}(2)})$ are invariant under $A(1)$ and $A(2)$, respectively.

Remark 3: A linear system can be decomposed into stable and unstable subspaces [2], [3]. Assumption 1 implies that the stable subspaces of the two subsystems span the whole R^r space. Assumption 2 implies that these two subsystems can be decomposed under the same transformation. These assumptions are reasonable. Actually, if there exists a subspace in which the corresponding eigenvalues of both $A(1)$ and $A(2)$ are positive, it is difficult to design a switching law to stabilize the system in such a subspace and no result is available to control such systems. A numerical example is given in Section V to show that such switched systems do exist. □

Let

$$T^{-1} = [X_{1,1}, \ldots, X_{1,\hat{\theta}(1)}, X_{2,1}, \ldots, X_{2,\hat{\theta}(2)}]. \tag{10}$$

From Assumptions 1 and 2, we have

$$TA(1)T^{-1} = \begin{bmatrix} \hat{A}_{11}(1) & \hat{A}_{12}(1) \\ 0 & \hat{A}_{22}(1) \end{bmatrix} \tag{11}$$

$$TA(2)T^{-1} = \begin{bmatrix} \hat{A}_{11}(2) & 0 \\ \hat{A}_{21}(2) & \hat{A}_{22}(2) \end{bmatrix} \tag{12}$$

One possible choice of Lyapunov function for a stable linear system can be made based on the result in [4].

Proposition 3: [4] Consider the Lyapunov equation

$$A^{\mathrm{T}}P + PA = -Q \tag{13}$$

where $Q = Q^{\mathrm{T}} > 0$ and A is stable.

Let $\mu(Q) = \lambda_{\min}(Q)/\lambda_{\max}(P)$, then $\mu(I) \geq \mu(Q), \forall Q = Q^{\mathrm{T}} > 0$.

From (7), we know that $\hat{A}_{11}(1)$ and $\hat{A}_{22}(2)$ are stable.

Using Proposition 3 , we set Q in (13) to be I. Obviously, there exist two positive–definite matrices $P(1)$ and $P(2)$ such that

$$\hat{A}_{11}^{\mathrm{T}}(1)P(1) + P(1)\hat{A}_{11}(1) = -I < 0 \tag{14}$$

$$\hat{A}_{22}^{\mathrm{T}}(2)P(2) + P(2)\hat{A}_{22}(2) = -I < 0 \tag{15}$$

We shall now give a result for the existence of a switching law for the stabilization of switched linear system (2) with $n = 2$.

Theorem 1: Consider switched linear system (2) with $n = 2$ satisfying Assumptions 1 and 2. There exists a switching law such that system (2) is asymptotically stable if

$$\begin{aligned}&(2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) \\ &\quad \times (2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|) \\ &\quad < (1 - \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)(1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|)\end{aligned} \tag{16}$$

$$\lambda_{\max}(P(1))\|\hat{A}_{12}(1)\| < 1 \tag{17}$$

$$\lambda_{\max}(P(2))\|\hat{A}_{21}(2)\| < 1 \tag{18}$$

where $\|A\|$ denotes the induced 2–norm for any matrix A in $R^{m \times n}$.

Proof: From (16), we can obtain the first equation shown at the bottom of the page. Now, choose α so that (19), shown at the bottom of the page, holds true.

We shall now analyze the linear system $\dot{x}(t) = (\alpha A(1) + (1 - \alpha)A(2))x(t)$. Using Proposition 1, this is equivalent to study the following linear system:

$$\dot{z}(t) = (\alpha TA(1)T^{-1} + (1 - \alpha)TA(2)T^{-1})z(t).$$

$$\frac{2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|}{1 - \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\| + 2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|} < \frac{1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|}{1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\| + 2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|}$$

$$\begin{aligned}&\frac{2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|}{1 - \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\| + 2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|} \\ &< \alpha \\ &< \frac{1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|}{1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\| + 2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|}\end{aligned} \tag{19}$$

To prove that it is asymptotically stable, we propose the Lyapunov function

$$V(z) = \tilde{V}(z_1, 1) + \tilde{V}(z_2, 2)$$

where

$$\tilde{V}(z_1, 1) = z_1^{\mathrm{T}} P(1) z_1; \quad \tilde{V}(z_2, 2) = z_2^{\mathrm{T}} P(2) z_2.$$

Note that

$$\begin{aligned}\frac{d\tilde{V}(z_1,1)}{dt} =& \frac{\partial \tilde{V}(z_1,1)}{\partial z_1}(\alpha \hat{A}_{11}(1) z_1 + (1-\alpha)\hat{A}_{11}(2) z_1 \\ &+ \alpha \hat{A}_{12}(1) z_2) \\ \le& -\alpha \|z_1\|^2 + (1-\alpha)\left\|\frac{\partial \tilde{V}(z_1,1)}{\partial z_1}\right\| \\ &\times \|\hat{A}_{11}(2)\| \|z_1\| + \alpha \|\hat{A}_{12}(1)\| \\ &\times \left\|\frac{\partial \tilde{V}(z_1,1)}{\partial z_1}\right\| \|z_2\| \\ \le& (-\alpha + 2(1-\alpha)\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\|)\|z_1\|^2 \\ &+ 2\alpha\lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|\|z_1\|\|z_2\| \\ \le& (-(1-\lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)\alpha \\ &+ 2(1-\alpha)\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\|)\|z_1\|^2 \\ &+ \alpha\lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|\|z_2\|^2\end{aligned}$$

and

$$\begin{aligned}\frac{d\tilde{V}(z_2,2)}{dt} =& \frac{\partial \tilde{V}(z_2,1)}{\partial z_2}(\alpha \hat{A}_{22}(1) z_2 + (1-\alpha)\hat{A}_{22}(2) z_2 \\ &+ (1-\alpha)\hat{A}_{21}(2) z_1) \\ \le& (\alpha - 1)\|z_2\|^2 + \alpha\left\|\frac{\partial \tilde{V}(z_2,2)}{\partial z_2}\right\| \\ &\times \|\hat{A}_{22}(1)\|\|z_2\| + 2(1-\alpha) \\ &\times \left\|\frac{\partial \tilde{V}(z_2,2)}{\partial z_2}\right\| \|\hat{A}_{21}(2)\|\|z_1\| \\ \le& ((\alpha-1)(1-\lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) \\ &+ 2\alpha\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\|)\|z_2\|^2 \\ &+ (1-\alpha)\lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|\|z_1\|^2\end{aligned}$$

Using inequality (19), it follows that

$$\begin{aligned}\frac{dV(z)}{dt} \le& ((\alpha-1)(1-\lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) \\ &+ \alpha(2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| \\ &\quad + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|))\|z_2\|^2 \\ &+ (-(1-\lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)\alpha \\ &\quad + (1-\alpha)(2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| \\ &\quad + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|))\|z_1\|^2 \\ <& 0\end{aligned}$$

In other words, $V(z)$ is a Lyapunov function for a convex combination of $TA(1)T^{-1}$ and $TA(2)T^{-1}$. Thus, there exists a stable convex combination.

Using Lemma 2, we know that there exists a switching law such that switched linear system (11) and (12) is asymptotically stable.

Using Proposition 1, we know that there exists a switching law such that the original switched linear system (2) is asymptotically stable. □

Remark 4: From the proof of Theorem 1, there exist a Lyapunov function $V(z)$ and an α satisfying $0 < \alpha < 1$ such that

$$\frac{\partial V(z)}{\partial z}(\alpha TA(1)T^{-1} z + (1-\alpha) TA(2) T^{-1} z) < 0.$$

For any t, there exists an $i \in \{1, 2\}$ such that

$$\frac{\partial V(z)}{\partial z} TA(i) T^{-1} z < 0. \tag{20}$$

Then, the switching law can be defined as follows.

A Switching Law: *Switched linear system (2) with $n = 2$ is switched to or stay at mode i* at time t *if (20) is satisfied at time t.* □

Remark 5: Let $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$, then from the proof of Theorem 1, we know that α_1 and α_2 satisfy

$$\begin{aligned}&-\alpha_2(1-\lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) \\ &+ \alpha_1(2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|) < 0 \\ &-\alpha_1(1-\lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|) \\ &+ \alpha_2(2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) < 0\end{aligned}$$

□

Remark 6: Theorem 1 can be generalized to the case of arbitrary n under the following assumptions.

Assumption 1': A basis for R^r can be selected from $\cup_{i=1}^n \hat{X}_i$. In other words, $\hat{\theta}(i)$ $(i = 1, 2, \ldots, n)$ vectors can be chosen respectively from $\hat{X}_i$, denoted as $X_{i,j} \in \hat{X}_i$ $(j = 1, 2, \ldots, \hat{\theta}(i))$, such that

$$\sum_{i=1}^{n} \hat{\theta}(i) = r \tag{21}$$

$$\mathrm{Span}(X_{1,1}, \ldots, X_{1,\hat{\theta}(1)}, X_{n,1}, \ldots, X_{n,\hat{\theta}(n)}) = R^r. \tag{22}$$

Without loss of generality, we assume that $\hat{\theta}(i) > 0$, since the ith subsystem will not be included in the switched system if $\hat{\theta}(i) = 0$.

Assumption 2': $\mathrm{Span}(X_{i,1}, \ldots, X_{i,\hat{\theta}(i)})$ $(i = 1, 2, \ldots, n)$ are invariant under $A(i)$ $(i = 1, 2, \ldots, n)$, respectively.

Assumption 3': There exist positive numbers $\alpha_i (1 \le i \le n)$ such that

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{23}$$

$$\begin{aligned}&-\left(1 - \sum_{1\le i\le n, i\ne l} \lambda_{\max}(P(l))\|\hat{A}_{li}(l)\|\right)\alpha_l \\ &+ \sum_{1\le j\le n, j\ne l} \alpha_j \left(\sum_{1\le i\le n, i\ne l} \|\hat{A}_{il}(j)\|\lambda_{\max}(P(i)) \right. \\ &\qquad + \left(2\|\hat{A}_{ll}(j)\| + \sum_{1\le i\le n, i\ne j, i\ne l} \|\hat{A}_{li}(j)\|\right) \\ &\qquad \left. \times \lambda_{\max}(P(l)) \right) < 0 \\ &1 \le l \le n\end{aligned} \tag{24}$$

where

$$\hat{A}_{ll}^{\mathrm{T}}(l) P(l) + P(l)\hat{A}_{ll}(l) = -I; \quad 1 \le l \le n \tag{25}$$

$$TA(1)T^{-1} = \begin{bmatrix} \hat{A}_{11}(1) & \hat{A}_{12}(1) & \ldots & \hat{A}_{1n}(1) \\ 0 & \hat{A}_{22}(1) & \ldots & \hat{A}_{2n}(1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{A}_{n2}(1) & \ldots & \hat{A}_{nn}(1) \end{bmatrix} \tag{26}$$

$$TA(2)T^{-1} = \begin{bmatrix} \hat{A}_{11}(2) & 0 & \ldots & \hat{A}_{1n}(2) \\ \hat{A}_{21}(2) & \hat{A}_{22}(2) & \ldots & \hat{A}_{2n}(2) \\ \hat{A}_{31}(2) & 0 & \ldots & \hat{A}_{3n}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{n1}(2) & 0 & \ldots & \hat{A}_{nn}(2) \end{bmatrix} \tag{27}$$

$$\cdots$$

$$TA(n)T^{-1} = \begin{bmatrix} \hat{A}_{11}(n) & \hat{A}_{12}(n) & \ldots & 0 \\ \hat{A}_{21}(n) & \hat{A}_{22}(n) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{n1}(n) & \hat{A}_{n2}(n) & \ldots & \hat{A}_{nn}(1) \end{bmatrix} \tag{28}$$

$$T^{-1} = [X_{1,1}, \ldots, X_{1,\hat{\theta}(1)}, \ldots, X_{n,1}, \ldots, X_{n,\hat{\theta}(n)}] \tag{29}$$

□

Remark 7: To use Theorem 1, the following algorithm is proposed to check if switched linear system (2) satisfies Assumptions 1 and 2.

Algorithm 1: Check if switched linear system (2) satisfies Assumptions 1 and 2.

Step 1) *Let* $IBS = \emptyset$.

Step 2) *Calculate* $\hat{X}_1$ *and* $\hat{X}_2$.

Step 3) *Find a basis of* R^r, $\{X_1, X_2, \ldots, X_r\} \notin IBS$, *where each* X_i $(i = 1, 2, \ldots, r)$ *is from* $\cup_{i=1}^{2} \hat{X}_i$.

Step 4) *Check if* $\{X_1, X_2, \ldots, X_r\}$ *satisfies Assumption 2. If not, then let* $IBS = IBS \cup \{X_1, X_2, \ldots, X_r\}$ and return to Step 3.

IV. Stabilization of Switched Nonlinear Systems

In this section, we shall use linear approximation method to consider the local stabilization of switched nonlinear system (1) with $n = 2$.

Let

$$A(m(t)) = \left.\frac{\partial f(x(t), m(t))}{\partial x(t)}\right|_{x(t)=0}$$

and

$$\tilde{f}(x(t), m(t)) = f(x(t), m(t)) - A(m(t))x(t).$$

It follows that

$$\lim_{\|x(t)\| \to 0} \frac{\|\tilde{f}(x(t), m(t))\|}{\|x(t)\|} = 0. \tag{30}$$

Suppose that $A(1)$ and $A(2)$ satisfy Assumptions 1 and 2. Similar to the linear case, we know that there exist two positive–definite matrices $P(1)$ and $P(2)$ such that $A(m(t))$ satisfies (14) and (15).

We shall now give a result for the existence of a switching law for local stabilization of a switched nonlinear system (1).

Theorem 2: Consider switched nonlinear system (1) with $n = 2$ and with the linear part satisfying Assumptions 1 and 2. There exists a switching law such that system (1) is locally asymptotically stable if the conditions of Theorem 1 hold.

Proof: Let

$$V(z) = z_1^{\mathrm{T}} P(1) z_1 + z_2^{\mathrm{T}} P(2) z_2$$

From (30), we know that there exists a $\delta > 0$ such that when $\|x(t)\| < \delta$, we have

$$\begin{aligned}
&\|\tilde{f}(T^{-1}z, i)\| \\
&\quad < \frac{\min\{-\Gamma_1, -\Gamma_2\}}{4\|T\|\|T^{-1}\| \max\{\lambda_{\max}(P(1)), \lambda_{\max}(P(2))\}} \|T^{-1}z\| \\
&\quad \le \frac{\min\{-\Gamma_1, -\Gamma_2\}}{4\|T\| \max\{\lambda_{\max}(P(1)), \lambda_{\max}(P(2))\}} \|z\|
\end{aligned}$$

where α satisfies (19) and

$$\begin{aligned}
\Gamma_1 =& ((\alpha - 1)(1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) \\
&+ \alpha(2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)) \\
\Gamma_2 =& (-(1 - \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)\alpha \\
&+ (1 - \alpha)(2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| \\
&+ \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|))
\end{aligned}$$

Note that when $V(z(t_0)) < \delta^2 \min\{\lambda_{\min}(P(1)), \lambda_{\min}(P(2))\}/\|T^{-1}\|^2$, we have

$$\begin{aligned}
\|x(t_0)\| \le& \|T^{-1}\|\|z(t_0)\| \\
\le& \|T^{-1}\| \sqrt{\frac{V(z(t_0))}{\min\{\lambda_{\min}(P(1)), \lambda_{\min}(P(2))\}}} \\
<& \delta.
\end{aligned}$$

Similar to the proof of Theorem 1, we have the equation shown at bottom of the page. Therefore, when $V(z(t_0)) < \delta^2 \min\{\lambda_{\min}(P(1)), \lambda_{\min}(P(2))\}/\|T^{-1}\|^2$, we have

$$\frac{dV(z)}{dt} < 0$$

The remaining proof is similar to that of Theorem 1 by using Lemma 1 and Proposition 2. □

Remark 8: From the proof of Theorem 2, there exist a Lyapunov function $V(z)$ and an α satisfying $0 < \alpha < 1$ such that

$$\frac{\partial V(z)}{\partial z}(\alpha T f(T^{-1}z, 1) + (1 - \alpha) T f(T^{-1}z, 2)) < 0.$$

For any t, there exists an $i \in \{1, 2\}$ such that

$$\frac{\partial V(z)}{\partial z} T f(T^{-1}z, i) < 0. \tag{31}$$

Then, the switching law can be defined as follows.

A Switching Law: *Switched nonlinear system* (1) *with* $n = 2$ *is switched to or stay at mode* i *at time* t *if (31) is satisfied at time* t. □

V. A Numerical Example

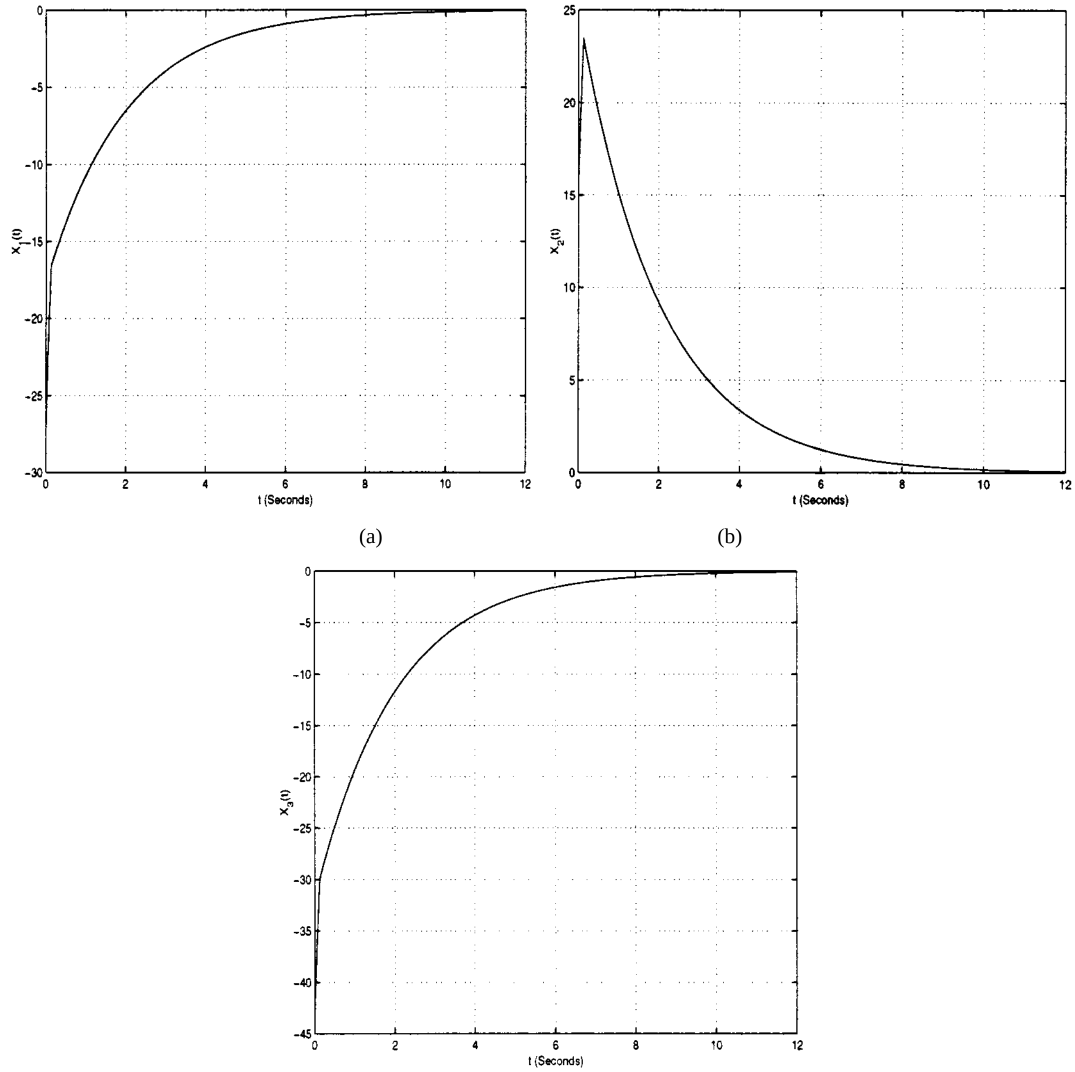
Consider a switched linear system composed of two subsystems given as follows:

$$\begin{aligned}
\dot{x}(t) =& A(1)x(t) = \begin{bmatrix} -0.2222 & -0.2963 & 1.7037 \\ -15.1111 & 0.5185 & 11.5185 \\ 1.7778 & -0.2963 & 0.7037 \end{bmatrix} x(t) \\
\dot{x}(t) =& A(2)x(t) = \begin{bmatrix} -0.7778 & 0.2963 & -1.7037 \\ 15.1111 & -1.5185 & -11.5185 \\ -1.7778 & 0.2963 & -1.7037 \end{bmatrix} x(t)
\end{aligned}$$

Note that $A(1)$ and $A(2)$ are unstable and

$$\hat{X}_1 = \{[1 \quad 6 \quad 0]^{\mathrm{T}}\}; \quad \hat{X}_2 = \{[2 \quad 9 \quad 3]^{\mathrm{T}}, [3 \quad 0.5 \quad 4]^{\mathrm{T}}\}$$

$$\begin{aligned}
\frac{dV(z)}{dt} \le& ((\alpha - 1)(1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) + \alpha(2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|))\|z_2\|^2 \\
&+ (-(1 - \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)\alpha + (1 - \alpha)(2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|))\|z_1\|^2 \\
&+ 2\max\{\lambda_{\max}(P(1)), \lambda_{\max}(P(2))\}\|z\|(\alpha\|T\|\|\tilde{f}(T^{-1}z, 1)\| + (1 - \alpha)\|T\|\|\tilde{f}(T^{-1}z, 2)\|) \\
\le& \frac{((\alpha - 1)(1 - \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|) + \alpha(2\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\| + \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|))}{2}\|z_2\|^2 \\
&+ \frac{(-(1 - \lambda_{\max}(P(1))\|\hat{A}_{12}(1)\|)\alpha + (1 - \alpha)(2\lambda_{\max}(P(1))\|\hat{A}_{11}(2)\| + \lambda_{\max}(P(2))\|\hat{A}_{21}(2)\|))}{2}\|z_1\|^2 \\
<& 0
\end{aligned}$$



(a) (b) (c)

Fig. 1. The time responses $x(t)$ of the system.

It can be shown that Assumptions 1 and 2 are satisfied. Choose

$$T^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 0.5 \\ 0 & 3 & 4 \end{bmatrix}$$

It follows that

$$TA(1)T^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$TA(2)T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

That is

$$\hat{A}_{11}(1) = [-2]; \quad \hat{A}_{22}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix};$$
$$\hat{A}_{12}(1) = [0 \quad 0]; \quad \hat{A}_{11}(2) = [1]$$
$$\hat{A}_{21}(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \hat{A}_{22}(2) = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

Thus

$$P(1) = [0.25]; \quad P(2) = \begin{bmatrix} 0.25 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

Note also that

$$4\lambda_{\max}(P(1))\lambda_{\max}(P(2))\|\hat{A}_{22}(1)\|\|\hat{A}_{11}(2)\| = 0.5 < 1$$

The switching law is chosen based on Remark 4 and it is given as follows:

For any t, the system is switched to or stay at mode i if

$$\frac{\partial V(x)}{\partial x} A(i)x < 0$$

where

$$V(x) = x^{\mathrm{T}} T^{\mathrm{T}} \begin{bmatrix} P(1) & 0 \\ 0 & P(2) \end{bmatrix} Tx$$

Using Theorem 1, the example can be asymptotically stabilized by the above switching law.

This can be illustrated in Fig. 1 with $x(0) = [-30, 15, -45]^{\mathrm{T}}$.

The switchings are very fast and Fig. 2 shows the switchings during periods$[0.12,\ 0.13]$ and$[19.98,\ 20]$.

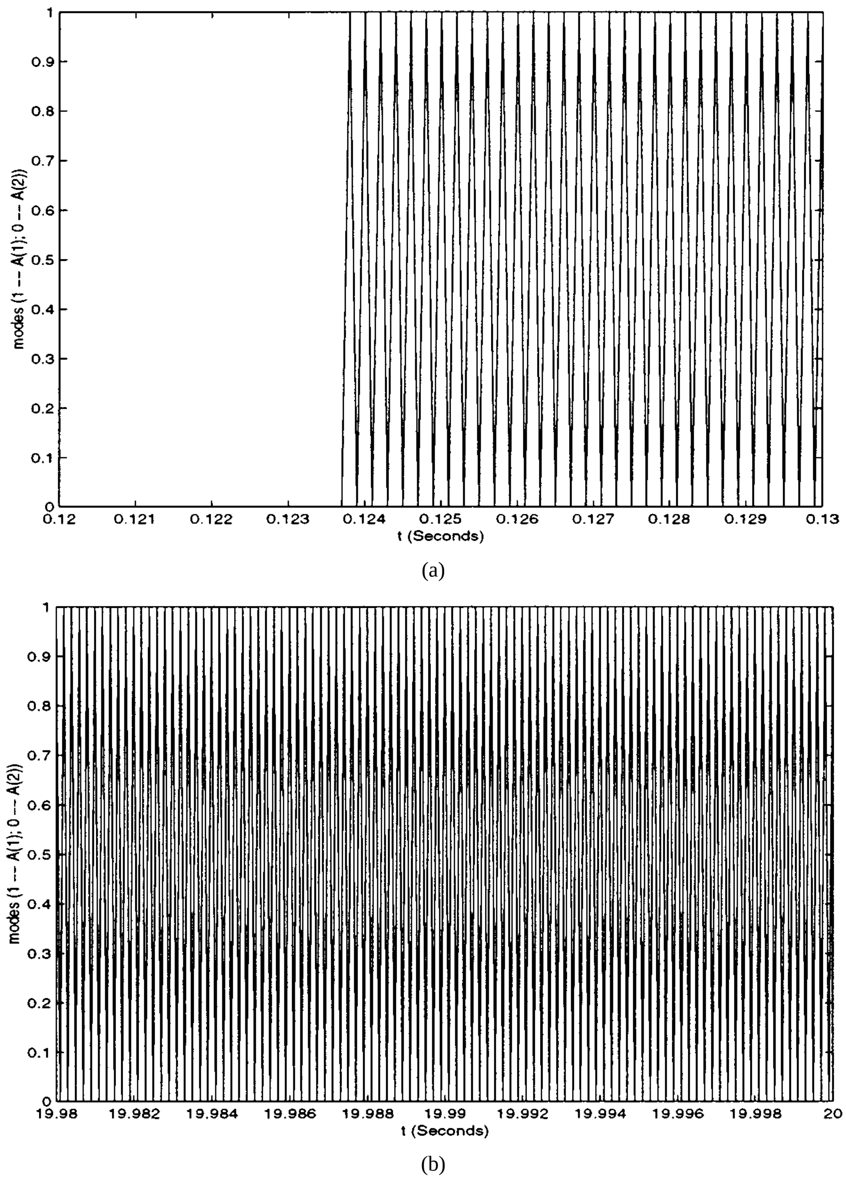


Fig. 2. The switching times of the system during $[0.12, 0.13]$ and $[19.98, 20]$.

VI. Conclusion

A linear state transformation was applied to study the stabilization of a class of switched systems. Under the state transformation, each subsystem can be decomposed into stable and unstable parts. For each stable part, there exists a Lyapunov function. Some sufficient conditions were derived to ensure the sum of these Lyapunov functions to be a Lyapunov function for a convex combination of the whole switched system. This ensured the existence of switching laws to stabilize the switched systems.

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