Looking on the analysis carried out in Section III and additional examples in [5], we may say that the following situation is typical: *the sampled zeros are mostly located neither close to the unit circle nor on the negative real axis close to the point* $z = -1$, *provided the process is not sampled unreasonably slow*. This is an attractive property since it is precisely such zero positions that can create difficulties in estimation, prediction, and control problems. Note that zeros close to the unit circle correspond to a spectrum with a deep notch. If a continuous-time process has a deep notch in the spectrum, then this notch is evened out after sampling, due to the frequency folding. This is an interpretation why sampled zeros are kept away from the unit circle. In Fig. 2(a) and (b), the zero is located close to the point $z = -1$ for $\omega h \approx \pi$. Note that such a slow sampling rate is not reasonable, since it introduces a lot of frequency folding. For appropriate sampling intervals with h at most equal to $1/\omega$, the zero location is always far away from the point $z = -1$.

V. Conclusions

We have investigated possible zero locations for sampled stochastic systems. The zero locations have a significant influence on estimation and prediction algorithms. The locations depend in a quite involved way on the continuous-time process and the sampling interval. The findings can be summarized as follows.

- Zeros can appear only in a *restricted area* within the unit circle. Hence, there are ARMA models (and, in fact, even AR models) that *cannot* be obtained by sampling a continuous-time process.
- For a very short sampling interval, the zeros cluster around $z = 1$ and, provided the continuous-time process has a pole excess larger than one, some positions at the negative real axis.
- A detailed analysis was carried out for second-order systems and all possible zero locations have been characterized. It has been shown how the zero location in this case depends on the continuous-time model parameters and the sampling interval.

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Adaptive Linear Control of Nonlinear Systems

CHANGYUN WEN AND DAVID J. HILL

Abstract—**The theory of stable parameter adaptive control has advanced to allow linear time-varying plants. However, a more honest view of such systems is that they are often derived from inexact linearization about a trajectory of a nonlinear system. Then standard adaptive control based on a linear model can be interpreted as one way to realize a nonlinear controller for a nonlinear plant. The implications of this view are studied. Analytically, the stability problem is seen to be equivalent to showing robustness to time-varying parameters and a locally bounded model uncertainty. It is shown that if the trajectory is known to be within a bound, a parameter estimator with projection can ensure boundedness of departures from the trajectory.**

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I. Introduction

Adaptive control is commonly presented as a technique, whereby a self-tuning linear controller is used on a linear plant with unknown parameters. The theory and design principles for such controllers are becoming quite well understood. The literature is enormous; some attempts to cover recent developments are provided by [1]-[4]. The theory of stable parameter adaptive control has advanced to allow linear slowly time-varying plants [5]-[8]. A more honest view is that the linear time-varying systems are derived from inexact linearization about a trajectory of a nonlinear system. We then see that adaptive linear control is one approach to deriving a nonlinear dynamic controller which can cope with nonlinear dynamics. It is a continuation of the linearization tradition in the control field aided by techniques for parameter estimation. In this note, we begin to study the implications of the nonlinear controller view and extend the results for stability in the presence of time-varying parameters [5]-[8] to allow for the higher order nonlinear terms in a linearized model. Also, our treatment of the time variation is somewhat new.

Thirty years ago, there were no general tools for designing nonlinear controllers. The adaptive viewpoint offered a way to apply conventional linear control techniques. The nonlinear system can be viewed [9], [10] as a time-varying linear part (with unknown parameters) and higher order terms which are ignored. We propose to analyze a typical adaptive linear control algorithm which has been applied to the time-varying linear part. We employ an algorithm along the lines used in [1]-[4] which has been subjected to extensive theoretical analysis in other situations. Implicit in the linearization view is our knowledge of the trajectory about which linearization occurs. Of course, this can only be known if the nonlinear system is known or at least has predictable behavior. Nevertheless, this is inherent in the adaptive linear solution which is used in practical control.

Conditions are given on the nonlinear system functions and certain bias signals which ensure useful boundedness properties of the overall adaptive system. It is shown that it is only necessary to know the nominal trajectory to within a bound.

The structure of the note is as follows. Section II gives the nonlinear problem formulation. Section III transforms the model to a perturbed nonlinear regressor form. Section IV describes the adaptive control scheme. The stability results are presented in Section V.

This note is based on the report [13] where more details can be found.

II. Nonlinear Control Problem

The class of plant to be considered in this note is described by the difference equation

$$\begin{aligned} y(t) = f\big(&y(t-1), y(t-2), \cdots, y(t-n),\\ &u(t-1), \cdots, u(t-n),\\ &\cdots, d(t-1), \cdots, d(t-p)\big) \end{aligned} \tag{2.1}$$

where y, u, and d represent output, input, and disturbance signals, respectively.

Assumption 2.1: n is a known integer.

In simpler notation, we rewrite (2.1) as

$$y(t) = f\big(v(t-1), \omega(t-1)\big) \tag{2.2}$$

where

$$v^T(t-1) := \left[y(t-1), y(t-2), \cdots, u(t-1), \cdots, u(t-n) \right]$$

$$\omega^T(t-1) := \left[d(t-1), \cdots, d(t-p) \right].$$

For any function $f\colon \mathbb{R}^{2n} \times \mathbb{R}^p \to \mathbb{R}$, signals u, d, and given initial conditions $v(0)$, $\omega(0)$, model (2.2) generates a unique solution y. We use v_i, $i = 1, \cdots, 2n$, and ω_i, $i = 1, \cdots, p$ to denote the components of the vectors v and ω.

The control problem is assumed to be regulation about a given desired output signal y^*. In many applications, function f is complicated, but known at least approximately. Otherwise, system behavior may be sufficiently understood that there is some knowledge of a nominal trajectory. We establish this property precisely as the following.

Assumption 2.2: For given y^* and d, $\exists$ an input signal u^* s.t.

$$y^*(t) = f\left(v^*(t-1), \omega(t-1)\right) \tag{2.3}$$

where v^* is derived from (u^*, y^*). Suppose that u_1 is known s.t.

$$\left\{ \sum_{t=i}^{n+i} \left(u_1(t) - u^*(t)\right)^2 \right\}^{1/2} \leqslant U \qquad \text{for all } i \tag{2.4}$$

and U is a constant.

At this stage, it is convenient to introduce the variables

$$\Delta u(t) := u(t) - u_1(t), \qquad \Delta y(t) := y(t) - y^*(t), \tag{2.5}$$

i.e., departures from the known nominal values. Our formulation describes precisely how standard adaptive (linear) control is implemented in a nonlinear real-world context. To aid clarification of this point, consider the scheme shown in Fig. 1. The nonlinear system modeled by (2.2) is controlled by two loops. The high level control establishes the nominal trajectory. For instance, in power systems, it represents the slower control actions due to a human operator, the governor, etc.; in aircraft control it could represent the action of the pilot. This loop senses the system state x. If it is acting successfully, y will be somewhat close to y^* under the action of u_1. Assumption 2.2 effectively says just this (although there is no constraint that U is small).

Of course, u_1 and y^* are known, so the departures $\Delta u, \Delta y$ are measurable. The second loop in Fig. 1 is an indirect parameter adaptive controller [1]. This serves to automatically improve the performance of the system on a faster time scale. In power systems, it could be an adaptive AVR loop. We will be concerned here with conditions under which this adaptive loop is stable.

Clearly, in order to analyze the adaptive regulator, we need to invoke an exact linearization model, i.e., one which relates $\Delta y(t)$ to $\Delta v(t-1)$ via a linear system and whatever else is needed to make the model exact.

III. Model Transformation

We consider the plant with representation (2.2) and use the Taylor expansion around a nominal trajectory to separate a linearized model from the nonlinear error term. The following assumption imposes some additional smoothness restrictions to those implied by Assumption 2.2 on the function $f\colon \mathbb{R}^{2n} \times \mathbb{R}^p \to \mathbb{R}$.

Assumption 3.1: The partial derivatives $f_{v_i} := \dfrac{\partial f}{\partial v_i}$ $i = 1, \cdots 2n$ exist and are locally Lipschitz in v and ω.

We let $\mathscr{B}(x_0, r) := \{x \mid \|x - x_0\| \leqslant r\}$ where $\|\cdot\|$ denotes the Euclidean norm. Also let $\mathscr{B}_r = \mathscr{B}(0, r)$.

We now rewrite model (2.2) in terms of the incremental variables (2.5). It is convenient to denote the signal vector (y^*, u_1) by $\underline{v}_1$. From Assumption 3.1, the system can be linearized about $\underline{v}_1$. Note that

$$\begin{aligned} \Delta y(t) &= y(t) - y^*(t) \qquad \text{from (2.5)} \\ &= f\left(v(t-1), \omega(t-1)\right) - f\left(v^*(t-1), \omega(t-1)\right) \\ &\qquad \text{using (2.2) and (2.3)} \\ &= \phi^T(t-1)\theta(t-1) + R(t) \end{aligned} \tag{3.1}$$

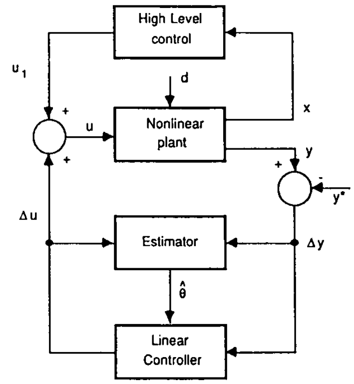

Fig. 1. Adaptive linear control scheme.

where

$$\theta^T(t) = \left[\theta_1(t), \cdots, \theta_{2n}(t) \right]$$

$$\phi^T(t-1) = \left[\Delta y(t-1), \cdots, \Delta y(t-n), \Delta u(t-1), \cdots, \Delta u(t-n) \right]$$

$$\theta_i(t) = f_{v_i}\left(v_1(t), \omega(t)\right) \qquad i = 1, \cdots, 2n \tag{3.2}$$

and $R(t)$ is the remainder term, made of two components

$$R(t) = R_1(t) + R_2(t)$$

where

$$R_1(t) = f\left(v(t-1), \omega(t-1) - f\left(\underline{v}_1(t-1), \omega(t-1)\right) - \phi^T(t-1)\theta(t-1)\right. \tag{3.3a}$$

$$R_2(t) = f\left(\underline{v}_1(t-1), \omega(t-1)\right) - f\left(v^*(t-1), \omega(t-1)\right). \tag{3.3b}$$

$R_1(t)$ will be referred to as the nonlinear error and $R_2(t)$ as the reference solution error.

Note that (3.1) has the so-called regression form commonly used in analysis of parameter estimation algorithms. We refer to θ and ϕ as the parameter and regression vectors, respectively. We now make a further assumption.

Assumption 3.2: The reference signals, disturbance, and derivatives f_{v_i} are such that $\theta(t) \in \mathscr{C} \forall t$ where $\mathscr{C} \subset \mathbb{R}^{2n}$ is a known compact convex region.

This assumption restricts the slope of f along $\underline{v}_1$ to be within finite limits.

As a consequence of Assumption 3.2, we note that

$$\|\theta_1 - \theta_2\| \leqslant k_c \tag{3.4a}$$

$$\|\theta\| \leqslant k_\theta \tag{3.4b}$$

$\forall \theta, \theta_1, \theta_2 \in \mathscr{C}$. k_c, k_θ are constants depending on the size of $\mathscr{C}$.

A key property of the linearization model (3.1) is that the remainder term R is linearly bounded by ϕ.

Fact 3: $\exists \epsilon_r$, $l \in \mathbb{R}_+$ s.t.

$$|R(t)| \leqslant \epsilon_r \|\phi(t-1)\| + l \qquad \forall \phi(t-1) \in \mathscr{B}_r. \tag{3.5}$$

Proof: From Assumption 3.1 and the mean value theorem applied to (3.3a)

$$R_1(t) = \left(\left.\frac{\partial f}{\partial v}\right|_{\bar{v}} \right)^T \left(v(t-1) - v_1(t-1)\right) - \left(\left.\frac{\partial f}{\partial v}\right|_{v_1} \right)^T \phi(t-1)$$

where $\bar{v} = \lambda v(t-1) + (1-\lambda)\underline{v}_1(t-1)$ for some $\lambda \in [0, 1]$

$$= \left(\left.\frac{\partial f}{\partial v}\right|_{\bar{v}} - \left.\frac{\partial f}{\partial v}\right|_{\underline{v}_1} \right)^T \phi(t-1).$$

Then let

$$\epsilon_r := \sup_{\substack{\bar{v}\text{ s.t.}\\ \|\bar{v}-\underline{v}_1\|\leqslant r}} \left\| \frac{\partial f}{\partial v}\bigg|_{\bar{v}} - \frac{\partial f}{\partial v}\bigg|_{\underline{v}_1} \right\| \tag{3.6}$$

and clearly, $|R_1(t)| \leqslant \epsilon_r \|\phi(t-1)\| \forall \phi(t-1) \in \mathscr{B}_r$. Similarly, we get from (2.4) $|R_2(t)| \leqslant LU$ where L is a constant (bounded by any Lipschitz constant of f). Taking $l := LU$, the result follows. □

Comments 3.1:

1) Clearly (3.6) links the bound in (3.5) to changes of the derivative $\frac{\partial f}{\partial v}$ in the ball $\mathscr{B}_r$. The value of parameter ϵ_r thus reflects the degree of nonlinearity of the plant. This is clearly demonstrated in (3.6). Loosely speaking, a more nonlinear plant will have larger ϵ_r for smaller r.

2) While being aware of its existence, we do not need to know the value of ϵ_r or l.

3) In the use of linear models [1], the system with unstructured disturbances is effectively described relative to signals $(0, u^*)$. In the design of adaptive control, we implicitly use $u_1 = 0$ to approximate u^*. The number U can be large.

We now make some assumptions which restrict the speeds of the signals u_1, y^*, and d. These will be passed through the smoothness Assumption 3.1 into the time variations of the linearized form (3.1).

Assumption 3.3: $\exists \delta_1, \delta_1', \delta'' \in \mathbb{R}_+$ s.t. y^*, d satisfy

$$|y^*(t) - y^*(t-1)| \leqslant \delta_1 \qquad \forall t$$

$$|d(t) - d(t-1)| \leqslant \delta_1' \qquad \forall t$$

and u_1 (Assumption 2.2) satisfies

$$|u_1(t) - u_1(t-1)| \leqslant \delta_1'' \qquad \forall t.$$

From Assumption 3.3, we easily conclude that there exists δ_v, $\delta_\omega > 0$ such that

$$\|\underline{v}_1(t) - \underline{v}_1(t-1)\| \leqslant \delta_v \qquad \forall t \tag{3.7a}$$

$$\|\omega(t) - \omega(t-1)\| \leqslant \delta_\omega \qquad \forall t. \tag{3.7b}$$

From Assumption 3.1, the f_{v_i} are locally Lipschitz continuous. Specifically, there exists $F_1, F_2 \geqslant 0$, and δ_2, δ_2' such that

$$|f_{v_i}(v, \omega) - f_{v_i}(\bar{v}, \omega)| \leqslant F_1 \|v - \bar{v}\| \tag{3.11a}$$

for a given ω, $v - \bar{v} \in \mathscr{B}_{\delta_2}$, $i = 1, \cdots, 2n$, and

$$|f_{v_i}(v, \omega) - f_{v_i}(v, \bar{\omega})| \leqslant F_2 \|\omega - \bar{\omega}\| \tag{3.11b}$$

for a given v, $\omega - \bar{\omega} \in \mathscr{B}_{\delta_2'}$, $i = 1, \cdots, 2n$.

Assumption 3.4: The signal variations in Assumption 3.3 are slow enough to ensure $\delta_2 \geqslant \delta_v$ and $\delta_2' \geqslant \delta_\omega$.

Combining inequalities (3.7) and (3.8), gives a restriction on the time variation of $\theta(t)$, i.e.,

$$\begin{aligned}|\theta_i(t) - \theta_i(t-1)| &= |f_{v_i}(v_1(t), \omega(t)) - f_{v_i}(v_1(t-1), \omega(t-1)|\\ &\leqslant |f_{v_i}(v_1(t), \omega(t)) - f_{v_i}(v_1(t-1), \omega(t))|\\ &\quad + |f\hat{v}_i(v_1(t-1), \omega(t))\\ &\quad - f_{v_i}(v_1(t-1), \omega(t-1))|\\ &\leqslant F_1\delta_v + F_2\delta_\omega \quad \text{using (3.7), (3.8), and Assumption 3.5.}\end{aligned}$$

It follows that there exists $\epsilon_\theta \geqslant 0$ such that

$$\|\theta(t) - \theta(t-1)\| \leqslant \epsilon_\theta \qquad \forall t. \tag{3.9}$$

Comments 3.2: The essential parameters of the nonlinearity in (2.2) are r, ϵ_r, and ϵ_θ. These are clearly related. In the bounds (3.5) and (3.9), ϵ_r and ϵ_θ will typically be larger for larger r.

IV. Adaptive Control Scheme

We study a certainty equivalence adaptive controller along the lines studied previously in many situations [1]. The parameter estimator is taken as the simple gradient scheme

$$\hat{\theta}(t) = \mathscr{P}\left\{\hat{\theta}(t-1) + \frac{\phi(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)}\right\} \tag{4.1}$$

where $\hat{\theta}(t)$ denotes the estimate of $\theta(t)$, $\mathscr{P}$ denotes the projection operator which ensures that $\hat{\theta}(t) \in \mathscr{C} \forall t$ [1], and

$$e(t) = \Delta y(t) - \phi^T(t-1)\hat{\theta}(t-1) \tag{4.2}$$

$e(t)$ is the prediction error.

The stability analysis relies on several key properties of the estimator. The following result can be proved using standard steps. It is related to results in [5], [14] but is closest in detail to the result in [15]; the main difference here is the need to accommodate the locally bounded error term $R(t)$. Suppose $r > r_0 \in \mathbb{R}_+$ is such that $\frac{l}{r_0} \leqslant \delta$.

Lemma 4.1: The estimator (4.1), (4.2), applied to system (2.3) [or (3.1)] has the properties:

1) $\exists k \in \mathbb{R}_+$ s.t. $|\tilde{e}(t)| \leqslant k$ (4.3)

provided $\phi(t-1) \in \mathscr{B}_r$ where

$$\tilde{e}(t) := \frac{e(t)}{\left(1 + \|\phi(t-1)\|^2\right)^{1/2}}$$

2) $\|\hat{\theta}(t) - \hat{\theta}(t-1)\| \leqslant |\tilde{e}(t)| \quad \forall t$ (4.4)

3) $\sum_{i=t_0+1}^{t} \tilde{e}(i) \leqslant \alpha_1 + \alpha_2(t - t_0) + \alpha_3(t - t_0)$ (4.5)

provided $\phi(i) \in \mathscr{B}_r$ and $\|\phi(i)\| > r_0$, $i = t_0, t_0 + 1, \cdots, t-1$ where α_1, α_2, and α_3 are given by

$$\alpha_1 = k_c^2 \qquad \alpha_2 = 0(\epsilon_r, \epsilon_\theta) \qquad \alpha_3 = 0(\delta). \tag{4.6}$$

Comments 4.1:

1) The form of (4.5) is special in that α_2 can be made arbitrarily small by reducing ϵ_r and ϵ_θ.

2) The more commonly used (in practice) least-squares estimator has similar properties, but the analysis is more tedious [1].

The estimator is combined with a conventional (nonadaptive) controller with the system parameters replaced by $\hat{\theta}(t)$. Here, we consider a pole assignment regulator of the form

$$\hat{L}(t-1)\Delta u(t) = -\hat{P}(t-1)\Delta y(t) \tag{4.7}$$

where $\hat{L}$, $\hat{P}$ are derived from

$$\hat{A}(t) = 1 - \hat{\theta}_1(t)q^{-1} - \cdots - \hat{\theta}_n(t)q^{-n} \tag{4.8a}$$

$$\hat{B}(t) = \hat{\theta}_{n+1}(t)q^{-1} + \cdots + \hat{\theta}_{2n}(t)q^{-n} \tag{4.8b}$$

$$\hat{A}(t-1)\hat{L}(t-1) + \hat{B}(t-1)\hat{P}(t-1) = A^*. \tag{4.9}$$

A^* is a given monic polynomial in shift operator q^{-1} of degree $2n$. Then $\hat{L}$ and $\hat{P}$ provide a strictly proper regulator. We express these in the form

$$\hat{L}(t) = 1 + \hat{l}_0(t)q^{-1} + \cdots + \hat{l}_{n-1}(t)q^{-n} \tag{4.10a}$$

$$\hat{P}(t) = \hat{p}_0(t)q^{-1} + \cdots + p_{n-1}(t)q^{-n}. \tag{4.10b}$$

Assumption 4.1: The polynomial $z^{2n}A^*$ is strictly (discrete-time) Hurwitz.

Just as in all previous discussions of indirect adaptive control, a technical difficulty (invariant under problem reformulation) arises in the solution of (4.9). We require $\|\hat{L}\|$, $\|\hat{P}\|$ to be bounded where $\|\cdot\|$

denotes the norm of the vector of polynomial coefficients. Thus, we require a further restriction on the system model [4], [7].

Assumption 4.2: The convex region $\mathscr{C}$ in Assumption 3.2 has the additional property that for all $\theta \in \mathscr{C}$, the linearized system model is uniformly stabilizable.

Comments 4.2:

1) An arbitrarily large region of the parameter space can be covered by the use of multiple convex regions [4]. The analysis presented below can be extended to this situation.

2) Assumption 4.2 effectively introduces a local controllability requirement on the nonlinear plant.

V. Stability Analysis

The main stability results establish that the overall adaptive system is locally bounded in a sense where all variables are departures from the nominal trajectory represented by v_1.

The establishment of a framework for analysis follows standard steps for certainty equivalence adaptive controllers [1]. Then the closed-loop system equations (4.2) and (4.7) can be combined to give

$$\phi(t+1) = A(t)\phi(t) + be(t+1) \tag{5.1}$$

where

$$A(t) := \left[\begin{array}{cccccc} \hat{\theta}_1(t) & \cdots & \hat{\theta}_n(t) & \hat{\theta}_{n+1}(t) & \cdots & \hat{\theta}_{2n}(t) \\ 1 & & 0 & 0 & & 0 \\ 0 & 1 & 0 & 0 & & 0 \\ \hline -\hat{p}_0(t) & \cdots & -\hat{p}_{n-1}(t) & -\hat{l}_0(t) & \cdots & -\hat{l}_{n-1}(t) \\ 0 & & 0 & 1 & & 0 \\ 0 & & 0 & 0 & 1 & 0 \end{array}\right] \tag{5.2a}$$

$$b^T := [1 \quad 0 \quad 0 \quad \cdots \quad 0]. \tag{5.2b}$$

The stability analysis of (5.1) makes use of the following lemma. It is a refinement of a result given by Kreisselmeier [5]; the proof in [13] shows it completely as a discrete-time counterpart of one by Middleton and Goodwin [7]. Many similar results exist in the adaptive control literature.

Lemma 5.1: Consider a linear time-varying system of the form

$$x(t+1) = A(t)x(t). \tag{5.3}$$

Suppose:

1) $A(t)$ is bounded,

2) $\sum_{\tau=t_0+1}^{t} \|A(\tau) - A(\tau-1)\|^2 \leqslant k_0 + k_1(t - t_0)$ for $t > t_0$ where k_0, k_1 are positive constants with k_1 sufficiently small,

3) $|\lambda_i(A(t))| < 1$ for all $t, i = 1, \cdots, n$.

Then the transition matrix of (5.3), denoted $\psi(t, \tau)$, satisfies

$$\|\psi(t,\tau)\| \leqslant c\mu^{t-\tau} \qquad \text{for } t \geqslant \tau \tag{5.4}$$

where $\mu \in (0, 1)$ and c is a constant.

Our first stability result can now be proved.

Theorem 5.1: Consider the adaptive scheme consisting of plant (2.2) [modeled by (3.2)], estimator (4.1), (4.3), and regulator (4.12)–(4.14). Under all above assumptions $\exists r^*$, ϵ_r^*, ϵ_θ^*, and r_0 s.t. $r \geqslant r^*$, $\epsilon_r \leqslant \epsilon_r^*$, $\epsilon_\theta \leqslant \epsilon_\theta^*$, and $\phi(0) \in \mathscr{B}_{r_0}$ ensures $\phi(t) \in \mathscr{B}_r \forall t$. In particular, the control signal $\Delta u(t)$ and tracking error $\Delta y(t)$ remain bounded.

Proof: Only an outline proof is given. Many of the details are similar to those given elsewhere by Wen and Hill (see [15] and the report [13]).

Divide the time sequence Z_+ into two subsequences

$$Z_1 = \{t \in Z_+ | \|\phi(t)\| > r_0\} \qquad Z_2 = \{t \in Z_+ | \|\phi(t)\| \leqslant r_0\}.$$

Recall $r > r_0$. Clearly, we only need to show $\|\phi(\tau)\| \leqslant r$ for $\tau \in Z_1$. Also, constrain the initial condition to satisfy $\|\phi(0)\| \leqslant r_0$.

We use an inductive proof by assuming $\|\phi(t)\| \leqslant M$, $\tau = 0, \cdots, t-1$ and choose t_0 so that $t_0 - 1 \in Z_2$ and $t_0, \cdots, t-1 \in Z_1$.

From Lemma 4.1, it is easy to check that $A(t)$ in (5.2a) satisfies all the conditions in Lemma 5.1 provided $\phi(\tau) \in \mathscr{B}_r$ for $\tau = 0, 1, \cdots, t-1$. In condition 2), we have $k_1 = k(\alpha_2 + \alpha_3)$ where k is an independent constant.

Thus, for the linear time-varying system

$$\phi(t+1) = A(t)\phi(t) \tag{5.5}$$

we conclude that the transition matrix $\Phi(t, \tau)$ satisfies

$$\|\Phi(t,\tau)\| \leqslant C\sigma^{t-\tau} \tag{5.6}$$

for $t \geqslant \tau$ if $\alpha_2 \leqslant \bar{\alpha}_2^*$, $r_0 \geqslant \bar{r}_0^*$ and $\phi(\tau) \in \mathscr{B}_r$ for $\tau = 0, 1, \cdots, t-1$. $\bar{\alpha}_2^*$ is a sufficiently small positive constant, $\bar{r}_0^*$ is a sufficiently large constant, and $\sigma \in (0, 1)$.

The general solution of (5.1) is

$$\phi(t) = \Phi(t, t_0)\phi(t_0) + \sum_{\tau=t_0}^{t-1} \Phi(t,\tau)be(\tau+1). \tag{5.7}$$

Using (5.6) and the Schwarz inequality in standard steps, we get

$$\sigma^{-t}\|\phi(t)\|^2 \leqslant S^2(t) + C_2 \sum_{\tau=t_0}^{t-1} \sigma^{-t}\|\phi(\tau)\|^2 |\bar{e}(\tau+1)|^2 \tag{5.8}$$

where

$$S^2(t) = 2C^2\sigma^t\|\phi(t_0)\|^2 + C_1. \tag{5.9}$$

Applying the discrete Grownwall lemma [12] to (5.9), using the theorem of the arithmetic and geometric means [12] and Lemma 4.1, we get [13]

$$\|\phi(t)\|^2 \leq 2C^2(\sigma_c^*)^t(1+kt)(\|\phi(t_0)\| + C_3)$$
$$\text{for } \alpha_2 \leqslant \bar{\bar{\alpha}}_2^* \text{ and } r_0 \geqslant \bar{\bar{r}}_0^* \tag{5.10}$$

where $\sigma < \sigma_c^* < 1$, $\bar{\bar{\alpha}}_2^*$ is a small constant, and $\bar{\bar{r}}_0^*$ is a sufficiently large constant.

Equation (5.10) gives

$$\|\phi(t)\|^2 \leqslant C_4\|\phi(t_0)\|^2 + C_3. \tag{5.11}$$

So if $r^2 > C_3$ and

$$\|\phi(t_0)\|^2 \leqslant \frac{r^2 - C_3}{C_4}, \tag{5.12}$$

we have

$$\|\phi(t)\|^2 \leqslant r^2,$$

i.e., $\phi(t) \in \mathscr{B}_r$. The induction proof is complete.

It remains to identify the constants in the Theorem statement. We have already $r^* = \max\{r_0, \sqrt{C_3}\}$, $r_0^2 = \max\{\bar{r}_0^*, \bar{\bar{r}}_0^*, \frac{r^2 - C_3}{C_4}\}$. Now set $\alpha_2^* = \min\{\bar{\alpha}_2^*, \bar{\bar{\alpha}}_2^*\}$ and select ϵ_θ^* and ϵ_r^* so that α_2, given by (4.6b), satisfies $\alpha_2 \leqslant \alpha_2^*$. □

Comments 5.1:

1) Referring back to Comment 3.3, we see that the bounds on ϵ_r, ϵ_θ, and also $r > r^*$ are simply restrictions on the class of nonlinearities allowed.

2) A further qualitative result that can be derived from the bounds concerns the degree of stability in A^*. A smaller σ allows larger values of ϵ_r and ϵ_θ, i.e., more severe nonlinearities. If the bound on $R(t)$ at (3.5) is global in the sense that ϵ_r, l are independent of r, a simpler result is easily seen to hold.

Theorem 5.2: Suppose the conditions of Theorem 5.1 are altered to allow $f(\cdot, \omega)$ to be globally Lipschitz. Then $\exists \epsilon_r^*$ and ϵ_θ^* s.t. $\epsilon_r \leqslant \epsilon_r^*$

and $\epsilon_\theta \leqslant \epsilon_\theta^*$ ensure the system is BIBS stable (about the nominal trajectory).

From the proof of Theorem 5.1, we can see that l does not affect stability in this case. This justifies our earlier Comment 3.1.3.

VI. Conclusions

The interpretation of adaptive linear control as a design technique for well-defined discrete-time nonlinear systems has been studied. The stability conditions essentially limit the gradient of the functional giving a difference equation description. For global stability, this functional is required to be globally Lipschitz.

The novel proof technique based on induction with respect to regressor boundedness does not have an immediate counterpart in continuous time. Nevertheless, it is interesting to speculate that a similar result holds.

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Stability Analysis of a Family of Matrices

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***Abstract*—In this note, we examine the stability of a family of matrices using the Lyapunov equation approach.**

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I. Introduction

The study of a family of matrices is primarily motivated by the need to check robust stability of a dynamical system which may be subjected to parameter perturbations. The perturbations in the system matrix can arise because of changes in operating conditions, linearization, data errors, etc. Hence, there has been considerable interest in the stability analysis of matrices with uncertain coefficients in the recent literature [1]–[12].

If there is no knowledge about the structural perturbation of the system matrix, then a simple sufficient condition for the stability of the system matrix is given by the bound on the singular value of the perturbed matrix [1], [2]. The bound on the perturbation matrix can be improved if the error matrix is highly structured [6]–[10]. In addition, state transformation [8] may be used to improve the result.

In this note, we shall extend the use of the Lyapunov equation method to check the stability of a family of matrices which is defined as the convex hull of a finite number of matrices. The family of interval matrices is a special case where each element has independent perturbation. While the stability of interval matrices cannot be inferred from the stability of its vertex matrices [11], [12], we shall show that the family of matrices will be stable if the symmetric matrices associated with the vertex matrices have matrix measures less than two.

II. Problem Formulation

Consider the following linear system:

$$\begin{aligned}\dot{x}(t) &= Ax(t)\\ &= [A^o + E]x(t) \end{aligned}\tag{2.1}$$

where $x(t) \to R^n$ is the state vector, A^o is the $n \times n$ time-invariant, nominally asymptotically stable system matrix, and E is the perturbation matrix.

Explicit bounds for robust stability of A under unstructured perturbations have been examined in [1] and [2]. In here, we are interested in the case where the perturbation matrix E is highly structured. In particular, we are interested in determining the stability of the family of matrices

$$S_A \triangleq \text{conv}\,\{A_i, \quad i = 1, 2, \cdots, m\} \tag{2.2}$$

where A_i are the vertex matrices of S_A. The set S_A may also be redefined as

$$S_A \triangleq A^o + \text{conv}\,\{E_i, \quad i = 1, 2, \cdots, m\} \tag{2.3}$$

where $A^o \in S_A$ defined in (2.2) and

$$E_i = A_i - A^o. \tag{2.4}$$

The choice of A^o is quite arbitrary, but a popular choice of A^o is given by

$$A^o = \frac{1}{m}\sum_{i=1}^{m} A_i. \tag{2.5}$$

An iterative approach to search for a better A^0 has been proposed in [15].

III. Mathematical Preliminaries

Conceptually, any robustness analysis method which is based on Lyapunov's method can be traced back to the following theorem.

Theorem 3.1: The matrix A in (2.1) is stable if

$$E^TP + PE < 2I \tag{3.1}$$

where P is the solution to the Lyapunov matrix equation

$$(A^o)^TP + PA^0 + 2I = 0. \tag{3.2}$$

Proof: The matrix A will be stable [13] if

$$A^TP + PA = -Q$$