# Correspondence

## Robust Adaptive Control of Proper Systems

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***Abstract*— To date, all the adaptive control algorithms have been proposed only for strictly proper systems. In this paper, a scheme is proposed to design an adaptive controller for proper systems. To study the robustness of the adaptive controller, both additive and multiplicative types of unmodeled dynamics are considered and can also be allowed to be proper, or even improper. Global bounded input bounded output stability is established. The achievement of a small in the mean tracking error and perfect tracking/rejection of deterministic trajectories/disturbances in the absence of system unmodeled dynamics are discussed. The results are also verified by simulation studies.**

***Index Terms*—Adaptive control, proper systems, robustness, stability, tracking.**

### I. Introduction

Stability results of adaptive control systems have been well established and understood [1]–[12]. However, all the adaptive control algorithms, from which system stability were established, were proposed only based on strictly proper transfer functions. As shown in [13], the stability properties can no longer be guaranteed in the presence of throughputs. Thus a scheme was proposed to consider systems with proper transfer functions in [13] and [14]. The idea is to cascade a first-order filter with the system such that a strictly proper system can be obtained. The adaptive controller is still designed from the resulting strictly proper transfer function. Under this scheme, the problem formulated is to force the filtered output to follow that of a reference model. Also in the context of robust adaptive control, no result has been obtained in the presence of proper additive unmodeled dynamics so far.

The strict properness restriction discussed above is perhaps due to the requirement that the order of the filter employed for controller design cannot be greater than that of the nominal system model ([3], [4], [8], [9], [11], [13], and [14]). In this paper, a scheme using higher order filters is proposed, and this allows us to design an adaptive controller directly from the nominal plant transfer function which may be proper. In the design, the plant output itself is formulated to track a given reference trajectory. The robustness of the adaptive controller is also examined. Both additive and multiplicative unmodeled dynamics considered can be proper, or even improper. By using an analysis technique similar to those in [4], it is shown that global stability is guaranteed in the presence of unmodeled dynamics and bounded external disturbances. The achievement of a small in the mean tracking error and perfect tracking/rejection of deterministic trajectories/disturbances in the absence of system unmodeled dynamics are discussed. Our simulation studies also show the effectiveness of the control scheme.

The remaining part of the paper is organized as follows. The mathematical model of the class of plants to be controlled is presented in Section II. The adaptive scheme is proposed in Section III. Stability is established in Section IV. In Section V, the control scheme is verified by simulation results. Finally, the paper is concluded in Section VI.

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### II. Plant Models

Let $p$ denote the differential operator $\frac{d}{dt}$, and let $y(t)$ and $u(t)$ be the plant output and input, respectively. Then the class of plants to be controlled is modeled as follows:

$$y(t) = \{H(p)[1 + \epsilon_1 H_1(p)] + \epsilon_2 H_2(p)\}u + \omega(t) \tag{1}$$

where $H(p)$ is the nominal transfer function of the plant defined as

$$H(p) = \frac{B(p)}{A(p)}$$
$$A(p) = p^n + a_{n-1}p^{n-1} + \cdots + a_0$$
$$B(p) = b_m p^m + b_{m-1}p^{m-1} + \cdots + b_0$$

$\epsilon_1, \epsilon_2$ are nonnegative constants, $H_1(p)$ and $H_2(p)$ are the multiplicative and additive unstructured uncertainties, respectively, and $\omega(t)$ denotes external disturbances.

For the plant, we make the following assumptions.

*Assumption 2.1:*

- A1) $n$ is known and $m$ is less than or equal to $n$.
- A2) The coefficients of the polynomials $A(p)$ and $B(p)$ are inside a known compact convex region in which $\hat{A}$ and $\hat{B}$ are uniformly coprime, where $\hat{A}, \hat{B}$ are the estimates of $A, B$.
- A3) $H_1$ and $H_2$ are stable and their relative degrees should be greater than or equal to $-1$.
- A4) $\omega(t)$ is bounded.

*Remark 2.1:* From the above assumption, we note that the nominal transfer function $H(p)$ can be proper and uncertainties $H_1, H_2$ can be improper. This implies that $|\epsilon_1 H_1(j\omega)|$ and $|\epsilon_2 H_2(j\omega)|$ may be large at high frequencies.

Model (1) is now transformed into the following form:

$$A(p)y(t) = B(p)u(t) + \eta(t) \tag{2}$$

where $\eta(t)$ denotes the effect of the modeling error and is given as

$$\eta(t) = [\epsilon_1 B H_1 + \epsilon_2 A H_2]u(t) + A\omega(t). \tag{3}$$

Similar to [17] in reducing the effects of the modeling errors including the bounded noise and high frequency unmodeled dynamics, a low-pass filter $\frac{1}{F}$ is introduced, where $F$ is a monic Hurwitz polynomial given as

$$F(p) = p^\gamma + f_{\gamma-1}p^{\gamma-1} + \cdots + f_0$$

and the order $\gamma$ is chosen to be greater than $n$. (If the relative degrees of $H_1$ and $H_2$ are equal to $-1$, $\gamma$ is chosen to be greater than $n+1$.)

*Remark 2.2:* As shown later, the order of the filter is a key point to relax the strict properness restriction of previous adaptive control schemes. Earlier in [3], [4], [8], [9], [11], [13], and [14], the order cannot be higher than $n$.

With this filter, the following filtered variables are defined:

$$y_f(t) = \frac{1}{F} y(t) \tag{4}$$

$$u_f(t) = \frac{1}{F} u(t) \tag{5}$$

$$\eta_f(t) = \frac{1}{F} \eta(t). \tag{6}$$

Now operating this filter on (2), we get

$$A y_f(t) = B u_f(t) + \eta_f(t). \tag{7}$$

To derive a suitable adaptive control algorithm, (7) is rewritten as

$$\begin{aligned} p^n y_f &= (p^n - A) y_f + B u_f(t) + \eta_f(t) \\ &= \phi^T(t)\theta_* + \eta_f(t) \end{aligned} \tag{8}$$

where

$$\phi^T(t) = [y_f(t), \cdots, p^{n-1} y_f(t), u_f(t), \cdots, p^m u_f(t)] \tag{9}$$

$$\theta_*^T = [-a_0, \cdots, -a_{n-1}, b_0, \cdots, b_m]. \tag{10}$$

Note that (8) is differentiator-free since the order of $F$, i.e., $\gamma$, is greater than $n$. Also from Assumption 2.1, we have $\theta_* \in \mathcal{C}$ where $\mathcal{C}$ is a compact convex set in $\Re^{n+m}$. Thus we have

$$\|\theta_1 - \theta_2\| \leq k_\theta \tag{11}$$

where $k_\theta$ is a constant depending on the size of $\mathcal{C}$ and $\|\cdot\|$ denotes the Euclidean norm.

From the stability of the unmodeled dynamics, we can readily obtain an overbounding function of the modeling error $\eta_f$. This is given as follows.

*Lemma 2.1:* For all members of the class of systems satisfying Assumption 2.1, there exists a constant $\epsilon \geq 0$ such that for all $t$

$$|\eta_f(t)| \leq \epsilon \sup_{0 \leq \tau \leq t} \|x(\tau)\| + d_0 \tag{12}$$

where

$$x^T(t) = \left[\frac{y}{F}, \cdots, p^{n-1}\frac{y}{F}, \frac{u}{F}, \cdots, p^{\gamma-1}\frac{u}{F}\right] \tag{13}$$

$d_0$ is a constant bounding $\frac{A}{F}\omega(t)$, and an exponentially decaying term depending on initial conditions.

*Proof:* Suppose $V(p)$ is a stable polynomial of the form

$$V(p) = p^{\gamma-1} + v_{\gamma-2}p^{\gamma-2} + \cdots + v_0.$$

From (6) and (3), we have

$$\begin{aligned} \eta_f(t) &= \frac{\epsilon_1 B H_1 + \epsilon_2 A H_2}{V} V \frac{u}{F} + \frac{A\omega}{F} \\ &= \frac{\epsilon_1 B H_1 + \epsilon_2 A H_2}{V} x_u + \frac{A\omega}{F} \end{aligned}$$

where

$$\begin{aligned} x_u &= v_0 \frac{u}{F} + v_1 p \frac{u}{F} + \cdots + v_{\gamma-2} p^{\gamma-2}\frac{u}{F} + p^{\gamma-1}\frac{u}{F} \\ &= v^T x \end{aligned} \tag{14}$$

where

$$v^T = [0, 0, \cdots, v_0, v_1, \cdots, v_{\gamma-2}, 1].$$

Clearly

$$|x_u(t)| \leq k_v \|x(t)\| \tag{15}$$

where $k_v$ is a constant depending on the coefficients of $V(p)$.

Then the result follows from Assumptions A3, A4, and (15). □

*Remarks 2.3:*

1) The constant $\epsilon$ in (12) can be made sufficiently small by reducing $\epsilon_1$ and $\epsilon_2$. While being aware their existence, we do not assume any knowledge of the constants $\epsilon$ and $d_0$.
2) Also note that vector $x(t)$ can be obtained from measurement. This vector will be used for the adaptive controller design in the next section.

Suppose $y^*$ is a given reference set-point for output $y$. The control problem is to design a controller for the class of plants satisfying Assumption 2.1 so that the closed-loop system is stable in the sense that all signals in the system are bounded for arbitrary bounded $y^*$ and initial conditions.

## III. Adaptive Control Scheme

An indirect adaptive control scheme is proposed to solve the control problem in this section. The adaptive controller can be obtained by the design of two independent modules: a parameter estimator and a linear controller designed based on the *Certainty Equivalence Principle* [1].

### A. Parameter Estimator

The following estimation algorithm with projection is introduced to the estimator:

$$\dot{\hat{\theta}}(t) = \mathcal{P}\left\{\beta \frac{\phi(t)e(t)}{1 + x^T(t)x(t)}\right\} \tag{16}$$

where $\beta$ is a positive constant denoting the adaptation gain, $\hat{\theta}$ is the estimate of $\theta_*$, $x(t)$ is defined in (13), $e(t)$ is the prediction error defined as

$$e(t) = p^n y_f(t) - \phi^T(t)\hat{\theta}(t) \tag{17}$$

and $\mathcal{P}\{\cdot\}$ denotes a projection operation proposed by Pomet and Praly in [15]. Such an operation can ensure that all the estimated parameter vector $\hat{\theta}(t) \in \mathcal{C}$ for all $t$ if $\hat{\theta}(0) \in \mathcal{C}$.

*Remark 3.1:* For the estimator in (16) and (17), the vectors $x(t)$ is used in the normalization. As seen from the stability analysis of the next section, $x(t)$ is the state vector of the closed-loop system. The normalization in the estimator is static.

Now some useful properties of the estimator in (16) and (17) can be stated as in the following lemma.

*Lemma 3.1:* Suppose $M_0$ is a positive constant s.t. $d_0/M_0 \leq \delta$ where $\delta$ is a sufficiently small positive constant. The estimator (16) and (17), applied to plants given in (1), has the following properties.

1) Define

$$\tilde{e}(t) = \frac{e(t)}{(1 + x^T(t)x(t))^{1/2}}.$$

If $\|x(t)\| \geq M_0$ and $\sup_{0 \leq \tau \leq t}\|x(\tau)\| = \|x(t)\|$ for all $t \geq t_0$, then

a)

$$|\tilde{e}(t)| \leq \beta(k_1 + \epsilon + \delta), \qquad \text{for } t \geq t_0 \tag{18}$$

where $k_1$ is a constant depending on $k_\theta$ in (11);

b)

$$\int_{t_0}^{t} \tilde{e}^2(\tau)\,d\tau \leq k_2 + \alpha_1(t - t_0) + \alpha_2(t - t_0), \qquad \text{for } t \geq t_0 \tag{19}$$

where

$$k_2 = \frac{1}{2\beta} k_\theta^2 \tag{20}$$

$$\alpha_1 = \beta(k_1 + 2\epsilon)\epsilon \tag{21}$$

$$\alpha_2 = \beta(k_1 + 2\delta)\delta \tag{22}$$

2)
$$\|\dot{\hat{\theta}}(t)\| \le \beta |\tilde{e}(t)|. \tag{23}$$

*Proof:* Comparing (9) with (13), we have

$$\|\phi(t)\| \le \|x(t)\|. \tag{24}$$

Once (24) is established, the results of the lemma follow from a similar analysis as in [4]. □

*Remark 3.2:* $\alpha_1$ in (21) and $\alpha_2$ in (22) can be made small by reducing $\epsilon$ and by making a sufficiently large number $M_0$, respectively. $M_0$ is used here for the purpose of stability analysis only. It is not a design parameter.

### *B. Control Law Synthesis*

Although there are many control schemes available [1], here we just employ the pole assignment strategy to tune the controller parameters based on the Certainty Equivalence Principle. The control $u(t)$ is then given by

$$\hat{L}\left(\frac{u}{F}\right) = -\hat{P}\left(\frac{y}{F}\right) + \hat{P}\left(\frac{y^*}{F}\right). \tag{25}$$

In (25), $\hat{L}$ and $\hat{P}$ are polynomials of the form

$$\hat{L}(p) = p^{n_l} + \hat{l}_{n_l-1} p^{n_l-1} + \cdots + \hat{l}_0$$
$$\hat{P}(p) = \hat{p}_{n_p} p^{n_p} + \hat{p}_{n_p-1} p^{n_p-1} + \cdots + \hat{p}_0$$

and are determined from the following Diophantine equation:

$$\hat{A}(t)\hat{L}(t) + \hat{B}(t)\hat{P}(t) = A^* \tag{26}$$

where $A^*$ is a monic polynomial of degree $n + \gamma$ and its zeros are chosen to be the required closed-loop poles according to guidelines in [16]. The degrees $n_l$ and $n_p$ are set to be $\gamma$ and $n - 1$, respectively.

The resulting controller can be implemented by transforming (25) to the following form:

$$u = (F - \hat{L})\left(\frac{u}{F}\right) - \hat{P}\left(\frac{y}{F}\right) + \hat{P}\left(\frac{y^*}{F}\right). \tag{27}$$

From Assumption A2, (26) gives a bounded solution for $\hat{L}, \hat{P}, \forall t$.

## IV. Stability Analysis

In this section, global stability of the closed-loop adaptive system is established. We now derive an equation to describe the closed-loop system. This can be achieved by considering the estimator and the controller equations. From (17), we get

$$\hat{A}\left(\frac{y}{F}\right) = \hat{B}\left(\frac{u}{F}\right) + e. \tag{28}$$

Then from (28) and (25), the closed-loop system can be described as

$$\dot{x}(t) = \hat{A}_c x(t) + b_1 e(t) + b_2 r(t) \tag{29}$$

where

$$b_1^T = [0, \cdots 0, 1, \cdots, 0], \quad b_2^T = [0, \cdots 0, 0, \cdots, 1], \quad r(t) = \hat{P}\frac{y^*}{F} \tag{30}$$

and (31), as shown at the bottom of the page.

As in [4], it can be shown, using Lemma 3.1, that $\exists c > 0, \sigma > 0$ such that the transition matrix of the homogeneous part of (29) $\Phi(t, \tau)$ satisfies

$$\|\Phi(t,\tau)\| \le c e^{-\sigma(t-\tau)}, \qquad \text{for } t \ge \tau \ge t_0 \tag{32}$$

if $\|x(t)\| \ge M_0$, $\sup_{0 \le \tau \le t} \|x(\tau)\| = \|x(t)\|$ $\forall t \ge t_0$, and for all $\epsilon \le \bar{\epsilon}^*, \delta \le \bar{\delta}^*$, where bounds $\bar{\epsilon}^*, \bar{\delta}^*$ are sufficiently small numbers to ensure $(\alpha_1 + \alpha_2) \le \alpha^*$. Here $\alpha_1, \alpha_2$ are given in (21) and (22) and $\alpha^*$ is a sufficiently small number. Then we can establish the system stability in a special case.

*Lemma 4.1:* Suppose that $\|x(t_0)\| = M_0$, $\|x(t)\| > M_0$ for all $t > t_0$ and $\sup_{0 \le \tau \le t} \|x(\tau)\| = \|x(t)\|$. Consider the adaptive system consisting of estimator (16) and (17) and controller (25)–(27). Under Assumption 2.1, there exists a constant $\epsilon_1^*$ such that for all $\epsilon \le \epsilon_1^*$ the closed-loop system ensures that

$$\|x(t)\| \le M \tag{33}$$

where $M = \sqrt{c_1 M_0^2 + c_2}$ and $c_1$, $c_2$ are positive generic constants.

*Proof:* From (29) and (32), we have

$$\begin{aligned} \|x(t)\| &\le c e^{-\sigma(t-t_0)} \|x(t_0)\| + c\int_{t_0}^t e^{-\sigma(t-\tau)} (|e(\tau)| + |r(\tau)|)\, d\tau \\ &\le c M_0 + c\int_{t_0}^t e^{-\sigma(t-\tau)} [|\tilde{e}(\tau)|(1 + \|x(\tau)\|^2)^{1/2} + |r(\tau)|]\, d\tau. \end{aligned} \tag{34}$$

Suppose the intermediate number $M_0$ is also such that

$$\|r(t)\|_\infty \le M_0.$$

Clearly such an $M_0$ always exists for any bounded $y^*$. Now squaring both sides of (34) and applying the Schwartz inequality, we get

$$\|x(t)\|^2 \le c_3 M_0^2 + c_3 \int_{t_0}^t e^{-\sigma(t-\tau)} |\tilde{e}(\tau)|^2 (1 + \|x(\tau)\|^2)\, d\tau \tag{35}$$

for a positive constant $c_3$. Multiplying both sides of (35) by $e^{\sigma t}$ gives

$$e^{\sigma t}\|x(t)\|^2 \le s^2(t) + c_3 \int_{t_0}^t e^{\sigma\tau} \|x(\tau)\|^2 |\tilde{e}(\tau)|^2\, d\tau \tag{36}$$

where

$$s^2(t) = e^{\sigma t} c_3 M_0^2 + c_3 \int_{t_0}^t e^{\sigma\tau} |\tilde{e}(\tau)|^2\, d\tau. \tag{37}$$

$$\hat{A}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & & 0 & & \\ & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & & & & \\ & & & & 1 & & & & & & \\ -\hat{a}_0 & \cdots & & & -\hat{a}_{n-1} & \hat{b}_0 & \cdots & \hat{b}_m, & 0 & \cdots & 0 \\ 0 & \cdots & & & & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & & \ddots & & & \vdots \\ & & & & & & & & & & 1 \\ -\hat{p}_0 & \cdots & & & -\hat{p}_{n-1} & -\hat{l}_0 & & & & & -\hat{l}_{\gamma-1} \end{bmatrix} \tag{31}$$

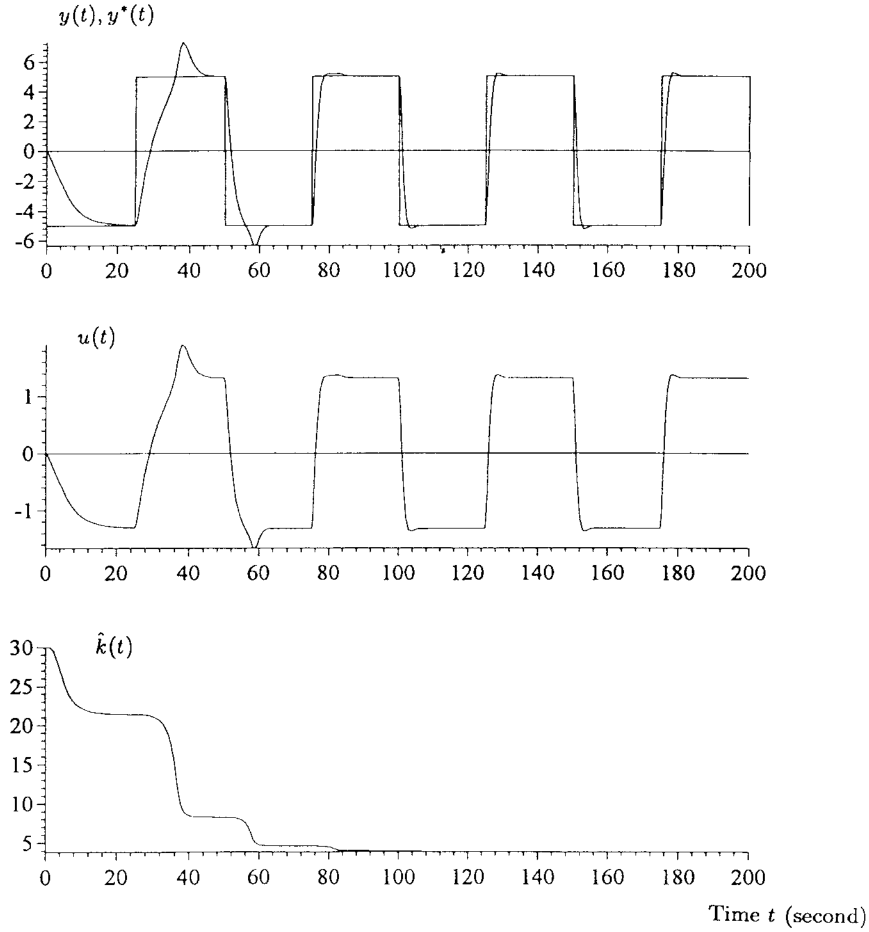


Fig. 1. System response when $H_2(p) = \frac{2(p+2)}{(P+5)}$.

Then applying the Bellman–Grownwall lemma to (36) and using Lemma 3.1, we have

$$\begin{aligned}\|x(t)\|^2 &\le e^{-\sigma t}s^2(t) + c_3\int_{t_0}^{t} e^{-\sigma t}|\tilde{e}(\tau)|^2 s^2(\tau) e^{\int_\tau^t c_3|\tilde{e}(\tau_1)|^2 d\tau_1}\,d\tau \\ &\le c_1 M_0^2 + c_2 \end{aligned} \tag{38}$$

for $\epsilon \le \bar{\bar{\epsilon}}^*$ and $\delta \le \delta^*$ where $\bar{\bar{\epsilon}}^*$ and $\delta^*$ are sufficiently small constants satisfying

$$c_3(\alpha_1^* + \alpha_2^*) < \sigma \tag{39}$$

with $\alpha_1^*, \alpha_2^*$ depending on $\bar{\bar{\epsilon}}^*$ and $\delta^*$. Taking $\epsilon_1^* = \min\{\bar{\epsilon}^*, \bar{\bar{\epsilon}}^*\}$, the result is proved.

Clearly, $c_1$ and $c_2$ are independent of $\epsilon$ if it is replaced by its bound $\epsilon_1^*$, which is a generic constant. □

To establish the stability result for the general case, we explore the parameter estimator further and this gives Lemma 4.2 as follows.

*Lemma 4.2:* If $\|x(t)\| > M_0$ for all $t \ge t_0$, $\|x(t)\| \le \sqrt{c_1 M_0^2 + c_2}$ for $t \in [0, t_1]$ and $\sup_{0\le\tau\le t}\|x(\tau)\| = \|x(t)\|$ for all $t \ge t_1$ where $t_0$ and $t_1$ are some constants satisfying $t_1 \ge t_0$, then

1)
$$|\tilde{e}(t)| = \beta(k_1 + \epsilon(\sqrt{c_1} + \sqrt{c_2}) + \delta), \qquad \text{for } t \ge t_0 \tag{40}$$

2)
$$\int_{t_0}^{t} \tilde{e}^2(\tau)\,d\tau \le k_2 + \alpha_1(\sqrt{c_1} + \sqrt{c_2})(t - t_0) + \alpha_2(t - t_0), \qquad \text{for } t \ge t_0. \tag{41}$$

*Proof:* By noting the condition of the lemma, the results can be established from a similar analysis as in Lemma 3.1. □

*Remark 4.1:* Note that the properties in the above lemma are quite similar to Lemma 3.1 except that the constants $c_1$ and $c_2$ appear here.

From Lemma 4.2, we get our main stability result as stated in the following theorem.

*Theorem 4.1:* Consider the adaptive system consisting of plant (1), estimator (16) and (17), and controller (25)–(27). Under Assumption 2.1 there exists a constant $\epsilon^*$ such that for all $\epsilon \le \epsilon^*$, the closed-loop system is globally stable in the sense that all signals remain bounded $\forall t$ for all finite initial states, any bounded $y^*$, and arbitrarily bounded external disturbances.

*Proof:* First, we consider the trajectory $\|x(t)\|$ and show that the constant $M$ in Lemma 4.1 is a uniform bound of $\|x(t)\|$. From (29), $\|x(\cdot)\|$ is continuous and thus we can divide the time interval $[0, \infty)$ into two subsequences $\Re_i^+ = [s_i, \tau_i]$ and $\Re_i^- = (\tau_i, s_{i+1})$ with $\tau_0 = 0$ such that

$$[0, \infty) = \left(\cup_{i=1}^{\infty} \Re_i^+\right) \cup \left(\cup_{i=0}^{\infty} \Re_i^-\right) \tag{42}$$

$$\|x(t)\| \ge M_0, t \in \Re_i^+; \qquad \|x(t)\| < M_0, t \in \Re_i^-. \tag{43}$$

In (43), $M_0$ is also satisfying $\|x(0)\| \le M_0$.

$\|x(t)\|$ can be ensured bounded if we can show that it is bounded in $\Re_i^+, \forall i \ge 1$, which can be done through induction. Thus we now consider $\|x(t)\|$ for $t \in \Re_1^+$. From the continuity of $\|x(t)\|$, $\exists t_1 \in \Re_1^+$ such that $\sup_{0\le\tau\le t}\|x(\tau)\| = \|x(t)\|$ for $t \le t_1$. Then using Lemma 4.1 and noting that $\|x(s_1)\| = M_0$, we can show that

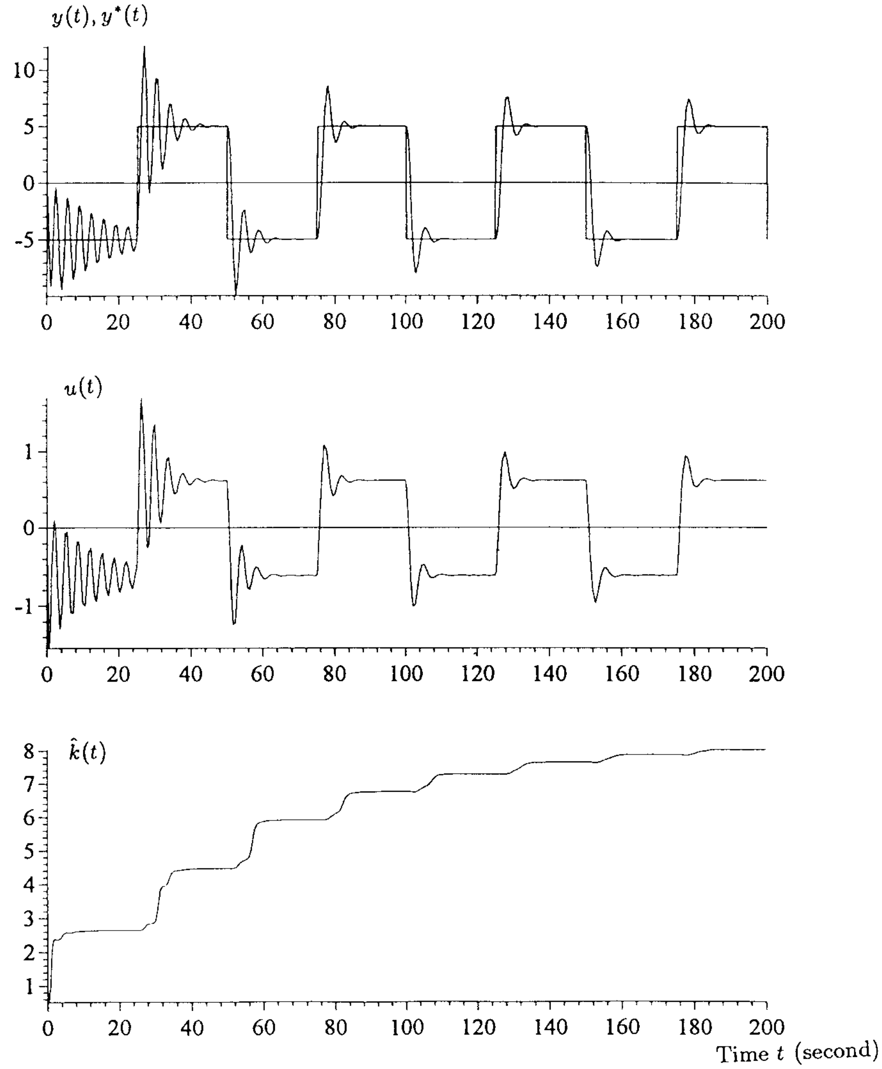


Fig. 2. System response when $H_2(p) = \frac{2(p-8)^2}{(p+5)^2}$.

for $t \leq t_1$

$$\sup_{0\leq\tau\leq t} \|x(\tau)\| \leq M \tag{44}$$

for all $\epsilon \leq \epsilon_1^*$. Suppose that $\sup_{0\leq\tau\leq t} \|x(\tau)\| \neq \|x(t)\|$ for $t \in [t_1, t_2] \subset \Re_1^+$. Then (44) automatically holds for $t \leq t_2$. Now if $\sup_{0\leq\tau\leq t} \|x(\tau)\| = \|x(t)\|$ for $t \geq t_2$ and $t \in \Re_1^+$, then following the same steps in the proof of Lemma 4.1 and applying Lemma 4.2 yields

$$\sup_{0\leq\tau\leq t} \|x(\tau)\| \leq M \tag{45}$$

for all $\epsilon \leq \epsilon^*$ where $\epsilon^* = \frac{\epsilon_1^*}{\sqrt{c_1}+\sqrt{c_2}}$. In this way, (45) can be shown to be true for all $t \in \Re_1^+$.

Then assuming (45) holds $\forall t \in \Re_k^+$, it can be shown that (45) is also true $\forall t \in \Re_{k+1}^+$ from the fact that $\|x(s_{k+1})\| = M_0$ and Lemma 4.2.

After establishing (45), we can have $\eta_f(t), \phi(t), e(t), \dot{\hat{\theta}}$, and $u$ bounded from (12), (24), (23), and (27). From (29), $\dot{x}$ is bounded. Consider the last row of (29). We can ensure $p(p^\gamma \frac{u}{F})$ bounded from the boundedness of $\hat{\theta}, \dot{\hat{\theta}}, \dot{x}$, and $x$. Thus $pu = pF\frac{u}{F}$ is bounded from the last row of (29). We now establish the boundedness of $y$. From (2) and (3), we get

$$\frac{A}{F}y(t) = \frac{B}{F}u(t) + \frac{\epsilon_1 BH_1 + \epsilon_2 AH_2}{F}u(t) + \frac{A}{F}\omega(t). \tag{46}$$

Since $pu$ is bounded, then $p^i(\frac{AS}{F}y)$, i.e., $p^{i+n}(\frac{y}{F}) + a_{n-1}p^{i+n-1}(\frac{y}{F}) + \cdots + a_0 p^i(\frac{y}{F})$ is bounded for $i = 0, 1, \cdots, \gamma - n$ from Assumption 2.1. From this fact and the boundedness of $p^k(\frac{y}{F}), k = 0, 1, \cdots, n$, we can successively show that $p^i\frac{y}{F}$ is bounded for $i = n+1, \cdots, \gamma$. Thus $y = F\frac{y}{F}$ is bounded. □

*Remarks 4.1:*

1) Note that some ideas used in [4] are applied to analyze the robustness of the proposed adaptive controller and thus to establish Theorem 4.1. However, due to the use of a higher order filter, the boundedness establishment of the input $u$ and output $y$ becomes much more involved. Also the technique is refined and improved here. Thus the presentation in this paper is more elegant and clearer than that in [4].
2) Suppose the disturbance $\omega(t)$ and the reference signal $y^*$ are purely deterministic. In other words, there exists a polynomial $S(p)$ such that

$$S(p)\omega(t) = 0$$
$$S(p)y^*(t) = 0.$$

In this case, $d_0$ at (6) is exponentially decaying. Thus having proven the boundedness of all states in the closed-loop system, it can be shown from (19) that the prediction error $e(t)$ satisfies

$$\int_{t_0}^{t} e^2(t) \leq \beta_1 + \beta_2 o(\epsilon)(t - t_0) \tag{47}$$

where $\beta_1$, $\beta_2$ are constants and $o(\epsilon)$ satisfies $\lim_{\epsilon \to 0} o(\epsilon) = 0$. If the internal model principle is employed in the controller synthesis, the tracking error $y - y^*$ can be shown to be $\epsilon$ small in the mean. In the absence of unmodeled dynamics, the tracking error tends to zero. This is shown in the following test example.

## V. An Example

The adaptive scheme is applied to control the following system:

$$y(t) = ku(t) + H_2(p)u(t). \tag{48}$$

In the design, the nominal transfer function $H(s) = k$ is used. The value of $k$ is unknown, but taken to be three for simulation studies. The required set point, $y^*$, is a square waveform of amplitude 5 and period 50 s. An integrator is introduced to achieve better tracking performance and thus the filter employed is $\frac{s}{(s+4)(s+0.1)}$. The estimator gain $\beta$ is chosen to be ten and we assume the unknown $k$ is within the interval $[0.1, 40]$. The required closed-loop characteristic polynomial $A^*$ is selected to be

$$A^* = s^2 + 2s + 2.$$

The following two cases with different additive unmodeled dynamics are simulated:

- $$H_2 = \frac{2(p+2)}{p+5}$$

The system response in this case is given in Fig. 1.

- $$H_2 = \frac{2(p-8)^2}{(p+5)^2}.$$

The system response in this case is presented in Fig. 2.

Comparing the results of the above two cases, we note that the system performance is degraded in the second case because unmodeled dynamics is more complicated and the overall plant is nonminimum phase. However, in both cases, the closed-loop system is stable and the performance is improved gradually as the adaptation continues.

## VI. Conclusion

In this paper, an adaptive control algorithm proposed is directly based on the nominal transfer function of the plant. The transfer function can be proper and both the additive and multiplicative unmodeled dynamics are allowed to be proper, or even improper. It has been shown that the proposed adaptive control scheme can globally stabilize the system with modeling errors due to unmodeled dynamics and bounded external disturbances. An example also shows the effectiveness of the adaptive control scheme.

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# One-Machine $n$-Part-Type Optimal Setup Scheduling: Analytical Characterization of Switching Surfaces

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***Abstract*—The authors consider optimal setup scheduling of a single reliable machine. Production flow of $n$ different part types and the setup process are described by differential equations. Setup change rates are control variables. Necessary conditions on optimal setup changes are characterized analytically, and optimal setup change times are derived for a given setup change sequence. The linearization of optimal setup switching surfaces is derived, indicating the existence of attractors observed in numerical optimal solutions. The approach developed in this paper establishes a strong basis for studying multimachine production systems and for constructing tractable near-optimal numerical solution techniques.**

***Index Terms*— Attractors, necessary setup conditions, one-machine scheduling, optimal control, switching surfaces.**

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