# Adaptive Output Control of Nonlinear Systems With Uncertain Dead-Zone Nonlinearity

J. Zhou, C. Wen, and Y. Zhang

***Abstract*—In this note, we present a new scheme to design adaptive controllers for uncertain systems preceded by unknown dead-zone nonlinearity. The control design is achieved by introducing a smooth inverse function of the dead-zone and using it in the controller design with backstepping technique. For the design and implementation of the controller, no knowledge is assumed on the unknown system parameters. It is shown that the proposed controller not only can guarantee stability, but also transient performance.**

***Index Terms*—Adaptive control, backstepping, dead-zone, nonlinear systems, stability.**

## I. Introduction

Adaptive control is popular in engineering and science. However, it still faces many important challenges, such as the handling of nonsmooth nonlinearity. Nonsmooth nonlinearity characteristics including dead-zone, backlash, and hysteresis are common in mechanical connection, hydraulic servo valves, piezoelectric translators, and electric servomotors. Several adaptive control schemes have recently been proposed to handle such nonlinearity; see, for example, [1]–[3]. In these papers, an adaptive inverse approach was presented to deal with nonsmooth nonlinearity in the design of continuous time model reference adaptive controllers. The proposed inverse functions are cascaded with the plant to cancel the effects of hysteresis in [3], backlash in [4], dead-zone in [5], and actuator failure in [6]. In the controller design, the term multiplying the control and the uncertain parameters of the system and nonsmooth nonlinearity must be within known bounded intervals. Dead-zone precompensation using neural network have also been used in feedback control systems [7]. With these developed schemes, the transient performance is usually not guaranteed due to their design methods. In [8]–[12], state feedback control was considered for nonlinear uncertain systems, where the hysteresis or dead-zone was treated in a similar way to disturbance. The characteristics of the nonlinearity were not considered in the controller design, so the performance, especially steady-state performance, may not be good enough. In [9] and [10], it is also assumed that the dead-zone slopes in both positive and negative sides must be the same.

In this note, we will address the output feedback control of similar class of nonlinear systems as in [8]–[11], in the presence of unknown dead-zone actuator nonlinearity. We take the dead-zone into account in our controller design unlike in [9], [10], and [12]. A new smooth inverse of the dead-zone will be introduced to compensate the effect of the dead-zone in controller design with backstepping approach. Note that ideas of using smooth inverse were also suggested in [15]. Such a smooth inverse can avoid chattering problems that may occur in the nonsmooth inverse approach proposed in [2], [5], and [13]. The specific treatment of the dead-zone may bring performance improvement. As system output feedback is employed, a state observer is required. To obtain such an observer, a new parametrization of the state observer for the plant is proposed to include two sets of parameters: one from the dead-zone nonlinearity and the other from the plant, similar to [16]. With our approach, *a priori* knowledge on system parameters is no longer needed. Besides showing stability of the system, the transient performance in terms of $L_2$ norm of the tracking error is derived to be an explicit function of design parameters and, thus, our scheme allows designers to obtain the closed-loop behavior by tuning design parameters in an explicit way.

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## II. Problem Statement

### A. System Model

We consider the same class of systems as in [8], [10], and [11]. For completeness, the system model is given as follows:

$$x^{(n)}(t) + a_1 Y_1(x(t)) + a_2 Y_2(x(t)) + \cdots + a_r Y_r(x(t)) = bu \tag{1}$$

$$y = x \quad u = DZ(v) \tag{2}$$

where $Y_i$ are known continuous linear or nonlinear functions, parameters $a_i$ and control gain $b$ are unknown constants, $v(t)$ is the output from the controller, $u(t)$ is the input to the system and $y(t)$ is the system output. The actuator nonlinearity $\mathrm{DZ}(v)$ is described as a dead-zone characteristic.

The control objective is to design an output feedback control law for $v(t)$ to ensure that all closed-loop signals are bounded and the plant output $y(t)$ tracks a given reference signal $y_r(t)$ under the following assumptions.

*Assumption 1:* The sign of $b$ is known, and $y_r(t)$ and its first $n$th derivatives are known and bounded.

*Assumption 2:* The dead-zone parameters $m_r$ and $m_l$ satisfy $m_r \geq m_{r0}$ and $m_l \geq m_{l0}$, where $m_{r0}$ and $m_{l0}$ are two small positive constants.

### B. Dead-Zone Characteristic

The dead-zone characteristic $\mathrm{DZ}(\cdot)$ can be represented as [15]

$$u(t) = \mathrm{DZ}(v(t)) = \begin{cases} m_r(v(t) - b_r) & v(t) \geq b_r \\ 0 & b_l < v(t) < b_r \\ m_l(v(t) - b_l) & v(t) \leq b_l \end{cases} \tag{3}$$

where $b_r \geq 0, b_l \leq 0$ and $m_r > 0, m_l > 0$ are constants. In general, the break-points $|b_r| \neq |b_l|$ and the slopes $m_r \neq m_l$. The essence of compensating dead-zone effect is to employ a dead-zone inverse as shown in [5], [13], and [15]. In this note, we propose a smooth inverse for the dead-zone as follows:

$$v(t) = \mathrm{DI}(u(t)) = \frac{u(t) + m_r b_r}{m_r}\phi_r(u) + \frac{u(t) + m_l b_l}{m_l}\phi_l(u) \tag{4}$$

where $\phi_r(u)$ and $\phi_l(u)$ are smooth continuous indicator functions defined as

$$\phi_r(u) = \frac{e^{u/e_0}}{e^{u/e_0} + e^{-u/e_0}} \quad \phi_l(u) = \frac{e^{-u/e_0}}{e^{u/e_0} + e^{-u/e_0}} \tag{5}$$

where $e_0 > 0$ is chosen by designer. Such an inverse is shown in Fig. 1.

*Remark 1:* Note that the use of smooth functions $\phi_r(u)$ and $\phi_l(u)$ can avoid possibly chattering phenomenon in the recursive backstepping control.

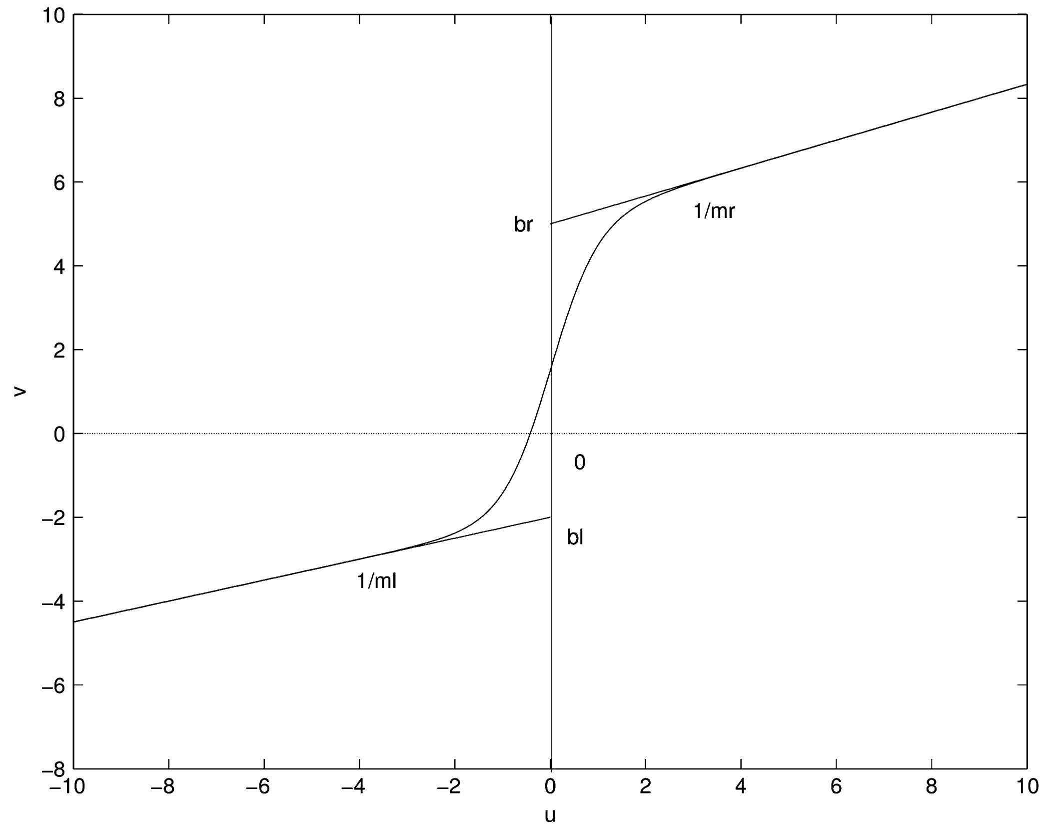


Fig. 1. Dead-zone inverse.

To design adaptive controller for the system, we parameterize the dead-zone as

$$u(t) = -\theta^T \omega \tag{6}$$

$$\text{where} \quad \theta = [m_r, m_r b_r, m_l, m_l b_l]^T$$
$$\omega(t) = [-\sigma_r(t)v(t), \sigma_r(t), -\sigma_l(t)v(t), \sigma_l(t)]^T \tag{7}$$

$$\sigma_r(t) = \begin{cases} 1, & \text{if } u(t) > 0 \\ 0, & \text{otherwise} \end{cases}$$
$$\sigma_l(t) = \begin{cases} 1, & \text{if } u(t) < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

As $\theta$ is unknown and $\omega$ is unavailable, the actual control input to the plant $u_d(t)$ is designed as

$$u_d(t) = -\hat{\theta}^T \hat{\omega}(t) \tag{9}$$

where $\hat{\theta}$ is an estimate of $\theta$, i.e.,

$$\hat{\theta} = [\widehat{m_r}, \widehat{m_r b_r}, \widehat{m_l}, \widehat{m_l b_l}]^T$$
$$\hat{\omega}(t) = [-\phi_r(v)v(t), \phi_r(v), -\phi_l(v)v(t), \phi_l(v)]^T. \tag{10}$$

Then corresponding control output $v(t)$ is given by

$$v(t) = \widehat{\mathrm{DI}}(u_d(t)) = \frac{u_d(t) + \widehat{m_r b_r}}{\widehat{m_r}} \phi_r(u_d) + \frac{u_d(t) + \widehat{m_l b_l}}{\widehat{m_l}} \phi_l(u_d(t)). \tag{11}$$

The resulting error between $u$ and $u_d$ is

$$u(t) - u_d(t) = (\hat{\theta} - \theta)^T \hat{\omega}(t) + d_N(t) \tag{12}$$

where $d_N(t) = \theta^T(\hat{\omega}(t) - \omega(t))$. The bound of $d_N(t)$ can be obtained as

$$|d_N(t)| = |\theta^T(\hat{\omega}(t) - \omega(t))|$$
$$\leq \begin{cases} \frac{1}{2}e^{-1}|m_r - m_l|e_0 + \frac{|m_r b_r - m_l b_l|}{e^{2b_r/e_0}+1} & v(t) \geq b_r \\ \max\{m_r, m_l\}|b_r - b_l| & b_l < v(t) < b_r \\ \frac{1}{2}e^{-1}|m_r - m_l|e_0 + \frac{|m_r b_r - m_l b_l|}{e^{-2b_l/e_0}+1} & v(t) \leq b_l \end{cases} \tag{13}$$

where we have used that $|v|e^{-|v|} \leq e^{-1}$. Note that when $b_l \leq v \leq b_r$ the bound of $d_N(t)$ decreases as $e_0$ increases, while outside this range the bound decreases as $e_0$ decreases. It has the desired properties that $d_N(t)$ is bounded for all $t \geq 0$ and $d_N(t)$ approaches to 0 as $\hat{\theta} \to \theta$ and $e_0 \to 0$.

## III. State Observer

As we consider output feedback, a state observer is required. To design such an observer, we rewrite plant (1) as

$$\dot{x} = Ax + a^T Y e_n + bue_n \quad y = cx \quad u = \mathrm{DZ}(v) \tag{14}$$

where

$$A = \begin{bmatrix} 0 & \\ \vdots & I_{n-1} \\ 0 & \cdots 0 \end{bmatrix} \quad a = \begin{bmatrix} -a_1 \\ \vdots \\ -a_r \end{bmatrix}$$
$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_r \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}^T \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

To construct an observer for (14), we choose $k = [k_1, \ldots, k_n]^T$ such that all eigenvalues of $A_0 = A - kc$ are at some desired stable locations. If the signal $u(t)$ were available then we would implement the following filters:

$$\hat{x}(t) = \xi_0 - \sum_{i=1}^{r} a_i \xi_i + b\eta \tag{15}$$

$$\begin{aligned} \dot{\eta} &= A_0 \eta + e_n u \quad \dot{\xi}_0 = A_0 \xi_0 + ky + \chi \\ \dot{\xi}_i &= A_0 \xi_i + Y_i e_n, \qquad i = 1, \ldots, r \end{aligned} \tag{16}$$

where $\chi$ is a design signal specified later. It can be shown that the state estimation error $\epsilon = x(t) - \hat{x}(t)$ satisfies $\dot{\epsilon} = A_0 \epsilon - \chi$.

Note that the signal $u(t)$ is not available. Thus the signal $\eta$ in (16) needs to be reparameterized. Let $p$ denote $(d)/(dt)$. With $\Delta(p) = \det(pI - A_0)$, we express $\eta(t)$ as

$$\begin{aligned} \eta(t) &= [\eta_1(t), \eta_2(t), \ldots, \eta_n(t)]^T \\ &= [q_1(p), q_2(p), \ldots, q_n(p)]^T \frac{1}{\Delta(p)} u(t) \end{aligned} \tag{17}$$

for some known polynomials $q_i(p), i = 1, \ldots, n$. Using (17) and $u(t) = -\theta^T \hat{\omega}(t) + d_N(t)$, we obtain

$$\eta_i(t) = -\theta^T \hat{\omega}_i(t) + d_i(t) \tag{18}$$

where

$$\hat{\omega}_i(t) = \frac{q_i(p) I_4}{\Delta(p)} \hat{\omega}(t) \quad d_i(t) = \frac{q_i(p)}{\Delta(p)} d_N(t) \tag{19}$$

where $I_4$ is a $4 \times 4$ identity matrix. Based on (18), $\hat{\omega}_i$ is available for controller design in place of $u$. Denoting the second component of $\xi_i$ as $\xi_{i2}, i = 0, \ldots, r$, we have

$$\hat{x}_2 = \xi_{02} - \sum_{i=1}^{r} a_i \xi_{i2} - b\theta^T \hat{\omega}_2(t) + bd_2(t) \tag{20}$$

$$\hat{\omega}_2(t) = \frac{(p + k_1) I_4}{p^n + k_1 p^{n-1} + \cdots + k_{n-1} p + k_n} \hat{\omega}(t). \tag{21}$$

## IV. Design of Adaptive Controllers

As usual in backstepping approach, the following change of coordinates is made:

$$\begin{aligned} z_1 &= y - y_r \\ z_i &= -\hat{\theta}^T \hat{\omega}_2^{(i-2)} - \hat{e} y_r^{(i-1)} - \alpha_{i-1}, \qquad i = 2, 3, \ldots, n \end{aligned} \tag{22}$$

where $\hat{e}$ is an estimate of $e = 1/b$ and $\alpha_{i-1}$ is the virtual control at the $i$th *step* and will be determined in later discussion. As in [11], we define functions $sg_i(z_i)$ and $f_i(z_i)$ as follows:

$$\begin{aligned} sg_i(z_i) &= \begin{cases} \frac{z_i}{|z_i|} & |z_i| \geq \delta_i \\ \frac{z_i^{(2q+1)}}{(\delta_i^2 - z_i^2)^{n-i+2} + |z_i|^{(2q+1)}} & |z_i| < \delta_i \end{cases} \\ f_i(z_i) &= \begin{cases} 1 & |z_i| \geq \delta_i \\ 0 & |z_i| < \delta_i \end{cases} \end{aligned} \tag{23}$$

where $\delta_i (i = 1, \ldots, n)$ is a positive design parameter and $q = \mathrm{round}\{(n - i + 2)/2\}$, where $\mathrm{round}\{x\}$ means the element of $x$ to the nearest integer. Clearly $2q + 1 \geq (n - i + 2)$. It can be shown that $sg_i(z_i)$ is at least $(n - i + 1)$th-order differentiable.

Even though the backstepping design procedures are similar to [11], the first and the last steps of the design are quite different and elaborated in details. The results of other steps, i.e., step $i, i = 2, \ldots, n-1$ are only presented without elaboration.

- *Step 1)* We start with the equation for the tracking error $z_1$ obtained from (14) and (20) to obtain

$$\dot{z}_1 = \xi_{02} + a^T \xi_2 + bz_2 + b\alpha_1 - b\tilde{\theta}^T \hat{\omega}_2(t) + d(t) + \epsilon_2 - b\tilde{e}\dot{y}_r \tag{24}$$

where $d(t) = bd_2(t), \tilde{\theta} = \theta - \hat{\theta}$. From (13) and (19), there exists a positive constant $D$ such that

$$|d(t)| \leq D.$$

*Remark 2:* The unknown bound $D$ of $d(t)$ will be estimated online and thus it is not assumed to be known in contrast with [5], [13], and [15]. In fact, bounded external disturbance can also be treated in the same way, even though disturbance is not considered explicitly in this note.

Now, select the first virtual control law $\alpha_1$ as

$$\begin{aligned} \alpha_1 &= \hat{e} \bar{\alpha}_1 \\ \bar{\alpha}_1 &= -\left(c_1 + \frac{\hat{b}^2}{4}\right)(|z_1| - \delta_1)^n sg_1 - \xi_{02} - \hat{a}^T \xi_2 \\ &\quad - \hat{D}_1 sg_1 - (\delta_2 + 1)\sqrt{\hat{b}^2 + \delta_0} \cdot sg_1 \end{aligned} \tag{25}$$

where $\delta_0$ is a small positive real number, $\hat{e}, \hat{a}$ and $\hat{b}$ are estimates of $e, a$, and $b, \hat{D}_1$ is an estimate of $D$

$$\begin{aligned} \dot{z}_1 = &-\left(c_1 + \frac{\hat{b}^2}{4}\right)(|z_1| - \delta_1)^n sg_1(z_1) \\ &+ \tilde{a}^T \xi_2 + bz_2 - b(\bar{\alpha}_1 + \dot{y}_r)\tilde{e} - b\tilde{\theta}^T \hat{\omega}_2(t) \\ &+ d(t) - \hat{D}_1 sg_1 + \epsilon_2 - (\delta_2 + 1)\sqrt{\hat{b}^2 + \delta_0} \cdot sg_1. \end{aligned} \tag{26}$$

We define a positive–definite function $V_1$ as

$$\begin{aligned} V_1 = &\frac{1}{n+1}(|z_1| - \delta_1)^{n+1} f_1 + \frac{1}{2}|b|\tilde{\theta}^T \Gamma_\theta^{-1} \tilde{\theta} \\ &+ \frac{1}{2}\tilde{a}^T \Gamma_a^{-1} \tilde{a} + \frac{|b|}{2\gamma_1}\tilde{e}^2 + \frac{1}{2\gamma_{d1}}\tilde{D}_1^2 + \frac{1}{2l_1}\epsilon^T P \epsilon \end{aligned} \tag{27}$$

where $\tilde{a} = a - \hat{a}, \tilde{e} = e - \hat{e}, \Gamma_\theta, \Gamma_a$ are positive definite matrices, $\gamma_1, \gamma_{d1}$ are positive constants, and $P = P^T > 0$ satisfies the equation $PA_0 + A_0^T P = -2I$. Let $\theta_i = e_i^T \theta, i = 1, \ldots, 4, e_i$ is identity $4 \times 1$ vector. We select the adaptive update laws as

$$\dot{\hat{\theta}}_i = e_i^T \tau_\theta, i = 2, 4 \quad \dot{\hat{\theta}}_i = \mathrm{Proj}\left(e_i^T \tau_\theta\right), i = 1, 3 \tag{28}$$

$$\begin{aligned} \tau_\theta &= -\mathrm{sign}(b)\Gamma_\theta \hat{\omega}_2(t)(|z_1| - \delta_1)^n f_1 sg_1 \\ \dot{\hat{e}} &= -\mathrm{sign}(b)\gamma_1(\bar{\alpha}_1 + \dot{y}_r)(|z_1| - \delta_1)^n f_1 sg_1 \end{aligned} \tag{29}$$

$$\dot{\hat{D}}_1 = \gamma_{d1}(|z_1| - \delta_1)^n f_1 \tag{30}$$

where $\mathrm{Proj}(\cdot)$ is a smooth projection operation to ensure the estimates $\hat{m}_r(t) \geq m_{r0}$ and $\hat{m}_l(t) \geq m_{l0}$. Such an operation can be found in [14].

Then from (26)–(29) and using $\dot{\epsilon} = A_0\epsilon - \chi$ and the property $-\tilde{\theta}^T\Gamma_\theta^{-1}\mathrm{Proj}(\tau_\theta) \le -\tilde{\theta}^T\Gamma_\theta^{-1}\tau_\theta$, we obtain the time derivative of $V_1$ as

$$\begin{aligned}
\dot{V}_1 =& (|z_1|-\delta_1)^n f_1 sg_1 \dot{z}_1 - |b|\tilde{\theta}^T\Gamma_\theta^{-1}\dot{\hat{\theta}} - \tilde{a}^T\Gamma_a^{-1}\dot{\hat{a}} \\
&- \frac{|b|}{\gamma_1}\tilde{e}\dot{\hat{e}} - \frac{1}{\gamma_{d1}}\tilde{D}_1\dot{\hat{D}}_1 + \frac{1}{l_1}\epsilon^T P\dot{\epsilon} \\
\le& -\left(c_1+\frac{\hat{b}^2}{4}\right)(|z_1|-\delta_1)^{2n} f_1 + |b|\tilde{\theta}^T \\
&\times\left(\mathrm{sign}(b)\hat{\omega}_2(|z_1|-\delta_1)^n f_1 sg_1 - \Gamma_\theta^{-1}\dot{\hat{\theta}}\right) \\
&+ \tilde{a}^T\left(\xi_2(|z_1|-\delta_1)^n f_1 sg_1 - \Gamma_a^{-1}\dot{\hat{a}}\right) \\
&- |b|\tilde{e}\left(\mathrm{sign}(b)(\bar{\alpha}_1+\dot{y}_r)(|z_1|-\delta_1)^n f_1 sg_1 + \frac{1}{\gamma_1}\dot{\hat{e}}\right) \\
&+ \tilde{D}_1(|z_1|-\delta_1)^n f_1 - \frac{1}{\gamma_{d1}}\tilde{D}_1\dot{\hat{D}}_1 \\
&+ \epsilon^T\left[e_2(|z_1|-\delta_1)^n f_1 sg_1 - \frac{1}{l_1}P\chi\right] - \frac{1}{l_1}\epsilon^T\epsilon \\
&+ (|z_1|-\delta_1)^n f_1 sg_1\left[bz_2 - (\delta_2+1)\sqrt{\hat{b}^2+\delta_0}sg_1\right] \\
\le& -\left(c_1+\frac{\hat{b}^2}{4}\right)(|z_1|-\delta_1)^{2n} f_1 + \tilde{a}^T(\tau_{a1} - \Gamma_a^{-1}\dot{\hat{a}}) \\
&+ \epsilon^T\left(\tau_{\chi 1} - \frac{1}{l_1}P\chi\right) - \frac{1}{l_1}\epsilon^T\epsilon \\
&+ (|z_1|-\delta_1)^n f_1 sg_1\left(bz_2 - (\delta_2+1)\sqrt{\hat{b}^2+\delta_0}sg_1\right)
\end{aligned} \tag{31}$$

$$\tau_{a1} = \xi_2(|z_1|-\delta_1)^n f_1 sg_1 \quad \tau_{\chi 1} = e_2(|z_1|-\delta_1)^n f_1 sg_1 \tag{32}$$

where $e_2 = [0,1,0,\ldots,0]^T$.

- *Step* $i, i = 2,\ldots,n$) As detailed in [11], we choose

$$\begin{aligned}
\alpha_i =& -(c_i+1)(|z_i|-\delta_i)^{n-i+1}sg_2 - \beta_i - (\delta_{i+1}+1)sg_i \\
&+ \frac{\partial\alpha_{i-1}}{\partial y}\hat{a}^T\xi_2 - \frac{\partial\alpha_{i-1}}{\partial y}\hat{\Theta}^T\hat{\omega}_2(t) \\
&+ \sqrt{\|\frac{\partial\alpha_{i-1}}{\partial y}\|^2+\delta_0}\cdot\hat{D}_i sg_i + \frac{\partial\alpha_{i-1}}{\partial\hat{a}}\Gamma_a\tau_{ai} \\
&+ \frac{\partial\alpha_{i-1}}{\partial\xi_0}l_1P^{-1}\tau_{\chi i} + \frac{\partial\alpha_{i-1}}{\partial\hat{\Theta}}\Gamma_\Theta\tau_{\Theta i} \\
&+ \sum_{k=2}^{i-1}(|z_k|-\delta_k)^{n-k+1}f_k sg_k\left[-\frac{\partial\alpha_{k-1}}{\partial\hat{a}}\frac{\partial\alpha_{i-1}}{\partial y}\xi_2\right. \\
&\left.- \frac{\partial\alpha_{k-1}}{\partial\xi_0}\frac{\partial\alpha_{i-1}}{\partial y}l_1P^{-1}e_2\right] \\
&- \sum_{k=3}^{i-1}(|z_k|-\delta_k)^{n-k+1}f_k sg_k\frac{\partial\alpha_{k-1}}{\partial\hat{\Theta}}\frac{\partial\alpha_{i-1}}{\partial y}\hat{\omega}_2
\end{aligned} \tag{33}$$

$$\begin{aligned}
\dot{\hat{b}} =& \gamma_2(|z_1|-\delta_1)^n f_1 sg_1 z_2 \\
\dot{\hat{D}}_i =& \gamma_{\mathrm{di}}\sqrt{\|\frac{\partial\alpha_{i-1}}{\partial y}\|^2+\delta_0}\cdot(|z_i|-\delta_i)^{n-i+1}f_i
\end{aligned} \tag{34}$$

$$\tau_{ai} = \tau_{ai-1} - \frac{\partial\alpha_{i-1}}{\partial y}\xi_2(|z_i|-\delta_i)^{n-i+1}f_i sg_i \tag{35}$$

$$\tau_{\chi i} = \tau_{\chi i-1} - \frac{\partial\alpha_{i-1}}{\partial y}(|z_i|-\delta_i)^{n-i+1}f_i sg_i e_2 \tag{36}$$

$$\tau_{\Theta i} = \tau_{\Theta i-1} - \frac{\partial\alpha_{i-1}}{\partial y}(|z_i|-\delta_i)^{n-i+1}f_i sg_i\hat{\omega}_2 \tag{37}$$

$$\begin{aligned}
V_i =& \sum_{k=1}^{i}\left[\frac{1}{n-k+2}(|z_k|-\delta_k)^{n-k+2}f_k\right. \\
&\left.+ \frac{1}{2\gamma_{dk}}\tilde{D}_k^2\right] + \frac{1}{2}|b|\tilde{\theta}^T\Gamma_\theta^{-1}\tilde{\theta} \\
&+ \frac{1}{2}\tilde{a}^T\Gamma_a\tilde{a} + \frac{|b|}{2\gamma_1}\tilde{e}^2 + \frac{1}{2}\tilde{\Theta}^T\Gamma_\Theta^{-1}\tilde{\Theta} \\
&+ \frac{1}{2\gamma_2}\tilde{b}^2 + \frac{1}{2l_1}\epsilon^T P\epsilon
\end{aligned} \tag{38}$$

where $\hat{\Theta}, \hat{D}_k$ are estimates of $\Theta = b\theta$ and $D, \tilde{\Theta} = \Theta - \hat{\Theta}, \tilde{b} = b - \hat{b}, \tilde{D}_k = D - \hat{D}_k, \beta_i$ contains all known terms, $\gamma_2, \gamma_{\mathrm{di}}, i = 1,\ldots,n$ are positive constants, $\Gamma_\Theta$ is a positive–definite matrix.

- *Step* $n$) Using (11) and (21), we have

$$\begin{aligned}
\hat{\theta}^T\hat{\omega}_2^{(n-1)} =& \hat{\theta}^T\frac{(p^n+k_1p^{n-1})I_4}{p^n+k_1p^{n-1}+\cdots+k_{n-1}p+k_n}\hat{\omega}(t) \\
=& -u_d(t)+\omega_0
\end{aligned} \tag{39}$$

where $\omega_0$ is given by

$$\omega_0 = -\frac{\left(k_2p^{n-2}+\cdots+k_{n-1}p+k_n\right)I_4}{p^n+k_1p^{n-1}+\cdots+k_{n-1}p+k_n}\hat{\omega}(t). \tag{40}$$

With this equation, the derivative of $z_n = -\hat{\theta}^T\hat{\omega}_2^{(n-2)} - \hat{e}y_r^{(n-1)} - \alpha_{n-1}$ is

$$\begin{aligned}
\dot{z}_n =& u_d + \beta_n - \frac{\partial\alpha_{n-1}}{\partial y}a^T\xi_2 + \frac{\partial\alpha_{n-1}}{\partial y}\Theta^T\hat{\omega}_2(t) \\
&- \frac{\partial\alpha_{n-1}}{\partial\hat{a}}\dot{\hat{a}} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{D}_j}\dot{\hat{D}}_j \\
&- \frac{\partial\alpha_{n-1}}{\partial\hat{\Theta}}\dot{\hat{\Theta}} - \frac{\partial\alpha_{n-1}}{\partial\xi_0}\chi - \frac{\partial\alpha_{n-1}}{\partial y}d(t) - \frac{\partial\alpha_{n-1}}{\partial y}\epsilon_2
\end{aligned} \tag{41}$$

where $\beta_n$ contains all known terms. Define a positive–definite Lyapunov function $V_n$ as

$$V_n = V_{n-1} + \frac{1}{2}(|z_n|-\delta_n)^2 f_n + \frac{1}{2\gamma_{dn}}\tilde{D}_n^2. \tag{42}$$

We choose the update laws for $\hat{a}, \hat{\Theta}, \hat{D}_n$

$$\begin{aligned}
\dot{\hat{a}} =& \Gamma_a\tau_{an}, \dot{\hat{\Theta}} = -\Gamma_\Theta\tau_{\Theta n} \\
\dot{\hat{D}}_n =& \gamma_{dn}\sqrt{\|\frac{\partial\alpha_{n-1}}{\partial y}\|^2+\delta_0}\cdot(|z_n|-\delta_n)f_n
\end{aligned} \tag{43}$$

and the design signal $\chi$ as

$$\chi = l_1P^{-1}\tau_{\chi n}. \tag{44}$$

Finally, the control law is given by

$$v(t) = \frac{u_d(t)+\widehat{m_rb_r}}{\widehat{m_r}}\phi_r(u_d) + \frac{u_d(t)+\widehat{m_lb_l}}{\widehat{m_l}}\phi_l(u_d) \quad u_d = \alpha_n \tag{45}$$

With this choice and similar steps in Step 1) for $\dot{V}_1$, the derivative of $V_n$ becomes

$$\dot{V}_n \leq -\sum_{i=1}^{n} c_i(|z_i|-\delta_i)^{2(n-i+1)} f_i - \frac{1}{l_1}\epsilon^T\epsilon. \tag{46}$$

From (46), we get the following Lemma.

*Lemma 1:* The adaptive controller designed above ensures that $z_1,\ldots,z_n,\hat{\theta},\hat{e},\hat{b},\hat{a},\hat{\Theta},\hat{D}_i,\epsilon$ are all bounded.

With Lemma 1, all the signals in the closed-loop can be shown to be bounded and a bound can be established for the tracking error, as stated in the following theorem.

*Theorem 1:* Consider the system consisting of the parameter estimators given by (28), (29), and (43), adaptive controllers designed using (45) with virtual control laws (25) and (33), and plant (1) with a dead-zone nonlinearity (3). The system is stable in the sense that all signals in the closed-loop are bounded. Furthermore, the following hold.

- The tracking error approaches $\delta_1$ asymptotically, i.e.,

$$\lim_{t\to\infty} |y(t)-y_r(t)| f_1 = \delta_1. \tag{47}$$

- The transient tracking error performance is given by

$$\begin{aligned}\| |y(t)-y_r(t)|-\delta_1\|_2 &\leq \frac{1}{c_1^{2n}}\left(\frac{1}{2}\tilde{a}(0)^T\Gamma_a^{-1}\tilde{a}(0)+\frac{|b|}{2\Gamma_\theta}\tilde{\theta}(0)^2+\frac{1}{2\Gamma_\Theta}\tilde{\Theta}(0)^2\right.\\ &\quad\left.+\frac{|b|}{2\gamma_1}\tilde{e}(0)^2+\sum_{i=1}^{n}\frac{1}{2\gamma_{\rm di}}\tilde{D}_i(0)^2+\frac{1}{2\gamma_2}\tilde{b}(0)^2+\frac{1}{2l_1}\epsilon(0)^2\right)^{1/2n}\end{aligned} \tag{48}$$

with $z_i(0)=0, i=1,\ldots,n$.

*Proof:* From Lemma 1, we have that $z_1,\ldots,z_n,\hat{\theta},\hat{e},\hat{b},\hat{a},\hat{\Theta},\hat{D}_i,\epsilon$ are bounded. The tracking error performance can be obtained from (46) following similar approaches to those in [11]. What we need to prove is the boundedness of state $x$, controller output $v$ and plant input $u$.

From state observers $\xi_i$ in (16), we have that $\xi_0,\ldots,\xi_r$ are bounded. Rewriting plant (1) as

$$p^n y+\sum_{i=1}^{r} a_i Y_i(y,py,\ldots,p^{n-1}y)=bu \tag{49}$$

and using (17), we have

$$\eta_2=\frac{q_2(p)}{\Delta(p)}u=\frac{p^n q_2(p)}{b\Delta(p)}y+\frac{q_2(p)}{b\Delta(p)}\sum_{i=1}^{r}a_iY_i(y). \tag{50}$$

Since $\Delta(p) = p^n + k_1p^{n-1}+\cdots+k_n$ is Hurwitz, so $(q_2(p))/(b\Delta(p))$ is stable. We have that $\eta_2$ is bounded because $y$ is bounded. From (18), we have

$$\eta_2=-\theta^T\hat{\omega}_2(t)+d_2(t) \tag{51}$$

As $d_2(t)\in L^\infty$, then $\theta^T\hat{\omega}_2\in L^\infty$.

Express (21) as

$$\begin{aligned}\hat{\omega}_2(t)&=\left[-\frac{q_2(p)}{\Delta(p)}\phi_r(v)v(t),\frac{q_2(p)}{\Delta(p)}\phi_r(v)(t),\right.\\ &\qquad\left.-\frac{q_2(p)}{\Delta(p)}\phi_l(v)v(t),\frac{q_2(p)}{\Delta(p)}\phi_l(v)(t)\right]^T\end{aligned} \tag{52}$$

$$\begin{aligned}\theta^T\hat{\omega}_2(t)&=-m_r\frac{q_2(p)}{\Delta(p)}\phi_r(v)v(t)+m_rb_r\frac{q_2(p)}{\Delta(p)}\phi_r(v)(t)\\ &\quad-m_l\frac{q_2(p)}{\Delta(p)}\phi_l(v)v(t)+m_lb_l\frac{q_2(p)}{\Delta(p)}\phi_l(v)(t).\end{aligned} \tag{53}$$

Because $\phi_r(v)\in L^\infty,\phi_l(v)\in L^\infty$ and $(q_2(p))/(\Delta(p))$ is stable, the terms $(q_2(p))/(\Delta(p))\phi_r(v)$ and $(q_2(p))/(\Delta(p))\phi_l(v)$ in (52) are bounded.

We now show that $\hat{\omega}_2$ is bounded in two cases.

Case 1) If $v(t)$ is bounded, $\hat{\omega}_2$ is bounded directly from (52).

Case 2) In case that $v(t)$ is unbounded, we divide $R^+=[0,\infty)$ into two subsequences $R^+=R_1\cup R_2$, where $R_1=\{t|v(t)\geq 0\}$ and $R_2=\{t|v(t)<0\}$. Then, the following two situations are considered.

i) $t\in R_1$. From (5) we get

$$\phi_l(v)\cdot v=\frac{e^{-v/e_0}}{e^{v/e_0}+e^{-v/e_0}}\cdot v=\frac{v}{1+e^{2v/e_0}}$$

Thus $\quad \phi_l(v)\cdot v\to 0,\quad$ when $v\to+\infty$ for $t\in R_1$. (54)

So, in (53), the third term $m_l(q_2(p))/(\Delta(p))\phi_l(v)v(t)\to 0$, with the boundedness of second term and fourth term and $\theta^T\hat{\omega}_2\in L^\infty$, we see that the first term $m_r(q_2(p))/(\Delta(p))\phi_r(v)v$ is bounded for $t\in R_1$.

ii) $t\in R_2$. Similarly, from (5), we can show that

$$\phi_r(v)\cdot v\to 0,\quad \text{when } v\to-\infty \text{ for } t\in R_2. \tag{55}$$

and the third term $m_l(q_2(p))/(\Delta(p))\phi_l(v)v$ is bounded for $t\in R_2$.

Combining i) and ii), we get that for all $t\in R^+,(q_2(p))/(\Delta(p))\phi_r(v)v$ and $(q_2(p))/(\Delta(p))\phi_l(v)v$ are bounded. Then $\hat{\omega}_2$ is bounded from (52).

In summary, from the two cases we obtain the boundedness of $\hat{\omega}_2$.

Since $\hat{\theta}^T\hat{\omega}_2$ and $z_2$ are bounded, from $z_2=-\hat{\theta}^T\hat{\omega}_2-\hat{e}\dot{y}_r-\alpha_1$ we can obtain the boundedness of $\alpha_1$. From (25), we have $\bar{\alpha}_1$ is bounded. From (33), $\alpha_2,\ldots,\alpha_n$ are bounded, and so is $\chi$. From (45) we have that $u_d(t)$ is bounded, and so are $v=\hat{\rm D}{\rm I}(u_d)$ and $u={\rm DI}(v)$. It following from (19) that $\hat{\omega}_i\in L^\infty, i=1,\ldots,n$. From (16), we have that $\eta$ is bounded. Then $\hat{x}$ is bounded from (15) and finally $x(t)=\hat{x}(t)+\epsilon(t)$ is bounded from (15)–(16). △△△

*Remarks 3:* From (47) and (48), we can discuss how the initial estimate errors and the choices of the adaptation gains affect the transient performance in terms of $L_2$ norm of the tracking error.

- The closer the initial estimates to the true values, the better the transient performance.
- We can decrease the effects of the initial error estimates on the transient performance by increasing the adaptation gains $\gamma_1,\gamma_{\rm di},\gamma_2$ and $\Gamma_a,\Gamma_\theta,\Gamma_\Theta$. However, increasing these gains may influence other performance such as $\dot{x}$, following similar discussion in [11].

## V. Simulation Studies

In this section, we illustrate the previous methodology on the following two examples.

*Example 1:* We consider the same system as in [8] and [9], which is described as

$$\begin{aligned}\dot{x}&=a\frac{1-e^{-x(t)}}{1+e^{-x(t)}}+bu(t)\\ u&={\rm DZ}(v)\end{aligned} \tag{56}$$

where $u$ represents the output of the dead-zone nonlinearity. The actual parameter values are $b=1$ and $a=1$, and the dead-zone parameter values are $m_r=1.05; m_l=1.05; b_r=0.3; b_l=-0.5$. The objective is to control the system state $x$ to follow a desired trajectory $y_r(t)=8.5\sin(2.5t)$. In the simulations, taking $c_1=4,\Gamma_a=0.1,\gamma_1=0.3,\gamma_2=0.2,\Gamma_\theta=[0.1,0.1,0.1,0.1]^T, e_0=1,\delta_1=0.02$ and the initial parameters $\hat{e}(0)=0.3,\hat{a}(0)=1.5,\hat{D}(0)=$

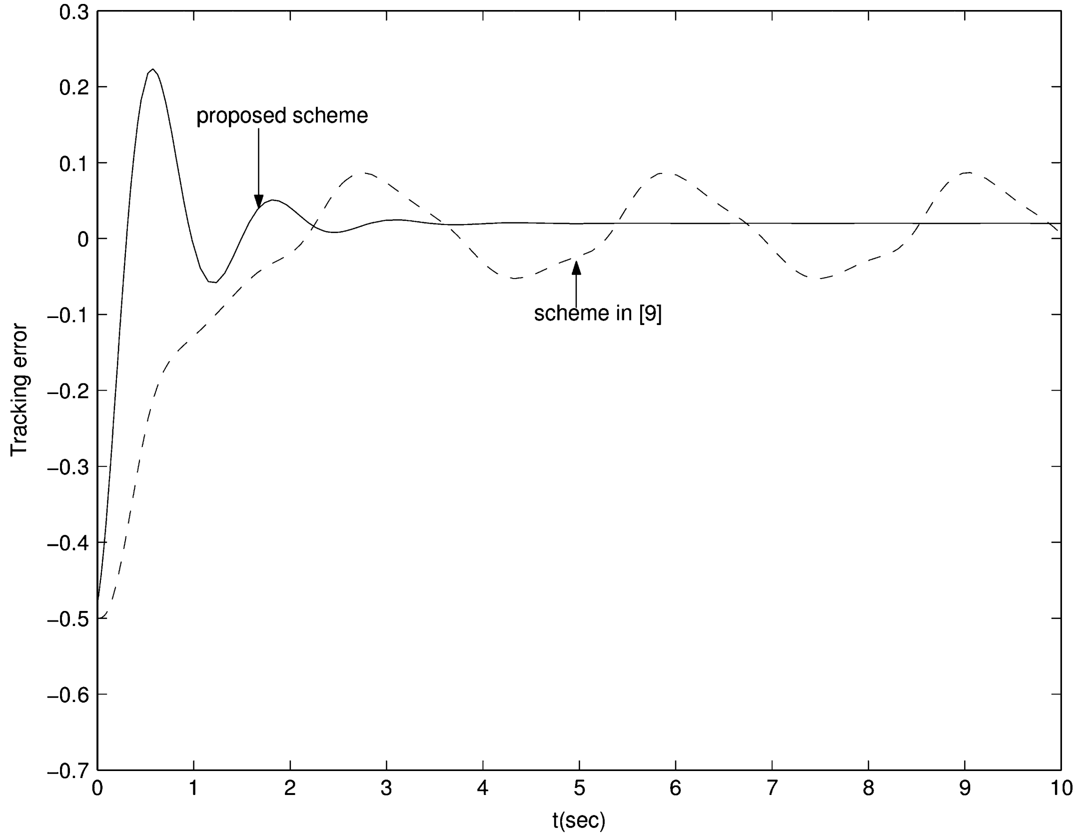


Fig. 2. Tracking error.

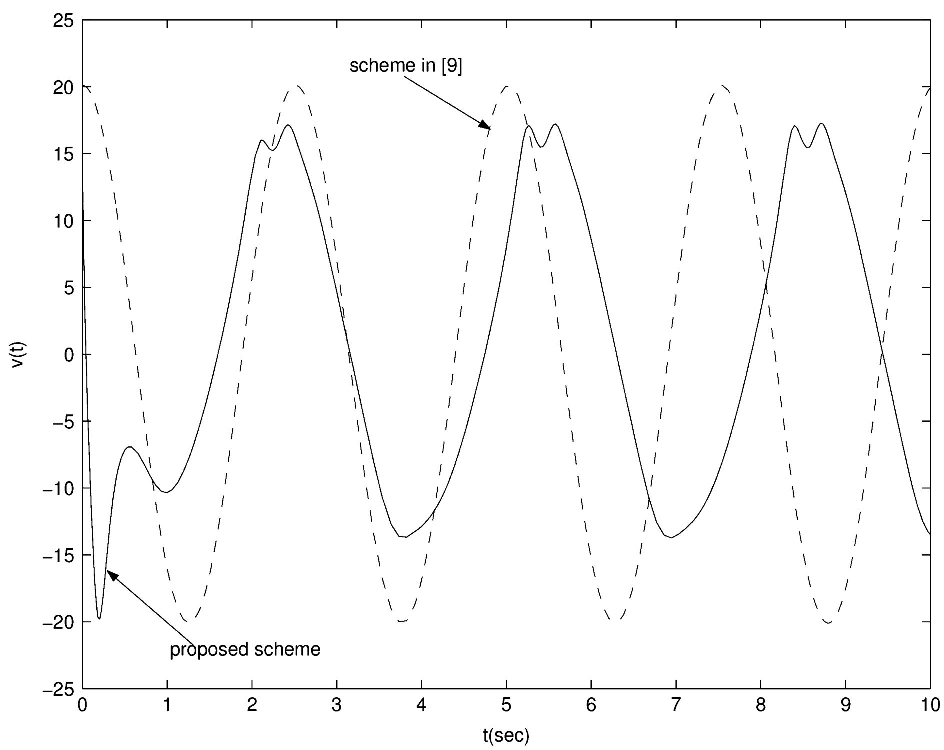


Fig. 3. Control signal $v(t)$.

$0.4, \hat{\theta}(0) = [1, 1, 0.2, -0.3]^T$. The initial state is chosen as $x(0) = -0.5$. The parameters and the initial states are the same as in [9]. For comparison, the scheme in [9] and our proposed scheme are both applied to the system. The simulation results presented in the Figs. 2 and 3 are the tracking error and the controller output $v(t)$. Clearly, the simulation results verify our theoretical findings and show the effectiveness of our control scheme. Also system performance is improved by our scheme.

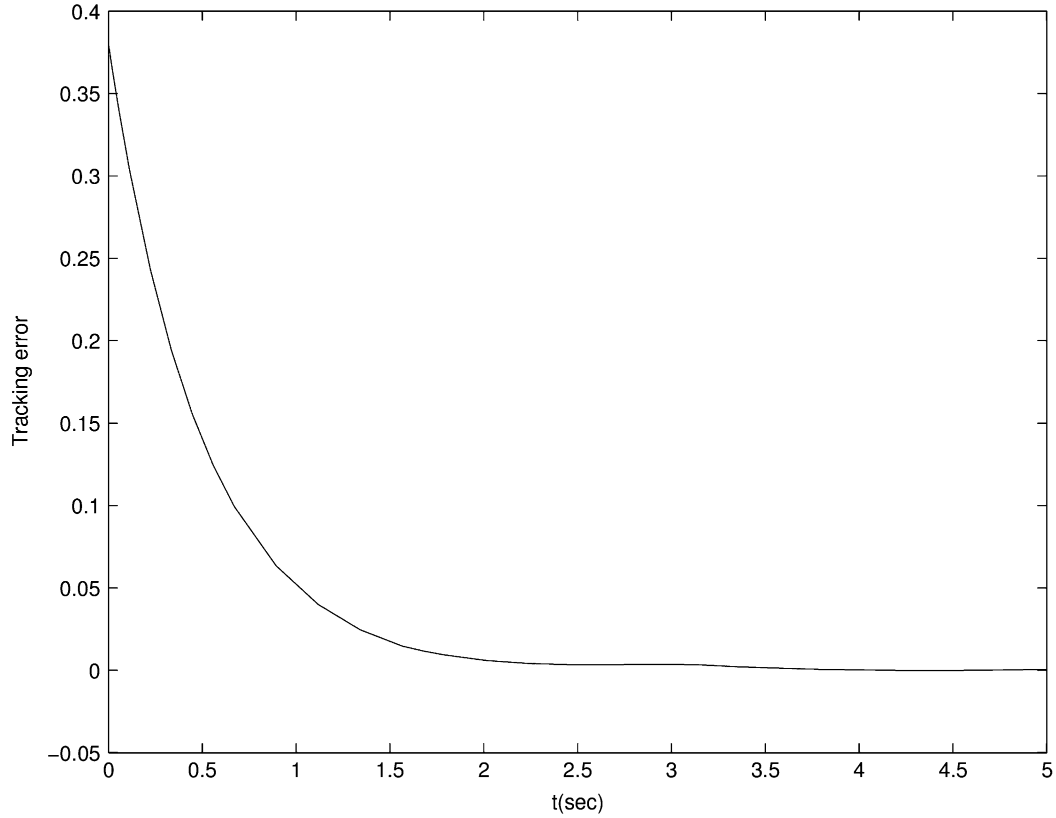


Fig. 4. Tracking error.

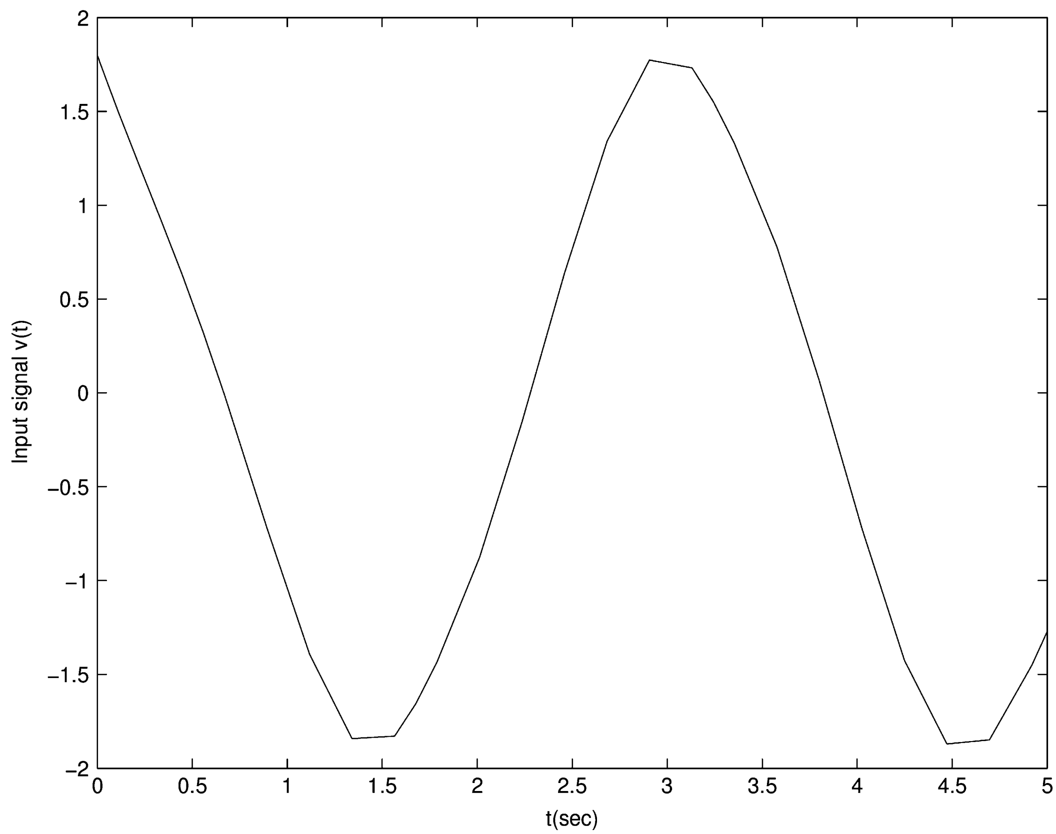


Fig. 5. Control signal $v(t)$.

*Example 2:* Consider the following system:

$$\ddot{x} = ax^2 + u \quad u = \mathrm{DZ}(v) \tag{57}$$

where $u$ represents the output of the dead-zone nonlinearity, parameter $a$ is unknown and dead-zone parameters $m_r, b_r, m_l, b_l$ are unknown, but $m_r \geq 0.1, m_l \geq 0.1$. The actual parameter values are chosen as $a = 1, m_r = 1, m_l = 1, b_r = 0.5, b_l = -0.5$. The objective is to control the system state $x$ to follow a desired trajectory $y_r(t) = 4\sin(2t)$. First, we choose the dead-zone inverse $v(t) = \hat{\mathrm{DI}}(u_d(t))$ as in (11) and the filters

$$\begin{aligned} \dot{\xi}_0 &= A_0\xi_0 + ky + \chi \quad \dot{\xi}_1 = A_0\xi_1 + Y_1 e_2 \\ \dot{\eta} &= A_0\eta + e_2 u \end{aligned} \tag{58}$$

$$\hat{\omega}_2 = \frac{p + k_1}{p^2 + k_1 p + k_2} I_4[\hat{\omega}] \tag{59}$$

where $Y_1 = x^2 \quad k = [k_1, k_2]^T = [1, 3]^T$

$$A_0 = \begin{bmatrix} -k_1 & 1 \\ -1_2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 0 \end{bmatrix}. \tag{60}$$

Then we apply our control design to the plant. In the simulations, taking $c_1 = c_2 = 2, \Gamma_a = 0.1, \gamma_2 = 0.2, \Gamma_\theta = [0.1, 0.1, 0.1, 0.1]^T, e_0 = 1, \delta_1 = 0.02$ and the initial parameters $\hat{a}(0) = 1.5, \hat{D}(0) = 0.4, \hat{\theta}(0) = [1, 1, 0.4, -0.4]^T$. The initial state is chosen as $x(0) = 0.4$. The tracking error and the controller output $v(t)$ are shown in Figs. 4 and 5. Clearly, the simulation results verify our theoretical findings and show the effectiveness of our control scheme.

## VI. Conclusion

This note presents an output feedback backstepping adaptive controller design scheme for a class of uncertain nonlinear single-input–single-output system preceded by uncertain dead-zone actuator nonlinearity. We propose a new smooth adaptive inverse to compensate the effect of the unknown dead-zone. Such an inverse can avoid possible chattering phenomenon which may be caused by nonsmooth inverse. The inverse function is employed in the backstepping controller design. For the design and implementation of the controller, no knowledge is assumed on the unknown system parameters. Besides showing stability, we also give an explicit bound on the $L_2$ performance of the tracking error in terms of design parameters. Simulation results illustrates the effectiveness of our schemes.

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# $H_\infty$ Control for Networked Systems With Random Communication Delays

Fuwen Yang, Zidong Wang, Y. S. Hung, and Mahbub Gani

***Abstract*—This note is concerned with a new controller design problem for networked systems with random communication delays. Two kinds of random delays are simultaneously considered: i) from the controller to the plant, and ii) from the sensor to the controller, via a limited bandwidth communication channel. The random delays are modeled as a linear function of the stochastic variable satisfying Bernoulli random binary distribution. The observer-based controller is designed to exponentially stabilize the networked system in the sense of mean square, and also achieve the prescribed $H_\infty$ disturbance attenuation level. The addressed controller design problem is transformed to an auxiliary convex optimization problem, which can be solved by a linear matrix inequality (LMI) approach. An illustrative example is provided to show the applicability of the proposed method.**

***Index Terms*—$H_\infty$ control, linear matrix inequalities (LMIs), networked systems, random communication delays, stochastic stability.**

## I. Introduction

Recent advances in network technology have led to more and more control systems whose feedback control loop is based on a network. This kind of control systems are called networked control systems (NCSs) [7], [10], [23]. The network itself is a dynamic system and induces possible delays via network communication due to limited

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