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Adaptive Backstepping Control Design for Systems with Unknown High-Frequency Gain

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***Abstract*—In this correspondence, Nussbaum gains are introduced in the backstepping design to obtain adaptive controllers for systems with unknown high-frequency gain. Two kinds of modified backstepping control design schemes are developed. It is shown that both schemes can give asymptotic tracking.**

***Index Terms*—Adaptive control, backstepping, high-frequency gain, Nussbaum gain.**

I. INTRODUCTION

High-frequency gain plays an important role in the adaptive control design. It has been a common concern on how we can design an adaptive controller without the knowledge of the sign of the system high-frequency gain [1], [2]. So far, some methods have been suggested to solve this problem [4]–[7]. One potential way is to use Nussbaum gains that usually take the forms of $\chi\cos(\chi)$ or $\chi^2\cos(\chi)$. This idea was first introduced in [3] to design an adaptive controller for first-order systems. Lately, in [4], it was extended to treat high-order systems with relative degrees less than two. The difficulty involved in using Nussbaum gains is the need to set up a subsystem to properly relate the variable χ to some other signals of the system so that it is feasible to establish the boundedness of χ and finally the stability of the whole adaptive system. This difficulty is increased further for high-order systems with relative degree larger than two. In [5], this problem was solved, but a complicated higher order subsystem was used to construct the Nussbaum gain. It is worth mentioning that all of these results were obtained for model reference adaptive controllers that are still based on the "certain equivalence" principle.

An alternative adaptive control design scheme is the backstepping technique proposed recently in [8]. This technique allows one to design adaptive controllers for a class of minimum phase linear systems with arbitrary relative degree. Because the adaptive law and the synthesis of the control law are carried out at the same time in the design procedure, the backstepping design technique provides a promising way to improve the transient performance of the adaptive systems. The sign of the high-frequency gain, however, has to be used in the control design of [8]. Because the structure of backstepping controller is totally different from that in [5], the Nussbaum gain design scheme of [5] cannot be applied to the backstepping design. Thus, it is of interest to devise a method to remove the requirement on the sign of the high-frequency gain in the backstepping design. Also, it is desired that the subsystem introduced for this purpose should be of low order and have a simple form whenever possible, for the sake of better transient performance.

This correspondence presents two modified backstepping design schemes by using Nussbaum gains to relax the requirement on the sign of high-frequency gain. In the first scheme, an auxiliary signal is added to the system output tracking error to carry out the first step of the backstepping design. As a result, a second-order auxiliary subsystem has to be employed to construct the Nussbaum gain. This, though slightly, increases the order of the controller. In the second scheme,

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only the output tracking error is used in the first step as in [8], but the regressor used for parameter estimation is properly augmented, and thus the order of the auxiliary subsystem is reduced to one. It is shown that the controllers designed by the two proposed schemes can achieve asymptotic tracking for minimum phase linear systems with unknown high-frequency gain and arbitrary relative degrees. Compared with the results in [3]–[5], the controllers proposed in this paper employ lower order subsystems to construct the Nussbaum gain. Particularly in the second scheme, the system order reaches its minimum level.

II. Problem Formulation

Consider linear systems described by

$$y = \frac{B(s)}{A(s)}\, u = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}\, u \tag{1}$$

where a_i and b_j $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$ are unknown constants.

The control objective is to force the system output to track a given reference signal $y_r(t)$. To this end, the following assumptions are made about the systems.

A.2) The system is minimum phase, i.e., all zeros of the polynomial $b_m s^m + \cdots + b_1 s + b_0$, are stable.

A.3) The relative degree $\rho = n - m$ is known.

A.4) $y_r, \dot{y}_r, \cdots, y_r^{(\rho)}$ are known bounded and piecewise continuous functions of time.

Remark 2.1: It is noted that the sign of the high-frequency gain, i.e., $\operatorname{sgn}(b_m)$, is not assumed *a priori* for the controller design as required in [8].

III. Backstepping Design with a Nussbaum Gain and an Auxiliary Signal

The desired adaptive controller can be obtained by performing the backstepping procedures [8] on the following system:

$$\dot{y} = \xi_2 + \omega^T\theta + \epsilon_2 = b_m v_{m,2} + \xi_2 + \overline{\omega}^T\theta + \epsilon_2 \tag{2}$$

$$\dot{v}_{m,i} = v_{m,i+1} - k_2 v_{m,1}, \qquad i = 2, 3, \cdots, \rho - 1 \tag{3}$$

$$\dot{v}_{m,\rho} = v_{m,\rho+1} - k_\rho v_{m,1} + u \tag{4}$$

where

$$\theta = [b_m, \cdots, b_0, a_{n-1}, \cdots, a_0]^T \tag{5}$$

$$\omega = \left[v_{m,2},\ v_{m-1,2},\ \cdots,\ v_{0,2},\ \Xi_{(2)} - y e_1^T\right]^T \tag{6}$$

$$\overline{\omega} = \left[0,\ v_{m-1,2},\ \cdots,\ v_{0,2},\ \Xi_{(2)} - y e_1^T\right]^T \tag{7}$$

with $v_{i,2} (i = m-1, \cdots, 0)$, $\Xi_{(2)}$, ξ_2, and ϵ_2 denoting, respectively, the second element of states $v_i (i = m-1, \cdots, 0)$, Ξ, ξ, and ϵ, which are defined by

$$v_i = A_0^i \lambda, \qquad i = 0, \cdots, m \tag{8}$$

$$\Xi = -\left[A_0^{n-1}\eta, \cdots, A_0\eta, \eta\right] \tag{9}$$

$$\xi = -A_0^n \eta \tag{10}$$

$$\dot{\epsilon} = A_0 \epsilon \tag{11}$$

$$\dot{\eta} = A_0\eta + e_n y \tag{12}$$

$$\dot{\lambda} = A_0\lambda + e_n u \tag{13}$$

$$\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_n]^T \tag{14}$$

$$k \triangleq [k_1, k_2, \cdots, k_n]^T \tag{15}$$

$$e_1 = [1, 0, \cdots, 0]^T \in R^{n\times 1} \tag{16}$$

$$e_n = [0, \cdots, 0, 1]^T \in R^{n\times 1} \tag{17}$$

$$A_0 = \begin{bmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{n-1} & 0 & 0 & \cdots & 1 \\ -k_n & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{18}$$

The vector k in (15) is chosen such that the matrix A_0 is strictly stable; i.e., all of the eigenvalues of A_0 have negative real parts.

In order to avoid using the sign of the high-frequency gain, we define

$$z_1 = y - y_r \tag{19}$$

$$z_i = v_{m,i} - y_r^{(i-1)} - \alpha_{i-1}, \qquad i = 2, 3, \cdots, \rho \tag{20}$$

where α_{i-1} is the virtual control at the ith step and will be determined in later discussions. In comparison with the coordinate transformation used in [8], the estimate of b_m^{-1} is no longer used here. To illustrate the backstepping procedures, the first two steps of the design are given in details as follows.

Step 1: For $i = 2$, it follows from (2), (19), and (20) that

$$\begin{aligned}\dot{z}_1 &= b_m v_{m,2} + \overline{\omega}^T\theta + \xi_2 + \epsilon_2 - \dot{y}_r \\ &= b_m(z_2 + \alpha_1 + \dot{y}_r) + \overline{\omega}^T\theta + \xi_2 + \epsilon_2 - \dot{y}_r \\ &= -c_1 z_1 + b_m\left(z_2 + \alpha_1 + \frac{b_m - 1}{b_m}\dot{y}_r + \overline{\omega}^T\left(\frac{1}{b_m}\theta\right)\right. \\ &\qquad \left. + \frac{1}{b_m}(c_1 z_1 + \xi_2)\right) + \epsilon_2. \end{aligned} \tag{21}$$

Define

$$\omega_1 = \left[\overline{\omega}^T,\ c_1 z_1 - \dot{y}_r + \xi_2\right]^T \tag{22}$$

$$\theta_1 = \left[\frac{\theta^T}{b_m},\ \frac{1}{b_m}\right]^T. \tag{23}$$

Then, (21) can be expressed as

$$\dot{z}_1 = -c_1 z_1 + b_m\left(z_2 + \alpha_1 + \omega_1^T\theta_1 + \dot{y}_r\right) + \epsilon_2. \tag{24}$$

To obtain the virtual adaptive control laws for this step without using the sign of b_m, we now introduce an auxiliary signal given by

$$\overline{z}_1 = z_1 + \delta \tag{25}$$

where

$$\dot{\delta} = -c_1\delta - \chi^2 \overline{z}_1 \tag{26}$$

$$\chi = \tfrac{1}{2}\overline{z}_1^2 + \gamma \tag{27}$$

$$\dot{\gamma} = \left(c_1 + \chi^2\right)\overline{z}_1^2. \tag{28}$$

Then, we take the following virtual control law α_1 and adaptive law $\hat{\theta}_1$ for estimating θ_1:

$$\alpha_1 = -\omega_1^T\hat{\theta}_1 - \dot{y}_r \tag{29}$$

$$\dot{\hat{\theta}}_1 = -N(\chi)\Gamma_1\omega_1\overline{z}_1 \tag{30}$$

where Γ_1 is a positive matrix of $R^{(n+2)\times(n+2)}$ and a Nussbaum gain $N(\chi)$ is chosen as

$$N(\chi) = \chi\cos(\chi). \tag{31}$$

With (24)–(29), it follows that

$$\begin{aligned}\dot{\overline{z}}_1 &= -c_1 z_1 + b_m\left(z_2 + \omega_1^T\tilde{\theta}_1\right) + \epsilon_2 - c_1\delta - \chi^2\overline{z}_1 \\ &= -\left(c_1 + \chi^2\right)\overline{z}_1 + b_m\left(z_2 + \omega_1^T\tilde{\theta}_1\right) + \epsilon_2\end{aligned} \tag{32}$$

where $\tilde{\theta}_1 \triangleq -\hat{\theta}_1 + \theta_1$.

Remark 3.1: Equation (32) actually gives the dynamics of the auxiliary signal $\overline{z}_1$. It is clear that if χ is bounded and $\overline{z}$ approaches zero as $t \to \infty$, z_1 also approaches zero. Thus, $\overline{z}_1$ can be used as an auxiliary signal reflecting the tracking error. In contrast with [8], $\overline{z}_1$ instead of z_1 is used as the first variable of coordinate transformation in the following backstepping design. As shown in later analysis, the boundedness of χ and all other signals in the system can be readily obtained after introducing $\overline{z}_1$.

To proceed, we define the Lyapunov function

$$V_1 = \tilde{\theta}_1^T \Gamma_1^{-1} \tilde{\theta}_1 + \frac{1}{4d_1} \epsilon^T P \epsilon \tag{33}$$

where P is a positive matrix such that $PA_0^T + A_0^T P = -I$ and d_1 is a real positive number to be determined later. Then, the derivative of V_1 along with the (29), (30), and (32) is given by

$$\dot{V}_1 = 2\tilde{\theta}_1^T \Gamma_1^{-1} \dot{\tilde{\theta}}_1 - \frac{1}{4d_1} \epsilon^T \epsilon = 2N(\chi) \tilde{\theta}_1^T \omega_1 \overline{z}_1 - \frac{1}{4d_1} \epsilon^T \epsilon. \tag{34}$$

Replacing $\tilde{\theta}_1^T \omega_1$ in (34) by using (32) gives

$$\begin{aligned} \dot{V}_1 = &\frac{2N(\chi)}{b_m} \overline{z}_1 \left(\dot{\overline{z}}_1 + \left(c_1 + \chi^2\right) \overline{z}_1 - b_m z_2 - \epsilon_2\right) \\ &- \frac{1}{4d_1} \epsilon_2^2 - \frac{1}{4d_1} \sum_{i=1,\, i\neq 2}^{n} \epsilon_i^2. \end{aligned} \tag{35}$$

From (27) and (28), we have

$$\overline{z}_1 \dot{\overline{z}}_1 = \dot{\chi} - \dot{\gamma} \tag{36}$$

$$\dot{\chi} = \overline{z}_1 \dot{\overline{z}}_1 + \left(c_1 + \chi^2\right) \overline{z}_1^2. \tag{37}$$

Thus

$$\begin{aligned} \dot{V}_1 =& \frac{2N(\chi)}{b_m} \dot{\chi} - 2N(\chi)\overline{z}_1 z_2 - \frac{2N(\chi)}{b_m} \overline{z}_1 \epsilon_2 \\ &- \frac{1}{4d_1} \epsilon_2^2 - \frac{1}{4d_1} \sum_{i=1,\, i\neq 2}^{n} \epsilon_i^2 \\ \leq& \frac{2N(\chi)}{b_m} \dot{\chi} - 2N(\chi)\overline{z}_1 z_2 + \left|\frac{2N(\chi)}{b_m} \overline{z}_1 \epsilon_2\right| - \frac{1}{4d_1} \epsilon_2^2 \\ \leq& \frac{2N(\chi)}{b_m} \dot{\chi} + \frac{|N(\chi)|^2 |\overline{z}_1|^2}{|b_m|} - \frac{1}{|b_m|} \left(|N(\chi)||\overline{z}_1| - |\epsilon_2|\right)^2 \\ &- 2N(\chi)\overline{z}_1 z_2 - \left(\frac{1}{4d_1} - \frac{1}{|b_m|}\right) \epsilon_2^2. \end{aligned} \tag{38}$$

Noting from (31) and (27), we have

$$|N(\chi)\overline{z}_1|^2 \leq \left|\chi^2 \overline{z}_1^2\right| \leq \dot{\gamma} = c_1 \overline{z}_1^2 + \chi^2 \overline{z}_1^2.$$

Then, it follows that

$$\dot{V}_1 \leq \frac{2N(\chi)}{b_m} \dot{\chi} + \frac{\dot{\gamma}}{|b_m|} - 2N(\chi)\overline{z}_1 z_2 \tag{39}$$

if $d_1 \leq |b_m|/4$.

Remark 3.2: Note that constant d_1 is used here only for analysis purpose. It is no longer a control design parameter as used in [8]. Thus, such a constant satisfying $d_1 \leq |b_m|/4$ always exists once the plant is given even if it is unknown. It should be mentioned that constants d_i $(i = 2, 3, \cdots, \rho)$, which appear later, are still control design parameters as in [8].

Step 2: Now, we evaluate the dynamic of the second state z_2. Differentiating both sides of (20) for $i = 2$ and using (3), we have

$$\dot{z}_2 = v_{m,3} - k_2 v_{m,1} - \ddot{y}_r - \dot{\alpha}_1. \tag{40}$$

Noting that α_1 is a function of $\hat{\theta}_1$, y, η, $v_{i,2}$, and y_r, it follows from the same analysis as in [8] that

$$\begin{aligned} \dot{z}_2 =& v_{m,3} - k_2 v_{m,1} - \ddot{y}_r - \frac{\partial \alpha_1}{\partial \eta} (A_0 \eta + e_n y) \\ &- \frac{\partial \alpha_1}{\partial y} \left(\xi_2 + \omega^T \theta + \epsilon_2\right) - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r \\ &- \sum_{j=1}^{m+1} \frac{\partial \alpha_1}{\partial \lambda_j} \left(-k_j \lambda_1 + \lambda_{j+1}\right) + \frac{\partial \alpha_1}{\partial \hat{\theta}_1} \dot{\hat{\theta}}_1 \end{aligned} \tag{41}$$

$$= v_{m,3} - \beta_2 - \ddot{y}_r - \frac{\partial \alpha_1}{\partial y} \left(\omega^T \theta + \epsilon_2\right) \tag{42}$$

where

$$\tilde{\theta} \triangleq \theta - \hat{\theta} \tag{43}$$

$$\begin{aligned} \beta_2 \triangleq& k_2 v_{m,1} + \frac{\partial \alpha_1}{\partial y} \xi_2 + \frac{\partial \alpha_1}{\partial \eta} (A_0 \eta + e_n y) + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r \\ &+ \sum_{j=1}^{m+1} \frac{\partial \alpha_1}{\partial \lambda_j} \left(-k_j \lambda_1 + \lambda_{j+1}\right) + \frac{\partial \alpha_1}{\partial \hat{\theta}_1} \dot{\hat{\theta}}_1. \end{aligned} \tag{44}$$

Substituting (20) with $i = 3$ into (42), we get

$$\begin{aligned} \dot{z}_2 &= \alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y} \left(\epsilon_2 + \omega^T \theta\right) + z_3 \\ &= \alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y} \left(\epsilon_2 + \omega^T \tilde{\theta}\right) + z_3 - \frac{\partial \alpha_1}{\partial y} \omega^T \hat{\theta}. \end{aligned} \tag{45}$$

Define the Lyapunov function for this step as

$$V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{4d_2} \epsilon^T P \epsilon + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}. \tag{46}$$

Using (30) and (45), it follows that

$$\begin{aligned} \dot{V}_2 =& \dot{V}_1 + z_2 \dot{z}_2 - \frac{1}{4d_2} \epsilon^T \epsilon + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \\ \leq& \frac{2N(\chi)}{b_m} \dot{\chi} + \frac{\dot{\gamma}}{|b_m|} - 2N(\chi)\overline{z}_1 z_2 - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \\ &+ z_2 \left(\alpha_2 - \beta_2 - \frac{\partial \alpha_1}{\partial y} \left(\omega^T \tilde{\theta} + \epsilon_2\right) + z_3 - \frac{\partial \alpha_1}{\partial y} \omega^T \hat{\theta}\right) \\ &- \frac{1}{4d_2} \epsilon^T \epsilon \\ \leq& \frac{2N(\chi)}{b_m} \dot{\chi} + \frac{\dot{\gamma}}{|b_m|} + z_2 z_3 + \tilde{\theta}^T \left(-\frac{\partial \alpha_1}{\partial y} \omega z_2 - \Gamma^{-1} \dot{\hat{\theta}}\right) \\ &+ z_2 \left[\alpha_2 - \beta_2 - 2N(\chi)\overline{z}_1 + d_2 \left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 - \frac{\partial \alpha_1}{\partial y} \omega^T \hat{\theta}\right] \\ &- d_2 \left(\frac{\partial \alpha_1}{\partial y} z_2 + \frac{1}{2d_2} \epsilon_2\right)^2. \end{aligned} \tag{47}$$

We take the virtual control for this step as

$$\alpha_2 = -c_2 z_2 + \beta_2 + 2N(\chi)\overline{z}_1 - d_2 \left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_2 + \frac{\partial \alpha_1}{\partial y} \omega^T \hat{\theta}. \tag{48}$$

Then

$$\dot{V}_2 \leq -c_2 z_2^2 + \frac{2N(\chi)}{b_m} \dot{\chi} + \frac{\dot{\gamma}}{|b_m|} + \tilde{\theta}^T \Gamma^{-1} \left(\tau_2 - \dot{\hat{\theta}}\right) \tag{49}$$

where

$$\tau_2 = -\Gamma \frac{\partial \alpha_1}{\partial y} \omega z_2. \tag{50}$$

Step i $(i = 3, \cdots, \rho)$*:* These steps are similar to those in [8], which include defining $V_i = V_{i-1} + (1/2) z_i^2 + (1/4d_i) \epsilon^T P \epsilon$, taking

$$\begin{aligned} \alpha_i =& -c_i z_i - d_i \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_i - z_{i-1} + \beta_i + \frac{\partial \alpha_{i-1}}{\partial y} \omega^T \hat{\theta} \\ &+ \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i - \left(\sum_{k=3}^{i-1} z_k \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}}\right) \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega \end{aligned}$$

and choosing

$$\tau_i = \tau_{i-1} - \Gamma \frac{\partial \alpha_{i-1}}{\partial y} w z_i.$$

Finally, the actual adaptive controller is given by

$$u(t) = \alpha_\rho \tag{51}$$

$$\dot{\hat{\theta}} = \tau_\rho. \tag{52}$$

The final Lyapunov function V_ρ satisfies

$$\dot{V}_\rho \leq -\sum_{k=2}^{\rho} c_k z_k^2 + \frac{2N(\chi)}{b_m}\,\dot{\chi} + \frac{\dot{\gamma}}{|b_m|}. \tag{53}$$

We now prove the boundedness of χ. To do this, we consider the integration of both sides of (53) over an interval $[0,\, t]$,

$$\int_0^t \dot{V}_\rho\, d\tau \leq -\int_0^t \sum_{k=2}^{\rho} c_k z_k^2\, d\tau + \frac{2}{b_m}\int_0^t N(\chi)\dot{\chi}\, d\tau + \int_0^t \frac{\dot{\gamma}}{|b_m|}\, d\tau. \tag{54}$$

Rearranging (54) and using the fact $\chi > \gamma$ [see (27)], we have

$$0 \leq V_\rho(t) + \int_0^t \sum_{k=2}^{\rho} c_k z_k^2\, d\tau \leq f(\chi(t)) - f(0) + V_\rho(0) \tag{55}$$

where

$$f(\chi(t)) = \frac{2}{b_m}\left(\cos(\chi(t)) + \chi\sin(\chi(t))\right) + \frac{1}{|b_m|}\,\chi(t) \tag{56}$$

$$f(0) = \frac{2}{b_m}\left(\cos(\chi(0)) + \chi\sin(\chi(0))\right) + \frac{1}{|b_m|}\,\gamma(0). \tag{57}$$

To come up with the conclusion, the following property of the function $f(\chi)$ is useful.

Lemma 1: If χ is unbounded, then for any constant C, an interval $[\chi^-,\, \chi^+]$ always exists such that

$$f(\chi) + C < 0, \qquad \forall\, \chi \in \left[\chi^-,\, \chi^+\right]. \tag{58}$$

Proof: Notice from the definition of $f(\chi(t))$ that

$$\lim_{\chi\to\infty}\left\{\frac{f(\chi(t)) - 2\chi(t)\sin(\chi(t))/b_m - \chi(t)/|b_m|}{\chi(t)}\right\} = 0. \tag{59}$$

If $\chi(t)$ is unbounded, then $\forall\, \varepsilon \in [0,\, (1/2|b_m|) - (1/B)]$, where B is a sufficiently large number, $X_1 > B|C|$ exists such that

$$\begin{aligned} &|f(\chi(t)) - 2\chi(t)\sin(\chi(t))/b_m - \chi(t)/|b_m||\\ &\quad < \varepsilon\chi(t), \qquad \forall\, \chi > \chi_1 > 0. \end{aligned} \tag{60}$$

To show (58), we consider two cases.

Case 1—$b_m > 0$: Because an interval $[\chi^-,\, \chi^+]$ always exists such that $\chi^- > \chi_1$ and $-1 - 2\sin(\chi(t)) \geq (1/2)$ for all $\chi(t) \in [\chi^-,\, \chi^+]$, we have

$$\left(\frac{1}{B} + \epsilon\right) b_m \leq \frac{1}{2} \leq -1 - 2\sin(\chi(t)). \tag{61}$$

Rearranging (61) yields

$$\varepsilon + \frac{1}{b_m} + \frac{2}{b_m}\,\sin(\chi(t)) \leq -\frac{1}{B}. \tag{62}$$

Thus, it follows from (60) that

$$\begin{aligned} f(\chi(t)) &< \left(\varepsilon + \frac{1}{b_m} + \frac{2}{b_m}\,\sin(\chi(t))\right)\chi(t)\\ &\leq -\frac{1}{B}\chi(t)\\ &\leq -|C| \end{aligned} \tag{63}$$

which confirms (58) because $f(\chi) + C \leq f(\chi) + |C|$.

Case 2—$b_m < 0$: In this case, we can always find an interval $[\chi^-,\, \chi^+]$ such that $\chi^- > \chi_1$ and $-1 + 2\sin(\chi(t)) \geq (1/2)$ for all $\chi(t) \in [\chi^-,\, \chi^+]$. Following the same analysis as in *Case 1*, it can be shown that (59) also holds for $b_m < 0$. Therefore, the conclusion of the Lemma 1 is valid. □

Taking $C = -f(0) + V_\rho(0)$ and using the result of Lemma 1, it is shown that $\chi(t)$ is bounded. Otherwise, it would result in a contradiction to (55) because its left side is always positive for all χ. With the boundedness of $\chi(t)$, it is straightforward to show from (55), (27), and (28) that z_i $(i = 2, \cdots, \rho)$, $\hat{\theta}$, γ, and $\overline{z}_1$ are bounded. From (32), $\dot{\overline{z}}_1$ is bounded. Also, it follows from (28) that

$$\int_0^t c_1\overline{z}_1^2\, d\tau \leq \int_0^t \dot{\gamma}\, d\tau = \gamma(t) - \gamma(0)$$

which implies $\overline{z}_1 \in L_2$. Therefore, it follows from Barbălat lemma that $\overline{z}_1 \to 0$ as $t \to \infty$. Using (26) and (25), it follows that $z_1 \to 0$ as $t \to \infty$. Finally, the stability of the whole system taking into account the zero dynamics can be established as in [8] because the system is minimum phase and thus the zero dynamics is stable.

TABLE I
ADAPTIVE BACKSTEPPING CONTROLLER WITH NUSSBAUM GAIN

Augmented error:	
$\bar{z}_1 = z_1 + \delta$	(T1.1)
$\dot{\delta} = -c_1\delta - \chi^2\bar{z}_1$	(T1.2)
Nussbaum Gain:	
$N(\chi) = \chi\cos(\chi)$	(T1.3)
$\chi = \frac{1}{2}\bar{z}_1^2 + \gamma$	(T1.4)
$\gamma = (c_1 + \chi^2)\bar{z}_1^2$	(T1.5)
Virtual Control Laws:	
$\alpha_1 = -\omega_1^T\hat{\theta}_1 - \dot{y}_r$ with $\omega_1 = [\bar{\omega}^T, c_1 z_1 - \dot{y}_r + \xi_2]^T$	(T1.6)
$\alpha_2 = -c_2 z_2 + \beta_2 + 2N(\chi)\bar{z}_1 + d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 + \frac{\partial\alpha_1}{\partial y}\omega^T\hat{\theta}$	(T1.7)
$\alpha_i = -c_i z_i - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_i - z_{i-1} + \beta_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \frac{\partial\alpha_{i-1}}{\partial y}\omega^T\hat{\theta} - \left(\sum_{k=3}^{i-1} z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right)\Gamma\frac{\partial\alpha_{i-1}}{\partial y}\omega,\ i = 3,\ldots,\rho$	(T1.8)
$u(t) = \alpha_\rho$	(T1.9)
Adaptive Laws:	
$\dot{\hat{\theta}}_1 = -N(\chi)\Gamma_1\omega_1 z_1$	(T1.10)
$\dot{\hat{\theta}} = \tau_\rho$	(T1.11)
with	
$\tau_2 = -\Gamma\frac{\partial\alpha_1}{\partial y}\omega z_2$	(T1.12)
$\tau_i = \tau_{i-1} - \Gamma\frac{\partial\alpha_{i-1}}{\partial y}\omega z_i,\ i = 3,\ldots,\rho$	(T1.13)

So far, we have obtained one backstepping design scheme with the Nussbaum gain chosen as (31). This scheme is now concisely presented in Table I. To conclude this section, the stability results presented above are summarized in the following theorem.

Theorem 1: Consider a minimum phase, linear, time-invariant system with known relative degree ρ. The adaptive controller presented in Table I can make the output of the system asymptotically track an arbitrary signal with bounded derivatives of up to order ρ, while all of the signals in the closed-loop system are bounded.

IV. BACKSTEPPING DESIGN WITHOUT AUXILIARY SIGNAL

It is noted from Table I that an augmented error $\overline{z}_1$ is used to construct a Nussbaum gain. Thus, a second-order auxiliary subsystem given by (T1.2) and (T1.5) has to be employed. This section presents an alternative way to construct the Nussbaum gain so that the order of the auxiliary subsystem reduces to one.

As in (31) and (28), we construct the following Nussbaum gain in the first step of the backstepping design:

$$N(\chi) = \chi\cos(\chi) \tag{64}$$

$$\chi = \tfrac{1}{2}z_1^2 + \gamma \tag{65}$$

$$\dot{\gamma} = (c_1 + \chi^2)z_1^2 \tag{66}$$

where the tracking error z_1 is directly used, instead of using the augmented error $\overline{z}_1$.

Introduce a new regressor ω_2 as

$$\omega_2 = \left[\overline{\omega}^T,\ c_1 z_1 - \chi^2 z_1 - \dot{y}_r + \xi_2\right]^T. \tag{67}$$

TABLE II
ADAPTIVE BACKSTEPPING CONTROLLER WITHOUT AUGMENTED ERROR

Nussbaum Gain:	
$N(\chi) = \chi\cos(\chi)$	(T1.1)
$\chi = \frac{1}{2}z_1^2 + \gamma$	(T1.2)
$\dot{\gamma} = (c_1 + \chi^2)z_1^2$	(T1.3)
Virtual Control Laws:	
$\alpha_1 = -\omega_2^T\hat{\theta}_1 - \dot{y}_r$ with $\omega_2 = [\bar{\omega}^T, c_1 z_1 + \chi^2 z_1 - \dot{y}_r + \xi_2]^T$	(T1.4)
$\alpha_2 = -c_2 z_2 + \beta_2 + 2N(\chi)z_1 + d_2\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_2 + \frac{\partial\alpha_1}{\partial y}\omega^T\hat{\theta}$	(T1.5)
$\alpha_i = -c_i z_i - d_i\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_i - z_{i-1} + \beta_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \frac{\partial\alpha_{i-1}}{\partial y}\omega^T\hat{\theta} - \left(\sum_{k=3}^{i-1} z_k\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right)\Gamma\frac{\partial\alpha_{i-1}}{\partial y}\omega, \quad i = 3,\ldots,\rho$	(T1.6)
$u(t) = \alpha_\rho$	(T1.7)
Adaptive Laws:	
$\dot{\hat{\theta}}_1 = -N(\chi)\Gamma_1\omega_1\bar{z}_1$	(T1.8)
$\dot{\hat{\theta}} = \tau_\rho$	(T1.9)
with	
$\tau_2 = -\Gamma\frac{\partial\alpha_1}{\partial y}\omega z_2$	(T1.10)
$\tau_i = \tau_{i-1} - \Gamma\frac{\partial\alpha_{i-1}}{\partial y}w z_i, \quad i = 3,\ldots,\rho$	(T1.11)

Then, (21) can be rewritten as

$$\dot{z}_1 = -(c_1 + \chi^2)z_1 + b_m\left(z_2 + \omega_2^T\tilde{\theta}_1\right) + \epsilon_2. \tag{68}$$

For this subsystem, we take the virtual control law α_1 and the updating law for $\hat{\theta}_1$ as follows:

$$\alpha_1 = -\omega_2^T\hat{\theta}_1 - \dot{y}_r \tag{69}$$
$$\dot{\hat{\theta}}_1 = N(\chi)\Gamma_1\omega_2 z_1. \tag{70}$$

Remark 4.1: Here, the tracking error z_1 is still used in the backstepping design as in [8], but a term $\chi^2 z_1$ is added in the regressor ω_2 instead of the regressor ω_1 used in the first scheme. By this way, only a first-order subsystem (66) is involved in the construction of the Nussbaum gain. Thus, the order of the controller is reduced by one.

We define a Lyapunov function as in (33)

$$V_1 = \tilde{\theta}_1^T\Gamma_1^{-1}\tilde{\theta}_1 + \frac{1}{4d_1}\epsilon^T P\epsilon. \tag{71}$$

The derivative of V_1 along with (68)–(70) is given by

$$\dot{V}_1 = 2\tilde{\theta}_1^T\Gamma_1^{-1}\dot{\tilde{\theta}}_1 - \frac{1}{4d_1}\epsilon^T\epsilon = 2N(\chi)\tilde{\theta}_1^T w_2 z_1 - \frac{1}{4d_1}\epsilon^T\epsilon. \tag{72}$$

Using (68), we have

$$\omega_2^T\tilde{\theta}_1 = \frac{\dot{z}_1 + (c_1 + \chi^2)z_1 - b_m z_2 - \epsilon_2}{b_m}. \tag{73}$$

Substituting the above equation into (72) gives

$$\begin{aligned}\dot{V}_1 = {} & \frac{2N(\chi)}{b_m} z_1\left(\dot{z}_1 + (c_1 + \chi^2)z_1 - b_m z_2 - \epsilon_2\right) \\ & - \frac{1}{4d_1}\epsilon_2^2 - \frac{1}{4d_1}\sum_{i=1,\, i\neq 2}^{n}\epsilon_i^2.\end{aligned} \tag{74}$$

On the other hand, it follows from (65) and (66) that

$$\dot{\chi} = z_1\dot{z}_1 + (c_1 + \chi^2)z_1^2. \tag{75}$$

Thus

$$\begin{aligned}\dot{V}_1 = {} & \frac{2N(\chi)}{b_m}\dot{\chi} - 2N(\chi)z_1 z_2 - \frac{2N(\chi)}{b_m}z_1\epsilon_2 \\ & - \frac{1}{4d_1}\epsilon_2^2 - \frac{1}{4d_1}\sum_{i=1,\, i\neq 2}^{n}\epsilon_i^2 \\ \leq {} & \frac{2N(\chi)}{b_m}\dot{\chi} - 2N(\chi)z_1 z_2 + \left|\frac{2N(\chi)}{b_m}z_1\epsilon_2\right| - \frac{1}{4d_1}\epsilon_2^2 \\ \leq {} & \frac{2N(\chi)}{b_m}\dot{\chi} + \frac{|N(\chi)|^2|z_1|^2}{|b_m|} - \frac{1}{|b_m|}(|N(\chi)\|z_1| - |\epsilon_2|)^2 \\ & - 2N(\chi)z_1 z_2 - \left(\frac{1}{4d_1} - \frac{1}{|b_m|}\right)\epsilon_2^2.\end{aligned} \tag{76}$$

Because $|N(\chi)z_1|^2 \leq \chi^2 z_1^2 \leq \dot{\gamma}$, we have

$$\dot{V}_1 \leq \frac{2N(\chi)}{b_m}\dot{\chi} + \frac{\dot{\gamma}}{|b_m|} - 2N(\chi)z_1 z_2 \tag{77}$$

if $d_1 \leq |b_m|/4$.

By replacing $\bar{z}_1$ in (39) with z_1, it is noted that (77) has the same form as (39) . Thus, the remaining design steps and the stability analysis are exactly the same as those presented in the previous section by replacing $\bar{z}_1$ with z_1. The final controller and the stability results are, respectively, summarized in Table II and Theorem 2.

Theorem 2: For a minimum phase, linear, time-invariant system with known relative degree ρ, the adaptive controller presented in Table II can ensure that the output of the system asymptotically tracks an arbitrary signal with bounded derivatives of up to order ρ and all of the signals in the closed-loop system are bounded.

V. CONCLUSION

In this correspondence, we have proposed an adaptive backstepping design procedure for systems with unknown high-frequency gain. The two proposed design schemes make use of the Nussbaum gains to relax the requirement on the sign of high-frequency gain. One scheme is to add an auxiliary signal to the system output tracking error, and the other is to properly augment the regressor that is used for parameter estimation into the first step of backstepping design. It is shown that the controllers obtained by both schemes can ensure that the output of the system asymptotically tracks a given signal and all of the signals in the adaptive control system remain bounded. Comparing these two schemes, the second one can yield adaptive controllers of lower order, but the regressor used for parameter estimation is somehow more complex than that in the first scheme.

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