Generalized Matrix Measure of Switched Nonlinear Systems

Z. G. Li, C. Y. Wen, Y. C. Soh, and W. X. Xie

Abstract—**A concept of generalized matrix measure for nonlinear systems is proposed to study the stability of switched nonlinear systems directly. Based on this concept, some sufficient conditions for robust stability of switched nonlinear systems are derived by using the methods of cycle analysis and contraction analysis.**

Index Terms—**Contraction, generalized matrix measure, robust stability, switched nonlinear systems.**

I. Introduction

Switched systems are composed of some continuous variable dynamic systems (CVDS) along with certain maps for switchings among them [2]. Recently, the stability of switched systems has been studied by many researchers. Liberzon and Morse [9] used Lie algebra to study the stability of switched systems. Branicky [2], Li *et al.* [8], and Johansson and Rantzer [5] used multiple Lyapunov functions to study the stability of switched systems. Since these methods are based on the Lyapunov stability theory, they need to find some implicit motion integrals which seems complicated [10]. It is desirable to provide a simple method to study the stability of switched nonlinear systems directly.

In this note, we shall provide such a method by introducing a new concept of generalized matrix measure for nonlinear systems. The generalized matrix measure is derived from the matrix measure, which is an effective tool for the stability and robustness analysis of linear systems [6]. Using the provided generalized matrix measure, the stability and robustness of nonlinear systems can be studied via a virtual displacement instead of Lyapunov functions. This simplifies the complexity of analysis. The method based on such a measure can be regarded as "contraction analysis method," which was firstly presented by [10] to consider the stability of a single nonswitched nonlinear systems. We shall also use this method to study the robust stability of switched nonlinear systems where the dwell time of each subsystem is in some given interval. However, the contraction analysis method cannot be directly used to study a switched nonlinear system because a switched nonlinear system is always composed of some unstable subsystems. To overcome this difficulty, the methods of contraction analysis and cycle analysis should be used together to study the stability and robustness of switched nonlinear systems.

The rest of the note is organized as follows. The problem is formulated in the following section. Generalized matrix measure is proposed in Section III and the main results are derived in Section IV. Section V contains a numerical example to illustrate the application of the main results. Finally, the note is concluded in Section VI.

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II. Problem Formulation

This note considers the following switched nonlinear systems which is composed of finite number of CVDS:

$$\dot{X}(t) = f(X(t), m(t)) + \Delta f(X(t), m(t)) \tag{1}$$

where $X(t) \in R^r$ is the continuous state and $m(t) \in \overline{M} = \{1, \ldots, n\}$ is the discrete state, $m(t)$ is left continuous with each i corresponding to a vector field $f(X(t), i)$. In (1), f, Δf: $R^r \times \overline{M} \to R^r$ are smooth vector fields, and $\Delta f(X(t), m(t))$ represents the perturbations of the model.

Let t_j denote the jth switching instant, t_j^- and t_j^+ represent respectively the time just before and just after t_j. When the trajectory of system (1) intersects the hypersurface

$$\begin{aligned} S_{m(t_j^-),\, m(t_j^+)} \\ = \{(X(t_j^-), t_j^-) | \phi(m(t_j^-), m(t_j^+), X(t_j^-), t_j^-) = 0\} \end{aligned} \tag{2}$$

some "switchings" will occur as follows [1]:

$$\begin{cases} X(t_j^+) = h(X(t_j^-), m(t_j^-), m(t_j^+)) \\ \quad +\Delta h(X(t_j^-), m(t_j^-), m(t_j^+)) \\ m(t_j^+) = \psi(t_j^-, X(t_j^-), m(t_j^-)) \end{cases} \tag{3}$$

where ϕ: $\overline{M} \times \overline{M} \times R^r \times R^+ \to R$, h: $R^r \times \overline{M} \times \overline{M} \to R^r$, ψ: $R^+ \times R^r \times \overline{M} \to \overline{M}$, h and Δh are smooth functions and $\Delta h(X(t_i^-), m(t_j^-), m(t_j^+))$ represents the perturbations of the reset map.

Equations (1) and (3) imply that a switching occurs when the states of the CVDS are in a corresponding hypersurface and the switching results in an abrupt change in the vector field f and a jump in the trajectory of $X(t)$. If $h(X(t_j^-), m(t_j^-), m(t_j^+)) = X(t_j^-)$ (i.e., h is an identity reset map) and $\Delta h(X(t_j^-), m(t_j^-), m(t_j^+)) = 0$, then the trajectory of the switched nonlinear system is continuous. In this case, there is no impulsive effect.

The system is said to be locally asymptotically stable with respect to a given trajectory if all trajectories in its neighborhood remain in the neighborhood and converge to the given trajectory. This given trajectory can be either an invariant set or an equilibrium. When we consider the robust stability of the switched system, we only consider the case of an equilibrium.

Let $t_{s,i}^k$ and $t_{f,i}^k$ denote respectively the kth starting time and the kth ending time of CVDS i. In this note, we suppose that

$$0 < \Delta_{1,i} = \inf_k \{t_{f,i}^k - t_{s,i}^k\} \le \sup_k \{t_{f,i}^k - t_{s,i}^k\} = \Delta_{2,i} < \infty. \tag{4}$$

Equation (4) implies that the dwell time of CVDS i is in a given interval $[\Delta_{1,i}, \Delta_{2,i}]$. This assumption has also been used in [9] and [11], and it is a quite common assumption.

The Objective: In this note, we shall study the local robust stability of the switched systems (1) and (3) satisfying (4) by using some simple and direct method, rather than finding some implicit motion integrals using Lyapunov theory as in [7] and [11].

III. Generalized Matrix Measure

In this section, we shall first introduce some basic notations. Suppose that matrices $A = [a_{ij}] \in R^{r \times r}$ and $B = [b_{ij}] \in R^{r \times r}$. As a notation, $A \ge B$ if and only if $a_{ij} \ge b_{ij}$ for all pairs (i, j) with $1 \le i, j \le r$.

The modulus matrix of A is denoted as $|A|_m$ and is given by $|A|_m = [|a_{ij}|]$ [6].

The system given by (1) and (3) can be regarded as an n-dimensional fluid flow, where $\dot{X}$ is the n-dimensional "velocity" vector at the n-dimensional position X. Then equations (1) and (3) without perturbations (i.e., $\Delta f \equiv 0,\ \Delta h \equiv 0$) yields the differentiable relations [10]

$$\frac{d\delta X(t)}{dt} = \frac{\partial f}{\partial X}(X(t), m(t))\delta X(t) \tag{5}$$

and

$$\delta X(t_j^+) = \frac{\partial h}{\partial X}(X(t_j^-), m(t_j^-), m(t_j^+))\delta X(t_j^-) \tag{6}$$

where δX is a virtual displacement which is an infinitesimal displacement `at fixed time`. Note that a virtual displacement, pervasive in physics and in the calculus of variation, is also a well defined mathematical object.

From (5) and (6), we have

$$\frac{d\delta X^T \delta X}{dt} = \delta X^T \left(\frac{\partial f^T}{\partial X} + \frac{\partial f}{\partial X}\right)\delta X$$

and

$$\delta X^T(t_j^+)\delta X(t_j^+) = \delta X^T(t_j^-)\frac{\partial h^T}{\partial X}\frac{\partial h}{\partial X}\delta X(t_j^-). \tag{7}$$

In this note, we derive some sufficient conditions for the stability of switched nonlinear systems by considering the characteristics of $\delta X^T\delta X$ along each type of cycle. To achieve this, we shall introduce the concept of the generalized matrix measure for nonlinear systems.

Definition 1: For any continuous differentiable function $f(X)$, the generalized matrix measure of the function, $\tilde{\mu}(f, CS)$, $\tilde{\mu}(|f|_m, CS)$, in a compact set CS, are of the forms

$$\tilde{\mu}(f, CS) = \sup_{X\in CS}\left\{\tilde{\mu}\left(\frac{\partial f}{\partial X}\right)\right\} \tag{8}$$

$$\tilde{\mu}(|f|_m, CS) = \sup_{X\in CS}\left\{\tilde{\mu}\left(\left|\frac{\partial f}{\partial X}\right|_m\right)\right\} \tag{9}$$

$$\tag{10}$$

and the generalized matrix norm, $\|f\|_{CS}$, $\||f|_m\|_{CS}$, in a compact set CS, are of the forms

$$\|f\|_{CS} = \sup_{X\in CS}\left\{\left\|\frac{\partial f}{\partial X}\right\|\right\};$$

$$\||f|_m\|_{CS} = \sup_{X\in CS}\left\{\left\|\left|\frac{\partial f}{\partial X}\right|_m\right\|\right\}. \tag{11}$$

In order to show the implication of Definition 1, we consider the case that $f(X) = AX$ and $CS = R^r$. From Definition 1, we have

$$\tilde{\mu}(A, R^r) = \mu(A); \qquad \tilde{\mu}(|A|_m, R^r) = \mu(|A|_m)$$

$$\|A\|_{R^r} = \|A\|; \qquad \||A|_m\|_{R^r} = \||A|_m\|.$$

Therefore, our definition reduces to the standard definition of matrix measure and matrix norm when $f(x)$ is a linear function. For more background about matrix measure and its application in robustness analysis of linear systems, please see [12] and [4].

We now derive some properties of the generalized matrix measure.

Lemma 1: For any continuously differentiable $g(x)$ and $f(x)$ satisfying $|\partial f/\partial X|_m \le A$ within a compact set CS and $a \ge 0$, we have

$$\tilde{\mu}(f+g, CS) \le \tilde{\mu}(f, CS) + \tilde{\mu}(g, CS) \tag{12}$$

$$\tilde{\mu}(f, CS) \le \|f\|_{CS} \tag{13}$$

$$\tilde{\mu}(af, CS) = a\tilde{\mu}(f, CS) \tag{14}$$

$$\tilde{\mu}(f, CS) \le \tilde{\mu}(|f|_m, CS) \le \mu(A) \tag{15}$$

$$\|f\|_{CS} \le \||f|_m\|_{CS} \le \|A\|. \tag{16}$$

Proof: We only prove inequality (12) as the other inequalities can be proved in a similar way. For any fixed $X \in CS$, we have

$$\tilde{\mu}\left(\frac{\partial f}{\partial X}(X) + \frac{\partial g}{\partial X}(X)\right) \le \tilde{\mu}\left(\frac{\partial f}{\partial X}(X)\right) + \tilde{\mu}\left(\frac{\partial g}{\partial X}(X)\right) \le \tilde{\mu}(f, CS) + \tilde{\mu}(g, CS).$$

It follows that:

$$\begin{aligned}\tilde{\mu}(f+g, CS) &= \sup_{X\in CS}\left\{\tilde{\mu}\left(\frac{\partial (f+g)}{\partial X}(X)\right)\right\}\\ &= \sup_{X\in CS}\left\{\tilde{\mu}\left(\frac{\partial f}{\partial X}(X) + \frac{\partial g}{\partial X}(X)\right)\right\}\\ &\le \tilde{\mu}(f, CS) + \tilde{\mu}(g, CS).\end{aligned}$$

□

It can be known from (5) and (6) and Definition 1 that the virtual displacement can be used to study the stability and robustness of nonlinear systems with the help of generalized matrix measure and its properties.

IV. Main Results

Before presenting the stability result in this note, we introduce the concept of cycles and give some supporting results.

A logical path in the switched systems (1) and (3) is a sequence $m(t_{i_1}^+), m(t_{i_1+1}^+), \ldots, m(t_{i_1+k}^+)$. A finite logical path $m(t_{i_1}^+), m(t_{i_1+1}^+), \ldots, m(t_{i_1+k}^+)$ is closed if $m(t_{i_1}^+) = m(t_{i_1+k}^+)$. A closed logical path $LC = m(t_{i_1}^+), m(t_{i_1+1}^+), \ldots, m(t_{i_1+k}^+)$ in which no state appears more than once except for the one that is the first and the last is a cycle. We can find all types of cycles by using graph theory [3]. In this note, we suppose that the total number of the types of cycles is θ_0 and we denote these cycles as $LC(1), LC(2), \ldots, LC(\theta_0)$. We now state some results about cycles.

Lemma 2 [7] (Cycle Lemma 1): Every closed path is composed of some cycles.

Lemma 3 [7] (Cycle Lemma 2): Suppose that $(l+1)$ discrete states $m(t_{i_1+s}^+)(0 \le s \le l)$ belong to a set which is composed of l different discrete states $j_k(1 \le k \le l)$. Then there exists at least one cycle in the logical path $m(t_{i_1}^+), m(t_{i_1+1}^+), \ldots, m(t_{i_1+l}^+)$.

For CVDS i, let $\widehat{CS}(i)$ denote the set

$$\widehat{CS}(i) = \left\{X \left| \frac{\partial f^T}{\partial X}(X, i) + \frac{\partial f}{\partial X}(X, i) \text{ is negative definite}\right.\right\}$$

and define

$$\overline{CS} = \bigcup_{i=1}^{n}\widetilde{CS}(i) \tag{17}$$

$$\widetilde{CS}(i) = \begin{cases}\widehat{CS}(i) & \widehat{CS}(i) \ne \emptyset\\ \emptyset & \text{Otherwise.}\end{cases} \tag{18}$$

In this note, we suppose that the following assumption holds.

Assumption 1: There exists a compact set $SCS \subseteq \overline{CS}$ such that SCS contains the given trajectory and for each $LC(j)(1 \le j \le \theta_0)$, we have

$$\tilde{\alpha}(j) = \sum_{i\in LC(j)}\tilde{\mu}(f(X, i), SCS)\Delta_{3,i} + \ln(\|h(X, i, l)\|_{S_{i,l}}) < 0 \tag{19}$$

where

$$\Delta_{3,i} = \begin{cases} \Delta_{1,i} & i \in \Gamma_2 \\ \Delta_{2,i} & i \in \Gamma_1 \end{cases} \tag{20}$$

$$\Gamma_1 = \{i | \tilde{\mu}(f(X, i), SCS) \geq 0\};$$
$$\Gamma_2 = \{i | \tilde{\mu}(f(X, i), SCS) < 0\}. \tag{21}$$

Remark 1: Assumption 1 implies that there exists a compact set such that $\delta X^T \delta X$ is nonincreasing along each type of cycle when the state stays in the set. The compact set is the whole R^r space if each subsystem is linear. Moreover, when $f(X, i) = A(i)X$, i.e., it is a linear function, $i \in \Gamma_1$ implies that the matrix measure of $A(i)$ is not less than 0 and $i \in \Gamma_2$ implies that the matrix measure of $A(i)$ is less than 0. □

Remark 2: We now give a general method to check Assumption 1. Without loss of generality, we suppose that the given trajectory is $\tilde{X}(t)(t \geq t_0)$ and define

$$\hat{\alpha}_j(\tilde{X}(t)) = \sum_{i \in LC(j)} \left(\frac{\partial f}{\partial X}(\tilde{X}(t))\Delta_{3,i} + \left\| \frac{\partial h}{\partial X} \right\|_{\tilde{X}(t)} \right). \tag{22}$$

Check if there exists a ξ_0 such that the set $\{X(t) | \|X(t) - \tilde{X}(t)\| < \xi_0\}$ is a subset of $\overline{CS}$ and if $\hat{\alpha}_j(\tilde{X}(t)) < 0$ holds for all j. If so, then Assumption 1 holds. The reason for this is presented as follows.

Since $\hat{\alpha}_j(\tilde{X}(t)) < 0$, then there exists a $\xi_j(1 \leq j \leq \theta_0)$ such that when $\|X(t) - \tilde{X}(t)\| < \xi_j$, we have

$$\hat{\alpha}_j(X(t)) < 0.$$

Let $\xi = (1/2)\min_{0 \leq j \leq \theta_0}\{\xi_j\}$ and

$$SCS = \{X(t) | \|X(t) - \tilde{X}(t)\| \leq \xi\}. \tag{23}$$

Then Assumption 1 holds with SCS given in (23).

For a given system, there may be some other better methods to find a larger compact set SCS to satisfy Assumption 1, for example, an alternative method presented in the numerical example in Section V.□

Proposition 1: Suppose that Assumption 1 holds. Consider a closed path CP_k with the starting time being t_{s,CP_k} and the ending time being t_{f,CP_k}. If $X(t) \in SCS$ holds for all $t \in [t_{s,CP_k}, t_{f,CP_k}]$, then we have

$$\|\delta X(t_{f,CP_k})\| \leq \alpha^{TNOC_{t_{s,CP_k}, t_{f,CP_k}}} \|\delta X(t_{s,CP_k})\| \tag{24}$$

and

$$\|\delta X(t_j^+)\| \leq \prod_{i \in \Gamma_1} e^{\tilde{\mu}(f(X,i), SCS)\Delta_{2,i}} \cdot (\max_{i,l}\{\max\{1, \|h(X, i, l)\|_{S_{i,l}}\}\})^n \cdot \|\delta X(t_{s,CP_k})\| \tag{25}$$

where $t_{s,CP_k} \leq t_j^+ \leq t_{f,CP_k}$, $TNOC_{t_{s,CP_k}, t_{f,CP_k}}$ is the total number of cycles in the closed path from $m(t_{s,CP_k})$ to $m(t_{f,CP_k})$ and

$$\alpha = \max_{1 \leq j \leq \theta_0} e^{\tilde{\alpha}(j)}. \tag{26}$$

□

Proof: We only consider the case that the closed path is composed of two cycles. The other cases can be shown in a similar way. Firstly, we shall show that (25) holds.

From Definition 1, we have

$$\|\delta X(t_{f,i}^k)\| \leq e^{\tilde{\mu}(f(X,i), SCS)\Delta_{1,i}} \|\delta X(t_{s,i}^k)\|; \qquad i \in \Gamma_2 \tag{27}$$

$$\|\delta X(t_{f,i}^k)\| \leq e^{\tilde{\mu}(f(X,i), SCS)\Delta_{2,i}} \|\delta X(t_{s,i}^k)\|; \qquad i \in \Gamma_1 \tag{28}$$

$$\|\delta X(t_j^+)\| \leq \|h(X, m(t_j^-), m(t_j^+))\|_{S_{m(t_j^-), m(t_j^+)}} \cdot \|\delta X(t_j^-)\|; \qquad \forall t_j. \tag{29}$$

From (27)–(29), we know that

$$\|\delta X(t_j^+)\| \leq \prod_{i \in ST(t_{s,CP_k}, t_j^+)} e^{\tilde{\mu}(f(X,i), SCS)\Delta_{3,i}} \cdot \|h(X, i, l)\|_{S_{i,l}} \|\delta X(t_{s,CP_k})\|$$

where l is the model next to i and

$$ST(t_{s,CP_k}, t_j^+) = \{i | i \text{ is in the path from } m(t_{s,CP_k}) \text{ to } m(t_j^+)\}.$$

To complete the proof of (25), we consider the following two cases.

a) There is no CVDS appearing twice in the path from $m(t_{s,CP_k})$ to $m(t_l^+)$ except for the one that is the first and the last. Then there exist at most n CVDS in the path. It follows that (25) holds.

b) There is one CVDS appearing twice in the path. Suppose that the cycle is cycle $LC(j_0)$. Using (19), we have

$$\begin{aligned} \|\delta X(t_j^+)\| &\leq e^{\tilde{\alpha}(j_0)} \prod_{i \in ST(t_{s,CP_k}, t_j^+) - ST(t_{s,LC(j_0)}, t_{f,LC(j_0)})} \\ &\quad \cdot e^{\tilde{\mu}(f(X,i), SCS)\Delta_{3,i}} \|h(X, i, l)\|_{S_{i,l}} \|\delta X(t_{s,CP_k})\|. \end{aligned}$$

Note that the total number of the CVDS in the set $(ST(t_{s,CP_k}, t_l^+) - ST(t_{s,LC(j_0)}, t_{f,LC(j_0)}))$ is less than or equal to n because there is no cycle in the set $(ST(t_{s,CP_k}, t_l^+) - ST(t_{s,LC(j_0)}, t_{f,LC(j_0)}))$. It follows that (25) holds.

We now show that (24) holds.

Note that

$$\begin{aligned} \|\delta X(t_{f,CP_k})\| &\leq \prod_{i \in ST(t_{s,CP_k}, t_{f,CP_k})} e^{\tilde{\mu}(f(X,i), SCS)\Delta_{3,i}} \\ &\quad \cdot \|h(X, i, l)\|_{S_{i,l}} \|\delta X(t_{s,CP_k})\| \\ &= \exp\left(\sum_{i \in ST(t_{s,CP_k}, t_{f,CP_k})} [\tilde{\mu}(f(X, i), SCS)\Delta_{3,i} + \ln(\|h(X, i, l)\|_{S_{i,l}})] \right) \|\delta X(t_{s,CP_k})\|. \end{aligned} \tag{30}$$

Suppose that the closed path is composed of cycles $LC(j_0)$ and $LC(j_1)$. Then rearrange the right side of (30) such that each cycle is a unit. Using (19), we have

$$\begin{aligned} \|\delta X(t_{f,CP_k})\| &\leq e^{\tilde{\alpha}(j_0)} e^{\tilde{\alpha}(j_1)} \|\delta X(t_{s,CP_k})\| \\ &\leq \alpha^2 \|\delta X(t_{s,CP_k})\|. \end{aligned}$$

It follows that (24) holds. □

We now consider the stability of a switched nonlinear system of the form (1) and (3) without perturbations.

Theorem 1: A switched nonlinear system of the form (1) and (3) without perturbations (i.e., $\Delta f \equiv 0$, $\Delta h \equiv 0$) is locally asymptotically stable with respect to a given trajectory if Assumption 1 holds.

Proof: Suppose that the radius of the largest ball in SCS is r. Let (31) hold true, as shown at the bottom of the page. We now show that the switched nonlinear system is asymptotically stable if $\|\delta X(t_0)\| \leq r_0$. We divide the proof into three steps.

Step 1: We prove that for any t_j^-, if $X(t) \in SCS$ holds for all $t_0 \le t \le t_j^-$, then

$$\|\delta X(t)\| \le \prod_{i\in\Gamma_1} e^{2\tilde{\mu}(f(X,i),\, SCS)\Delta_{2,\,i}} \cdot \left(\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,\,l}}\}\}\right)^{2n-2} \|\delta X(t_0)\| \quad (32)$$

holds for any $t_0 \le t \le t_j^+$.

Since there are infinite switchings in switched systems and the CVDS is finite, there exists at least two CVDS which appear infinite times. Suppose that i_0 is the first of the two such CVDS. Using Proposition 1, we know that

$$\|\delta X(t_{s,\,i_0}^{k+1})\| \le \alpha^{TNOC_{t_{s,\,i_0}^k,\, t_{s,\,i_0}^{k+1}}} \|\delta X(t_{s,\,i_0}^k)\|; \qquad k \ge 1 \quad (33)$$

where $TNOC_{t_{s,\,i_0}^k,\, t_{s,\,i_0}^{k+1}}$ is the total number of the cycles in the path from $m(t_{s,\,i_0}^k)$ to $m(t_{s,\,i_0}^{k+1})$.

We now show that

$$\|\delta X(t_{s,\,i_0}^1)\| \le \alpha^{TNOC_{t_0,\, t_{s,\,i_0}^1}} \prod_{i\in\Gamma_1} e^{\tilde{\mu}(f(X,i),\, SCS)\Delta_{2,\,i}} \cdot \left(\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,\,l}}\}\}\right)^{n-2} \|\delta X(t_0)\|. \quad (34)$$

a) If there is no state appearing twice in the path from $m(t_0)$ to $m(t_{s,\,i_0}^1)$, from Lemma 3, we know that there exist at most $(n-2)$ different CVDS in the path from $m(t_0)$ to $m(t_{s,\,i_0}^1)$. By inequalities (27) and (28), we know that (34) holds.
b) If there exist some CVDS appearing twice in the path from $m(t_0)$ to $m(t_{s,\,i_0}^1)$. Similar to the proof of Proposition 1, we know that (34) holds.

From (33), (34), and Proposition 1, we know that

$$\|\delta X(t)\| \le \alpha^{TNOC_{t_0,\,t}} \prod_{i\in\Gamma_1} e^{2\tilde{\mu}(f(X,i),\, SCS)\Delta_{2,\,i}} \cdot \left(\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,\,l}}\}\}\right)^{2n-2} \|\delta X(t_0)\|. \quad (35)$$

It follows that (32) holds.

Step 2: We show that $X(t) \in SCS$ for any t if $\|\delta X\| \le r_0$ by induction.

A) Consider the case that $t_0 \le t \le t_1^+$. Suppose that there exists a t' such that $X(t') \notin SCS$, i.e.,

$$\|\delta X(t')\| > r.$$

Note that $X(t_0) \in SCS$. Then, there exists a t'' such that

$$\|\delta X(t'')\| = r; \quad \|\delta X(t)\| \le r; \quad t \in [t_0,\, t''] \quad (36)$$

and

$$\frac{\partial f^T}{\partial X}(X(t),\, m(t_0)) + \frac{\partial f}{\partial X}(X(t),\, m(t_0)) \le 2\tilde{\mu}(f(X,\, m(t_0)),\, SCS)I \quad (37)$$

holds for all $t_0 \le t \le t''$.

From (37), we know that

$$\|\delta X(t'')\| \le e^{\tilde{\mu}(f(X,\,m(t_0)),\, SCS)(t''-t_0)} \|\delta X(t_0)\| < r.$$

Clearly, this contradicts with (36). Thus, $X(t) \in SCS$ holds for $t_0 \le t \le t_1^-$. It follows that:

$$\|\delta X(t_1^-)\| \le e^{\tilde{\mu}(f(X,\,m(t_0)),\, SCS)(t_1^- t_0)} \|\delta X(t_0)\| < \frac{r}{\left(\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,l}}\}\}\right)^{2n-1}}$$

and

$$\|\delta X(t_1^+)\| \le \|h(X,\, m(t_1^-),\, m(t_1^+)\|_{S_{m(t_1^-),\, m(t_1^+)}} \|\delta X(t_1^-)\| < r.$$

In other words, $X(t_1^+) \in SCS$.

B) Suppose that $X(t) \in SCS$ holds for all $t_0 \le t \le t_N^+$.

We consider the case that $t_N^+ \le t \le t_{N+1}^+$. From Assumption 1 and the first step, we know that as shown in the equations at the bottom of the page and

$$\|\delta X(t_N^+)\| \le \|h(X,\, m(t_N^-),\, m(t_N^+))\|_{S_{m(t_N^-),\, m(t_n^+)}} \|\delta X(t_l^-)\| \le \frac{r}{e^{\tilde{\mu}(f(X,\, m(t_N^+)),\, SCS)\Delta_{2,\, m(t_N^+)}}}.$$

Thus, $X(t_N^+) \in SCS$. Similar to A), we know that $X(t) \in SCS$ holds for $t_N^+ \le t < t_{N+1}^+$.

By induction, we know that $X(t) \in SCS$ for any t.

Step 3: We shall show that the theorem holds.

Note that $TNOC_{t_0,\,t} \to \infty$ as $t \to \infty$, thus, from (35), we have

$$\lim_{t\to\infty} \|\delta X(t)\| = 0.$$

That is, the result holds. □

$$r_0 = \frac{r}{\prod_{i\in\Gamma_1} e^{2\tilde{\mu}(f(x,i),\, SCS)\Delta_{2,i}} (\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,\,l}}\}\})^{2n-1} e^{\max_{i\in\Gamma_1} \tilde{\mu}(f(X,\,i),\, SCS)\Delta_{2,\,i}}}. \quad (31)$$

$$\|\delta X(t_N^-)\| \le \prod_{i\in\Gamma_1} e^{2\tilde{\mu}(f(X,i),\, SCS)\Delta_{2,\,i}} \left(\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,\,l}}\}\}\right)^{2n-2} \|\delta X(t_0)\| \le \frac{r}{\max_{i,\,l}\{\max\{1,\, \|h(X,\,i,\,l)\|_{S_{i,\,l}}\}\} e^{\tilde{\mu}(f(X,\, m(t_N^+)),\, SCS)\Delta_{2,\, m(t_N^+)}}}$$

Consider the case that $h(X, i, l) = X$, that is, there is no impulsive switchings in the switched nonlinear systems. In this case, Assumption 1) becomes

Assumption 1′: There exists a compact set $SCS \subseteq \overline{CS}$ such that SCS contains the given trajectory and

$$\tilde{\alpha}(j) = \sum_{i \in LC(j)} \tilde{\mu}(f(X, i), SCS)\Delta_{3, i} < 0; \qquad 1 \le j \le \theta_0. \quad (38)$$

From Theorem 1, we can obtain the following corollary.

Corollary 1: A nonimpulsive switched system of the form (1) without perturbations [i.e., $\Delta f(X, i) \equiv 0 (1 \le i \le n)$] is locally asymptotically stable with respect to a given trajectory if Assumption $1'$ holds. □

We now consider the robust stability of the switched nonlinear systems (1) and (3). Suppose that the system satisfies the following assumptions.

Assumption 2: Δf satisfies the bound

$$\left| \frac{\partial \Delta f(X(t), m(t))}{\partial X} \right|_m \le qB(m(t)); \qquad X(t) \in SCS; \ \forall t$$

where q is a real-positive number and $B(m(t))$ is a known nonnegative matrix.

Assumption 3: Δh satisfies that

$$\left| \frac{\partial \Delta h(X(t_i^-), m(t_j^-), m(t_j^+))}{\partial X} \right|_m \le qC(m(t_j^-), m(t_j^+)); \qquad X(t_j^-) \in S_{m(t_j^-), m(t_j^+)}$$

where $C(m(t_j^-), m(t_j^+))$ is a known nonnegative matrix.

Remark 3: These two assumptions are reasonable because $\partial \Delta f(X(t), m(t))/\partial X$ and $\partial \Delta h(X(t), m(t_j^-), m(t_j^+))/\partial X$ are continuous functions of $X(t)$ and SCS and $S_{m(t_j^-), m(t_j^+)}$ are compact sets. Moreover, they imply that the perturbations are bounded by some known nonnegative matrices. □

Theorem 2: Suppose that a switched nonlinear system given by (1) and (3) without perturbations [i.e., $\Delta f(X, i) \equiv 0$ and $\Delta h(x, i, l) \equiv 0$ $(1 \le i, l \le n)$] satisfies the condition of Theorem 1. Then the switched system with perturbations is still locally asymptotically stable if $q < \min\{\min_{1 \le j \le \theta_0}\{p_0(j)\}, \min_{i \in \Gamma_2}\{p_2(i)\}\}$, where $p_0(j) (1 \le j \le \theta_0)$ are, respectively, the solutions of the following equations $(1 \le j \le \theta_0)$:

$$\sum_{i \in LC(j)} \left[\ln \left(1 + \frac{\|C(i)\|}{\left\| \dfrac{\partial h(X, i, l)}{\partial X} \right\|_{S_{i, l}}} p_0(j) \right) + p_0(j) \sum_{i \in LC(j)} \Delta_{3, i}\mu(B(i)) \right] + \sum_{i \in LC(j)} \left[\ln \left\| \frac{\partial h(X, i, l)}{\partial X} \right\|_{S_{i, l}} + \Delta_{3, i}\tilde{\mu}(f(X, i), SCS) \right] = 0 \qquad (39)$$

and

$$p_2(i) = -\frac{\tilde{\mu}(f(X, i), SCS)}{\mu(B(i))}.$$

Proof: We shall first show that $\tilde{\mu}(f(X, i) + \Delta f(X, i), SCS) < 0$ holds for all $i \in \Gamma_2$ when $q < \min_{i \in \Gamma_2}\{p_2(i)\}$. From inequalities (12) and (15), we know that

$$\begin{aligned} &\tilde{\mu}(f(X, i) + \Delta f(X, i), SCS) \\ &\quad \le \tilde{\mu}(f(X, i), SCS) + \tilde{\mu}(\Delta f(X, i), SCS) \\ &\quad \le \tilde{\mu}(f(X, i), SCS) + \tilde{\mu}(|\Delta f(X, i)|_m, SCS) \\ &\quad \le \tilde{\mu}(f(X, i), SCS) + q\mu(B(i)). \end{aligned}$$

It follows that $\tilde{\mu}(f(X, i) + \Delta f(X, i), SCS) < 0$ holds when $q < \min_{i \in \Gamma_2}\{p_2(i)\}$.

Then, we need to show that

$$\sum_{i \in LC(j)} [\ln \|h(X, i, l) + \Delta h(X, i, l)\|_{S_{i, l}} + \Delta_{3, i}\tilde{\mu}(f(X, i) + \Delta f(X, i), SCS)] < 0$$

holds for all $1 \le j \le \theta_0$.

From inequalities (13), (15), (16), and (14), we have

$$\begin{aligned} &\sum_{i \in LC(j)} [\ln \|h(X, i, l) + \Delta h(X, i, l)\|_{S_{i, l}} - \ln \|h(X, i, l)\|_{S_{i, l}} + \tilde{\mu}(\Delta_{3, i}\Delta f(X, i), SCS)] \\ &\le \sum_{i \in LC(j)} [\ln(\|h(X, i, l)\|_{S_{i, l}} + \|\Delta h(X, i, l)\|_{S_{i, l}}) - \ln \|h(X, i, l)\|_{S_{i, l}} + \Delta_{3, i}\tilde{\mu}(|\Delta f(X, i)|_m, SCS)] \\ &\le \sum_{i \in LC(j)} [\ln(\|h(X, i, l)\|_{S_{i, l}} + \||\Delta h(X, i, l)|_m\|_{S_{i, l}}) - \ln \|h(X, i, l)\|_{S_{i, l}} + \Delta_{3, i}\tilde{\mu}(|\Delta f(X, i)|_m, SCS)] \\ &\le \sum_{i \in LC(j)} [\ln(\|h(X, i, l)\|_{S_{i, l}} + q\|C(i, l)\|) - \ln \|h(X, i, l)\|_{S_{i, l}} + q\Delta_{3, i}\mu(B(i)))] \\ &= \sum_{i \in LC(j)} \left[\ln \left(1 + q\frac{\|C(i, l)\|}{\|h(X, i, l)\|_{S_{i, l}}} \right) + q\Delta_{3, i}\mu(B(i))) \right]. \end{aligned}$$

It follows that:

$$\begin{aligned} &\sum_{i \in LC(j)} [\ln \|h(X, i, l) + \Delta h(X, i, l)\|_{S_{i, l}} + \tilde{\mu}(\Delta_{3, i}(f(X, i) + \Delta f(X, i), SCS)] \\ &\le \sum_{i \in LC(j)} [\ln \|h(X, i, l) + \Delta h(X, i, l)\|_{S_{i, l}} - \ln \|h(X, i, l)\|_{S_{i, l}} + \Delta_{2, i}\tilde{\mu}(\Delta f(X, i), SCS)] \\ &\quad + \sum_{i \in LC(j)} [\ln \|h(X, i, l)\|_{S_{i, l}} + \Delta_{3, i}\tilde{\mu}(f(X, i), SCS)] \\ &\le \sum_{i \in LC(j)} \left[\ln \left(1 + q\frac{\|C(i, l)\|}{\|h(X, i, l)\|_{S_{i, l}}} \right) + q\Delta_{3, i}\mu(B(i))) \right] \\ &\quad + \sum_{i \in LC(j)} [\ln \|h(X, i, l)\|_{S_{i, l}} + \Delta_{3, i}\tilde{\mu}(f(X, i), SCS)] \\ &= \psi(q). \end{aligned}$$

Note that

$$\frac{d\psi(q)}{dq} = \sum_{i \in LC(j)} \left[\frac{\dfrac{\|C(i, l)\|}{\|h(X, i, l)\|_{S_{i, l}}}}{1 + \dfrac{\|C(i, l)\|}{\|h(X, i, l)\|_{S_{i, l}}} q} + \Delta_{3, i}\mu(B(i)) \right].$$

Obviously, $d\psi(q)/dq > 0$. Therefore, when $0 < q < \min_{1\le j\le \theta_0}\{p_0(j)\}$, we have

$$\sum_{i\in LC(j)} [\ln \|h(X, i, l) + \Delta h(X, i, l)\|_{S_{i,l}} + \Delta_{3,i}\tilde{\mu}(f(X, i) + \Delta f(X, i), SCS)] < 0.$$

From Theorem 1, we know that the perturbed switched system is locally asymptotically stable. □

Remark 4: The Conditions of Theorem 2 imply that $\delta X^T \delta X$ is still nonincreasing along each type of cycle in the presence of uncertainties. □

If there is no impulsive switchings in the switched systems, then we have the following.

Corollary 2: Suppose that the nonimpulsive switched system of the form (1) without perturbations [i.e., $\Delta f(X, i) \equiv 0$ $(1 \le i \le n)$] satisfies the condition of Corollary 1. Then the perturbed nonimpulsive switched system is locally asymptotically stable in the sense of Lyapunov if $q < \min_{1\le j\le\theta_0}\{p_3(j)\}$, where

$$p_3(i) = \frac{-\sum_{i\in LC(j)} \Delta_{3,i}\tilde{\mu}(f(X, i), SCS)}{\sum_{i\in LC(j)} \Delta_{3,i}\mu(B(i))}; \qquad 1 \le j \le \theta_0. \quad (40)$$

□

Remark 5: If Assumptions 1), 2), and 3) hold globally, then the corresponding global stability and robustness results can be established. □

V. A Numerical Example

In this section, we use an example to illustrate the results obtained. Consider the following nonimpulsive switched nonlinear system composed of two CVDS

$$f(X, 1) = \begin{bmatrix} -2X_1 + 3X_1^2 \\ -3X_2 + X_2^2 \end{bmatrix}; \qquad \Delta f(X, 1) = \begin{bmatrix} qX_1^2 \\ qX_2^2 \end{bmatrix}$$

$$f(X, 2) = \begin{bmatrix} X_1 + X_1^2 \\ X_2/2 + 2X_2^3 \end{bmatrix}; \qquad \Delta f(X, 2) = \begin{bmatrix} qX_1 \\ qX_2^2 \end{bmatrix}.$$

The dwell time of CVDS 1 and CVDS 2 are 6 and 2.5, respectively.

Note that

$$\text{CVDS 1:} \quad \frac{d\delta X(t)}{dt} = \begin{bmatrix} -2 + 6X_1 & 0 \\ 0 & -3 + 2X_2 \end{bmatrix} \delta X(t) + q \begin{bmatrix} 2X_1 & 0 \\ 0 & 2X_2 \end{bmatrix} \delta X(t)$$

$$\text{CVDS 2:} \quad \frac{d\delta X(t)}{dt} = \begin{bmatrix} 1 + 2X_1 & 0 \\ 0 & 1/2 + 6X_2^2 \end{bmatrix} \delta X(t) + q \begin{bmatrix} 1 & 0 \\ 0 & 2X_2 \end{bmatrix} \delta X(t).$$

It follows that:

$$\widehat{CS}(1) = \{X | X_1 < 1/3; X_2 < 3/2\}; \qquad \widehat{CS}(2) = \emptyset.$$

Thus, $\overline{CS} = \widehat{CS}(1)$. Let $\xi_0 = 1/6$. Obviously, the set $\{X | \|X\| < 1/6\}$ is a subset of $\overline{CS}$. It can also be checked that $\hat{\alpha}(0) = -2 \times 6 + 2.5 \times 1 < 0$. Similar to Remark 2, Assumption $1'$ can be checked as follows.

Note that $\hat{\alpha}(X) < 0$ when $\|X\| < \xi_1 \triangleq 1/6$. Consider the following compact set:

$$SCS = \{X | \|X\| \le 1/12\}.$$

From Definition 1, we know that

$$\tilde{\mu}(f(\cdot, 1), SCS) = -1.5; \tilde{\mu}(f(\cdot, 2), SCS) = 1/6.$$

It can be shown that

$$6\tilde{\mu}(f(\cdot, 1), SCS) + 2.5\tilde{\mu}(f(\cdot, 2), SCS) < 0.$$

That is, Assumption $1'$ holds. From Corollary 1, we know that the switched nonlinear system without perturbations is locally asymptotically with respect to $X_e = 0$.

It is also possible to find some other type of compact sets to satisfy Assumption $1'$. Actually, consider the following compact set:

$$SCS = \{X | -1 \le X_1 \le 1/6; -1/2 \le X_2 \le 1/2\}. \quad (41)$$

From Definition 1, we know that

$$\tilde{\mu}(f(\cdot, 1), SCS) = -1; \qquad \tilde{\mu}(f(\cdot, 2), SCS) = 2.$$

It can be shown that

$$6\tilde{\mu}(f(\cdot, 1), SCS) + 2.5\tilde{\mu}(f(\cdot, 2), SCS) = -1 < 0.$$

That is, Assumption $1'$ holds.

We shall now consider the robust stability with SCS given in (41). From Assumption 2, we obtain

$$B(1) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad B(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From Corollary 2, we know that when $q < 1/14.5$, the switched nonlinear system is locally asymptotically stable with respect to $X_e = 0$.

VI. Conclusion

We have proposed a new concept of generalized matrix measure for nonlinear systems to study the stability of switched nonlinear systems directly. Based on this concept, we have derived some robust stability conditions for switched nonlinear systems by using the cycle analysis method and contraction analysis method, rather than through finding some implicit motion integrals in Lyapunov theory.

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