can be taken to be zero and hence $\sigma^2(x) = c_0^2\,(2\mu - \sigma_0^2)x^2$, with $c_0^2 > 1$, $P = 1$, and $z = 0$.

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Input-to-State Stabilization of Switched Nonlinear Systems

Wenxiang Xie, Changyun Wen, and Zhengguo Li

***Abstract*—This note derives some sufficient conditions to ensure that the whole switched nonlinear system is input-to-state stabilizable (ISS) when each mode is ISS. Both cases that the switchings of system modes coincide exactly and do not coincide with those of the corresponding controllers are considered. For the latter, a model-based identification scheme is used to identify the system modes. The proposed scheme can overcome the finite escape time that may happen in this case.**

***Index Terms*—Input-to-state stabilization, switched nonlinear systems.**

II. INTRODUCTION

Input-to-state stability is an important property of nonlinear systems besides asymptotical stability. So far, the study of such a property was mostly limited to a single nonlinear system (see [1], [2]–[5], and the references therein), and the property has been employed from robust control to nonlinear small-gain theorems [6]–[8]. But it is also of interest to consider such a property for switched nonlinear systems.

Switched nonlinear systems, a class of hybrid dynamical systems, can arise in many practical processes. An electric arc furnace (EAF) control system is such an example [9]. After raw material is melted down and oxidized in an EAF, it is sent to a ladle furnace or a continuous caster machine for some other processing. The above processes keep repeating. Raw material being melted down and steel being oxidized are some continuous variable dynamical systems (CVDSs), each of which can be represented by a differential equation and the whole process in the EAF can be modeled by a switched nonlinear system.

As we know, a switched system might not be stable even if each mode is stable. It is necessary to impose other proper additional conditions to ensure so. Liberzon, in [10], mentioned that the input-to-state stabilizable (ISS) property is preserved under switching if the intervals between switching instants are large enough under the assumption that each mode is ISS with a very special form. However, the general case has not been solved yet. The purpose of this note is to derive some sufficient conditions to input-to-state stabilize the whole switched system when each mode is input-to-state stabilizable in a general form. To achieve this objective, we design a switched controller which consists of two parts: local controllers for all the modes and their switching law. We consider both the case where switchings of the controllers coincide exactly with those of system modes and the case where the switchings of the controllers do not coincide exactly with those of the system. For simplicity, we call the former as a synchronous case and the latter as an asynchronous case, respectively.

The synchronous case is simpler and will be considered first. In practice, however, the switchings of the controllers may not coincide exactly with those of system modes, because we may not know the initial mode and also the subsequent modes of the system in advance. Thus, we do not know which controller should be initially used, and which controller and when it should be switched into action. For this case, as pointed out in [11], it is difficult to design a switched controller for a switched nonlinear system because of possibility of finite escape time. That is, if a wrong controller is used over a specified amount of time, the solution to the system might escape to infinity before a correct controller is switched into action. In this note, we discuss a model-based identification scheme which is used to identify the initial mode and the subsequent modes of the system and then determine the corresponding controllers to be switched into action, i.e., the switching law of the switched controller. The proposed scheme can avoid the problem of fininte escape time. It is shown that the switched nonlinear systems can be input-to-state stabilized by switched controllers for the above two cases.

The rest of this note is organized as follows. In Section II we introduce some preliminaries. The synchronous case and the asynchronous case are considered respectively in Section III and in Section IV. A numerical example is used to illustrate our results in Section V. Concluding Remarks are given in Section VI.

III. PRELIMINARIES

First, we introduce the following notations:

R	field of real numbers;
R_+	field of nonnegative-real numbers;
R^r	r-dimensional real vector space;
$\lvert\cdot\rvert$	Euclidean norm;
$\lVert u(t_0,\, t)\rVert$	$ess\sup\{\lvert u(s)\rvert,\, s \in [t_0,\, t]\}$;
$r(*)$	largest integer less than or equal to $*$;
$a \boxplus b$	$\max\{a,\, b\}$ for any $a,\, b \in R$.

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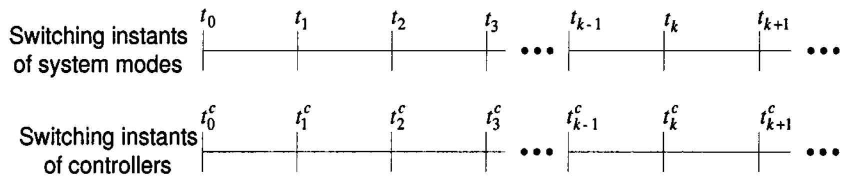


Fig. 1. An illustrative diagram for switching instants in the synchronous case.

In this note, we consider the input-to-state stabilization of switched nonlinear systems modeled by

$$\dot{X}(t) = f(X(t), v(t), m(t)) \tag{1}$$

where

$X(t) \in R^r$ and $v(t) \in R^p$	continuous state and the control input;
$m(t) \in \overline{M} = \{1, \ldots, n\}$	index for the discrete states;
$\{f: R^r \times R^p \times \overline{M} \to R^r\}$	family of sufficiently regular functions.

Each $i \in \overline{M}$ stands for a location where the system dynamics is governed by the corresponding vector field $f(X(t), v(t), i)$, called a mode, with $f(0, 0, i) = 0$.

A continuous function $\gamma: R_+ \to R_+$ is a $\mathcal{K}$ function if it is strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$ function and also $\gamma(r) \to \infty$ as $r \to \infty$. A function $\beta: R_+ \times R_+ \to R_+$ is a $\mathcal{KL}$ function if for each fixed s the function $\beta(r, s)$ is a $\mathcal{K}$ function with respect to r, and for each fixed r the function $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Definition 1 [2]: System (1) is said to be ISS if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that for any $X(t_0)$ and for any locally essentially bounded input $v(\cdot)$ on $[0, \infty)$ the solution satisfies

$$|X(t)| \le \beta(|X(t_0)|, t - t_0) + \gamma(\|v(t_0, t)\|) \tag{2}$$

for all t_0 and t such that $t \ge t_0 \ge 0$.

Remark 1: In inequality (2), let $t = t_0$ and $v(t) = 0$, then we have $|X(t_0)| \le \beta(|X(t_0)|, 0)$. That is, the following property holds for any $\mathcal{KL}$ function β satisfying (2) and any $s \in R_+$

$$\beta(s, 0) \ge s. \tag{3}$$

□

Definition 2: System (1) is said to be ISS if there exists an input $v(t) = K(X(t), u(t), m(t))$ with $u(t)$ being the reference input such that $\dot{X}(t) = \overline{f}(X(t), u(t), m(t)) = f(X(t), K(X(t), u(t), m(t)), m(t))$ is ISS.

Many research results on input-to-state stabilization of single nonlinear systems are available (e.g., see [3], [2] and the references therein), so it is natural to assume the following.

Assumption 1: For each mode $i (i \in \overline{M})$ of system (1), there exists an input $v(t) = K(X(t), u(t), i)$ such that for any locally essentially bounded input $u(.)$, we have

$$|X(t)| \le \beta_i(|X(t_0)|, t - t_0) + \gamma_i(\|u(t_0, t)\|); \qquad t \ge t_0 \ge 0 \tag{4}$$

where β_i is a $\mathcal{KL}$ function, γ_i is a $\mathcal{K}$ function.

Remark 2: Condition (4) implies that each mode is ISS. Note that the input-to-state stabilization of switched systems was also considered in [10]. However, β_i is in a special form of $cX(0)e^{-\lambda t}$ in [10]. When β_i is in a general form as in (4), the design and analysis will be much more difficult. □

The objective of this note is to derive proper conditions to input-to-state stabilize system (1). To this end, we need the following two supporting results.

Lemma 1: Suppose that $\phi_i (i = 1, 2, \ldots, l)$ are $\mathcal{KL}$ functions, $\rho_i (i = 1, 2, \ldots, l)$ are $\mathcal{K}$ functions and $a_i (i = 1, 2, \ldots, k) \in R_+$, then $\sum_{i=1}^l \phi_i$ is a $\mathcal{KL}$ function, $\sum_{i=1}^l \rho_i$ is a $\mathcal{K}$ function, and

$$\rho_i(a_1 \boxplus a_2 \boxplus \cdots \boxplus a_k) = \rho_i(a_1) \boxplus \rho_i(a_2) \boxplus \cdots \boxplus \rho_i(a_k) \tag{5}$$

$$\phi_i(a_1 \boxplus a_2 \boxplus \cdots \boxplus a_k, s) = \phi_i(a_1, s) \boxplus \phi_i(a_2, s) \boxplus \cdots \boxplus \phi_i(a_k, s). \tag{6}$$

Proof: The results can be obtained from the definitions of $\mathcal{K}$ and $\mathcal{KL}$ functions. ■

Lemma 2: Suppose that $\phi_i (i = 1, 2, \ldots, l)$ are $\mathcal{KL}$ functions. For any positive constants a and $b < 1$, let

$$\beta(s, t) = \sum_{i, j=1 (i \ne j)}^{l} a\phi_i\left(a\phi_j\left(ab^{h(t, t_0)}s, 0\right), 0\right) \tag{7}$$

where $h(t, t_0)$ is an increasing function of t and $h(t, t_0) \to \infty$ as $t \to \infty$. Then, $\beta(s, t)$ is also a $\mathcal{KL}$ function.

Proof: For any fixed t, it is clear that $\beta(s, t)$ is a $\mathcal{K}$ function.

For any fixed s, note that $b < 1$ and $h(t, t_0)$ is an increasing function of t. Thus, from (7) we know that $\beta(s, t)$ is a decreasing function of t. Note also that $h(t, t_0) \to \infty$ as $t \to \infty$. It follows that $\beta(s, t) \to 0$ as $t \to \infty$. Therefore, $\beta(s, t)$ is a $\mathcal{KL}$ function. ■

In the following section, we will consider both the synchronous case and the asynchronous case.

IV. Synchronous Controller Switchings

In the synchronous case, the switching instance is illustrated as in Fig. 1. In the figure, t_k denotes the kth switching instant of modes of system (1), while t_k^c denotes the kth switching instant of the controllers. In this case, $t_k = t_k^c$.

We recall that switched systems might become unstable even if all modes are stable. In general, a proper switching law of system modes is required to guarantee the stability of the considered switched system. Similarly, we also need such requirements on the switching law of system modes to input-to-state stabilize system (1) even if Assumption 1 holds.

Let $t_{s,i}^k$ and $t_{e,i}^k$ denote respectively the kth starting time and the kth ending time of mode i $(i \in \overline{M})$. We require the switching law of system modes to satisfy

$$\inf_k \{t_{e,i}^k - t_{s,i}^k\} = \Delta T_i > 0$$

and similar to [10], we suppose that $\Delta T_i (i = 1, \ldots, n)$ are large enough such that for any $s \in R_+$, we have

$$\beta_i(2\beta_j(2s, \Delta T_j), \Delta T_i) \le \lambda s < s \qquad \forall i, j \in \overline{M} \tag{8}$$

where $0 < \lambda < 1$ and β_i $(i \in \overline{M})$ satisfies condition (4).

A possible method to verify (8) is to calculate the following limit:

$$\lim_{\Delta T_i, \Delta T_j \to \infty} \frac{\beta_i(2\beta_j(2s, \Delta T_j), \Delta T_i)}{s}; \qquad \forall i, j \in \overline{M}.$$

If all the results are less than one, then (8) holds for some large values of ΔT_i and ΔT_j based on the definition of limit.

Remark 3: Under Assumption 1 and the above switching law of system modes, it can be easily shown that (1) is ISS if the number of switchings is finite. Thus, we only consider the case where the number of switchings is infinite. □

Theorem 1: Consider system (1) satisfying Assumption 1. Suppose that the switchings of the controllers coincide exactly with those of system modes satisfying (8). Then, the system is ISS and

$$|X(t)| \le \overline{\beta}(|X(t_0)|, t-t_0) + \overline{\gamma}(\|u(t_0, t)\|) \qquad t \ge t_0 \ge 0 \tag{9}$$

where

$$\begin{aligned} \overline{\beta}(|X(t_0)|, t-t_0) &= \sum_{i, j=1 (i \ne j)}^{n} \beta_i(2\beta_j(2\lambda^l |X(t_0)|, 0), 0) \\ \overline{\gamma}(\|u(t_0, t)\|) &= \tilde{\gamma_0} + \gamma_0 \\ \tilde{\gamma_0} &= \sum_{i, j=1 (i \ne j)}^{n} \beta_i(2\beta_j(2\gamma_0, 0), 0), \\ \gamma_0 &= \sum_{i=1}^{n} \gamma_i(\|u(t_0, t)\|), \\ l &= r\left(\frac{k}{2}\right) \end{aligned} \tag{10}$$

and k denotes the total number of switchings of system modes from t_0 to t.

Proof: For ease of presentation, we let $m_k = m(t_k^c)$, $\gamma_{m_k} = \gamma_{m_k}(\|u(t_k^c, t_{k+1}^c)\|)$. In the following proof, we shall use the fact that

$$\Gamma(r_1 + r_2, s) \le \Gamma(2r_1, s) \boxplus \Gamma(2r_2, s) \tag{11}$$

for any $\mathcal{KL}$ function Γ and any nonnegative constants r_1, r_2. From Lemma 1, Assumption 1, condition (8), and the property expressed in (3), we have

$$\begin{aligned} |X(t_1^c)| &\le \beta_{m_0}(|X(t_0)|, t_1^c - t_0^c) + \gamma_{m_0} \\ &\le \beta_{m_0}(|X(t_0)|, \Delta T_{m_0}) + \gamma_{m_0} \\ |X(t_2^c)| &\le \beta_{m_1}(|X(t_1^c)|, t_2^c - t_1^c) + \gamma_{m_1} \\ &\le \beta_{m_1}(\beta_{m_0}(|X(t_0)|, \Delta T_{m_0}) + \gamma_{m_0}, \Delta T_{m_1}) + \gamma_{m_1} \\ &\le \beta_{m_1}(2\beta_{m_0}(|X(t_0)|, \Delta T_{m_0}), \Delta T_{m_1}) \\ &\quad \boxplus \beta_{m_1}(2\gamma_{m_0}, \Delta T_{m_1}) + \gamma_{m_1} \\ &\le \lambda |X(t_0)| \boxplus \beta_{m_1}(2\gamma_{m_0}, \Delta T_{m_1}) + \gamma_{m_1}; \\ |X(t_3^c)| &\le \beta_{m_2}(|X(t_2^c)|, t_3^c - t_2^c) + \gamma_{m_2} \\ &\le \beta_{m_2}(\lambda |X(t_0)| \\ &\quad \boxplus \beta_{m_1}(2\gamma_{m_0}, \Delta T_{m_1}) + \gamma_{m_1}, \Delta T_{m_2}) + \gamma_{m_2} \\ &\le \beta_{m_2}(2\lambda |X(t_0)|, \Delta T_{m_2}) \\ &\quad \boxplus \beta_{m_2}(2\beta_{m_1}(2\gamma_{m_0}, \Delta T_{m_1}), \Delta T_{m_2}) \\ &\quad \boxplus \beta_{m_2}(2\gamma_{m_1}, \Delta T_{m_2}) + \gamma_{m_2} \\ &\le \beta_{m_2}(2\lambda |X(t_0)|, \Delta T_{m_2}) \boxplus \lambda\gamma_{m_0} \\ &\quad \boxplus \beta_{m_2}(2\gamma_{m_1}, \Delta T_{m_2}) + \gamma_{m_2}. \end{aligned}$$

Using the inductive method, we can get the following inequalities when the number of switchings of system modes k is even and $k > 3$

$$\begin{aligned} |X(t_k^c)| &\le \beta_{m_{k-1}}(|X(t_{k-1}^c)|, t_k^c - t_{k-1}^c) + \gamma_{m_{k-1}} \\ &\le \lambda^l |X(t_0)| \boxplus \beta_{m_{k-1}}(2\lambda^{l-1}\gamma_{m_0}, \Delta T_{m_{k-1}}) \\ &\quad \boxplus \lambda^{l-1}\gamma_{m_1} \boxplus \beta_{m_{k-1}}(2\lambda^{l-2}\gamma_{m_2}, \Delta T_{m_{k-1}}) \\ &\quad \boxplus \lambda^{l-2}\gamma_{m_3} \boxplus \cdots \\ &\quad \boxplus \beta_{m_{k-1}}(2\lambda\gamma_{m_{k-4}}, \Delta T_{m_{k-1}}) \boxplus \lambda\gamma_{m_{k-3}} \\ &\quad \boxplus \beta_{m_{k-1}}(2\gamma_{m_{k-2}}, \Delta T_{m_{k-1}}) + \gamma_{m_{k-1}} \end{aligned}$$

where $l = r(k/2)$.

Thus, for any $t \in [t_k^c, t_{k+1}^c]$, we have

$$\begin{aligned} |X(t)| &\le \beta_{m_k}(|X(t_k^c)|, t - t_k^c) + \gamma_{m_k}(\|u(t_k^c, t)\|) \\ &\le \beta_{m_k}(2\lambda^l |X(t_0)|, t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\beta_{m_{k-1}}(2\lambda^{l-1}\gamma_{m_0}, \Delta T_{m_{k-1}}), t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\lambda^{l-1}\gamma_{m_1}, t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\beta_{m_{k-1}}(2\lambda^{l-2}\gamma_{m_2}, \Delta T_{m_{k-1}}), t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\lambda^{l-2}\gamma_{m_3}, t - t_k^c) \boxplus \cdots \\ &\quad \boxplus \beta_{m_k}(2\beta_{m_{k-1}}(2\lambda\gamma_{m_{k-4}}, \Delta T_{m_{k-1}}), t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\lambda\gamma_{m_{k-3}}, t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\beta_{k-1}(2\gamma_{m_{k-2}}, \Delta T_{m_{k-1}}), t - t_k^c) \\ &\quad \boxplus \beta_{m_k}(2\gamma_{m_{k-1}}, t - t_k^c) + \gamma_{m_k}(\|u(t_k^c, t)\|). \end{aligned}$$

From (10), we replace γ_{m_i}, $i = 0, \ldots, k$, with γ_0, and notice that for any $a, b \in R_+$, $a \boxplus b = a$ if $a \ge b$ and $a \boxplus b \le a + b$. Thus, we can further obtain

$$\begin{aligned} |X(t)| &\le \beta_{m_k}(2\lambda^l |X(t_0)|, 0) \\ &\qquad \boxplus \beta_{m_k}(2\beta_{m_{k-1}}(2\gamma_0, 0), 0) + \gamma_0 \\ &\le \beta_{m_k}(2\lambda^l |X(t_0)|, 0) \\ &\quad + \beta_{m_k}(2\beta_{m_{k-1}}(2\gamma_0, 0), 0) + \gamma_0 \\ &\le \overline{\beta}(|X(t_0)|, t - t_0) + \tilde{\gamma} + \gamma_0 \\ &= \overline{\beta}(|X(t_0)|, t - t_0) + \overline{\gamma}(\|u(t_0, t)\|). \end{aligned} \tag{12}$$

In a similar way, it can be shown that (12) holds in the case where k is odd and $k > 3$.

Note that $l = r(k/2)$ is an increasing function of t and $l \to \infty$ as $t \to \infty$. From Lemma 2, $\overline{\beta}(|X(t_0)|, t - t_0)$ is a $\mathcal{KL}$ function. Therefore, system (1) is ISS in this case. ■

Remark 4: It should be emphasized that we use inequality (11) in the above proof. Note that the whole derivation cannot be proceeded if we employ the following fact, which is usually used:

$$\Gamma(r_1 + r_2, s) \le \Gamma(2r_1, s) + \Gamma(2r_2, s).$$

□

V. Asynchronous Controller Switchings

In practice, the switchings of the controllers may not generally coincide exactly with those of system modes since we do not know the ini-

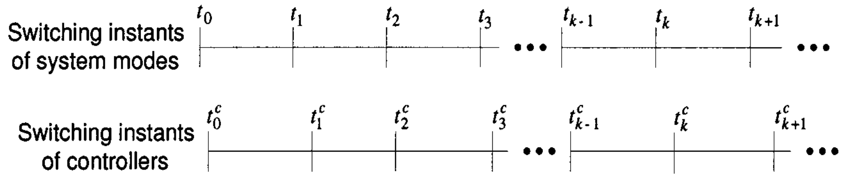


Fig. 2. An illustrative diagram for switching instants in the asynchronous case.

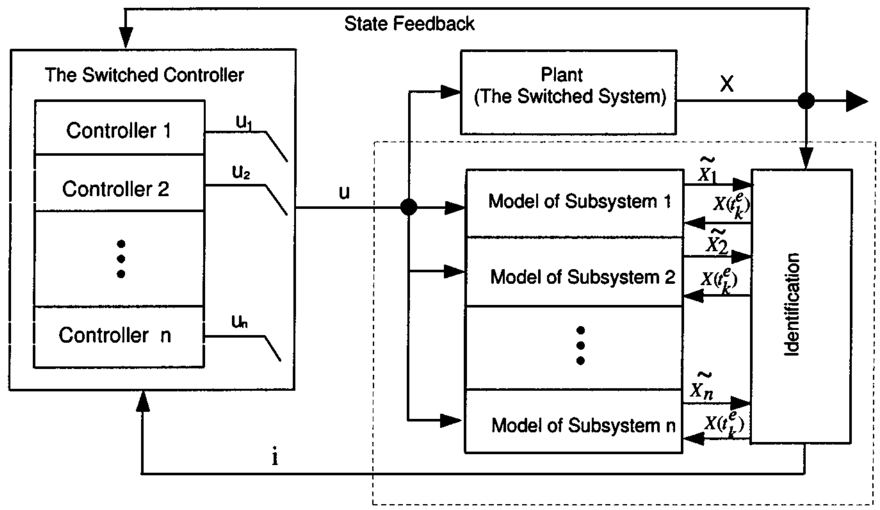


Fig. 3. Overall system diagram.

tial mode and the subsequent modes of the system in advance. Thus, it is necessary to identify them and then switch from the present controller to the corresponding controllers. As expected, the design and analysis are much more involved than the synchronous case, since we need to identify the initial mode and the subsequent modes of the system. To achieve this, we impose some delay on the switchings of subcontrollers, that is, as shown in Fig. 2, $t_k^c > t_k$ $(k = 0, 1, 2, \ldots)$. Intervals $[t_k, t_k^c]$ $(k = 0, 1, 2, \ldots)$ are used to do the identification. Once the active mode is known, the corresponding sub-controller is switched to.

Similar to [12], we use a model-based scheme, as illustrated in Fig. 3, to do the identification. We assume that there is only one mode model whose state is equal to the state of system (1) for any control input and any interval if system (1) and all the models of the modes have the same initial state and there is no measurement noise or disturbance. Without loss of generality, we also suppose that $X(t) \neq 0$ for all $t \geq t_0$.

Since t_k is unknown, we also need to estimate it. Thus, the whole task is composed of two steps: estimate the kth switching instant of system modes and identify the kth active system mode. These are given in details as follows.

Step 1) Estimate the kth switching instant of system modes.

In Fig. 3, t_k^e and $\tilde{X}_i$ denote the estimate of the kth switching instant of system modes and the state of the model of mode $i \in \overline{M}$ respectively. Then, $t_0^e = t_0$ and $t_k^e (k \geq 1)$ are determined by

$$t_k^e = \sup_t \Big\{ t > t_{k-1}^c | X(t) = \tilde{X}_{m(t_{k-1}^c)}(t) \quad \text{and} \quad |X(t)| \leq \beta_{m(t_{k-1}^c)}(|X(t_{k-1}^c)|, t - t_{k-1}^c) + \gamma_{m(t_{k-1}^c)}(\|u(t_{k-1}^c, t)\|) \Big\}. \tag{13}$$

Step 2) Identify the kth system modes.

To identify the kth active system mode, $X(t_k^e)$ is fed back to each mode model to ensure that system (1) and all mode models have the same state at time point t_k^e. To avoid that the states of (1) escape into infinity before a proper controller is switched into action, t_k^c is defined as

$$t_k^c = \sup_t \{ t_k^e \leq t \leq t_k^e + \Delta t \, | \, |X(t)| \leq a |X(t_k^e)| \} \tag{14}$$

where $t_k^e = t_0$, $a > 1$ and $\Delta t = \min_{1 \leq i \leq n} ((\Delta T_i - \Delta T_i')/2)$. Here, we can determine a and $\Delta T_i'$ $(i = 1, \ldots, n)$ by letting

$$H(p, t_i, t_j) = p\beta_i(2p\beta_j(2s, t_j), t_i); \quad p, t_i, t_j \in R_+, \; i, j \in \overline{M}.$$

Obviously, $H(p, t_i, t_j)$ is a continuous function of p, t_i and t_j. From (8), we have

$$H(1, \Delta T_i, \Delta T_j) \leq \lambda s < s.$$

It follows that there exist $a > 1$ and $\overline{\lambda}(0 < \overline{\lambda} < 1)$, $\Delta T_i' < \Delta T_i$ and $\Delta T_j' < \Delta T_j$ such that:

$$H(a, \Delta T_i', \Delta T_j') \leq \overline{\lambda} s < s. \tag{15}$$

Thus, the present active mode can be obtained from the state of system (1) and all the models of the modes within $[t_k^e, t_k^c]$. Based on the above discussion, we can also show that (1) under $v(t) = K(X(t), u(t), m(t))$ $(m(t) = 1, \ldots, n)$ is ISS in the asynchronous case, and

$$|X(t)| \leq \hat{\beta}(|X(t_0)|, t - t_0) + \hat{\gamma}(\|u(t_0, t)\|) \tag{16}$$

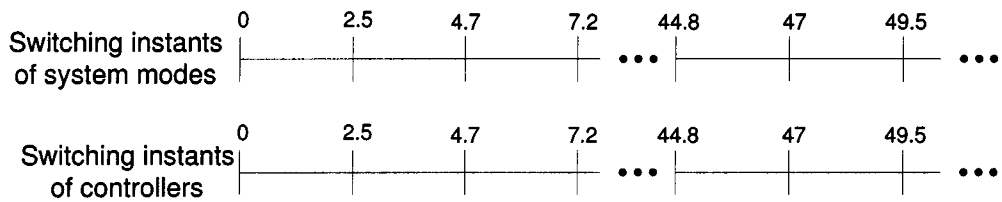


Fig. 4. The switching instants of system modes.

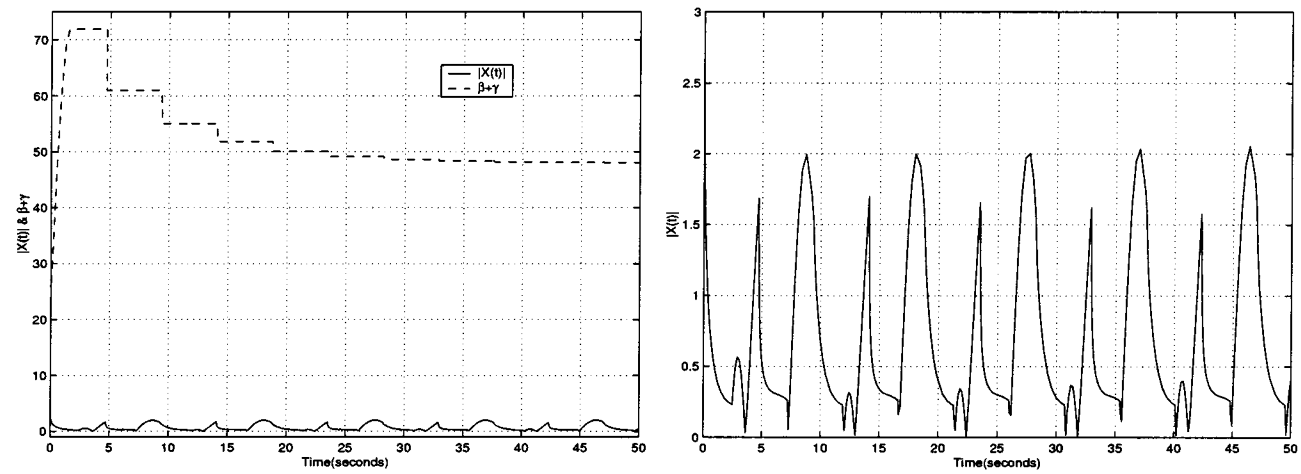


Fig. 5. Simulation results in the synchronous case.

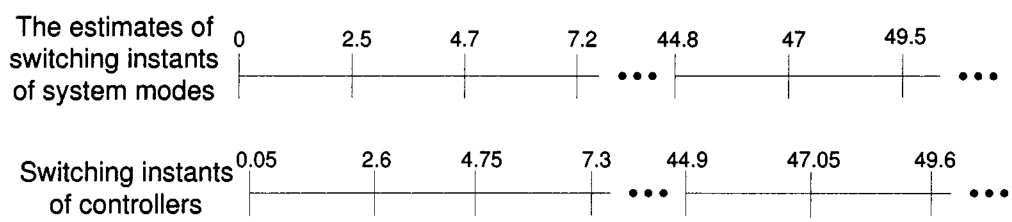


Fig. 6. The switching instants of system modes and controllers.

where

$$\hat{\beta}(|X(t_0)|, t-t_0) = \sum_{i,\,j=1(i\neq j)}^{n} a\beta_i\left(2a\beta_j(2\overline{\lambda}^l a|X(t_0)|, 0), 0\right)$$

$$\hat{\gamma}(\|u(t_0, t)\|) = \tilde{\gamma_0} + \gamma_0$$

$$\tilde{\gamma_0} = \sum_{i,\,j=1(i\neq j)}^{n} a\beta_i(2a\beta_j(2\gamma_0), 0), 0)$$

$$\gamma_0 = \sum_{i=1}^{n} a\gamma_i(\|u(t_0, t)\|),$$

$$l = r\left(\frac{k}{2}\right)$$

and k denotes the total number of switchings of system modes from t_0 to t.

VI. A Numerical Example

Consider a switched nonlinear system consisting of the following two one-dimensional (1-D) modes.

$$Mode1: \quad \dot{X}(t) = X^3(t) - \frac{X^3(t)v^2(t)}{2}$$

$$Mode2: \quad \dot{X}(t) = X(t) + v(t)$$

where $X(t) \in R$ and $v(t) \in R$.

It can be shown that Assumption 1 holds with $v(t) = \sqrt{2+u^2(t)}$, $\beta_1(r, s) = r/\sqrt{2r^2 s + 1}$ and $\gamma_1(s) = s$ for mode 1, and $v(t) = -2X(t) + u(t)$, $\beta_2(r, s) = re^{-s}$ and $\gamma_2(s) = s$ for mode 2. Moreover, it can be checked that

$$\lim_{\Delta T_1, \Delta T_2 \to \infty} \frac{\beta_1(2\beta_2(2s, \Delta T_2), \Delta T_1)}{s} = \lim_{\Delta T_1, \Delta T_2 \to \infty} \frac{\beta_2(2\beta_1(2s, \Delta T_1), \Delta T_2)}{s} = 0.$$

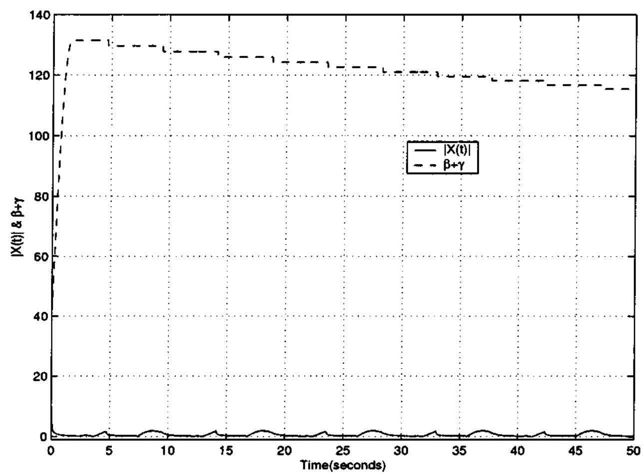
Thus, (8) holds for some large ΔT_1 and ΔT_2. For example, if $\Delta T_1 = \Delta T_2 = 2s$, then $\lambda = 4e^{-2}$ and (8) holds. Also note that the results in [10] cannot be used to study this example.

Now, we consider the synchronous case, i.e., the case where the switchings of controllers coincide exactly with those of system modes. Using Theorem 1, we know that

$$|X(t)| \leq 8\lambda^l |X(t_0)| + 16\|u(t_0, t)\|; \qquad t \geq t_0 \geq 0$$

where l is defined in Theorem 1. For simulation studies, take the switching instance of system modes as the values shown in Fig. 4, mode 1 as the initial mode, and let $u(t) = 3\sin(t)$ and $X(0) = 3$. The simulation result, illustrated in Fig. 5 with $\beta + \gamma$ standing for $8\lambda^l |X(t_0)| + 16\|u(t_0, t)\|$, indicates that the considered system is input-to-state stabilized.

We next discuss the asynchronous case, i.e., the case where the switchings of controllers do not coincide with those of



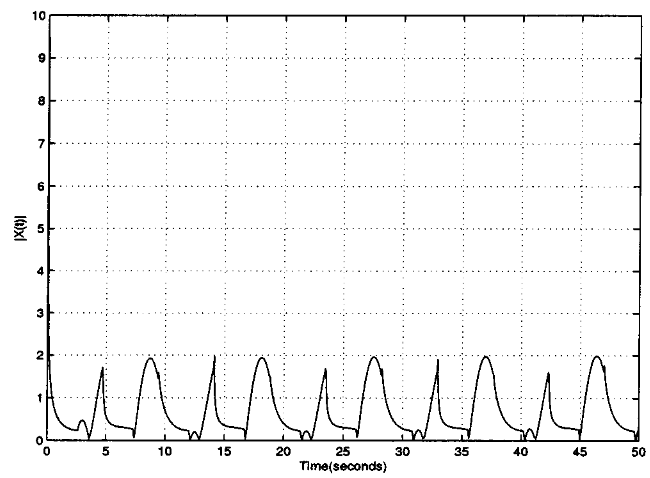


Fig. 7. Simulation results in the asynchronous case.

system modes. From (16), we can determine the $\mathcal{KL}$ function $\hat{\beta}(|X(t_0)|, t - t_0) = 8\overline{\lambda}^l a^3 |X(t_0)|$ and the $\mathcal{K}$ function $\hat{\gamma}(\|u(t_0, t)\|) = (16a^3 + 2a)\|u(t_0, t)\|$. That is

$$|X(t)| \leq 8\overline{\lambda}^l a^3 |X(t_0)| + (16a^3 + 2a)\|u(t_0, t)\|; \quad t \geq t_0 \geq 0.$$

From (15), it can be computed that $a = 1.2$, $\overline{\lambda} = 5.76e^{-1.8}$ and $\Delta T_1' = \Delta T_2' = 1.8s$. As a simulation example, the estimates of the switching instance of system modes and the switching instants of controllers are given in Fig. 6. The result shown in Fig. 7 also demonstrates that the system with the same initial conditions as above is ISS in the asynchronous case. In the figure, $\beta + \gamma$ stands for $8\overline{\lambda}^l a^3 |X(t_0)| + (16a^3 + 2a)\|u(t_0, t)\|$.

VII. Conclusion

This note has investigated the issue on the input-to-state stabilization of switched nonlinear systems. The ideal case that the switchings of the system modes coincide exactly with those of the corresponding controllers is first considered. Some sufficient conditions are then derived to input-to-state stabilize the whole switched nonlinear system. In general, the switchings of the controllers cannot coincide exactly with those of the corresponding modes, since we do not know the initial mode and the subsequent modes of the system beforehand. If a wrong controller is used over a specified amount of time, the solution to the system might escape to infinity before a correct controller is switched into action. In this case, a model-based identification scheme is discussed for the identification of the system modes such that the corresponding controllers can be determined. The whole switched nonlinear system can also be ISS in this case.

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